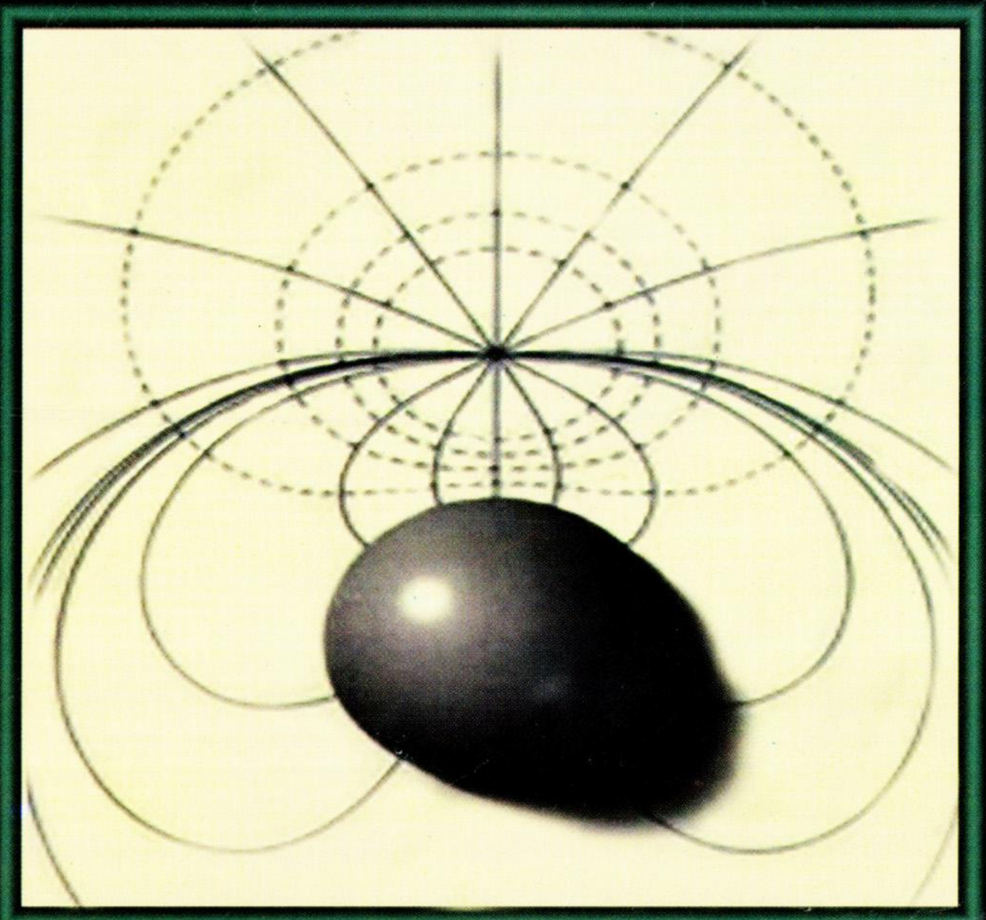
ELECTRICITY & MAGNETISM



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Himalaya Publishing House

ELECTRICITY AND MAGNETISM

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ISBN : 978-81-83189-33-0
Revised Edition : 2009

Published by : **Mrs. Meena Pandey**
for HIMALAYA PUBLISHING HOUSE,
"Ramdoot", Dr. Bhalerao Marg, Girgaon, Mumbai-400 004.
Phones : 23860170/23863863 Fax : 022-23877178
Email : himpub@vsnl.com
Website : www.himpub.com

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Phone : 0484-2378012, 2378016

Printed at : A to Z Printers, Daryaganj, New Delhi-110002

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1 VECTORS

INTRODUCTION

Electromagnetics (EM) or electromagnetism is a branch of physics or engineering in which electric and magnetic phenomena are studied. It may be regarded as the study of interactions between electric charges at rest and in motion. The field is electric only known as *electrostatic* when the charges are at rest and the branch is known as electrostatics. The field may be electric as well as magnetic when the charges are in motion. The EM principles find applications in almost all sciences. EM devices have become the need of everyday. The design of EM devices requires through knowledge of the laws and principles of EM.

1.1 SCALARS AND VECTORS

The mathematics is the language of physics. The physical formulae are the applications of laws of physics. Vector analysis provides an efficient way for representing physical laws. Particularly in electromagnetism, vector analysis brings to the forefront the physical ideas involved in equations. Since most students taking this course may not have required exposure to vector analysis, special attention is given to it, as a first chapter. The vector algebra and calculus have been given mainly in Cartesian coordinates.

The physical quantities which are completely defined by magnitude alone, are called *scalar quantities or scalars.* They are represented by Latin or Greek letters. They obey the ordinary laws of algebra. Examples are mass, charge, potential, time, temperature, etc. In addition, there are other physical quantities which are completely defined by a magnitude and a direction in space and follow the parallelogram law of addition. They are called vector quantities or vectors. In printing, they are represented by bold face letters or Greek letters with arrow over them. In ordinary writing a vector quantity is represented by putting an arrow over the Greek letter or by putting a wavy line underneath the letter. They are represented geometrically by an arrow pointing in the direction of the vector and of length equal to its magnitude. Examples are velocity, displacement, force, momentum, electric and magnetic fields, etc. The vectors may be classified as : (1) polar vector and (2) axial vector. For the quantity, in which more linear action in a particular direction is involved, the vector is called *polar vector* (*e.g.*, displacement, velocity and force). For the quantity, in which rotary action of some kind takes place about an axis, the vector is called *axial vector* (*e.g.*, angular velocity, angular acceleration). The polar vectors change their direction by reflection or inversion. The axial vectors do not change by reflection or inversion. Such vectors are called *Pseudovectors* (pseudo means false). The scalars have no direction associated with them. They should not change their sign on reflection or inversion. Volume of a parallelopiped may be expressed as a scalar triple product, which is a scalar quantity and changes to –V on inversion **A** → –**A**, **B** → –**B** and **C** → –**C**. Such a scalar quantity which changes their sign on reflection or inversion is called *Pseudoscalar*.

A vector which has no magnitude is called a *null* or *zero vector*. If it is added with any vector **A**, the vector **A** will always remain unchanged, *i.e.*, $\mathbf{0} + \mathbf{A} = \mathbf{A}$. The vectors other than null vectors are called *proper vectors*.

A vector whose module is unity is called *unit vector*. It is represented by letter with a cap over it. On multiplying a scalar with the unit vector we get a vector in the direction of the unit vector, *i.e.*, $\mathbf{A} = A\hat{A}$.

Two vectors **A** and **B** are said to be equal if and only if they have the same magnitude and the same direction, *i.e.*,

$$\mathbf{A} = \mathbf{B} \quad \text{or}\ A\hat{A} = B\hat{B},\ \text{if}\ A = B\ \text{and}\ \hat{A} = \hat{B}. \qquad \text{...(1)}$$

Vector Addition : The vector sum of two vectors can be obtained as follows : Draw the vectors **A** and **B**, as in Fig. 1.1, at any point, with the initial point of **B** (tail of **B**) at the end point of **A** (head of **A**). The vector sum **C** is the vector from the initial point of **A** upto the end point of **B**. We thus have **C** = **A** + **B**. **C** is the diagonal of the parallelogram with sides **A** and **B** and its magnitude and direction can be found by trigonometry. In the three dimensional cartesian coordinates system.

$$C_x = A_x + B_x,\ C_y = A_y + B_y \text{ and } C_z = A_z + B_z.$$

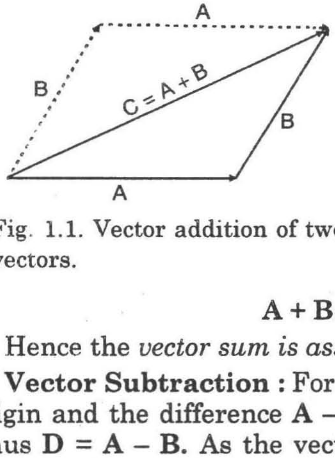


Fig. 1.1. Vector addition of two vectors.

The resultant **C** may also be obtained by adding vector **B** with vector **A** (*i.e.*, **B** + **A** = **C**). Thus the *vector addition is commutative*. This method may be extended for addition of three or more vectors with the polygon law of addition according to which a polygon is drawn by placing the tail end of second vector at the head end of first, then the tail end of third vector at the head end of second and so on. The resulting vector is drawn from the tail end of first to the head end of the last. The summation is independent of the order and grouping of terms, *e.g.*,

$$\mathbf{A} + \mathbf{B} + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) = (\mathbf{A} + \mathbf{B}) + \mathbf{C}. \qquad \text{...(2)}$$

Hence the *vector sum is associative.*

Vector Subtraction : For *vector subtraction,* vectors **A** and **B** are drawn from the common origin and the difference **A** – **B** is the vector from the head of **B** to head of **A** [Fig. 1.2 (*a*)]. Thus **D** = **A** – **B**. As the vector –**B** is the vector of magnitude *B* but possesses an opposite direction, hence the difference **A** – **B** may be obtained by adding to the vector **A** a vector –**B** [Fig. 1.2 (*b*)].

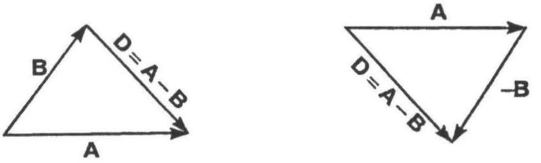


Fig. 1.2. Vector subtraction of two vectors.

If **A** and **B** are parallel vectors, then the magnitude of **C** is equal to the scalar sum of the magnitudes of **A** and **B** and the magnitude of **D** is equal to the scalar difference.

Vector Components : A vector **C** may be resolved into two vectors **A** and **B,** such that **C** is the diagonal of a parallelogram of the adjacent sides **A** and **B** [Fig. 1.3 (*a*)]. Generally **A** and **B** are chosen to be at right angles. Any vector in space may be resolved into three components parallel to the axes of Cartesian co-ordinates.

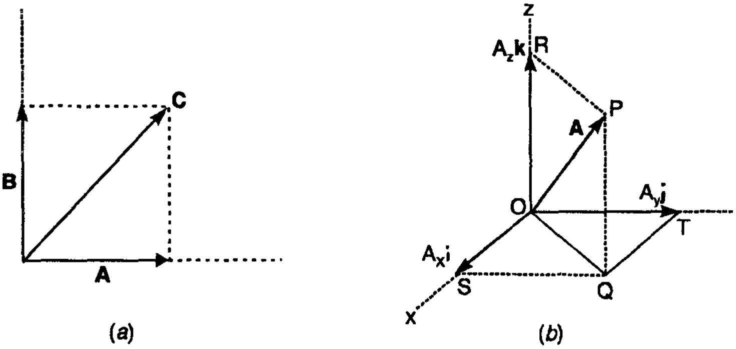


Fig. 1.3. Resolution of a vector.

If A_x, A_y and A_z are the magnitudes of components of the vector **A** parallel to the axes *x*, *y*, and *z* respectively, then

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k},$$

where **i**, **j** and **k** are vectors of unit length along the axes *x*, *y* and *z* respectively, Fig. 1.3 (*b*). since $OP^2 = OQ^2 + OR^2 = OS^2 + OT^2 + OR^2$. Therefore the magnitude of **A** may be written as $A = |\mathbf{A}| = (A_x^2 + A_y^2 + A_z^2)^{1/2}$. The sum of two vectors may thus be written as

$$\mathbf{A} + \mathbf{B} = (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k}.$$

Direction Cosines : The magnitude of the component vectors may be written as $A_x = |\mathbf{A}| \cos(\mathbf{A}, \mathbf{i})$, $A_y = |\mathbf{A}| \cos(\mathbf{A}, \mathbf{j})$ and $A_z = |\mathbf{A}| \cos(\mathbf{A}, \mathbf{k})$. Here $\cos(\mathbf{A}, \mathbf{i})$, $\cos(\mathbf{A}, \mathbf{j})$ and $\cos(\mathbf{A}, \mathbf{k})$ are the cosines of the angles which **A** makes with the *x*–, *y*– and *z*-axes respectively, and are therefore called the *direction cosines* of **A**. These are generally represented by the letters *l*, *m* and *n*, such as $l = \cos(\mathbf{A}, \mathbf{i})$, $m = \cos(\mathbf{A}, \mathbf{j})$ and $n = \cos(\mathbf{A}, \mathbf{k})$. Thus we have

$$A^2 = A_x^2 + A_y^2 + A_z^2 = |\mathbf{A}|^2 [\cos^2(\mathbf{A}, \mathbf{i}) + \cos^2(\mathbf{A}, \mathbf{j}) + \cos^2(\mathbf{A}, \mathbf{k})]$$

or $$l^2 + m^2 + n^2 = 1, \qquad \ldots(3)$$

i.e., the sum of the squares of the three direction cosines of a vector is unity.

$$\therefore \qquad \hat{A} = \frac{\mathbf{A}}{|\mathbf{A}|} = \frac{A_x}{|\mathbf{A}|}\mathbf{i} + \frac{A_y}{|\mathbf{A}|}\mathbf{j} + \frac{A_z}{|\mathbf{A}|}\mathbf{k}$$

$$= \cos(\mathbf{A}, \mathbf{i})\,\mathbf{i} + \cos(\mathbf{A}, \mathbf{j})\,\mathbf{j} + \cos(\mathbf{A}, \mathbf{k})\,\mathbf{k}. \qquad \ldots(4)$$

Position Vector : The distance from the origin *O* to a point *P* [Fig. 1.4(*a*)] is specified by the vector *OP*. It is represented by a vector **r** and is known as *position vector of point P* relative to origin *O*. If the coordinates of point *P* be (*x*, *y*, *z*), then we have

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

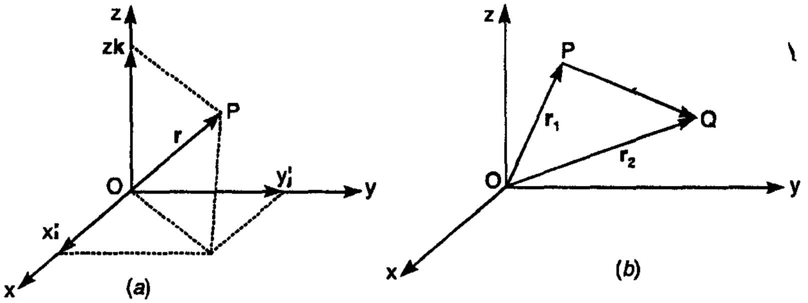


Fig. 1.4. Position vectors.

If there are two points P and Q [Fig. 1.4(*b*)] with coordinates (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, then their position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to origin O may be given as

$$\mathbf{r}_1 = x_1\,\mathbf{i} + y_1\,\mathbf{j} + z_1\,\mathbf{k} \text{ and } \mathbf{r}_2 = x_2\,\mathbf{i} + y_2\,\mathbf{j} + z_2\,\mathbf{k},$$

$$\therefore \quad \mathbf{PQ} = \mathbf{r}_2 - \mathbf{r}_1 = (x_2 - x_1)\,\mathbf{i} + (y_2 - y_1)\,\mathbf{j} + (z_2 - z_1)\,\mathbf{k}. \qquad \text{...(6)}$$

This PQ is called distance or separation vector.

Collinear Vectors : Two vectors **A** and **B** are collinear or parallel if their respective direction cosines are equal, *i.e.*,

$$\cos(\mathbf{A}, \mathbf{i}) = \cos(\mathbf{B}, \mathbf{i}); \cos(\mathbf{A}, \mathbf{j}) = \cos(\mathbf{B}, \mathbf{j}); \cos(\mathbf{A}, \mathbf{k}) = \cos(\mathbf{B}, \mathbf{k})$$

$$\therefore \quad \frac{A_x}{|\mathbf{A}|} = \frac{B_x}{|\mathbf{B}|}, \frac{A_y}{|\mathbf{A}|} = \frac{B_y}{|\mathbf{B}|} \text{ and } \frac{A_z}{|\mathbf{A}|} = \frac{B_z}{|\mathbf{B}|},$$

$$\text{or} \quad \frac{A_x}{B_x} = \frac{A_y}{B_y} = \frac{A_z}{B_z} = \frac{|\mathbf{A}|}{|\mathbf{B}|} \qquad \text{...(7)}$$

This is the necessary condition for the two vectors to be collinear or parallel.

Coplaner Vectors : The vectors are said to be coplaner if these are confined to the same plane. For vectors **A** and **B**, we have $\mathbf{R} = n_1\mathbf{A} + n_2\mathbf{B}$. The vector **R** is in the plane containing **A** and **B**, therefore called a vector coplaner with **A** and **B** for all values of n_1 and n_2.

Three or more vectors are called coplaner vectors, if they are parallel to the same plane. Any plane parallel to it is called plane of vectors. Three vectors **A, B** and **C** are the coplaner vectors, if they satisfy the condition

$$n_1\,\mathbf{A} + n_2\,\mathbf{B} + n_3\,\mathbf{C} = 0. \qquad \text{...(8)}$$

In three dimensional coordinates system, the above equation for coplanarity reduces to

$$n_1 A_x + n_2 B_x + n_3\,C_x = 0,\ n_1\,A_y + n_2\,B_y + n_3{}'C_y = 0$$

and $n_1\,A_z + n_2\,B_z + n_3\,C_z = 0$.

On eliminating n_1, n_2 and n_3, the above equations reduce to

$$A_x\,(B_y\,C_z - B_z\,C_y) + A_y\,(B_z\,C_x - B_x\,C_z) + A_z\,(B_x\,C_y - B_y\,C_x) = 0. \qquad \text{...(9)}$$

We will see that the L.H.S. represents the volume of a parallelopiped whose sides are **A, B** and **C**.

1.2 MULTIPLICATION OF VECTORS

When a vector **A** is multiplied by a scalar b, new vector **C** is formed whose direction is the same as that of the original vector **A** and whose magnitude is the product of the scalar b and the magnitude **A** of the original vector **A**. Thus

$$\mathbf{C} = b\mathbf{A} = b[A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}]. \qquad \text{...(10)}$$

$$\text{Similarly} \quad (a + b)\,\mathbf{A} = a\mathbf{A} + b\mathbf{A}. \qquad \text{...(11)}$$

Following two types of vector multiplication have been defined.

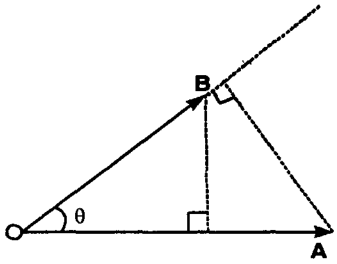


Fig. 1.5. Scalar product of two vectors.

(*i*) Scalar Product : The scalar product of two vectors **A** and **B** is a scalar quantity. It is indicated by a dot, placed between them and thus is also known as *dot-product.* Its magnitude is equal to the product of the magnitudes of two vectors and the cosine of the angle between them. [Fig. 1.5].

$$\mathbf{A{\cdot}B} = AB\cos\theta = BA\cos\theta = \mathbf{B{\cdot}A} \qquad \text{...(12)}$$

This relation shows that the scalar product is commutative. As $B\cos\theta$ = projection of **B** along **A**, hence the scalar product may be defined as *the scalar quantity numerically equal to the magnitude of one vector multiplied by the component of the other along the first.*

The scalar product of two perpendicular vectors is zero. Thus for the unit vectors **i**, **j**, and **k**, we have

$$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0 \text{ and } \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1. \quad ...(13)$$

$$\therefore \quad \mathbf{A}.\mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) . (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

$$= A_xB_x + A_yB_y + A_zB_z. \quad ...(14)$$

Relation (12) shows that

$$\mathbf{A}\cdot\mathbf{B} = AB\cos 0 = AB, \quad \text{if } \mathbf{A} \parallel \mathbf{B}$$

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\pi = -AB \quad \text{if } \theta = \pi$$

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\pi/2 = 0 \text{ if either } \theta = \pi/2$$

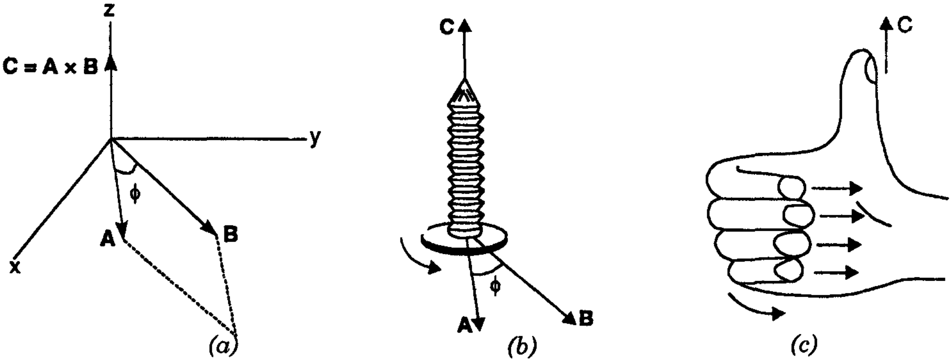
The length of a vector A is thus given as

$$|\mathbf{A}| = (\mathbf{A}\cdot\mathbf{A})^{1/2}. \quad ...(15)$$

***(ii)* Vector Product** : The *vector product of two vectors is a vector quantity.* It is indicated by placing a cross between them, hence is also called *cross product.* Its magnitude is equal to the product of magnitudes of two vectors and the sine of the angle between them. *Its direction is ⊥ to the plane containing these vectors and is governed by the right handed screw if turned from the first vector towards the second vector* [Fig. 1.6(*b*)]. If vectors **A** and **B** are in $x - y$ plane, the resultant $\mathbf{C} = \mathbf{A} \times \mathbf{B}$ (read **A** cross **B**) is ⊥ to both **A** and **B** and is therefore in the z-direction.

$$\mathbf{C} = \mathbf{A} \times \mathbf{B} = AB\sin\phi\,\hat{n} = -\mathbf{B} \times \mathbf{A}. \quad ...(16)$$

where $\hat{n}$ indicates the direction of normal to both **A** and **B**.



(*a*) (*b*) (*c*)

Fig. 1.6. Vector product of two vectors and its direction.

Since $AB\sin\phi$ represents the area of the parallelogram having A and B as adjacent sides. Hence vector product of two vectors may be defined as a vector of magnitude equal to the area of the parallelogram with these vectors as its adjacent sides and in the direction ⊥ to this area [Fig. 1.6(*a*)].

Another way to obtain the direction of a vector product is the *right hand rule.* If we wrap fingers of the right hand around the axis, which is ⊥ to the plane of vectors **A** and **B**, so that the curled fingers follow the rotation of **A** into **B**, the extended right thumb will give the direction of the vector product **A** × **B** [Fig. 1.6(*c*)].

Since $\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A}$, hence the vector product is not *commutative.* Relation (16) shows that

$$\mathbf{A} \times \mathbf{B} = 0 \quad \text{if } \mathbf{A} \parallel \mathbf{B} \text{ or } \mathbf{A} = \mathbf{B}$$

$$\mathbf{A} \times \mathbf{B} = AB\hat{n} \quad \text{if } \mathbf{A} \perp \mathbf{B}.$$

For unit vectors, we have

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}, \quad \mathbf{j} \times \mathbf{i} = -\mathbf{k}, \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

and

$$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0. \quad ...(17)$$

The cross products of unit vectors are obtained using cyclic permutation. The right hand or right handed screw rule is satisfied in a right handed coordinate system.

In cartesian coordinates, the vector product may be expressed as

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$= \mathbf{i}(A_yB_z - A_zB_y) + \mathbf{j}(A_zB_x - A_xB_z) + \mathbf{k}(A_xB_y - A_yB_x) \qquad ...(18)$$

1.3 MULTIPLE PRODUCT

(*i*) $\mathbf{A}\cdot(\mathbf{B} + \mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$ ...(19)

(*ii*) $\mathbf{A} \times (\mathbf{B} + \mathbf{C}) = \mathbf{A} \times \mathbf{B} + \mathbf{A} \times \mathbf{C}$ ...(20)

(*iii*) $(\mathbf{A}\cdot\mathbf{B})\mathbf{C} = (AB \cos \theta)\ \mathbf{C} = ABC \cos \theta$ along vector **C**. ...(21)

(*iv*) Scalar Triple Product : Consider three vectors **A**, **B** and **C** which form the sides of a parallelopiped, as shown in Fig. 1.7. The vectors **B** and **C** form the base of the parallelopiped. Its area = $BC \sin \phi$. It will be the magnitude of the vector product **B** × **C**, whose direction will be perpendicular to the base. The dot product of **A** with (**B** × **C**) will be a scalar quantity, with a magnitude equal to the product of this area and the projection of **A** along **B** × **C**. Therefore

$$\begin{aligned} \mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) &= A \cos \theta .\ BC \sin \phi \\ &= \text{Vertical height } h \times \text{area of the base} \\ &= \text{Volume of the parallelopiped.} \end{aligned}$$

Any face of the parallelopiped may be taken as a base, but we will have the same volume of the parallelopiped. We thus conclude that

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) = \mathbf{C}\cdot(\mathbf{A} \times \mathbf{B}).$$

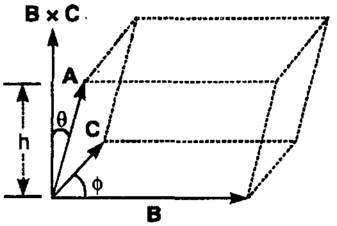


Fig. 1.7. Scalar triple product.

Here the vectors **A, B** and **C** must be arranged in the cyclic order, otherwise the volume will be of negative sign. In cartesian coordinates, we may write the above relation as

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = \mathbf{B}\cdot(\mathbf{C} \times \mathbf{A}) = \mathbf{C}\cdot(\mathbf{A} \times \mathbf{B}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} \qquad ...(22)$$

Here we see that this vector product is a scalar quantity and is therefore called *Scalar-triple product.* It is unchanged by an exchange of dot and cross or by a cyclic permutation of the three vectors and therefore may be written as [ABC]. The scalar triple product has the following properties.

(*a*) If **A**, **B** and **C** are coplaner or parallel to the same plane.

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = 0.$$

(*b*) If **A**, **B** and **C** are orthogonal to each other,

$$\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = ABC.$$

(*c*) If any two of the three are identical or parallel.

$$\mathbf{A}\cdot(\mathbf{A} \times \mathbf{C}) = \mathbf{A}\cdot(\mathbf{B} \times \mathbf{A}) = \mathbf{A}\cdot(\mathbf{B} \times \mathbf{B}) = 0.$$

(*v*) Vector Triple Product : The quantity **A** × (**B** × **C**) will be a vector quantity in a direction perpendicular to the plane containing **A** and (**B** × **C**), *i.e.,* in the plane of **B** and **C**. It is therefore called the *vector triple product.* Similarly **C** × (**A** × **B**) lies in the plane of **A** and **B**. It shows that **A** × (**B** × **C**) ≠ (**A** × **B**) × **C**. Therefore the *Vector multiplication is not associative.* To find the value, let us write these vectors in cartesian coordinates as :

Let $\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k},\ \mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j}$ and $\mathbf{C} = C_x\,\mathbf{i}$

$\therefore$ $\mathbf{B} \times \mathbf{C} = -B_y\,C_x\mathbf{k}$

and

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}) \times (-B_y\,C_x)\,\mathbf{k} \\ &= A_x\,B_y\,C_x\,\mathbf{j} - A_y\,B_y\,C_x\,\mathbf{i}.\end{aligned}$$

On adding and subtracting $A_x\,B_x\,C_x\,\mathbf{i}$ on R.H.S., we get,

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= A_x\,B_x\,C_x\,\mathbf{i} + A_x\,B_y\,C_x\,\mathbf{j} - A_x\,B_x\,C_x\,\mathbf{i} - A_y\,B_y\,C_x\,\mathbf{i} \\ &= A_x\,C_x\,(B_x\,\mathbf{i} + B_y\,\mathbf{j}) - (A_x\,B_x + A_y\,B_y)\,C_x\,\mathbf{i} \\ &= (\mathbf{A}\cdot\mathbf{C})\,\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C} \qquad \text{...(23)}\end{aligned}$$

One can prove very easily that

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \times (\mathbf{C} \times \mathbf{A}) + \mathbf{C} \times (\mathbf{A} \times \mathbf{B}) = 0 \qquad \text{...(24)}$$

(*vi*)

$$(\mathbf{A} \times \mathbf{B})\cdot(\mathbf{C} \times \mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})\,(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})\,(\mathbf{B}\cdot\mathbf{C}) \qquad \text{...(25)}$$

(*vii*)

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times (\mathbf{C} \times \mathbf{D}) &= [\mathbf{A}\ \mathbf{C}\ \mathbf{D}]\,\mathbf{B} - [\mathbf{B}\ \mathbf{C}\ \mathbf{D}]\,\mathbf{A} \\ &= [\mathbf{A}\ \mathbf{B}\ \mathbf{D}]\,\mathbf{C} - [\mathbf{A}\ \mathbf{B}\ \mathbf{C}]\,\mathbf{D} \qquad \text{...(26)}\end{aligned}$$

1.4 DIVISION OF A VECTOR

The division of a vector by a scalar can be defined as the multiplication by the reciprocal of a scalar. The resultant will be a parallel vector. The division of a vector by another vector is possible only if two vectors are parallel. We can not divide a scalar or a vector by any vector, as no unique solution is obtained in such case.

Reciprocal Vectors : It A, B and C are non-coplaner vectors. Their reciprocal vectors are A′, B′ and C′ respectively, such that

$$\mathbf{A}\cdot\mathbf{A}' = \mathbf{B}\cdot\mathbf{B}' = \mathbf{C}\cdot\mathbf{C}' = 1.$$

Since $\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})$ is a scalar quantity, therefore the reciprocal vectors may be defined as

$$\mathbf{A}' = \frac{\mathbf{B} \times \mathbf{C}}{\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})},\quad \mathbf{B}' = \frac{\mathbf{C} \times \mathbf{A}}{\mathbf{B}\cdot(\mathbf{C} \times \mathbf{A})} \text{ and } \mathbf{C}' = \frac{\mathbf{A} \times \mathbf{B}}{\mathbf{C}\cdot(\mathbf{A} \times \mathbf{B})}$$

or

$$\mathbf{A}' = \frac{\mathbf{B} \times \mathbf{C}}{\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})},\quad \mathbf{B}' = \frac{\mathbf{C} \times \mathbf{A}}{\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})} \text{ and } \mathbf{C}' = \frac{\mathbf{A} \times \mathbf{B}}{\mathbf{A}\cdot(\mathbf{B} \times \mathbf{C})} \qquad \text{...(27)}$$

These quantities may also be written as

$$\mathbf{A} = \frac{\mathbf{B}' \times \mathbf{C}'}{\mathbf{A}'\cdot(\mathbf{B}' \times \mathbf{C}')},\ \mathbf{B} = \frac{\mathbf{C}' \times \mathbf{A}'}{\mathbf{A}'\cdot(\mathbf{B}' \times \mathbf{C}')} \text{ and } \mathbf{C} = \frac{\mathbf{A}' \times \mathbf{B}'}{\mathbf{A}'\cdot(\mathbf{B}' \times \mathbf{C}')} \qquad \text{...(28)}$$

1.5 DIFFERENTIATION AND INTEGRATION OF VECTORS

Vector quantities are often expressed as functions of scalar variables. The vector may be differentiated and integrated with respect to these variables. Thus if $\mathbf{F}(u)$ is a vector function of the scalar variable u, the differential of $\mathbf{F}$ with respect to a scalar variable u is defined as

$$\frac{d\mathbf{F}}{du} = \underset{\Delta u \to 0}{\text{limit}}\,\frac{\Delta\mathbf{F}}{\Delta u} = \underset{\Delta u \to 0}{\text{limit}}\,\frac{\mathbf{F}(u + \Delta u) - \mathbf{F}(u)}{\Delta u}. \qquad \text{...(29)}$$

The derivative is a vector in the direction of $\Delta\mathbf{F}$ as Δu approaches zero. The derivative of the sum of two vectors is the sum of the derivatives. The derivative of a product is the same as that for the derivative of a product of scalars, with due regard for the change in sign if the order of the vectors is changed in the case of the vector product.

If $\mathbf{C} = \mathbf{A} + \mathbf{B}$, then

$$\frac{d\mathbf{C}}{dt} = \frac{d\mathbf{A}}{dt} + \frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{\mathbf{A}} + \dot{\mathbf{B}} \qquad \text{...(30)}$$

If $\mathbf{C} = b\mathbf{A}$, then

$$\frac{d\mathbf{C}}{dt} = \frac{db}{dt}\mathbf{A} + b\,\frac{d\mathbf{A}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{b}\,A + b\,\dot{\mathbf{A}}. \qquad \text{...(31)}$$

If $\mathbf{C} = \mathbf{A} \cdot \mathbf{B}$, then $$\frac{d\mathbf{C}}{dt} = \frac{d\,\mathbf{A}}{dt} \cdot \mathbf{B} + \mathbf{A} \cdot \frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{\mathbf{A}} \cdot \mathbf{B} + \mathbf{A} \cdot \dot{\mathbf{B}}.$$

If $\mathbf{C} = \mathbf{A} \times \mathbf{B}$, then $$\frac{d\mathbf{C}}{dt} = \frac{d\mathbf{A}}{dt} \times \mathbf{B} + \mathbf{A} \times \frac{d\mathbf{B}}{dt} \text{ or } \dot{\mathbf{C}} = \dot{\mathbf{A}} \times \mathbf{B} + \mathbf{A} \times \dot{\mathbf{B}}. \qquad ...(33)$$

We can differentiate the scalar triple product or vector triple product with the ordinary processes of differentiation, keeping the order of vectors as unchanged.

The process of integration is the limit approached in a summation of simple products. When a force **F** acts for a small distance $d\mathbf{l}$, the work done $dW = \mathbf{F}.d\mathbf{l}$. Total work done over a large distance is given by

$$W = \int \mathbf{F} \cdot d\mathbf{l} = \int F \cos\theta \, dl. \qquad ...(34)$$

This integral is known as line integral of **F** along the curve. Similarly we have surface and volume integrals.

1.6 SCALAR AND VECTOR FIELDS

A physical quantity which can be expressed as a continuous function of the position of the point in a region of space is known as a *point or position function*. The field is a function that specifies a particular quantity everywhere in a region. If the quantity is a scalar (or vector), the field is called scalar (or vector) field. Thus a scalar function φ which has values throughout a region constitutes a field known as *scalar field*. Examples of scalar fields are temperature distribution in a block or building, sound intensity in a hall, electric or gravitational potential in a region of electric or gravitational field and refractive index of a medium. Vector function **F** which has values throughout a region constitutes a field known as *vector field*. Examples are the *velocity field* or the *force field*.

1.7 THE GRADIENT OF A SCALAR FIELD

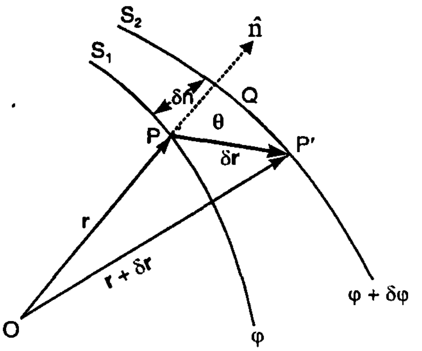


Fig. 1.8. Gradient of a scalar field.

Let φ be a scalar point function, which is continuous in the given region of space. In this region, consider two level surfaces (the level surface of a scalar point function is a surface for all points of which the function has the same value, *e.g.*, isothermal or equipotential surfaces) S_1 and S_2 very close together, Fig. 1.8. If these surfaces are characterised by scalar functions φ and $\varphi + \delta\varphi$ respectively. Reference to the origin *O*, we then have

$$\mathbf{OP} = \mathbf{r}, \ \mathbf{OP'} = \mathbf{r} + \delta\mathbf{r}, \text{ hence } \mathbf{PP'} = \delta\mathbf{r}.$$

The rate of increase of φ at point *P* in the direction **PP′** is $\delta\varphi/\delta r$ with the limit $\delta r \to 0$. The least distance between the surfaces is *PQ* in the direction of unit normal vector $\hat{n}$ at *P* to the level surface of φ in the direction of φ increasing. Thus the maximum rate of increase of φ is $\partial\varphi/\partial n$ and is along $\hat{n}$. If the normal at *P* makes an angle θ with **PP′**, then we have

$$\delta n = \hat{n} . \delta\mathbf{r} \text{ or } \delta n = \delta r \cos\theta.$$

$$\therefore \qquad \frac{\delta\varphi}{\delta r} = \frac{\delta\varphi}{\delta n} \cdot \frac{\delta n}{\delta r} = \frac{\delta\varphi}{\delta n} \cos\theta.$$

As $\cos\theta \le 1$, hence $\delta n \le \delta r$ and $\delta\varphi/\delta r \le \delta\varphi/\delta n$,

or $$\lim_{\delta r \to 0} \frac{\delta\varphi}{\delta r} < \frac{\partial\varphi}{\partial n}. \qquad ...(35)$$

Thus the vector $(\partial\varphi/\partial n)\,\hat{n}$ gives the greatest rate of increase of φ at the point P and is known as *gradient* or *grad of a scalar point function* φ. Hence

$$\text{grad } \varphi = (\partial\varphi/\partial n)\,\hat{n}\,. \qquad \text{...(36)}$$

It shows that the gradient of a scalar field is a *vector field. This vector has magnitude equal to the most rapid rate of change of the scalar field and is in the direction in which this rate is maximum.*

As
$$\frac{\partial\varphi}{\partial n} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial n} + \frac{\partial\varphi}{\partial y}\frac{\partial y}{\partial n} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial n}$$

∴
$$\text{grad } \varphi = \frac{\partial\varphi}{\partial n}\hat{n} = \frac{\partial\varphi}{\partial x}\frac{\partial x}{\partial n}\hat{n} + \frac{\partial\varphi}{\partial z}\frac{\partial y}{\partial n}\hat{n} + \frac{\partial\varphi}{\partial z}\frac{\partial z}{\partial n}\hat{n}$$

$$= \frac{\partial\varphi}{\partial x}\mathbf{i} + \frac{\partial\varphi}{\partial y}\mathbf{j} + \frac{\partial\varphi}{\partial z}\mathbf{k}, \qquad \text{...(37)}$$

where $(\partial x/\partial n)\,\hat{n}$, $(\partial y/\partial n)\,\hat{n}$ and $(\partial z/\partial n)\,\hat{n}$ are the unit vectors **i, j** and **k** along the axes x, y and z respectively.

∴
$$\text{grad } \varphi = \left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right]\varphi = \nabla\varphi, \qquad \text{...(38)}$$

where the differential operator ∇ is pronounced as **del** or **nabla.** It is defined as

$$\nabla \equiv \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}. \qquad \text{...(39)}$$

As a differential operator the ∇ vector by itself can be characterized neither by magnitude nor by direction. It only assumes a definite value when it is applied to some field quantity. If it is applied to a scalar φ, then we have $\nabla\varphi \equiv \text{grad } \varphi$. If the scalar function represents temperature, then $\nabla\varphi = \text{grad } \varphi$ is a temperature gradient. It is a vector quantity. Its direction being that in which temperature changes most rapidly.

Similarly the intensity of electric field at any point is given by the relation $\mathbf{E} = -\nabla\varphi$.

The gradient is an intrinsic property of a scalar field. It is independent of any particular system of coordinates, therefore is *invariant.*

Important Deductions

(1) Gradient of sum of two scalar functions is the vector sum of their gradients, *i.e.,*

$$\nabla(u + v) = \nabla u + \nabla v.$$

(2) $\nabla(uv) = u\,\nabla v + v\,\nabla u.$

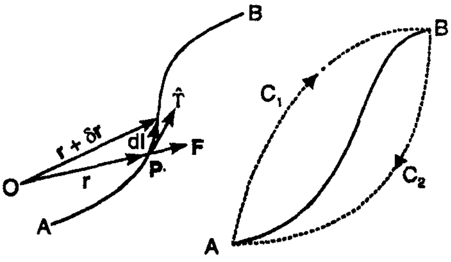
(3) $\nabla f(u) = f'(u)\,\nabla u.$ ...(40)

1.8 LINE INTEGRAL

Let **F** (x, y, z) be a vector function defined throughout some region of space with two positions A and B anywhere in this region (Fig. 1.9). The line integral of **F** alone the curve of some path that runs between A and B is defined as the integral of tangential component of **F** alone the curve. Thus

$$\text{line integral} = \int_A^B \mathbf{F}\cdot d\mathbf{l}. \qquad \text{...(41)}$$

This means : Divide the path into short segments, each segment is represented by a vector $d\mathbf{l}$ and take the scalar product of the path segment vector with the vector **F**. Now add these products for the whole path. If the segments are made shorter, this sum can be represented by the integral from one end to other.



(a) (b)

Fig. 1.9. Line integral of a vector.

If $\hat{T}$ is a unit vector tangent to the curve at any point P with a position vector **r**. Its value is given as

$$\hat{T} = d\mathbf{r}/dl.$$

$$\therefore \quad \text{Line integral} = \int \mathbf{F}\cdot d\mathbf{l} = \int \mathbf{F}\cdot\hat{T}dl = \int \mathbf{F}\cdot\frac{d\mathbf{r}}{dl}dl = \int \mathbf{F}\cdot d\mathbf{r}. \qquad \text{...(42)}$$

Thus we see that $d\mathbf{l}$ is equivalent to $d\mathbf{r}$. For the three dimensional case, we have

$$\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k} \text{ and } d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k}.$$

$$\therefore \quad \int \mathbf{F}\cdot d\mathbf{r} = \int (F_x dx + F_y dy + F_z dz), \qquad \text{...(43)}$$

Conservative Field : The line integral may or may not depend upon the path integration. When it is independent of the path of integration, the field defined by **F** is known as *conservative field*. In this case.

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{C_2} \mathbf{F}\cdot d\mathbf{r}. \qquad \text{...(44)}$$

where C_1 and C_2 are the two curves with the same end points A and B [Fig. 1.9 (*b*)]. If we travel from A to B *through curve* C_1 and come back to A through C_2, then for conservative field **F**, we get

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} = 0. \qquad \text{...(45)}$$

Let $\mathbf{F} = \nabla\varphi$, therefore

$$\int_B^A \mathbf{F}\cdot d\mathbf{r} = \int_B^A \nabla\varphi\cdot d\mathbf{r} = \int_A^B d\varphi = \varphi_B - \varphi_A, \qquad \text{...(46)}$$

which is independent of path but depends only on the end points A and B.

For any closed path the above relation reduces to

$$\oint \mathbf{F}\cdot d\mathbf{r} = \int_A^B \mathbf{F}\cdot d\mathbf{r} + \int_B^A \mathbf{F}\cdot d\mathbf{r}$$

$$= \varphi_B - \varphi_A + \varphi_A - \varphi_B = 0. \qquad \text{...(47)}$$

Thus a conservative field is that field which may be defined as a gradient of a scalar field and for which the closed line integral is always zero. Such fields are called *irrotational fields*. As these are obtained from the scalar by taking gradients, hence are called *scalar potential fields*. Since space is divided up into layers or laminae by the level or equipotential surfaces of the function φ, hence the fields are also called the *lamellar vector fields*.

1.9 SURFACE INTEGRAL

If **F** represents a velocity vector **v** of a fluid, the dot product $\mathbf{v}.\delta\mathbf{S}$ represents the rate of flow of fluid through an area $\delta\mathbf{S}$ (whose direction is perpendicular to surface) and is also known *as normal velocity flux*. Consider some vector field **F** in space and in this space some arbitrary closed surface. Now divide the whole surface into little patches which are so small that over any one patch the surface is practically flat and the vector field is same at all points on it (Fig. 1.10). The dot product $\mathbf{F}\cdot\delta\mathbf{S}$ is the flux through the patch of surface area $\delta\mathbf{S}$. If we add all these products for all patches we get the flux Φ through the entire surface.

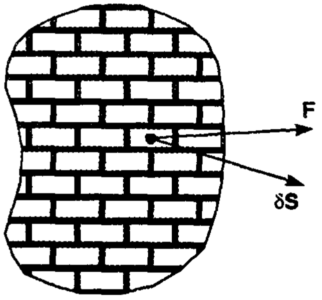


Fig. 1.10. Normal vector flux.

$$\Phi = \Sigma\mathbf{F}\cdot\delta\mathbf{S}. \qquad \text{...(48)}$$

If the patches are of very small area, the flux Φ can be written as the surface integral, *i.e.*,

$$\Phi = \int_{\text{Entire surface}} \mathbf{F}\cdot d\mathbf{S}. \qquad \text{...(49)}$$

As the surface is a two dimensional quantity, we generally use double integral, such as

$$\Phi = \iint_S \mathbf{F}\cdot d\mathbf{S}. \qquad \text{...(50)}$$

Since $d\mathbf{S} = \mathbf{i}dS_x + \mathbf{j}dS_y + \mathbf{k}dS_z$, hence we may express the above relation as

$$\Phi = \iint (F_x dS_x + F_y dS_y + F_z dS_z). \qquad \text{...(51)}$$

1.10 THE DIVERGENCE OF A VECTOR FIELD

For a physical concept of the divergence of a vector point function **v**, let us consider an infinitesimal element of volume with sides Δx, Δy and Δz parallel to the axes of x, y and y, respectively (Fig. 1.11). Let the vector point function **v** at the middle of this element has components of magnitude v_x, v_y and v_z along x–, y– and z–axes respectively.

Consider the two parallel faces of the volume element each with area $\Delta x\, \Delta z$ perpendicular to the y-axis. On the left hand face of the volume, the value of v_y at the middle of the face becomes

$$v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y.$$

Since the face area is infinitesimally small, hence the above value may be taken as the value all over the face. Similarly, on the right hand face, the value of v_y is

$$v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y.$$

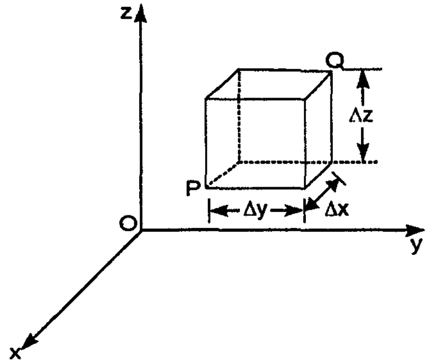


Fig. 1.11. Parallelopiped *PQ* flux.

We define flux of a vector field through any face as the scalar product of the vector area of the face and vector **v**, *i.e.*, the product of the area of the face and normal component of the vector upon it. Thus the flux entering the element in the y-direction

$$= \left(v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\Delta z,$$

and flux leaving the element in the y-direction

$$= \left(v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\Delta z.$$

∴ Excess of flux leaving the element over that entering it in the y-direction is

$$\left(v_y + \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\Delta z - \left(v_y - \frac{1}{2}\frac{\partial v_y}{\partial y}\Delta y\right)\Delta x\Delta z.$$

$$= (\partial v_y/\partial y)\,\Delta y\,\Delta x\,\Delta z = (\partial v_y/\partial y)\,\Delta x\,\Delta y\,\Delta z.$$

Similarly the net outward flux in the x-direction $= (\partial v_x/\partial x)\,\Delta x\,\Delta y\,\Delta z.$
and in the z-direction, it is

$$= (\partial v_z/\partial z)\,\Delta x\,\Delta y\,\Delta z.$$

∴ Total net outward flux diverging from or leaving the element

$$= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)\Delta x\,\Delta y\,\Delta z.$$

The amount of this flux per unit volume is defined as the divergence of the vector point function **v** and is thus written as

$$\text{div}\,\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}. \qquad \text{...(52)}$$

Since the divergence is the amount of net flux, it is essentially a scalar. Since

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right).\,(\mathbf{i}A_x + \mathbf{j}A_y + \mathbf{k}A_z)$$
$$= \nabla\cdot\mathbf{A}.$$

Hence
$$\text{div}\,\mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y'} + \frac{\partial v_z}{\partial z} = \nabla\cdot\mathbf{v}. \qquad \text{...(53)}$$

Since the operator ∇ is invariant, *i.e.*, independent of any system of coordinates, hence the divergence of any vector is also *invariant*. If the divergence exists at a point in a fluid of density ρ_m, the quantity $[\nabla\cdot(\rho_m\mathbf{v})]$ of fluid is known as the strength of the **v** flow or simply the flux of **v**. This represents the excess of the outward over the inward flow or the divergence of the fluid. The quantity $-\nabla\cdot(\rho_m\mathbf{v})$ represents the excess of the inward over the outward flow, or the convergence of the fluid. Due to this reason $\nabla\cdot(\rho_m\mathbf{v})$ or $\nabla\cdot\mathbf{F}$ is named divergence of $\rho_m\mathbf{v}$ or **F**, denoted by *div*.

In general we can write it for a vector function **F** as

$$\text{div}\,\mathbf{F} = \nabla\cdot\mathbf{F} = \partial F_x/\partial x + \partial F_y/\partial y + \partial F_z/\partial z. \qquad \text{...(54)}$$

Equation (54) may be written as

$$\text{div}\,\mathbf{F} = \underset{V\to 0}{\text{limit}}\,\frac{1}{V}\int \mathbf{F}\cdot d\mathbf{S}. \qquad \text{...(55)}$$

Thus the divergence of a vector is the limit of its surface integral per unit volume as the volume V enclosed by the surface goes to zero. It is a scalar quantity representing the net amount of flux coming out of a volume element.

The divergence of the fluid is positive at any point if either the fluid is expanding and its density is decreasing or there is a *source of fluid at that point.*

The divergence is negative if the fluid is contracting and its density is increasing or the point is a *sink.* If the net flow of fluid is zero divergence is zero.

If there is no source or sink and the density of the fluid is not changing, the fluid is called *incompressible.* The vector field satisfying this condition is said to be *solenoidal field.* Hence for incompressible fluid or for a solenoidal field, divergence of a vector field **F** is always zero, *i.e.*, div $\mathbf{F} = \nabla\cdot\mathbf{F} = 0$.

The flux may be flow of liquid, flow of heat or flow of electric lines of force. Thus in the electric flux, the existence of a finite positive value of the divergence at a point shows that there must be a positive charge at the concerned point.

1.11 GAUSS'S DIVERGENCE THEOREM

This theorem states that *the surface integral of the normal component of a vector point function* **F** *taken over a closed surface S enclosing a volume V is equal to the volume integral of the divergence of the vector* **F** *taken through the volume V, i.e.,*

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iiint_V \text{div}\,\mathbf{F}\,dV. \qquad \text{...(56)}$$

In rectangular co-ordinates $\mathbf{F} = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$, $dV = dx\,dy\,dz$,

$$dS_x = dy\,dz,\ dS_y = dx\,dz \text{ and } dS_z = dx\,dy.$$

The equation (56) may thus be written as

$$\iint_S (F_x dS_x + F_y dS_y + F_z dS_z) = \iiint_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz \qquad ...(57)$$

To prove this theorem, let us first integrate the second term of right hand side with respect to y. For this we consider a rectangular prism PQ with cross sectional area $dx.dz$ and edges parallel to y-axis, as shown in Fig. 1.12. We thus get

$$\iiint_V \frac{\partial F_y}{\partial y} dx\,dy\,dz = \iint [F_y(x, y_2, z) - F_y(x, y_1, z)]\,dxdz. \qquad ...(58)$$

Here x, y_1, z are the coordinates of end P and x, y_2, z that of the end Q.

Let $\hat{n}_P$ and $\hat{n}_Q$ be the unit vectors along the normals to the elements of areas dS_P and dS_Q, the areas where the elementary prism PQ meets the close surface. The area dS has a projection on the x-z plane of value $dS_y = dx.\,dz$. As $\hat{n}_P$ makes an obtuse angle with the y-axis, hence the projection of dS_y on x-z plane

$$= (-\mathbf{i}\cdot\hat{n}_P)\,dS_P = dx\,dz = -(dS_y)_P.$$

Similarly for the area dS_Q the projection on x-z plane

$$= (\mathbf{i}\cdot\hat{n}_Q)dS_Q = dx\,dz = (dS_y)_Q.$$

Hence the equation (58) becomes

$$\iiint_V \frac{\partial F_y}{\partial y} dx\,dy\,dz = \iint [(F_y dS_y)_Q + (F_y dS_y)_P].$$

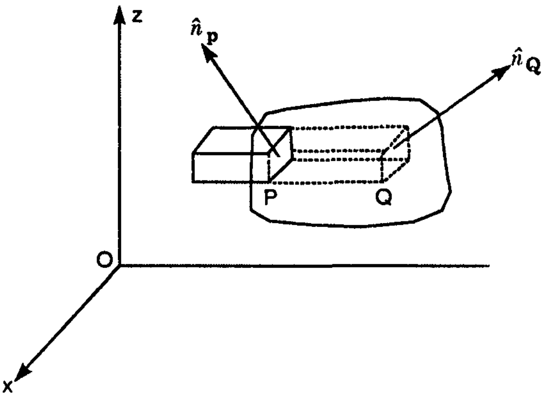


Fig. 1.12. Rectangular prism PQ.

Thus dividing the whole volume into the large number of rectangular prisms with edges parallel to y-axis and of small cross sectional areas and adding all contributions we get equation (58) as

$$\iiint_V \frac{\partial F_y}{\partial y} dy\,dx\,dz = \iint F_y dS_y. \qquad ...(59)$$

Similarly we can show that

$$\iiint_V \frac{\partial F_x}{\partial x} dx\,dy\,dz = \iint F_x dS_x, \text{ and } \iiint_V \frac{\partial F_z}{\partial z} dz\,dx\,dy = \iint F_z dS_z. \qquad ...(60\text{-}61)$$

On adding these equations, we get equation (57) as

$$\iiint_V \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right) dx\,dy\,dz = \iint_V (F_x dS_x + F_y dS_y + F_z dS_z).$$

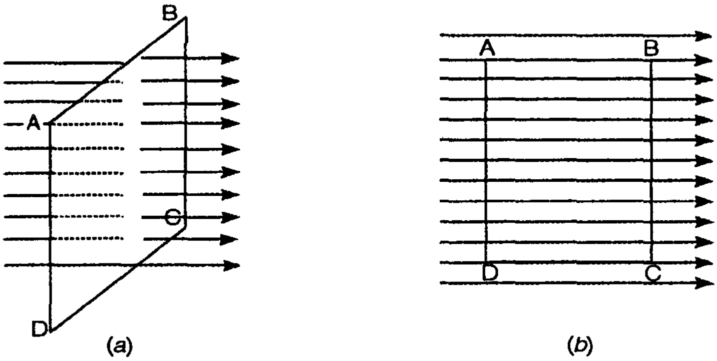
This is *Gauss's divergence theorem.* It is very useful when we transform the surface integral of vector function over a closed surface into a volume integral. This theorem also shows that if the surface integral of a vector **F** is equal to the volume integral of a scalar function P over the volume enclosed by the surface, then we get

$$P = div\mathbf{F}. \qquad ...(62)$$

1.12 ROTATIONAL VECTOR FIELDS, CURL OF A VECTOR

We have read that for irrotational fields the line integral around a closed path is zero. There are also other vector fields for which the line integral around a closed path is not zero, such fields are known as *rotational vector fields*.

The curl is an important property for these vector fields. Let us first understand the *physical significance* of a curl of a vector field.



(*a*) (*b*)

Fig. 1.13. Rectangular plane in a vector field.

Consider a small rectangular plane *ABCD* placed in a very small region of rotational vector field. When the plane is in such a way that the vector field is normal to it [Fig. 1.13(*a*)], *i.e.*, normal to each side of this plane, the line integral along its boundary is

$$\oint \mathbf{F}\cdot d\mathbf{l} = \int_A^B \mathbf{F}\cdot d\mathbf{l} + \int_B^C \mathbf{F}\cdot d\mathbf{l} + \int_C^D \mathbf{F}\cdot d\mathbf{l} + \int_D^A \mathbf{F}\cdot d\mathbf{l}$$

$$= 0 + 0 + 0 + 0 = 0 \qquad \text{...(63)}$$

If this plane is rotated to 90° such that the vector field is parallel to it [Fig. 1.13(*b*)], *i.e.*, to the sides *AB* and *CD*, then the line integral

$$\oint \mathbf{F}\cdot d\mathbf{l} = \int_A^B \mathbf{F}\cdot d\mathbf{l} + \int_B^C \mathbf{F}\cdot d\mathbf{l} + \int_C^D \mathbf{F}\cdot d\mathbf{l} + \int_D^A \mathbf{F}\cdot d\mathbf{l}$$

$$= \int_A^B \mathbf{F}\cdot d\mathbf{l} + \int_C^D \mathbf{F}\cdot d\mathbf{l}. \qquad \text{...(64)}$$

As **F** is continuously varying with the position, hence **F** for *AB* is different than that for *CD* and the net closed line integral is of a finite value. Thus we see that the closed line integral along *ABCD* depends upon its orientation with respect to the vector field. There is a certain orientation of the area for which the line integral will be maximum. This maximum line integral at any point in such a vector field around a closed curve, expressed for unit area is called the **curl of a vector field** at that point. It is a vector quantity directed along the normal to the test area, towards which a right handed screw moves when turned in the sense in which the line integral along the boundary of the test area is maximum. Hence curl **F** is defined as

$$\text{curl } \mathbf{F} = \underset{\Delta S \to 0}{\text{limit}} \frac{1}{\Delta S} \oint \mathbf{F}\cdot d\mathbf{l}. \qquad \text{....(65)}$$

To understand clearly the curl of a vector and its direction, let us consider a small paddle wheel in the path of the flow of water in a canal (Fig. 1.14). On account of the viscous property of the water, the velocity of the water layer decreases downwards, it is nearly zero at the bottom and is maximum at the top.

The wheel will thus turn in the clockwise direction, as shown in Fig. 1.14, showing that there is a small circulation of water round about the wheel. The axis about which the wheel rotates gives the direction of the **curl.**

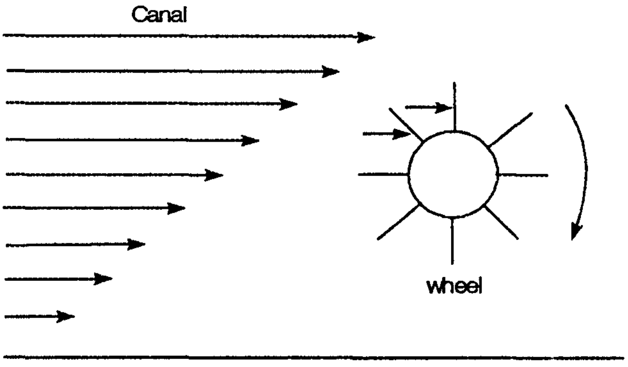


Fig. 1.14. Paddle wheel in a canal.

Let us find out the curl of a vector field **F** in terms of the cartesian components. Consider a rectangular area *ABCD* perpendicular to the axis of y, having sides Δx and Δz along the x- and z- directions respectively (Fig. 1.15). Because of the small sides we can assume that the numerical value of the component of **F** at the middle point of any side is the average value along that side. Let F_x, F_y, and F_z be the components of vector **F** along x-, y- and z-axes respectively at the origin *O*. Thus average values along sides *AB, BC, CD* and *DA* are

$$\left[F_x + \frac{\partial F_x}{\partial z}\left(-\frac{1}{2}\Delta z\right)\right], \left[F_z + \frac{\partial F_z}{\partial x}\left(-\frac{1}{2}\Delta x\right)\right], \left[F_x + \frac{\partial F_x}{\partial z}\left(\frac{1}{2}\Delta z\right)\right]$$

$$\text{and} \left[F_z + \frac{\partial F_z}{\partial x}\left(\frac{1}{2}\Delta x\right)\right] \text{ respectively.}$$

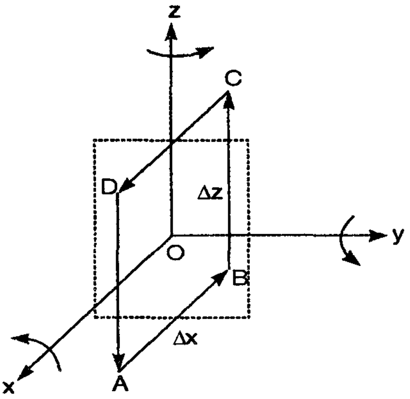


Fig. 1.15. Rectangular plane in a vector field.

As
$$\int_A^B \mathbf{F}\cdot d\mathbf{l} = \left[F_x + \frac{\partial F_x}{\partial z}\left(-\frac{1}{2}\Delta z\right)\right](-\Delta x),$$

$$\int_B^C \mathbf{F}\cdot d\mathbf{l} = \left[F_z + \frac{\partial F_z}{\partial x}\left(-\frac{1}{2}\Delta x\right)\right](\Delta z).$$

$$\int_C^D \mathbf{F}\cdot d\mathbf{l} = \left[F_x + \frac{\partial F_x}{\partial z}\left(\frac{1}{2}\Delta z\right)\right](\Delta x).$$

and
$$\int_D^A \mathbf{F}\cdot d\mathbf{l} = \left[F_z + \frac{\partial F_z}{\partial x}\left(\frac{1}{2}\Delta x\right)\right](-\Delta z).$$

Hence the total line integral of **F** along the boundary *ABCD*

$$\oint \mathbf{F}\cdot d\mathbf{l} = \int_A^B \mathbf{F}\cdot d\mathbf{l} + \int_B^C \mathbf{F}\cdot d\mathbf{l} + \int_C^D \mathbf{F}\cdot d\mathbf{l} + \int_D^A \mathbf{F}\cdot d\mathbf{l}$$
$$= (\partial F_x/\partial z - \partial F_z/\partial x)\,\Delta x\,\Delta z. \qquad \ldots(66)$$

The value of this integral per unit area is $(\partial F_x/\partial z - \partial F_z/\partial x)$. It is in the direction normal to the area *ABCD, i.e.,* in the y-direction. It is thus the y-component of the curl of a vector function **F**. Hence

$$(\text{curl } \mathbf{F})_y = (\partial F_x/\partial z - \partial F_z/\partial x)\,\mathbf{j},$$

where **j** is the unit vector along y-axis.

Similarly we have

$$(\text{curl } \mathbf{F})_x = (\partial F_z/\partial y - \partial F_y/\partial z)\,\mathbf{i} \text{ and } (\text{curl } \mathbf{F})_z = (\partial F_y/\partial x - \partial F_x/\partial y)\mathbf{k}.$$

$$\therefore \quad (\text{curl } \mathbf{F}) = (\text{curl } \mathbf{F})_x + (\text{curl } \mathbf{F})_y + (\text{curl } \mathbf{F})_z$$

$$= \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k} \qquad \text{...(67)}$$

This may be conveniently written in the form of determinant as

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix} \qquad \text{...(68)}$$

Since the vector product of the del operator and the vector **F** may be written as

$$\nabla \times \mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (F_x\mathbf{i} \times F_y\mathbf{j} + F_z\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\therefore \text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \text{Del cross } \mathbf{F}. \qquad \text{...(69)}$$

Since the operation is independent of the system of axes, hence the curl of a vector is *invariant*.

The name curl suggests that a vector field **F** at a point of space has circulation there. As the curl is associated in hydrodynamics with the rotation of a fluid hence is sometimes called *rotation* or simply *rot*. *For any conservative vector field the curl is zero at all points of space, as the line integral for a closed path is zero for such a field.*

1.13 STOKE'S THEOREM

This theorem states that *the line integral of tangential component of a vector field* **F** *around a closed curve is equal to the surface integral of the normal component of its curl over the surface S bounded by that curve C, i.e.,*

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \oint_S \text{curl } \mathbf{F}\cdot d\mathbf{S} \qquad \text{...(70)}$$

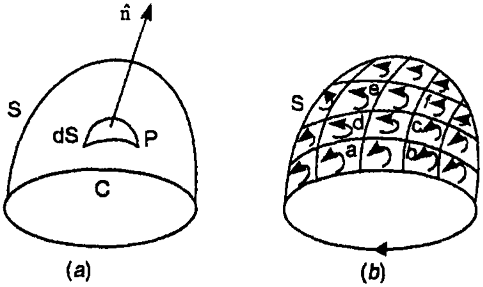
Consider a surface area dS at any point P on the surface S having closed boundary C [(Fig. 1.16(*a*)]. If $\hat{n}$ is the unit normal vector on this area, then $d\mathbf{S} = \hat{n}dS$ and the Stoke's theorem may also be expressed as

$$\oint_C \mathbf{F}\cdot d\mathbf{l} = \iint_S \hat{n}\cdot(\text{curl } \mathbf{F})\, dS. \qquad \text{...(71)}$$

Let the surface S be subdivided by sets of curves lying on the surface and joining the points on the curve C so as to form a network as shown in Fig. 1.16(*b*).

If we consider the line integrals of **F** around the two adjacent meshes we see that the line integrals along the common side of the two meshes cancel each other as one has to travel in opposite directions. In the case of meshes *abcd* and *dcfe*, the common side *dc* does not contribute to the line integral and the line integral along *abcd* + line integral along *dcfe* = line integral along *abfe*. In general every mesh has a side common with its neighbouring meshes, hence the sum of the line integrals of **F** around all the meshes forming a network on the whole surface S is the line integral of **F** along the contour of the curve C, *i.e.*,

$$\int_c \mathbf{F}\cdot d\mathbf{l} = \Sigma \oint \mathbf{F}\cdot d\mathbf{l}. \qquad \text{...(72)}$$



(*a*) (*b*)

Fig. 1.16. Surface S with closed boundary C.

Let us consider a single mesh *ABCD* which is a parallelogram with the sides *AB* and *AD* represented by infinitesimal vectors $\delta\mathbf{a}$ and $\delta\mathbf{b}$ as shown in Fig. 1.17. Let the mean values of vector field **F** along the sides *AB, BC, CD* and *DA* be $\mathbf{F}_1$, $\mathbf{F}_2$, $\mathbf{F}_3$ and $\mathbf{F}_4$ respectively, then

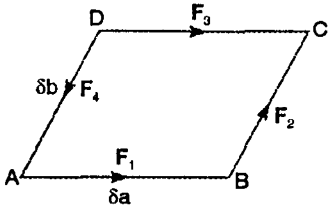


Fig. 1.17. Mesh in a vector field.

$$\begin{aligned}\oint \mathbf{F}\cdot d\mathbf{l} &= \int_A^B \mathbf{F}\cdot d\mathbf{l} + \int_B^C \mathbf{F}\cdot d\mathbf{l} + \int_C^D \mathbf{F}\cdot d\mathbf{l} + \int_D^A \mathbf{F}\cdot d\mathbf{l}\\ &= \mathbf{F}_1\cdot\delta\mathbf{a} + \mathbf{F}_2\cdot\delta\mathbf{b} - \mathbf{F}_3\cdot\delta\mathbf{a} - \mathbf{F}_4\cdot\delta\mathbf{b}\\ &= \delta\mathbf{b}\cdot(\mathbf{F}_2 - \mathbf{F}_4) - \delta\mathbf{a}\cdot(\mathbf{F}_3 - \mathbf{F}_1). \qquad \text{...(73)}\end{aligned}$$

We know that the change in **F** corresponding to a displacement $\delta\mathbf{r}$ is given by

$$\delta\mathbf{F} = \left(\delta x\frac{\partial}{\partial x} + \delta y\frac{\partial}{\partial y} + \delta z\frac{\partial}{\partial z}\right)\mathbf{F} = (\delta\mathbf{r}\,.\,\nabla)\mathbf{F}. \qquad \text{...(74)}$$

Hence we can write

$$\mathbf{F}_2 - \mathbf{F}_4 = (\delta\mathbf{a}\cdot\nabla)\,\mathbf{F}.$$

and
$$\mathbf{F}_3 - \mathbf{F}_1 = (\delta\mathbf{b}\cdot\nabla)\,\mathbf{F}. \qquad \text{...(75)}$$

Substituting these values in equation (73), we get

$$\begin{aligned}\oint \mathbf{F}\cdot d\mathbf{l} &= \delta\mathbf{b}\cdot(\delta\mathbf{a}\cdot\nabla)\mathbf{F} - \delta\mathbf{a}\cdot(\delta\mathbf{b}\cdot\nabla)\mathbf{F}\\ &= [\delta\mathbf{b}(\delta\mathbf{a}\cdot\nabla) - \delta\mathbf{a}(\delta\mathbf{b}\cdot\nabla)].\,\mathbf{F}.\end{aligned}$$

Using vector identities

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = (\mathbf{c}\cdot\mathbf{a})\,\mathbf{b} - (\mathbf{c}\cdot\mathbf{b})\mathbf{a}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$,

we get

$$\begin{aligned}\oint \mathbf{F}\cdot d\mathbf{l} &= [(\delta\mathbf{a}\times\delta\mathbf{b})\times\nabla]\cdot\mathbf{F} = (\delta\mathbf{a}\times\delta\mathbf{b})\cdot(\nabla\times\mathbf{F})\\ &= (\nabla\times\mathbf{F})\cdot(\delta\mathbf{a}\times\delta\mathbf{b}). \qquad \text{...(76)}\end{aligned}$$

Putting $\delta\mathbf{a}\times\delta\mathbf{b} = \delta\mathbf{S}$ = Area of *ABCD*, we get

$$\oint \mathbf{F}\cdot d\mathbf{l} = (\nabla\times\mathbf{F})\cdot\delta\mathbf{S}. \qquad \text{...(77)}$$

Summing these results for all the meshes, we get

$$\oint_c \mathbf{F}\cdot d\mathbf{l} = \sum_{\text{all the meshes}}\oint \mathbf{F}\cdot d\mathbf{l} = \iint_S (\nabla\times\mathbf{F}).\,d\mathbf{S}.$$

or
$$\oint_c \mathbf{F}\cdot d\mathbf{l} = \iint_S \hat{n}\cdot\text{curl}\,\mathbf{F}\,dS. \qquad \text{...(78)}$$

Hence the Stoke's theorem.

Like the divergence theorem this theorem is very useful in the evaluation of integrals.

1.14 CURVILINEAR COORDINATES

Sometimes, it is more convenient to use coordinate systems other than cartesian. A point in space can be described by three parameters *l*, *m* and *n*. If we keep *m* = const and *n* = const and change *l*, then we obtain a smooth curve, called the *l*–curve. Similarly we have *m*-curve and *n*-curve. We define $\mathbf{l}_0$, $\mathbf{m}_0$ and $\mathbf{n}_0$ the unit vectors along the tangents to these curves. Thus a position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ may be defined in terms of *l, m, n* as

$$\mathbf{r} = x(l, m, n)\,\mathbf{i} + y(l, m, n)\,\mathbf{j} + z(l, m, n)\,\mathbf{k} \qquad \text{...(79)}$$

$$\therefore \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$$

$$d\mathbf{r} = \left(\frac{\partial x}{\partial l}dl + \frac{\partial x}{\partial m}dm + \frac{\partial x}{\partial n}dn\right)\mathbf{i} + \left(\frac{\partial y}{\partial l}dl + \frac{\partial y}{\partial m}dm + \frac{\partial y}{\partial n}dn\right)\mathbf{j} + \left(\frac{\partial z}{\partial l}dl + \frac{\partial z}{\partial m}dm + \frac{\partial z}{\partial n}dn\right)\mathbf{k}$$

or $$\frac{\partial \mathbf{r}}{dl} = \frac{\partial x}{\partial l}\mathbf{i} + \frac{\partial y}{\partial l}\mathbf{j} + \frac{\partial z}{\partial l}\mathbf{k}.$$

This is a vector along the direction of l_0. Therefore

$$\mathbf{l}_0 = \frac{\partial \mathbf{r}/\partial l}{|\partial \mathbf{r}/\partial l|} = \frac{(\partial x/\partial l)\,\mathbf{i} + (\partial y/\partial l)\,\mathbf{j} + (\partial z/\partial l)\,\mathbf{k}}{\sqrt{(\partial x/\partial l)^2 + (\partial y/\partial l)^2 + (\partial z/\partial l)^2}}.$$

The denominator may be written as h_l. Thus the length of elementary arc ds, when only l changes is given by

$$ds = |(d\mathbf{r})_{m,n}| = h_l\, dl.$$

Similarly we can deduce

$$\mathbf{m}_0 = [(\partial x/\partial m)\,\mathbf{i} + (\partial y/\partial m)\,\mathbf{j} + (\partial z/\partial m)\,\mathbf{k}]/h_m$$

and $$\mathbf{n}_0 = [(\partial x/\partial n)\,\mathbf{i} + (\partial y/\partial n)\,\mathbf{j} + (\partial z/\partial n)\,\mathbf{k}]/h_n,$$

where $h_m = \sqrt{\left(\frac{\partial x}{\partial m}\right)^2 + \left(\frac{\partial y}{\partial m}\right)^2 + \left(\frac{\partial z}{\partial m}\right)^2}$ and $h_n = \sqrt{\left(\frac{\partial x}{\partial n}\right)^2 + \left(\frac{\partial y}{\partial n}\right)^2 + \left(\frac{\partial z}{\partial n}\right)^2}$

$\therefore$ $$d\mathbf{r} = h_l\, dl\, \mathbf{l}_0 + h_m\, dm\, \mathbf{m}_0 + h_n\, dn\, \mathbf{n}_0 \qquad ...(80)$$

and $$ds = |d\mathbf{r}| = [h_l^2\, dl^2 + h_m^2\, dm^2 + h_n^2\, dn^2]^{1/2} \qquad ...(81)$$

There are three important coordinate systems : rectangular, cylindrical and spherical.

***(i)* Rectangular Coordinate System :** The position of any point P in space is specified in terms of cartesian coordinates, x, y and z [Fig. 1.18 (*a*)]. The infinitesimal increments in the lengths along the $\hat{x}, \hat{y}$ and $\hat{z}$ directions are dx, dy and dz respectively, as shown in Fig. 1.18 (*b*).

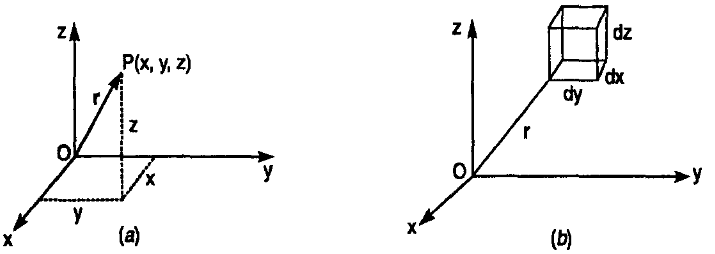


Fig. 1.18. Rectangular coordinates.

The position vector $$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z}. \qquad ...(82)$$

The line element $$d\mathbf{r} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z}. \qquad ...(83)$$

The volume element $$dV = dx\, dy\, dz. \qquad ...(84)$$

***(ii)* Cylindrical Coordinate System :** The position of any point P is specified in cylindrical coordinates by ρ, ϕ, and z, as shown in Fig. 1.19 (*a*). The unit vectors $\hat{\rho}, \hat{\phi}$ and $\hat{z}$ are perpendicular to each of the three surfaces at P. $\hat{\rho}$ is perpendicular to the surface ρ = constant, $\hat{\phi}$ is perpendicular to the surface ϕ = constant and $\hat{z}$ is perpendicular to the surface

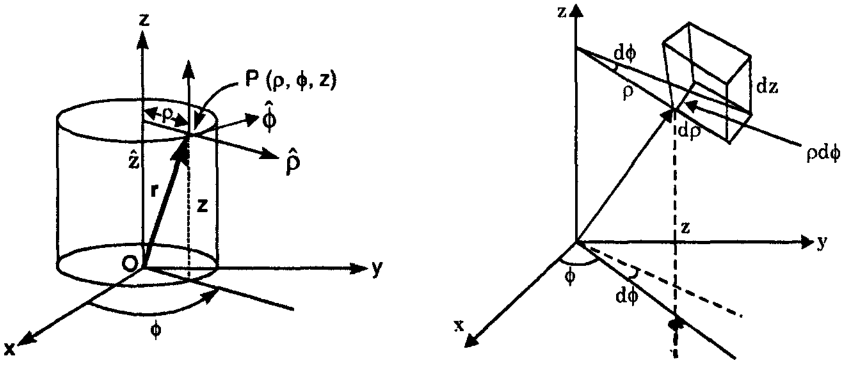
z = constant. These are in the directions of increasing ρ, ϕ and z respectively. Fig. 1.19 (*b*) shows that the line element

$$d\mathbf{r} = d\rho\,\hat{\rho} + (\rho d\phi)\,\hat{\phi} + dz\,\hat{z} \qquad \text{...(85)}$$

and the volume element $\quad dV = (d\rho)\,(\rho d\phi)\,(dz) = \rho d\rho\, d\,\phi\, dz. \qquad$ ...(86)

The cartesian and cylindrical coordinates of a point are related as

$$x = \rho\cos\phi,\ y = \rho\sin\phi \text{ and } z = z. \text{ and } \hat{\rho} = \hat{x}\cos\phi + \hat{y}\sin\phi,\ \hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi,\ \hat{z} = \hat{z}. \qquad \text{...(87)}$$



(*a*) (*b*)

Fig. 1.19. Cylindrical coordinates.

(*iii*) Spherical Coordinate System : The position of any point P is specified in spherical coordinates r, θ and ϕ, as shown in Fig. 1.20 (*a*). $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ are the unit vectors which are perpendicular to each other and form a right handed triad of unit vectors. Fig. 1.20 (*b*) shows that the line element

$$d\mathbf{r} = dr\,\hat{r} + (rd\theta)\,\hat{\theta} + (r\sin\theta\, d\phi)\,\hat{\phi}. \qquad \text{...(88)}$$

and the volume element

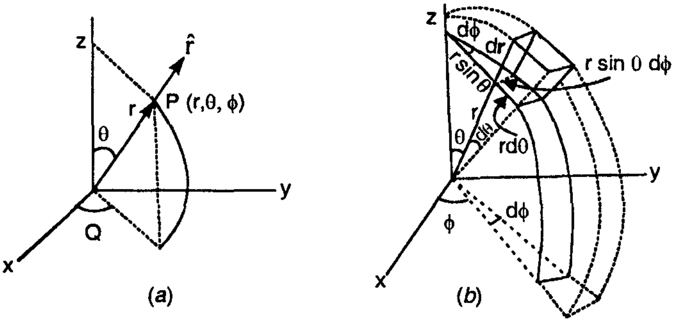
$$dV = dr\,(rd\theta)\,(r\sin\theta\, d\phi) = r^2\sin\theta\, dr\, d\,\theta\, d\,\phi \qquad \text{...(89)}$$

Here we see that the angle ϕ is the same angle in both the cylindrical and spherical coordinate systems. But in the order of the coordinates, ϕ appears in the second position in cylindrical (r, ϕ, z) and the third position in spherical (r, θ, ϕ). We generally use ρ in place of r in cylindrical system as (ρ, ϕ, z), as symbol r here measures the distance from the z-axis in a plane normal to the z-axis, while in the spherical system r measures the distance from the origin to the point.

In all these coordinate systems, the each unit vector is normal to its coordinate surface and is in the direction in which the coordinate increases. All these systems are right handed and thus we have

$$\hat{x}\times\hat{y} = \hat{z},\ \ \hat{\rho}\times\hat{\phi} = \hat{z} \text{ and } \hat{r}\times\hat{\theta} = \hat{\phi}.$$

The cartesian and spherical coordinates are related as $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $\hat{r} = \hat{x}\sin\theta\cos\phi + \hat{y}\sin\theta\sin\phi + \hat{z}\cos\theta$, $\hat{\theta} = \hat{x}\cos\theta\cos\phi + \hat{y}\cos\theta\sin\phi - \hat{z}\sin\theta$, $\hat{\phi} = -\hat{x}\sin\phi + \hat{y}\cos\phi$ ...(90)



(*a*) (*b*)

Fig. 1.20. Spherical coordinates.

1.15 GRAD, DIV AND CURL IN CURVILINEAR COORDINATES

There are various curvilinear coordinates. We here restrict ourselves to the cylindrical and spherical system of coordinates.

***(i)* Cylindrical coordinates :** We know that a scalar function $\varphi(\mathbf{r})$ may be written in this system as $\varphi\,(\rho, \phi, z)$. Thus by partial differentiation, we get

$$d\varphi = \frac{\partial\varphi}{\partial\rho}d\rho + \frac{\partial\varphi}{\partial\phi}d\phi + \frac{\partial\varphi}{\partial z}dz = d\mathbf{r}\cdot\text{grad }\varphi. \qquad \text{...(91)}$$

The line element $dl = d\rho\,\hat{\rho} + \rho\, d\phi\,\hat{\phi} + dz\,\hat{z}$. ...(85)

Since grad φ is a vector, thus we may write it as

$$\text{grad }\varphi = \hat{\rho}\,(\text{grad }\varphi)_\rho + \hat{\phi}\,(\text{grad }\varphi)_\phi + \hat{z}\,(\text{grad }\varphi)_z.$$

$$\therefore \qquad d\varphi = d\rho\,(\text{grad }\varphi)_\rho + \rho d\phi\,(\text{grad }\varphi)_\phi + dz\,(\text{grad }\varphi)_z. \qquad \text{...(92)}$$

On comparing Eqs. (91) and (92), we get

$$(\text{grad }\varphi)_\rho = \partial\varphi/\partial\rho,\ (\text{grad }\varphi)_\phi = (1/\rho)\,\partial\varphi/\partial\phi,\ (\text{grad }\varphi)_z = \partial\varphi/\partial z.$$

$$\therefore \qquad \text{grad }\varphi = \nabla\varphi = \hat{\rho}\,\frac{\partial\varphi}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial\varphi}{\partial\phi} + \hat{z}\,\frac{\partial\phi}{\partial z} \qquad \text{...(93)}$$

Therefore the del operator in cylindrical coordinates is given by

$$\nabla = \hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial}{\partial\phi} + \hat{z}\,\frac{\partial}{\partial z}. \qquad \text{...(94)}$$

To obtain div **A** in cylindrical coordinates, we use Eq. (94) and the relation

$$\mathbf{A}(\mathbf{r}) = \mathbf{A}\,(\rho, \phi, z) = \hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z.$$

$$\text{div }\mathbf{A} = \nabla\cdot\mathbf{A} = \left(\hat{\rho}\,\frac{\partial}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial}{\partial\phi} + \hat{z}\,\frac{\partial}{\partial z}\right).\,(\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z) \qquad \text{...(95)}$$

From or Eq. (87), one can prove that $\partial\hat{\rho}/\partial\phi = \hat{\phi}$, and $\partial\hat{\phi}/\partial\phi = -\hat{\rho}$. ...(96)

Using the orthogonal property of $\hat{\rho}, \hat{\phi}$ and $\hat{z}$ along with above relations while solving Eq. (95), we get

$$\nabla\cdot\mathbf{A} = \frac{\partial A_\rho}{\partial\rho} + \frac{1}{\rho}A_\rho + \frac{1}{\rho}\,\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \ = \frac{1}{\rho}\,\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\,\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}. \qquad \text{...(97)}$$

The curl of **A** may also be obtained in the similar manner as :

$$\begin{aligned}
\nabla\times\mathbf{A} &= \left(\hat{\rho}\,\frac{\partial}{\partial\rho} + \hat{\phi}\,\frac{1}{\rho}\,\frac{\partial}{\partial\phi} + \hat{z}\,\frac{\partial}{\partial z}\right)\times(\hat{\rho}A_\rho + \hat{\phi}A_\phi + \hat{z}A_z) \\
&= \hat{\rho}\times\left[\hat{\rho}\frac{\partial A_\rho}{\partial\rho} + \hat{\phi}\frac{\partial A_\phi}{\partial\rho} + \hat{z}\frac{\partial A_z}{\partial\rho}\right] + \frac{\hat{\phi}}{\rho}\times\left[\frac{\partial\hat{\rho}}{\partial\phi}A_\rho + \hat{\rho}\frac{\partial A_\rho}{\partial\phi} + \frac{\partial\hat{\phi}}{\partial\phi}A_\phi + \hat{\phi}\frac{\partial A_\phi}{\partial\phi} + \hat{z}\frac{\partial A_z}{\partial\phi}\right] \\
&\qquad + \hat{z}\times\left[\hat{\rho}\,\frac{\partial A_\rho}{\partial z} + \hat{\phi}\,\frac{\partial A_\phi}{\partial z} + \hat{z}\frac{\partial A_z}{\partial z}\right] \\
&= \hat{z}\frac{\partial A_\phi}{\partial\rho} - \hat{\phi}\frac{\partial A_z}{\partial\rho} + \frac{1}{\rho}\left[-\hat{z}\,\frac{\partial A_\rho}{\partial\phi} + \hat{z}A_\phi + \hat{\rho}\frac{\partial A_z}{\partial\phi}\right] + \hat{\phi}\frac{\partial A_\rho}{\partial z} - \hat{\rho}\frac{\partial A_\phi}{\partial z} \\
&= \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right] + \hat{z}\left[\frac{\partial A_\phi}{\partial\rho} - \frac{1}{\rho}\frac{\partial A_\rho}{\partial\phi} + \frac{A_\phi}{\rho}\right]
\end{aligned}$$

$$= \hat{\rho}\left[\frac{1}{\rho}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}\right] + \hat{\phi}\left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial \rho}\right] + \frac{\hat{z}}{\rho}\left[\frac{\partial}{\partial \rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial \phi}\right] \qquad ...(98)$$

***(ii)* Spherical coordinates** : The scalar function $\varphi(\mathrm{r})$ may be written in this system as (r, θ, ϕ). Thus by partial differentiation, we get

$$d\varphi = \frac{\partial \varphi}{\partial r}dr + \frac{\partial \varphi}{\partial \theta}d\theta + \frac{\partial \varphi}{\partial \phi}d\phi = \text{grad } \varphi . d\mathbf{r}. \qquad ...(99)$$

The line element $d\mathbf{r} = \hat{r}\, dr + \hat{\theta} r\, d\theta + \hat{\phi}\,(r \sin\theta\, d\,\phi)$. (88)

$$\therefore \qquad d\varphi = (\text{grad } \varphi)_r\, dr + (\text{grad } \varphi)_\theta\, (r d\theta) + (\text{grad } \varphi)_\phi\, (r \sin\theta\, d\,\phi). \qquad ...(100)$$

On comparing Eqs. (99) and (100), we get

$$(\text{grad } \varphi)_r = \partial\varphi/\partial r,\ (\text{grad } \varphi)_\theta = (1/r)\ \partial\varphi/\partial\theta \text{ and } (\text{grad } \varphi)_\phi = (1/r \sin\theta)\ \partial\varphi/\partial\phi.$$

$$\therefore \qquad \text{grad } \varphi = \hat{r}\frac{\partial \varphi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial \varphi}{\partial \theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial \varphi}{\partial \phi}. \qquad ...(101)$$

The del operator in spherical coordinate is thus given by

$$\nabla = \hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}. \qquad ...(102)$$

The vector **A** in the spherical coordinates may be expressed as

$$\mathbf{A(r)} = \mathbf{A}\,(r, \theta, \phi) = \hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi.$$

$$\therefore \qquad \text{div } \mathbf{A} = \nabla \cdot \mathbf{A} = \left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\right] . \left[\hat{r}A_r + \hat{\theta}A_\theta + \hat{\phi}A_\phi\right]$$

From Eq. (90), one can prove that

$\partial\hat{r}/\partial\theta = \hat{\theta},\ \partial r/\partial\phi = \hat{\phi}\sin\theta,\ \partial\hat{\theta}/\partial\theta = -\hat{r}, \partial\,\hat{\theta}/\partial\phi = \hat{\phi}\cos\theta,$ and $\hat{\partial}\phi/\partial\phi = -\hat{r}\sin\theta - \hat{\theta}\cos\theta.$

Using these relations and the orthogonality property of $\hat{r}, \hat{\theta}$ and $\hat{\phi}$ we get,

$$\text{div } \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 A_r\right) + \frac{1}{r \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta A_\theta\right) + \frac{1}{r \sin\theta}\frac{\partial A_\phi}{\partial \phi} \qquad ...(103)$$

curl $\mathbf{A} = \nabla \times \mathbf{A}$

$$= \left[\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\phi}\frac{1}{r \sin\theta}\frac{\partial}{\partial \phi}\right] \times \left[\hat{r}\ A_r + \hat{\theta}A_\theta + \hat{\phi}\ A_\phi\right]$$

$$= \frac{\hat{r}}{r \sin\theta}\left[\frac{\partial}{\partial \theta}(\sin\theta\ A_\phi) - \frac{\partial A_\theta}{\partial \phi}\right] + \frac{\hat{\theta}}{r}\left[\frac{1}{\sin\theta}\frac{\partial A_r}{\partial \phi} - \frac{\partial}{\partial r}(r\ A_\phi)\right] + \frac{\hat{\phi}}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial \theta}\right]$$

...(104)

1.16 LAPLACIAN OPERATOR

The divergence of a gradient of a scalar function φ is given by

$$\text{div grad } \varphi = \nabla . (\nabla\varphi) = \nabla^2\varphi. \qquad ...(105)$$

The operator $\nabla^2 = \nabla \cdot \nabla$ is known as Laplacian operator. It, just like ∇, $\nabla\ \cdot$, $\nabla\ \times$, makes no reference to any particular coordinate system. In various coordinate systems, ∇^2 may be easily calculated by first taking the gradient of φ and then taking its divergence.

In **cartesian coordinates**, we have

$$\nabla^2 = \nabla\cdot\nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

$$= \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2. \qquad \text{...(106)}$$

In **cylindrical coordinates**, we have

$$\nabla^2 = \nabla\cdot\nabla = \left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)$$

$$= \hat{\rho}\frac{\partial}{\partial\rho}\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right) + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)$$

$$+ \hat{z}\frac{\partial}{\partial z}\left(\hat{\rho}\frac{\partial}{\partial\rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial}{\partial\phi} + \hat{z}\frac{\partial}{\partial z}\right)$$

$$= \frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}. \qquad \text{...(107)}$$

In **spherical coordinates**, we have

$$\nabla^2 = \nabla\cdot\nabla = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)\cdot\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$\nabla^2 = \hat{r}.\frac{\partial}{\partial r}\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) + \frac{\hat{\theta}}{r}.\frac{\partial}{\partial\theta}\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$+ \frac{\hat{\phi}}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right)$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}\frac{\cos\theta}{\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}. \qquad \text{...(108)}$$

1.17 GREEN'S THEOREM

Green's theorem can be derived from Gauss's divergence theorem as follows. Substituting vector field **F** as the product of one scalar φ and grad of another scalar ψ, *i.e.*, $\mathbf{F} = \varphi\nabla\psi$, we get

$$\nabla\,.\,\mathbf{F} = \frac{\partial}{\partial x}\left(\varphi\frac{\partial\psi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\varphi\frac{\partial\psi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\varphi\frac{\partial\psi}{\partial z}\right)$$

$$= \varphi\left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}\right) + \frac{\partial\varphi}{\partial x}\frac{\partial\psi}{\partial x} + \frac{\partial\varphi}{\partial y}\frac{\partial\psi}{\partial y} + \frac{\partial\varphi}{\partial z}\frac{\partial\psi}{\partial z}$$

$$= \varphi\nabla^2\psi + \nabla\varphi\cdot\nabla\psi. \qquad \text{...(109)}$$

Hence Gauss's theorem reduces to

$$\iiint_V (\varphi\nabla^2\psi + \nabla\varphi\cdot\nabla\psi)dV = \iint_S (\varphi\nabla\psi)\cdot d\mathbf{S}. \qquad \text{...(110)}$$

This is known as the first form of Green's theorem. After inter-changing φ and ψ, we have

$$\iiint_V (\psi\nabla^2\varphi + \nabla\psi\cdot\nabla\varphi)\,dV = \iint_S (\psi\nabla\varphi)\cdot d\mathbf{S}. \qquad \text{...(111)}$$

On substracting Eq. (111) from Eq. (110), we get

$$\iiint_V (\varphi\nabla^2\psi + \psi\nabla^2\varphi)\, dV = \iint_S (\varphi\nabla\psi - \psi\nabla\varphi)\cdot d\mathbf{S}. \qquad ...(112)$$

This is referred to as the *second form of Green's theorem*. These are of extreme importance in the fields of electrodynamics and hydrodynamics.

Some Useful Vector Relations involving the vector ∇

$$(\mathbf{u}\cdot\nabla)\,\varphi = \mathbf{u}\cdot(\nabla\varphi)$$

$$(\mathbf{u}\cdot\nabla)\,\mathbf{r} = \mathbf{u}.$$

$$\text{div}\,(\mathbf{u}+\mathbf{v}) = \nabla\cdot(\mathbf{u}+\mathbf{v}) = \nabla\cdot\mathbf{u} + \nabla\cdot\mathbf{v}$$

$$\text{curl}\,(\mathbf{u}+\mathbf{v}) = \nabla\times(\mathbf{u}+\mathbf{v}) = \nabla\times\mathbf{u} + \nabla\times\mathbf{v}.$$

$$\text{div}\,(\varphi\mathbf{u}) = \nabla\cdot(\varphi\mathbf{u}) = \varphi\,\text{div}\,\mathbf{u} + \mathbf{u}\cdot(\text{grad}\,\varphi).$$

$$\text{curl}\,(\varphi\mathbf{u}) = \nabla\times(\varphi\mathbf{u}) = \varphi\,\text{curl}\,\mathbf{u} + (\text{grad}\,\varphi)\times\mathbf{u}.$$

$$\text{div}\,(\mathbf{u}\times\mathbf{v}) = \mathbf{v}\cdot\text{curl}\,\mathbf{u} - \mathbf{u}\cdot\text{curl}\,\mathbf{v}.$$

$$\text{curl}\,(\mathbf{u}\times\mathbf{v}) = (\mathbf{v}\cdot\text{grad})\,\mathbf{u} - (\mathbf{u}\cdot\text{grad})\,\mathbf{v} + \mathbf{u}\,\text{div}\,\mathbf{v} - \mathbf{v}\,\text{div}\,\mathbf{u}.$$

$$\text{grad}\,(\mathbf{u}\cdot\mathbf{v}) = \mathbf{u}\times\text{curl}\,\mathbf{v} + \mathbf{v}\times\text{curl}\,\mathbf{u} + (\mathbf{u}\cdot\text{grad})\,\mathbf{v} + (\mathbf{v}\cdot\text{grad})\,\mathbf{u}.$$

Second order differential operators

$$\text{div grad}\,\varphi = \nabla^2\varphi.$$

$$\text{div curl}\,\mathbf{u} = 0$$

$$\text{curl grad}\,\varphi = 0$$

$$\text{curl curl}\,\mathbf{u} = \text{grad div}\,\mathbf{u} - \nabla^2\mathbf{u}.$$

EXERCISES

Example 1. *Prove that the four points*

$4\mathbf{i} + 5\mathbf{j} + \mathbf{k}$, $-(\mathbf{j} + \mathbf{k})$, $3\mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$ *and* $4(-\mathbf{i} + \mathbf{j} + \mathbf{k})$ *are coplaner.*

Let the given vectors represent the points *A, B, C* and *D* respectively, then

$$\mathbf{a} = \mathbf{AB} = \text{position vector of B} - \text{Position vector of A}$$

$$= -(\mathbf{j} + \mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -(4\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}).$$

Similarly $\mathbf{b} = \mathbf{AC} = 3\mathbf{i} + 9\mathbf{j} + 4\mathbf{k} - (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}.$

and $\mathbf{c} = \mathbf{AD} = 4(-\mathbf{i} + \mathbf{j} + \mathbf{k}) - (4\mathbf{i} + 5\mathbf{j} + \mathbf{k}) = -8\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$

If the points *A, B, C,* and *D* are coplaner, then vectors **a**, **b** and **c** will also be coplaner. It is possible only when $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = 0$.

Now
$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} = \begin{vmatrix} -4 & -6 & -2 \\ -1 & 4 & 3 \\ -8 & -1 & 3 \end{vmatrix} = 0$$

Hence the points *A, B, C, D are coplaner.*

Example 2. *Three field quantities are given by*

$$\mathbf{A} = 2\mathbf{i} - \mathbf{k},\ \mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k},\ \mathbf{C} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}.$$

Determine (*a*) $(\mathbf{A} + \mathbf{B})\times(\mathbf{A} - \mathbf{B})$; (*b*) $\mathbf{B}\cdot\mathbf{C}\times\mathbf{A}$, (*c*) $\sin\theta_{BC}$.

(*a*)
$$(\mathbf{A} + \mathbf{B})\times(\mathbf{A} - \mathbf{B}) = \mathbf{A}\times(\mathbf{A} - \mathbf{B}) + \mathbf{B}\times(\mathbf{A} - \mathbf{B})$$

$$= \mathbf{A}\times\mathbf{A} - \mathbf{A}\times\mathbf{B} + \mathbf{B}\times\mathbf{A} - \mathbf{B}\times\mathbf{B}$$

$$= \mathbf{0} - \mathbf{A}\times\mathbf{B} - \mathbf{A}\times\mathbf{B} + \mathbf{0} = -2\mathbf{A}\times\mathbf{B}$$

$$(\mathbf{A} + \mathbf{B})\times(\mathbf{A} - \mathbf{B}) = -2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & -1 \\ 2 & -1 & 2 \end{vmatrix}$$

$$= -2\,(0 - 1)\,\mathbf{i} - 2\,(-2 - 4)\,\mathbf{j} - 2\,(-2 - 0)\,\mathbf{k} = 2\,\mathbf{i} + 12\,\mathbf{j} + 4\,\mathbf{k}$$

(*b*)
$$\mathbf{B}\cdot\mathbf{C}\times\mathbf{A} = (2\mathbf{i}-\mathbf{j}+2\mathbf{k})\cdot\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & -3 & 1\\ 2 & 0 & -1\end{vmatrix} = \begin{vmatrix}2 & -1 & 2\\ 2 & -3 & 1\\ 2 & 0 & -1\end{vmatrix}$$
$$= 2(3-0) -1(+2+2) +2(0+6) = 6-4+12 = 14$$

(*c*)
$$\sin\theta_{BC} = \frac{|\mathbf{B}\times\mathbf{C}|}{|\mathbf{B}||\mathbf{C}|} = \frac{|(2\mathbf{i}-\mathbf{j}+2\mathbf{k})\times(2\mathbf{i}-3\mathbf{j}+\mathbf{k})|}{[2^2+1^2+2^2]^{1/2}\,(2^2+3^2+1^2)^{1/2}}$$
$$= \frac{|5\mathbf{i}+2\mathbf{j}-4\mathbf{k}|}{3\sqrt{14}} = \frac{\sqrt{45}}{3\sqrt{14}} = \sqrt{\frac{5}{14}} = 0.5976.$$

Example 3. *Derive the cosine formula $a^2 = b^2 + c^2 - 2bc \cos A$ and the sine formula $a/\sin A = b/\sin B = c/\sin B$, using dot product and cross product, respectively.*

Consider a triangle as shown in 1.21. From the figure we have

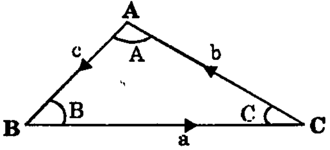


Fig. 1.21. Triangle ABC.

$$\mathbf{a}+\mathbf{b}+\mathbf{c} = 0$$

or
$$\mathbf{b}+\mathbf{c} = -\mathbf{a}$$

$\therefore$
$$(-\mathbf{a})\cdot(-\mathbf{a}) = (\mathbf{b}+\mathbf{c})\cdot(\mathbf{b}+\mathbf{c})$$

or
$$a^2 = \mathbf{b}\cdot\mathbf{b}+\mathbf{c}\cdot\mathbf{c}+2\mathbf{b}\cdot\mathbf{c} = b^2+c^2+2bc\cos A,$$

where A is the angle between **b** and **c**.

The area of a triangle is half of the product of its height and base. Hence

$$\left|\tfrac{1}{2}\,\mathbf{a}\times\mathbf{b}\right| = \left|\tfrac{1}{2}\,\mathbf{b}\times\mathbf{c}\right| = \left|\tfrac{1}{2}\,\mathbf{c}\times\mathbf{a}\right|$$

or
$$ab\sin C = bc\sin A = ca\sin B.$$

Dividing by abc gives

$$\frac{\sin C}{c} = \frac{\sin B}{b} = \frac{\sin A}{a}$$

or
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{b}{\sin C}.$$

Example 4. For a position *vector* $\mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, *show that*
(*a*) *div* $\mathbf{r} = 3$, (*b*) *div* $(r^n\,\mathbf{r}) = (3+n)r^n$, (*c*) *curl* $\mathbf{r} = 0$, (*d*) *curl* $(\mathbf{r}/r^3) = 0$.
(*a*) We know that

$$\text{div}\,\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

In the present case $\mathbf{F} = \mathbf{r} = \mathbf{i}x + \mathbf{j}y + \mathbf{k}z$, therefore

$$\text{div}\,\mathbf{r} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 1+1+1 = 3.$$

(*b*) In this case $\mathbf{F} = r^n\mathbf{r} = r^n\,(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$,

$\therefore$
$$\text{div}\,\mathbf{F} = \text{div}\,(r^n\mathbf{r}) = \frac{\partial}{\partial x}(r^n x) + \frac{\partial}{\partial y}(r^n y) + \frac{\partial}{\partial z}(r^n z)$$
$$= 3r^n + nr^{n-1}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

Since $r^2 = x^2 + y^2 + z^2$, hence $\partial r/\partial x = x/r$, $\partial r/\partial y = y/r$ and $\partial r/\partial z = z/r$,

$\therefore$
$$\text{div}\,r^n\mathbf{r} = 3r^n + nr^{n-1}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right] = 3r^n + nr^{n-1}\,\frac{r^2}{r}$$
$$= (3+n)r^n.$$

(*c*) Since curl of a vector field **F** is given as

$$\text{curl } \mathbf{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right)\mathbf{k}$$

In this case $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, hence

$$\text{curl } \mathbf{F} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k}.$$

Since x, y, z are independent coordinates, hence

$$\frac{\partial x}{\partial y} = \frac{\partial y}{\partial x} = \frac{\partial z}{\partial y} = \frac{\partial y}{\partial z} = \frac{\partial x}{\partial z} = \frac{\partial z}{\partial x} = 0 \text{ and curl } \mathbf{r} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

(*d*) In this case

$$\mathbf{F} = \frac{\mathbf{r}}{r^3} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{r^3} = \left(\frac{x}{r^3}\right)\mathbf{i} + \left(\frac{y}{r^3}\right)\mathbf{j} + \left(\frac{z}{r^3}\right)\mathbf{k}.$$

$$\begin{aligned}\therefore\ \text{curl}\left(\frac{\mathbf{r}}{r^3}\right) &= \mathbf{i}\left[\frac{\partial}{\partial y}\left(\frac{z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{y}{r^3}\right)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}\left(\frac{x}{r^3}\right) - \frac{\partial}{\partial x}\left(\frac{z}{r^3}\right)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}\left(\frac{y}{r^3}\right) - \frac{\partial}{\partial y}\left(\frac{x}{r^3}\right)\right] \\ &= \mathbf{i}\left[-\frac{3}{r^4}\left(z\frac{\partial r}{\partial y} - y\frac{\partial r}{\partial z}\right)\right] + \mathbf{j}\left[-\frac{3}{r^4}\left(x\frac{\partial r}{\partial z} - z\frac{\partial r}{\partial x}\right)\right] + \mathbf{k}\left[-\frac{3}{r^4}\left(y\frac{\partial r}{\partial x} - x\frac{\partial r}{\partial y}\right)\right] \\ &= \mathbf{i}\left[-\frac{3}{r^4}\left(\frac{zy}{r} - \frac{yz}{r}\right)\right] + \mathbf{j}\left[-\frac{3}{r^4}\left(\frac{xz}{r} - \frac{zx}{r}\right)\right] + \mathbf{k}\left[-\frac{3}{r^4}\left(\frac{yx}{r} - \frac{xy}{r}\right)\right] = 0.\end{aligned}$$

This shows that $\mathbf{F} = \mathbf{r}/r^3$ is irrotational in nature.

Example 5. *If* **r** *is a position vector and* **a** *is a constant vector then prove that* $\nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a}$.
Using the vector identity
grad $(\mathbf{A}\cdot\mathbf{B}) = \mathbf{A} \times \text{curl } \mathbf{B} + \mathbf{B} \times \text{curl } \mathbf{A} + (\mathbf{A}\cdot \text{grad})\,\mathbf{B} + (\mathbf{B}\cdot \text{grad})\,\mathbf{A}$,
we have

$$\nabla(\mathbf{a}\cdot\mathbf{r}) = \mathbf{a} \times (\nabla \times \mathbf{r}) + \mathbf{r} \times (\nabla \times \mathbf{a}) + (\mathbf{a}\cdot\nabla)\mathbf{r} + (\mathbf{r}\cdot\nabla)\mathbf{a}. \qquad ...(i)$$

As **a** is a constant vector, hence curl **a** = 0. We know the vector identities curl **r** = 0 and

$$(\mathbf{A}\cdot\nabla)\mathbf{B} = A_x\frac{\partial \mathbf{B}}{\partial x} + A_y\frac{\partial \mathbf{B}}{\partial y} + A_z\frac{\partial \mathbf{B}}{\partial z}. \qquad ...(ii)$$

$$\therefore \quad (\mathbf{a}\cdot\nabla)\mathbf{r} = a_x\frac{\partial \mathbf{r}}{\partial x} + a_y\frac{\partial \mathbf{r}}{\partial y} + a_z\frac{\partial \mathbf{r}}{\partial z} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = \mathbf{a}.$$

$$\text{and} \quad (\mathbf{r}\cdot\nabla)\mathbf{a} = x\frac{\partial \mathbf{a}}{\partial x} + y\frac{\partial \mathbf{a}}{\partial y} + z\frac{\partial \mathbf{a}}{\partial z} = 0. \text{ (as } \mathbf{a} \text{ is a constant vector)}$$

Substituting these values in Eq. (*i*), we get

$$\nabla(\mathbf{a}\cdot\mathbf{r}) = 0 + 0 + \mathbf{a} + 0 = \mathbf{a}. \text{ Hence the result.}$$

Example 6. *Find the angle at which line* $x = y = 2z$ *intersects the ellipsoid* $x^2 + y^2 + 2z^2 = 10$.

The line $x = y = 2z$ can be represented by

$$r(\lambda) = 2\lambda a_x + 2\lambda a_y + \lambda a_z,$$

where λ is a parameter.

Let the line and the ellipsoid meet at angle θ. The line and the ellipsoid will meet, when

$$(2\lambda)^2 + (2\lambda)^2 + 2(\lambda)^2 = 10 \quad \text{or} \quad \lambda = \pm 1.$$

Thus the point of intersection is 2, 2, 1, where $\mathbf{r} = 2\mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

The surface of the ellipsoid is defined as

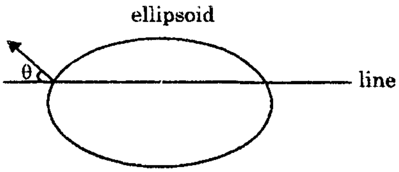


Fig. 1.18.

$$f(x, y, z) = x^2 + y^2 + 2z^2 - 10$$

Its gradient $\nabla f = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 4z\,\mathbf{k}$.

At the point of intersection (2, 2, 1), we have

$$\nabla f = 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}.$$

Hence the unit vector normal to the ellipsoid at the point of intersection is

$$\hat{n} = \pm\frac{\nabla f}{|\nabla f|} = \pm\frac{4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}}{\sqrt{48}} = \pm\frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}$$

Taking the positive sign, the angle between $\hat{n}$ and $\mathbf{r}$ is given by

$$\cos\theta = \frac{\hat{n}\cdot\mathbf{r}}{|\hat{\mathbf{n}}||\mathbf{r}|} = \frac{2+2+1}{\sqrt{3}\sqrt{9}} = \frac{5}{3\sqrt{3}}$$

Since $\theta + \phi = 90°$, therefore $\phi = \sin(90 - \theta) = \cos\theta = \dfrac{5}{3\sqrt{3}}$

$\therefore \qquad \phi = 74.21°$.

Different angles are obtained for different combinations of + or – for λ and $\hat{n}$.

Example 7. *(a) Calculate grad* $\varphi(\mathbf{r})$. *(b) Prove that grad is ⊥ to the surface for which* φ *is constant.*

(*a*) grad $\varphi = \mathbf{i}\,\partial\varphi/\partial x + \mathbf{j}\,\partial\varphi/\partial y + \mathbf{k}\,\partial\varphi/\partial z$,

where $\dfrac{\partial\varphi}{\partial x} = \dfrac{\partial\varphi}{\partial r}\dfrac{\partial r}{\partial x} = \varphi'.\dfrac{\partial r}{\partial x}$; $\dfrac{\partial\varphi}{\partial y} = \varphi'.\dfrac{\partial r}{\partial y}$ and $\dfrac{\partial\varphi}{\partial z} = \varphi'.\dfrac{\partial r}{\partial z}$.

As $\qquad r = (x^2 + y^2 + z^2)^{1/2}$, hence

$$\partial r/\partial x = 2x/2(x^2 + y^2 + z^2)^{1/2} = x/r,\ \partial r/\partial y = y/r \text{ and } \partial r/\partial z = z/r.$$

$$\therefore \qquad \text{grad}\,\varphi(r) = \mathbf{i}\varphi'\frac{x}{r} + \mathbf{j}\varphi'\frac{y}{r} + \mathbf{k}\varphi'\frac{z}{r} = \varphi'\frac{\mathbf{r}}{r} = \frac{\partial\varphi}{\partial r}\frac{\mathbf{r}}{r} = \frac{\partial\varphi}{\partial r}\hat{r}.$$

$$(b) \qquad d\varphi = \frac{\partial\varphi}{\partial x}dx + \frac{\partial\varphi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz = (\text{grad}\,\varphi).\,d\mathbf{r}.$$

Let $d\mathbf{r}$ lies along the surface for which φ = const, then obviously $d\varphi = 0$ and (grad φ). $d\mathbf{r} = 0$. It shows that grad φ and $d\mathbf{r}$ are mutually ⊥ to each other for a surface φ = const.

Example 8. *Prove the vector identities :*

(i) div curl $\mathbf{F} = 0$. *(ii) curl grad* $\varphi = 0$.

$$(i) \qquad \text{div curl}\,\mathbf{F} = \frac{\partial}{\partial x}(\text{curl}\,\mathbf{F})_x + \frac{\partial}{\partial y}(\text{curl}\,\mathbf{F})_y + \frac{\partial}{\partial z}(\text{curl}\,\mathbf{F})_z$$

$$= \frac{\partial}{\partial x}\left[\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y}\right]$$

$$= \frac{\partial^2 F_z}{\partial x\partial y} - \frac{\partial^2 F_y}{\partial x\partial z} + \frac{\partial^2 F_x}{\partial y\partial z} - \frac{\partial^2 F_z}{\partial y\partial x} + \frac{\partial^2 F_y}{\partial z\partial x} - \frac{\partial^2 F_x}{\partial z\partial y} = 0.$$

It means that vector field whose divergence is everywhere zero can be expressed as the curl of some other suitable vector field. The fields whose divergence is zero have their field lines always forming closed curves or the vector field is solenoidal.

(*ii*)
$$(\text{curl grad } \varphi)_x = \left[\frac{\partial}{\partial y}(\text{grad } \varphi)_z - \frac{\partial}{\partial z}(\text{grad } \varphi)_y\right]\mathbf{i}$$
$$= \left[\frac{\partial}{\partial y}\left(\frac{\partial \varphi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \varphi}{\partial y}\right)\right]\mathbf{i} = 0.$$

Similarly we can prove that y– and z– components are also zero.

$$\therefore \qquad \text{curl grad } \varphi = 0.$$

It means that a vector field whose curl is everywhere zero can be expressed as the gradient of another suitable scalar field. The field is known as *non-curl, irrotational, or scalar potential field.* All the conservative fields are of this type.

Example 9. *A rigid body is rotating with constant angular velocity* ω *about a fixed axis, if* **v** *is the velocity of a point of the body, prove that curl* **v** = 2ω. *Give its physical significance.*

Consider a rigid body which is rotating with an angular velocity ω about an axis *OA*, where *O* is a fixed point in the body [Fig. 1.18]. Then any point *P* on the body moves in a circular path about *OA* with a tangential linear velocity **v** (= ω × **r**), where **r** is the distance of point *P* from the fixed point *O*. Since the angular velocity ω is a vector constant for all points on the body and can be written

$\omega = \omega_x\mathbf{i} + \omega_y\mathbf{j} + \omega_z\mathbf{k}$, hence the components $\omega_x, \omega_y, \omega_z$ are independent of the coordinates **r** of the point.

$$\therefore \qquad \mathbf{v} = \omega \times \mathbf{r} = (\omega_y z - \omega_z y)\mathbf{i} + (\omega_z x - \omega_x z)\mathbf{j} + (\omega_x y - \omega_y x)\mathbf{k}$$

$$\text{and } \nabla \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ (\omega_y z - \omega_z y) & (\omega_z x - \omega_x z) & (\omega_x y - \omega_y x) \end{vmatrix} = 2\omega.$$

Hence when the rigid body is in motion, the curl of its linear velocity at any point is equal to twice its angular velocity.

Thus we see that if a motion of a rigid body is such that the velocity has a curl, the ultimate particles of the body are in rotation with an angular velocity, which is equal to half of that curl. The motion is described as *rotational.*

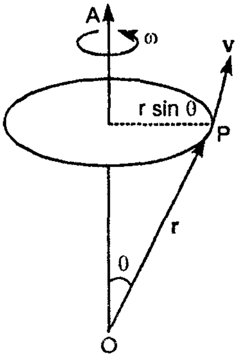


Fig. 1.19. Rotating Body.

Example 10. *For a position vector* **r** = **i***x* + **j***y* + **k***z*, *find the values of (i) grad (1/r) and (ii) grad r^m.*

$$\text{grad}\frac{1}{r} = \nabla\left(\frac{1}{r}\right) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right)$$
$$= \mathbf{i}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial x} + \mathbf{j}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial y} + \mathbf{k}\left(-\frac{1}{r^2}\right)\frac{\partial r}{\partial z} = -\frac{1}{r^2}\left[\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right]$$
$$= -\frac{1}{r^2}\left[\mathbf{i}\frac{x}{r} + \mathbf{j}\frac{y}{r} + \mathbf{k}\frac{z}{r}\right] = -\frac{\mathbf{r}}{r^3}$$

Similarly

$$\text{grad } r^m = \nabla(r^m) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(r^m) = mr^{m-1}\left[\mathbf{i}\frac{\partial r}{\partial x} + \mathbf{j}\frac{\partial r}{\partial y} + \mathbf{k}\frac{\partial r}{\partial z}\right]$$
$$= mr^{m-2}\,[(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z] = mr^{m-2}\mathbf{r}.$$

Example 11. *Prove that curl* **A** = *grad div* **A** – ∇^2**A**.

We know that

$$\text{curl}\,\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}.$$

$$\therefore \quad \text{curl}\,\mathbf{A} = \nabla \times \nabla \times \mathbf{A}$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \left[\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{k}\right]$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{k} - \frac{\partial}{\partial x}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{j} - \frac{\partial}{\partial y}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{k}$$

$$+ \frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\mathbf{i} + \frac{\partial}{\partial z}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\mathbf{j} - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\mathbf{i}.$$

Let us consider only the x component and call it curl_x curl A.

$$\text{curl}_x\,\text{curl}\,\mathbf{A} = \left[\frac{\partial}{\partial y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\right]\mathbf{i}$$

$$= \left[\frac{\partial^2 A_y}{\partial y \partial x} - \frac{\partial^2 A_x}{\partial y \partial y} - \frac{\partial^2 A_x}{\partial z \partial z} + \frac{\partial^2 A_z}{\partial z \partial x}\right]\mathbf{i}.$$

Adding and subtracting $\partial^2 A_x/\partial x^2$ in this expression, we get

$$\text{curl}\,x\,\text{curl}\,\mathbf{A} = \left[\frac{\partial^2 A_x}{\partial x \partial x} + \frac{\partial^2 A_y}{\partial y \partial x} + \frac{\partial^2 A_z}{\partial z \partial x} - \left(\frac{\partial^2 A_x}{\partial x^2} + \frac{\partial^2 A_x}{\partial y^2} + \frac{\partial^2 A_x}{\partial z^2}\right)\right]\mathbf{i}$$

$$= \mathbf{i}\frac{\partial}{\partial x}\,(\text{div}\,\mathbf{A}) - \nabla^2\,(A_x\mathbf{i}) = \text{grad}_x\,(\text{div}\,\mathbf{A}) - \nabla^2(A_x\mathbf{i})$$

Similarly, we have

$$\text{curl}_y\,\text{curl}\,\mathbf{A} = \text{grad}_y\,(\text{div}\,\mathbf{A}) - \nabla^2\,(A_y\mathbf{j})$$

and $\quad \text{curl}_z\,\text{curl}\,\mathbf{A} = \text{grad}_z\,(\text{div}\,\mathbf{A}) - \nabla^2\,(A_z\mathbf{k})$

$$\therefore \quad \text{curl}\,\text{curl}\,\mathbf{A} = \text{grad}\,\text{div}\,\mathbf{A} - \nabla^2\mathbf{A}.$$

This relation can also be derived by using the identity $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}\,(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$.

as $\quad \nabla \times \nabla \times \mathbf{A} = \nabla\,(\nabla\cdot\mathbf{A}) - (\nabla\cdot\nabla)\,\mathbf{A}$

$$= \text{grad}\,\text{div}\,\mathbf{A} - \nabla^2\mathbf{A}.$$

Example 12. *Express the unit vector which points from $z = h$ on the z-axis toward $(r, \phi, 0)$ in the cylindrical coordinates system.*

The point on the z-axis has $r = 0$ and $z = h$.

The vector **R** is the difference of two vectors

$$\mathbf{R} = (r - 0)\,\hat{r} + (\phi - \phi)\,\hat{\phi} + (0 - h)\,\hat{z} = r\hat{r} - h\hat{z}$$

$$\therefore \quad \text{Unit vector along } \mathbf{R} = \hat{R} = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{r\,\hat{r} - h\,\hat{z}}{\sqrt{r^2 + h^2}}$$

Here the angle ϕ does not appear, because **R** and $\hat{R}$ both vary with ϕ through $\hat{r}$.

Example 13. *Find the area of the strip $\alpha \le \theta \le \beta$ on the spherical shell of radius a, using the spherical coordinates.*

The differential surface element in spherical coordinate system is given by

$$dS = (rd\theta)\,(r\sin\theta\,d\phi)$$

$$\therefore \qquad S \text{ or Area } A = \int_0^{2\pi} d\phi \int_\alpha^\beta a^2 \sin\theta \, d\theta$$

$$= 2\pi a^2 (\cos\alpha - \cos\beta).$$

Example 14. *Transform the vector relation*

$$\mathbf{A} = y\mathbf{a}_x + x\mathbf{a}_y + [x^2/\sqrt{x^2+y^2}\,]\mathbf{a}_z$$

from cartesian to cylindrical coordinate system.

Referring to the Fig. 1.19 for the cylindrical system, we get

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ \rho = (x^2+y^2)^{1/2}.$$

Therefore the given relation becomes

$$\therefore \qquad \mathbf{A} = \rho\sin\phi\,\mathbf{a}_x + \rho\cos\phi\,\mathbf{a}_y + \rho\cos^2\phi\,\mathbf{a}_z.$$

Since the projections of the cartesian unit vectors on $\mathbf{a}_\rho$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$ are obtained as

$$\mathbf{a}_x.\mathbf{a}_\rho = \cos\phi,\ \mathbf{a}_x.\mathbf{a}_\phi = -\sin\phi \text{ and } \mathbf{a}_x.\mathbf{a}_z = 0$$

$$\mathbf{a}_y.\mathbf{a}_\rho = \sin\phi,\ \mathbf{a}_y.\mathbf{a}_\phi = \cos\phi. \text{ and } \mathbf{a}_y.\mathbf{a}_z = 0$$

$$\mathbf{a}_z.\mathbf{a}_\rho = 0,\ \mathbf{a}_z.\mathbf{a}_\phi = 0 \text{ and } \mathbf{a}_z.\mathbf{a}_z = 1$$

$$\therefore \qquad \mathbf{a}_x = \cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi$$

$$\mathbf{a}_y = \sin\phi\,\mathbf{a}_\rho + \cos\phi\,a_\phi \text{ and } \mathbf{a}_z = \mathbf{a}_z$$

$$\therefore \qquad \mathbf{A} = \rho\sin\phi\,(\cos\phi\,\mathbf{a}_\rho - \sin\phi\,\mathbf{a}_\phi) + \rho\cos\phi\,(\sin\phi\,\mathbf{a}_\rho + \cos\phi\,\mathbf{a}_\phi) + \rho\cos^2\phi\,\mathbf{a}_z$$

$$= 2\rho\sin\phi\cos\phi\,\mathbf{a}_\rho + (\rho\cos^2\phi - \rho\sin^2\phi)\,\mathbf{a}_\phi + \rho\cos^2\phi\,\mathbf{a}_z.$$

Example 15. *Determine the divergence of the vector field* **F** *given in the spherical coordinates, as*

$$\mathbf{F} = (1/r^2)\cos\theta\,\mathbf{a}_r + r\sin\theta\cos\phi\,\mathbf{a}_\theta + \cos\theta\,\mathbf{a}_\phi.$$

We know that the relation for the divergence in the spherical polar coordinate is given by

$$\nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(F_\phi).$$

In the given problem $F_r = (1/r^2)\cos\theta$, $F_\theta = r\sin\theta\cos\phi$, $F_\phi = \cos\theta$.

$$\therefore \qquad \nabla\cdot\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2.\frac{1}{r^2}\cos\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta.\,r\sin\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)$$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}(\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(r.\sin^2\theta\cos\phi) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\cos\theta)$$

Since $\partial/\partial r$, $\partial/\partial\theta$ and $\partial/\partial\phi$ are the partial derivatives with respect to r, θ and ϕ respectively, therefore

$$\frac{\partial}{\partial r}(\cos\theta) = 0,\ \frac{\partial}{\partial\theta}(r\sin^2\theta\cos\phi) = r\cos\phi\,(2\sin\theta\cos\theta),\ \frac{\partial}{\partial\phi}(\cos\theta) = 0.$$

$$\therefore \qquad \nabla\cdot\mathbf{F} = \frac{1}{r^2}\times 0 + \frac{1}{r\sin\theta}\,2r\cos\phi\sin\theta\cos\theta + \frac{1}{r\sin\theta}\times 0$$

$$= 2\cos\theta\cos\phi.$$

Example 16. *If* $\mathbf{A} = \rho\cos\phi\,\mathbf{a}_\rho + \sin\phi\,\mathbf{a}_\phi$, *evaluate its line integral around the path shown in Fig. 1.20. Confirm this using Stokes's theorem.*

The closed line integral of **A** around the path *abcd* is

$$\oint \mathbf{A}\cdot d\mathbf{l} = \int_a^b \mathbf{A}\cdot d\mathbf{l} + \int_b^c \mathbf{A}\cdot d\mathbf{l} + \int_c^d \mathbf{A}\cdot d\mathbf{l} + \int_d^a \mathbf{A}\cdot d\mathbf{l},$$

where path L has been divided into segments *ab, bc, cd* and *da*.

Along ab, $\phi = 30°$ and $d\mathbf{l} = d\rho\,\mathbf{a}_\rho$, hence

$$\int_a^b \mathbf{A}\cdot d\mathbf{l} = \int_{\rho=2}^{5} \rho\cos\phi\, d\rho = \cos 30° \int_2^5 \rho d\rho = \frac{\sqrt{3}}{2}\left[\frac{1}{2}\rho^2\right]_2^5 = \frac{21\sqrt{3}}{4}.$$

Along bc, $\rho = 5$ and $d\mathbf{l} = \rho d\phi\,\mathbf{a}_\phi$. Hence

$$\int_b^c \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=30°}^{60°} \rho\sin\phi\, d\phi = 5\left[-\cos\phi\right]_{30°}^{60°} = \frac{5}{2}\left(\sqrt{3}-1\right)$$

Along cd, $\phi = 60°$, and $d\mathbf{l} = d\rho\,\mathbf{a}_\rho$, hence

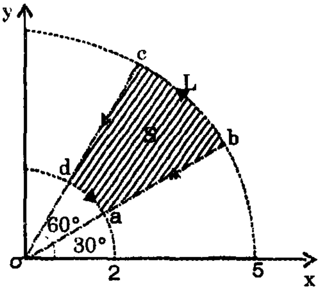


Fig. 1.20.

$$\int_c^d \mathbf{A}\cdot d\mathbf{l} = \int_{\rho=5}^{2} \rho\cos\phi\, d\rho = \cos 60°\left[\frac{\rho^2}{2}\right]_5^2 = -\frac{21}{4}$$

Along da, $\rho = 2$ and $d\mathbf{l} = \rho d\phi\,\mathbf{a}_\phi$, hence

$$\int_d^a \mathbf{A}\cdot d\mathbf{l} = \int_{\phi=60°}^{30°} \rho\sin\phi\, d\phi = 2\left[-\cos\phi\right]_{60°}^{30°} = -\left(\sqrt{3}-1\right)$$

$$\therefore \quad \oint \mathbf{A}\cdot d\mathbf{l} = \text{Sum of above four values}$$

$$= \frac{21\sqrt{3}}{4} + \frac{5}{2}\left(\sqrt{3}-1\right) - \frac{21}{4} - \left(\sqrt{3}-1\right) = 4.94.$$

From Stokes's theorem $\oint_L \mathbf{A}\cdot d\mathbf{l} = \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S}$.

Here $d\mathbf{S} = \rho d\phi d\rho\,\mathbf{a}_z$, therefore

$$\nabla\times\mathbf{A} = \left[\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right]\mathbf{a}_\rho + \left[\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right]\mathbf{a}_\phi + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}\left(\rho A_\phi\right) - \frac{\partial A_\rho}{\partial\phi}\right]\mathbf{a}_z$$

$$= (0-0)\mathbf{a}_\rho + (0-0)\mathbf{a}_\phi + \frac{1}{\rho}(1+\rho)\sin\phi\,\mathbf{a}_z$$

$$\therefore \quad \int_S (\nabla\times\mathbf{A})\cdot d\mathbf{S} = \int_{\phi=30°}^{60°}\int_{\rho=2}^{5}\frac{1}{\rho}(1+\rho)\sin\phi\,\rho d\rho d\phi = \int_{30°}^{60°}\sin\phi\, d\phi \int_2^5 (1+\rho)\, d\rho$$

$$= \left[-\cos\phi\right]_{30°}^{60°}\left[\rho + \frac{\rho^2}{2}\right]_2^5 = \frac{27}{4}\left(\sqrt{3}-1\right) = 4.941.$$

Hence proved the theorem.

Example 17. *Find the Laplacian of the scalar field*

$$\Phi = Ar\sin^2\theta\cos\phi.$$

Since the scalar field is given in spherical coordinates, therefore

$$\nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}$$

Since $\frac{\partial\Phi}{\partial r.} = A\sin^2\theta\cos\phi$, $\frac{\partial\Phi}{\partial\theta} = Ar\sin 2\theta\cos\phi$, and $\frac{\partial\Phi}{\partial\theta} = -Ar\sin^2\theta\cos\phi$.

$$\therefore \nabla^2\Phi = \frac{1}{r^2}\frac{\partial}{\partial r}(Ar^2\sin^2\theta\cos\phi) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(Ar\sin 2\theta\sin\theta\cos\phi) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}(-Ar\sin^2\theta\sin\phi)$$

$$= \frac{2A}{r}\sin^2\theta\cos\phi + \frac{2A}{r}\cos 2\theta\cos\phi + \frac{2A}{r}\cos^2\theta\cos\phi - \frac{A}{r}\cos\phi$$

$$= \frac{A\cos\phi}{r}[2\sin^2\theta + 2\cos 2\theta + 2\cos^2\theta - 1] = \frac{A\cos\phi}{r}(1 + 2\cos{}^2\theta).$$

ORAL QUESTIONS

1. Whether is the current a vector quantity ?
2. Identify which of the following quantities is not a vector : (*a*) forces, (*b*) work, (*c*) momentum, (*d*) energy, (*e*) accelerator, (*f*) weight.
3. Differentiate scalar, pseudoscalar and pseudovector.
4. Give the necessary condition for the two vectors to be Collinear and Coplaner.
5. What is vector field ?
6. Is it correct that the gradient is invariant ?
7. The field is incompressible and is called .... field.
8. Which of these is correct ?
   (*a*) $\mathbf{A} \times \mathbf{A} = |\mathbf{A}|^2$, (*b*) $\mathbf{A} \times \mathbf{B} + \mathbf{B} \times \mathbf{A} = 0$, (*c*) $\mathbf{a}_x . a_y = \mathbf{a}_z$.
9. Which of the following statements are meaningless ?
   (*a*) $\mathbf{A} \cdot \mathbf{A} + \mathbf{B} \cdot \mathbf{B} = 0$; (*b*) $\mathbf{A}\,(\mathbf{A} + \mathbf{B}) = 2$, (*c*) $\mathbf{A} \cdot \mathbf{B} + 2\mathbf{B} = 6$.
10. At every point in space $\mathbf{a}_\theta . \mathbf{a}_\phi = 1$, false or true.
11. Which of the following is a meaningless combination : (*a*) grad div, (*b*) div curl, (*c*) curl grad, (*d*) curl div.
12. If $\mathbf{r} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$, the position vector of point (x, y, z), which of the following is incorrect ?
    (*a*) $\nabla . \mathbf{r} = 1$, (*b*) $\nabla \times \mathbf{r} = 0$; (*c*) $\nabla r = \mathbf{r}/r$.
13. The net outward flux of some vector quantity through a closed surface can be represented by (*a*) divergence of that quantity, (*b*) curl of that quantity or (*c*) Stoke's Theorem.
14. For three vectors **A**, **B** and **C**, which of the following is/are not well defined ?
    (*a*) $m\mathbf{A} + n\mathbf{B}$. (*b*) $m\mathbf{A} \cdot n\mathbf{B}$, (*c*) $\mathbf{A} \cdot \mathbf{B} \times \mathbf{C}$, (*d*) $\mathbf{A} \times \mathbf{B} \times \mathbf{C}$
15. Show how three equal magnitude vectors would have to be oriented if they were to add to give zero. Can this be done with three unequal vectors ? Two unequal vectors ?
16. Find the dot and cross products of two similar vectors ?
17. Why is the work a scalar quantity and the moment of the force a vector quantity ?
18. If ∇ operates **v** *when* **u** is assumed as constant, then which of the following is the correct form $(\nabla \cdot \mathbf{u})\mathbf{v}$ or $(\mathbf{u} \cdot \nabla)\mathbf{v}$?
19. Find the surface integral of a curl of *a* vector field when the surface is closed.
20. If two vectors **A** and **B** are not parallel in question $a\mathbf{A} + b\mathbf{B} = 0$, then find the values of *a* and *b*.

Ans. 1. No; **2.** *b*; **6.** yes; **7.** solenoidal; **8.** (*b*); **9.** *b*, *c*; **10.** false; **11.** (*d*); **12.** (*a*); **13.** (*a*) **14.** (*d*).

PROBLEMS

1. A river flows south east at 10 km/hr and a boat flows upon it with its bow pointed in the direction of travel. A man walks upon the deck at 2 km/hr in a direction to the right and perpendicular to the direction of the boat's movement. Calculate the absolute velocity of the man.
   (10.2 km/hr at 56.3° south of east)
2. Prove that the line joining the vertex of a parallelogram and the mid–point of the opposite side trisects the diagonal.
3. Prove that $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$, $\mathbf{a} \times \mathbf{a} = 0$ and $(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) = 2\,(\mathbf{a} \times \mathbf{b})$.
4. Show that the vectors $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{B} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ and $\mathbf{C} = 3\mathbf{i} - 4\mathbf{j} - 4\mathbf{k}$ form the sides of a right triangle.
5. Find two unit vectors parallel to the diagonal of a parallelogram whose sides are $2\mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$ and $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$. Find the area of the parallelogram. $\left[\frac{1}{7}(3\mathbf{i} + 6\mathbf{j} - 2\mathbf{k}), \frac{1}{\sqrt{69}}(-\mathbf{i} - 2\mathbf{j} + 8\mathbf{k}), 11\sqrt{5}\right]$
6. If $|\mathbf{A} + \mathbf{B}| = |\mathbf{A} - \mathbf{B}|$, prove that **A** and **B** are mutually perpendicular.
7. Given $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$. What is the angle between $\mathbf{A} \times \mathbf{B}$ and the axis of **z** ?
8. If **A** and **B** are the sides of a parallelogram and **C** and **D** are the diagonals, show that $C^2 + D^2 = 2(A^2 + B^2)$.

9. Prove that the components of **F** in the plane of **a** and **F**, along and perpendicular to **a** are $\frac{\mathbf{a}.\mathbf{F}}{a^2}\,\mathbf{a}$ and $\frac{(\mathbf{a}\times\mathbf{F}\times\mathbf{a})}{a^2}$ respectively.
10. Find grad v, where $v = r^m$ and $r = (x^2 + y^2 + z^2)^{1/2}$ $[mr^{m-2}]$
11. Find div **F** and curl **F**, where $\mathbf{F} = \text{grad}\,(x^3 + y^3 + z^3 - 3xyz)$. $[6(x + y + z)$ and $0]$
12. Evaluate $\nabla^2 r$, where $r = (x^2 + y^2 + z^2)^{1/2}$. . $[2r^{-1}]$
13. Show that the potential function $\varphi = q(x^2 + y^2 + z^2)^{-1/2}$ satisfies the Laplace's equation.
14. Convert the vector $\mathbf{A} = z \sin\phi\,\mathbf{a}_\rho - \rho\cos\phi\,\mathbf{a}_\phi + 2\rho\, z\,\mathbf{a}_z$ in cartesian coordinates.

$$\left[\frac{xyz}{x^2+y^2}+\frac{xy}{(x^2+y^2)^{1/2}}\right]\mathbf{a}_x+\left(\frac{y^2z}{x^2+y^2}-\frac{x^2}{(x^2+y^2)^{1/2}}\right)\mathbf{a}_y+2z\,(x^2+y^2)^{1/2}\mathbf{a}_z$$

15. If **r** is the position vector of point (x, y, z) and $r = |\,\mathbf{r}\,|$, prove that : (*a*) $\nabla(\log r) = \mathbf{r}/r$, (*b*) $\nabla^2(\log r) = 1/r^2$.
16. Verify the divergence theorem for $\mathbf{A} = xy^2\mathbf{i} + y^3\mathbf{j} + y^2z\mathbf{k}$ and the surface of the cuboid defined by $0 < x < 1$, $0 < y < 1$, $0 < z < 1$.
17. Show that $\int_V r^n dV = \frac{1}{n+3}\oint_c r^n\mathbf{r}\cdot d\mathbf{r}$.
18. Find the distance between $(1, \pi/4, 0)$ and $(1, 3\pi/4, \pi)$, where the points are given in spherical coordinates. (2.0)
19. Use the cylindrical coordinates to find the area of the curved surface and the volume of a right circular cylinder of radius a and height h. $(2\pi ah, \pi a^2 h)$
20. Using spherical coordinates to express the differential volume, obtain the volume defined by $1 \le r \le 2m$, $0 \le \theta \le \pi/2$ and $0 \le \phi \le \pi/2$. $(2\pi/6\ m^3)$
21. Determine the divergence of the vector $\mathbf{B} = \rho z \sin\phi\,\mathbf{a}_\rho + 3\rho z^2\cos\phi\,\mathbf{a}_\phi$ and evaluate at point $(5, \pi/2, 1)$. $[(2 - 3z)\,z\sin\phi, -1]$

2

PARTIAL DERIVATIVES & REPEATED INTEGRALS

2.1 FUNCTIONS OF TWO OR THREE VARIABLES

The physical quantities are, in general, functions of space and time, the quantities may be described uniquely in space in a suitable manner. This requires a choice of appropriate coordinate system. The coordinate system is called *orthogonal* if the coordinates are mutually perpendicular. The examples are: *cartesian*, *cylindrical* and *spherical*, which are generally used.

In the cartesian coordinate system, the variables are x, y and z, each may range from $-\infty$ to $+\infty$. In cylindrical coordinate system, the variables are ρ, ϕ and z (also written sometimes as r, θ, z) with ranges

$$0 \le \rho \le \infty, \quad 0 \le \phi \le 2\pi, \quad -\infty < z < \infty.$$

In spherical coordinate system, the variables are r, θ and ϕ, with ranges,

$$0 \le r \le \infty, \quad 0 \le \theta \le \pi, \quad 0 \le \phi \le 2\pi.$$

If we consider a set of n independent variables $x, y, z, \ldots t$ and one dependent variable ψ, the equation

$$\psi = \psi\,(x, y, z \ldots t) \qquad \ldots(1)$$

is called a *functional relation*, where the dependent variable ψ is said to be the function of the independent variables $x, y, z, \ldots t$. The arbitrarily assigned values of the independent variables will give corresponding values of the dependent variable ψ. It is rarely possible to obtain an equation expressing one of the variables explicity in terms of the others. Thus most of the functions with more then one variables are implicit functions.

If x and y are two independent variables and a variable ψ depends for its values on the values of these x and y by a functional relation

$$\psi = f\,(x, y) \qquad \ldots(2)$$

Then ψ is called *explicit function* of x and y. The ordered pair of numbers (x, y) is called a *point* and the aggregate of the pairs of numbers (x_i, y_i) is said to be the *domain of definition*.

When the domain of definition is bounded by a closed curve C, it is said to be closed if f is defined for all points within and on the curve C, but open or *unclosed* if f is defined for points with in but not on the curve C.

The function f is said to be continuous at a point (a, b) of its domain of definition if

$$\operatorname*{Limit}_{\substack{x\to a\\ y\to b}} f(x,y) = f(a,b). \qquad ...(3)$$

The position of a point in a space is generally represented by three variables, the coordinates (x, y, z in cartesian, ρ, ϕ and z in cylindrical, and r, θ, ϕ in spherical coordinate systems). Therefore the position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or $r = f(x, y, z)$. The temperature of the rod, heated at one end depends upon the distance from the hot end and the time, *i.e.* $T = f(x, t)$. This is the example of a function of two variables x and t.

In electricity, the nonrelativistic physical quantities are generally the functions of two or three variables.

2.2 PARTIAL DERIVATIVES

The ordinary derivative of a function of several variables with respect to one of the independent variables, keeping all other independent variables constant is called the partial derivative of the function with respect to this variable. Thus for a function $y = f(x, y)$. The partial derivative of this function with respect to x is the ordinary derivative of ψ when y is assumed as constant. It is denoted by $\partial\psi/\partial x$ or $\partial f/\partial x$ or f_x. The partial derivative with respect to y is denoted by $\partial\psi/\partial y$ or $\partial f/\partial y$ or f_y. Whenever we take the case of two independent variables, we generally assume them to be x and y and the dependent variable or function as z., *i.e.*, $z = f(x, y)$. The partial derivatives are expressed as

$$\frac{\partial f}{\partial x} = \operatorname*{Limit}_{\delta x\to 0} \frac{f(x+\delta x,y)-f(x,y)}{\delta x}$$

and

$$\frac{\partial f}{\partial y} = \operatorname*{Limit}_{\delta y\to 0} \frac{f(x,y+\delta y)-f(x,y)}{\delta y}. \qquad ...(4)$$

The partial derivatives at a particular point P with coordinates (a, b) are often denoted by $(\partial f/\partial x)_{a,\,b}$ or $f_x(a, b)$ and $(\partial f/\partial y)_{a,\,b}$ or $f_y(a, b)$.

$$\therefore \quad f_x(a, b) = \operatorname*{Limit}_{h\to 0} \frac{f(a+h,b)-f(a,b)}{h}$$

and

$$f_y(a, b) = \operatorname*{Limit}_{k\to 0} \frac{f(a,b+k)-f(a,b)}{k} \qquad ...(5)$$

in case the limit exists.

The derivative functions f_x, f_y are in general functions of x and y and hence may themselves have partial derivatives with respect to x and y. Thus we may have the second order of partial derivatives, such as

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx} = f_{x^2}, \quad \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy} = f_{y^2}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\partial y} = f_{xy} \text{ and } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\partial x} = f_{yx}. \qquad ...(6)$$

The higher order partial derivatives may be obtained in the similar manner.

The second order partial derivatives at a particular point (a, b) are denoted by

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_{a,b} = \frac{\partial^2 f(a,b)}{\partial x^2} = f_{xx}(a,b) = f_{x^2}(a,b)$$

$$= \operatorname*{Limit}_{h\to 0} \frac{f_x(a+h,b)-f_x(a,b)}{h} \qquad ...(7)$$

$$
\begin{aligned}
\left(\frac{\partial^2 f}{\partial x \partial y}\right)_{a,b} &= \frac{\partial^2 f(a,b)}{\partial x \partial y} = f_{xy}(a,b) \\
&= \operatorname*{Limit}_{h\to 0} \frac{f_y(a+h,b) - f_y(a,b)}{h} \\
&= \operatorname*{Limit}_{h\to 0} \frac{1}{h}\left[\operatorname*{Limit}_{k\to 0} \frac{f(a+h,b+k) - f(a+h,b)}{k} - \operatorname*{Limit}_{k\to 0} \frac{f(a,b+k) - f(a,b)}{k}\right] \\
&= \operatorname*{Limit}_{h\to 0} \operatorname*{Limit}_{k\to 0} \left[\frac{f(a+h,b+k) - f(a+h,b) - f(a,b+k) + f(a,b)}{hk}\right] \\
&= \operatorname*{Limit}_{h\to 0} \operatorname*{Limit}_{k\to 0} \frac{\phi(h,k)}{hk} \qquad \text{...(8)}
\end{aligned}
$$

2.3 GEOMETRICAL INTERPRETATION OF PARTIAL DERIVATIVES OF FUNCTIONS OF TWO VARIABLES

The simple example of the function of two variables is the position vectors in a plane. The position of any point A on a x-y plane may be represented by its position coordinates (x, y). If the point is displaced to point B on the same plane, with position coordinates $(x + \Delta x, y + \Delta y)$. We may reach point B from point A either via point D or point C. In the former case we first move a distance Δx along the x-axis and then Δy along the y-axis and in the later case we first move Δy along y-axis and then Δx along x-axis. Thus we see that the resultant position remains the same and independent of the order.

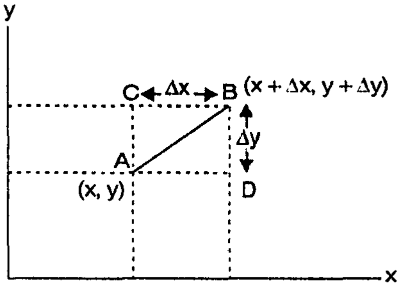


Fig. 2.1. Geometrical interpretation of partial derivatives.

For a continuous and differentiable function f, the partial derivative has the same value in whatever order the different operations are performed, *i.e.*,

$$f_{xy} = f_{yx}, f_{xyx} = f_{xxy} = f_{yxx}.$$

The equality of f_{xy} and f_{yx} may or may not occur, if the conditions of continuity and differentiability are not satisfied. Thus the condition of equality of $f_{xy} = f_{yx}$ is sufficient but not necessary.

2.4 DIFFERENTIAL OF A FUNCTION OF TWO VARIABLES

Let $z = f(x, y)$ be a function of two independent variables x and y, defined in a domain N and let it be differentiable at a point (x, y) of the domain.

The *first differential* of a function z, denoted by dz is given by

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy. \qquad \text{...(9)}$$

Here we see that dz remains same whether we change it first along x and then along y or a vice versa.

If dx and dy are regarded as constants and if $\partial z/\partial x$ and $\partial z/\partial y$ are differentiable at (x, y), then dz is a function of x and y and is itself differentiable at (x, y). The differential of dz is denoted by d^2z and is called *second differential* of z. It is calculated in the same way, *i.e.*,

$$d^2z = d\,(dz) = d\left(\frac{\partial z}{\partial x}\right)dx + d\left(\frac{\partial z}{\partial y}\right)dy$$

Since $$d = \left(\frac{\partial}{\partial x}\right)dx + \left(\frac{\partial}{\partial y}\right)dy, \text{ Therefore } d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)dy$$

or $$d\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}dx + \frac{\partial^2 z}{\partial y \partial x}dy.$$

Similarly $$d\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)dy = \frac{\partial^2 z}{\partial x \partial y}dx + \frac{\partial^2 z}{\partial y^2}dy.$$

Since $\partial z/\partial x$ and $\partial z/\partial y$ are differentiable, therefore $\partial^2 z/\partial x\partial y = \partial^2 z/\partial y\partial x$,

$$\therefore \qquad d^2 z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x\partial y}dxdy + \frac{\partial^2 z}{\partial y^2}dy^2. \qquad ...(10)$$

Again $d^2 z$ is differentiable at (x, y) if all the second order partial derivatives $\partial^2 z/\partial x^2$, $\partial^2 z/\partial x\partial y$, $\partial^2 z/\partial y^2$ are differentiable at (x, y). Thus we will get the *third differential* of as

$$d^3 z = \frac{\partial^3 z}{\partial x^3}dx^3 + 3\frac{\partial^3 z}{\partial x^2\partial y}dx^2dy + 3\frac{\partial^2 z}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3 z}{\partial y^3}dy^3. \qquad ...(11)$$

Proceeding in this manner, one may find successive differentials. Thus the differential of n^{th} order may be obtained as

$$d^n z = \frac{\partial^n z}{\partial x^n}dx^n + n\frac{\partial^n z}{\partial x^{n-1}\partial y}dx^{n-1}dy + \frac{n(n-1)}{2!}\frac{\partial^n z}{\partial x^{n-2}\partial y^2}dx^{n-2}dy^2 + \ldots\frac{\partial^2 z}{\partial y^n}dy^n \qquad ...(12)$$

$$= \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy\right)^n z, \qquad ...(13)$$

where $\partial/\partial x$ and $\partial/\partial y$ are the differential operators operating a function z, not dx and dy.

2.5 DIFFERENTIAL OF A FUNCTION OF THREE VARIABLES

Let $\psi = f(x, y, z)$ be a function of three independent variables x, y and z, defined in a domain N and let it be differentiable at a point (x, y, z) of the domain. The first differential of a function ψ, denoted by $d\psi$ is given by

$$d\psi = \frac{\partial\psi}{\partial x}dx + \frac{\partial\psi}{\partial y}dy + \frac{\partial\varphi}{\partial z}dz, \qquad ...(14)$$

Here we see that $d\psi$ remains the same and is independent of the order of independent variables taken for the purpose.

If dx, dy and dz are regarded as constants and if $\partial\psi/\partial x$, $\partial\psi/\partial y$ and $\partial\psi/\partial z$ are differentiable at (x, y, z), then $d\psi$ will also be differentiable at (x, y, z). The differential of $d\psi$ is denoted by $d^2\psi$ and is called second differential of ψ. It is calculated in the same way as discussed in the proceeding article.

$$d^2\psi = d\,(d\psi) = d\left(\frac{\partial\psi}{\partial x}\right)dx + d\left(\frac{\partial\psi}{\partial y}\right)dy + d\left(\frac{\partial\psi}{\partial z}\right)dz.$$

Since $$d = \left(\frac{\partial}{\partial x}\right)dx + \left(\frac{\partial}{\partial y}\right)dy + \left(\frac{\partial}{\partial z}\right)dz,$$ therefore

$$d\left(\frac{\partial\psi}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial x}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial x}\right)dz = \frac{\partial^2\psi}{\partial x^2}dx + \frac{\partial^2\psi}{\partial y\,\partial x}dy + \frac{\partial^2\psi}{\partial z\,\partial x}dz.$$

$$d\left(\frac{\partial\psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial y}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial y}\right)dz = \frac{\partial^2\psi}{\partial x\partial y}dx + \frac{\partial^2\psi}{\partial y^2}dy + \frac{\partial^2\psi}{\partial z\,\partial y}dz.$$

and $$d\left(\frac{\partial\psi}{\partial z}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial z}\right)dx + \frac{\partial}{\partial y}\left(\frac{\partial\psi}{\partial z}\right)dy + \frac{\partial}{\partial z}\left(\frac{\partial\psi}{\partial z}\right)dz = \frac{\partial^2\psi}{\partial x\,\partial z}dx + \frac{\partial^2\psi}{\partial y\,\partial z}dy + \frac{\partial^2\psi}{\partial z^2}dz.$$

$$\therefore \qquad d^2\psi = \frac{\partial^2\psi}{\partial x^2}dx^2 + \frac{\partial^2\psi}{\partial y\,\partial x}dydx + \frac{\partial^2\psi}{\partial z\,\partial x}dzdx + \frac{\partial\psi}{\partial x\,\partial y}dxdy + \frac{\partial^2\psi}{\partial y^2}dy^2 + \frac{\partial^2\psi}{\partial z\,\partial y}dzdy$$
$$+ \frac{\partial^2\psi}{\partial x\,\partial z}dxdz + \frac{\partial^2\psi}{\partial y\,\partial z}dydz + \frac{\partial^2\psi}{\partial z^2}dz^2.$$

Since $\partial\psi/\partial x$, $\partial\psi/\partial y$, and $\partial\psi/\partial z$ are differentiable, therefore $\partial^2\psi/\partial y\partial x = \partial^2\psi/\partial x\partial y$, $\partial^2\psi/\partial z\partial x = \partial^2\psi/\partial x\partial z$ and $\partial^2\psi/\partial y\,\partial z = \partial^2\psi/\partial z\partial y$,

$$\therefore \quad d^2\psi = \frac{\partial^2\psi}{\partial x^2}dx^2 + 2\frac{\partial^2\psi}{\partial x\partial y}dxdy + \frac{\partial^2\psi}{\partial y^2}dy^2 + 2\frac{\partial^2\psi}{\partial x\partial z}dxdz + 2\frac{\partial^2\psi}{\partial y\partial z}dydz + \frac{\partial^2\psi}{\partial z^2}dz^2. \quad ...(15)$$

Again $d^2\psi$ is differentiable at x, y, z if all the second order partial derivatives are differentiable at x, y, z. Thus we will get the third order differential of ψ as

$$d^3\psi = \frac{\partial^3\psi}{\partial x^3}dx^3 + 3\frac{\partial^3\psi}{\partial x^2\partial y}dx^2dy + 3\frac{\partial^3\psi}{\partial x\partial y^2}dxdy^2 + \frac{\partial^3\psi}{\partial y^3}dy^3 + 3\frac{\partial^3\psi}{\partial x^2\partial z}dx^2dz + 3\frac{\partial^3\psi}{\partial x\partial z^2}dxdz^2$$

$$+ 3\frac{\partial^3\psi}{\partial y^2\partial z}dy^2dz + 3\frac{\partial^3\psi}{\partial y\partial z^2}dydz^2 + \frac{\partial^3\psi}{\partial z^3}dz^3 \quad ...(16)$$

Proceeding in this manner, the differential of n^{th} order may be obtained as

$$d^n\psi = \left(\frac{\partial}{\partial x}dx + \frac{\partial}{\partial y}dy + \frac{\partial}{\partial z}dz\right)^n \psi. \quad ...(17)$$

For n variables function, we have

$$d^n\psi = \left(\frac{\partial}{\partial x_1}dx_1 + \frac{\partial}{\partial x_2}dx_2 + \frac{\partial}{\partial x_3}dx_3 + \ldots\ldots \frac{\partial}{\partial x_n}dx_n\right)^n \psi \quad ...(18)$$

2.6 DERIVATIVES OF COMPOSITE FUNCTIONS

If z is a differentiable function of x and y, which are differentiable functions of two independent variables u and v, then z possesses continuous partial derivatives with respect to u and v.

$$x = f(u, v) \quad \text{and} \quad y = f(u, v), \text{ therefore}$$

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv \quad \text{and} \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv. \quad ...(19)$$

$$\therefore \quad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = \frac{\partial z}{\partial x}\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right) + \frac{\partial z}{\partial y}\left(\frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv\right)$$

$$= \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}\right)du + \left(\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}\right)dv. \quad ...(20)$$

Since z is a differentiable function of x and y and x and y are differentiable functions of u and v. Thus z is a differentiable function of u and v and

$$dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv. \quad ...(21)$$

From equations (20) and (21), we get

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} \quad \text{and} \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}. \quad ...(22)$$

Thus we see that $\partial z/\partial u$ and $\partial z/\partial v$ are continuous functions of u and v.

2.7 APPLICATIONS

(*a*) Extreme Values (Maxima and Minima) : The function f is said to have an extreme values at c if $f(c)$ is either a maximum or a minimum value. Thus at an extreme point c, $f(x) - f(c)$ keeps the same sign for all values of x in a small neighbourhood of c. The value is maximum if $f(c)$ is the greatest value of the function and is minimum if $f(c)$ is the least value of the function in a small neighbourhood of c.

The *necessary condition* for the function $f(c)$ to be maximum or minimum (or of extreme value) is that $f'(c) = 0$, *i.e., if the derivative exists, it must vanish at the extreme value.*

In case of two variable system, let (a, b) be a point of the domain of definition of a function f, then $f(a, b)$ is an extreme value of f, if for every point (x, y) of some neighbourhood of (a, b), the difference $f(x, y) - f(a, b)$ keeps the same sign. It is called a maximum or minimum value according to the sign of $f(x, y) - f(a, b)$ is negative or positive.

If the partial derivatives exist, the *necessary condition* for $f(x, y)$ to have an extreme value at (a, b) is that $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

The general rule is that if $f(a, b)$ is an extreme value of $f(x, y)$, if $f_x(a, b) = f_y(a, b) = 0$ and $f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$ and this extreme value is a maximum or a minimum according as $f_{xx}(a, b)$ or $f_{yy}(a, b)$ is negative or positive respectively. The further investigation is necessary if $f_{xx}(a, b)\, f_{yy}(a, b) - [f_{xy}(a, b)]^2 = 0$.

In case of three variable system, *i.e.*, a function $f(x, y, z)$ of three independent variables sufficient conditions for (a, b, c) to be an extreme point are that

$$df = f(x, y, z) - f(a, b, c) = f_x dx + f_y dy + f_z dz, \text{ so that}$$

$$f_x = f_y = f_z = 0 \qquad \text{...(23)}$$

and

$$d^2f(a, b, c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2 + 2f_{xy}dxdy + 2f_{yz}dydz + 2f_{zx}dzdx \qquad \text{...(24)}$$

keeps the same sign. The extreme point being a maxima or minima according to as d^2f is negative or positive. The further investigation is required if d^2f does not keep the same sign.

For a function of n-variables, the point $(a_1, a_2 \ldots a_n)$ is said to be an extreme point and $f(a_1, a_2, \ldots a_n)$ an extreme value of a function f, if for every point $(x_1, x_2 \ldots x_n)$, other than this extreme point in its neighbourhood, the difference

$$f(x_1, x_2, \ldots x_n) - f(a_1, a_2, \ldots a_n)$$

keeps the same sign. The extreme value is a maximum or a minimum value according as the sign is negative or positive respectively.

The necessary conditions for $f(a_1, a_2, \ldots a_n)$ to be an extreme value of the function f are that all the partial derivatives $f_{x_1}, f_{x_2}, \ldots f_{x_n}$, if they exist, vanish at $(a_1, a_2, \ldots a_n)$. The point $(a_1, a_2, \ldots a_n)$ is called a stationary point if all the first order partial derivatives of the function vanish at that point, such as

$$f_{x_1}(x_1, x_2, \ldots x_n) = f_{x_2}(x_1, x_2, \ldots x_n) = \ldots f_{x_n}(x_1, x_2, \ldots x_n) = 0. \qquad \text{...(25)}$$

Therefore the differential df of the function vanishes at a stationary point.

***(b)* Change of Variables** : The change of variables is frequently required to transform a particular expression involving a combination of derivatives with respect to one set of variables into the derivatives with respect to another set of variables. In physics change of variables is needed generally when we require the physical function in another coordinate system, such as shifting of origin, rotation of the axes, change in the system of coordinates (cartesian to cylindrical or spherical).

If z is a function $f(x, y)$ of the independent variables x and y, which are changed to new independent variables u, v by assuming $x = \psi(u, v)$, $y = \psi(u, v)$. Let us express the derivatives of z with respect to x, y in terms of u, v.

As discussed earlier, the derivatives of z with respect to (u, v) are given as

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v} \qquad \text{...(22)}$$

Solving these for $\partial z/\partial x$ and $\partial z/\partial y$, we get

$$\frac{\partial z}{\partial x} = A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}, \quad \frac{\partial z}{\partial y} = C\frac{\partial z}{\partial u} + D\frac{\partial z}{\partial v}, \qquad \text{...(26)}$$

where $A = (\partial y/\partial v)/J$, $B = (-\partial y/\partial u)/J$, $C = (-\partial x/\partial v)/J$, $D = (\partial x/\partial u)/J$,

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} \text{ called the Jacobian.} \qquad \text{...(27)}$$

Thus the Jacobian is the functional determinant of the differentiable functions with respect to variables.

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial(y,z)}{\partial(u,v)}\Big/\frac{\partial(x,y)}{\partial(u,v)} \text{ and } \frac{\partial z}{\partial y} = -\frac{\partial(z,x)}{\partial(u,v)}\Big/\frac{\partial(x,y)}{\partial(u,v)} \qquad \text{...(28)}$$

Equation (26) expresses $\partial z/\partial x$ and $\partial z/\partial y$ in terms of A, B, C, D, $\partial z/\partial u$ and $\partial z/\partial v$, all of which are functions of u, v and not containing x, y.

For the second order derivative of z with respect to u, v, we use relations (26) as

$$\frac{\partial}{\partial x} = A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v} \quad \text{and} \quad \frac{\partial}{\partial y} = C\frac{\partial}{\partial u} + D\frac{\partial}{\partial v}.$$

Thus
$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(A\frac{\partial}{\partial u} + B\frac{\partial}{\partial v}\right)\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$
$$= A\frac{\partial}{\partial u}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right) + B\frac{\partial}{\partial v}\left(A\frac{\partial z}{\partial u} + B\frac{\partial z}{\partial v}\right)$$
$$= A^2\frac{\partial^2 z}{\partial u^2} + AB\frac{\partial^2 z}{\partial u \partial v} + BA\frac{\partial^2 z}{\partial v du} + B^2\frac{\partial^2 z}{\partial v^2} + \left(A\frac{\partial A}{\partial u} + B\frac{\partial A}{\partial v}\right)\frac{\partial z}{\partial u} + \left(A\frac{\partial B}{\partial u} + B\frac{\partial B}{\partial v}\right)\frac{\partial z}{\partial v} \quad ...(29)$$

$\dfrac{\partial^2 z}{\partial y^2}$ and $\dfrac{\partial^2 z}{\partial x \partial y}$ may be found in the same way as :

$$\frac{\partial^2 z}{\partial y^2} = C^2\frac{\partial^2 z}{\partial u^2} + 2CD\frac{\partial^2 z}{\partial u \partial v} + D^2\frac{\partial^2 z}{\partial v^2} + \left(C\frac{\partial C}{\partial u} + D\frac{\partial C}{\partial v}\right)\frac{\partial z}{\partial u} + \left(C\frac{\partial D}{\partial u} + D\frac{\partial D}{\partial v}\right)\frac{\partial z}{\partial v} \quad ...(30)$$

and
$$\frac{\partial^2 z}{\partial x \partial y} = AC\frac{\partial^2 z}{\partial u^2} + AD\frac{\partial^2 z}{\partial u \partial v} + BC\frac{\partial^2 z}{\partial v \partial u} + BD\frac{\partial^2 z}{\partial v^2} + \left(A\frac{\partial C}{\partial u} + B\frac{\partial C}{\partial v}\right)\frac{\partial z}{\partial u} + \left(A\frac{\partial D}{\partial u} + B\frac{\partial D}{\partial v}\right)\frac{\partial z}{\partial v} \quad ...(31)$$

2.8 ITERATED (REPEATED) INTEGRALS OF MORE THAN ONE VARIABLES

An iterated integral is an integral of the form

$$I = \int_a^b dx \int_{\psi_1}^{\psi_2} f(x,y)\,dy,$$

where ψ_1 or ψ_2 or both are functions of x or constants. This form of integral shows that for each fixed x between a and b, the integral

$$\int_{\psi_1}^{\psi_2} f(x,y)\,dy = F(x)$$

is calculated and then the integral $\int_a^b F(x)\,dx$ is solved.

$$\therefore \quad I = \int_a^b F(x)\,dx = \int_a^b \left[\int_{\psi_1}^{\psi_2} f(x,y)\,dy\right]dx = \int_a^b dx \int_{\psi_1}^{\psi_2} f(x,y)\,dy \quad ...(32)$$

Another form of iterated (repeated) integral may be

$$I = \int_c^d dy \int_{\phi_1}^{\phi_2} f(x,y)\,dx = \int_c^d \left[\int_{\phi_1}^{\phi_2} f(x,y)\,dx\right]dy, \quad ...(33)$$

where ϕ_1 or ϕ_2 or both are the functions of y or constants.

2.9 DOUBLE INTEGRALS

Let us consider a curve whose equation is $y = f(x)$ and let a, $f(a)$, and b, $f(b)$ be the coordinates of the points A and B on it, Fig. 2.2. Consider two closed points P and Q with coordinates in the two dimensional cartesian coordinates as (x, y) and $(x + \Delta x, y + \Delta y)$ on this curve. The area of the elementary strip $PQRS$ between the limits $y = 0$ and $y = f(x)$ is given by

$$\underset{\delta y \to 0}{\text{Limit}} \sum \delta x \delta y = \left[\int_0^{f(x)} dy\right]\delta x$$

Hence the area $ABCD$ = Sum of elementary strips

$$\therefore \text{ Area } ABCD = \underset{\delta x \to 0}{\text{Limit}} \sum \left[\int_0^{f(x)} dy\right]\delta x = \int_a^b \left[\int_0^{f(x)} dy\right]dx$$
$$= \int_a^b \int_0^{f(x)} dx dy. \quad ...(34)$$

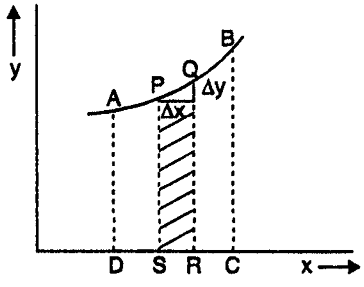


Fig. 2.2.

Here the right hand element indicates the first integration. The area is thus obtained by double integration.

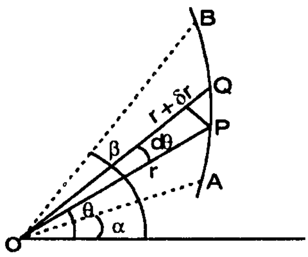


Fig. 2.3.

Let the curve AB be represented in polar coordinates with polar coordinates of A and B points on the curve as α and β respectively, Fig. 2.3. If the radius vector of point P is given by $r = f(\theta)$, the area of

$$OPQ = \underset{\delta r \to 0}{\text{Limit}} \left[\sum r\delta r\right] \delta\theta = \left[\int_0^{f(\theta)} r dr\right] \delta\theta$$

$$\therefore \quad \text{Area } OAB = \underset{\delta\theta \to 0}{\text{Limit}} \sum \left[\int_0^{f(\theta)} r dr\right] \delta\theta = \int_\alpha^\beta \left[\int_0^{f(\theta)} r dr\right] d\theta$$

$$= \int_\alpha^\beta \int_0^{f(\theta)} r d\theta dr, \qquad \text{...(35)}$$

Here the right hand element dr indicates first integration.

For a rectangle R (a, b, c, d), the double integration

$$I = \iint_R f\, dxdy \qquad \text{...(36)}$$

If $\int_c^d f dy$ *also exists for each fixed x in (a, b). Then the iterated integral* $\int_a^b dx \int_c^d f dy$ *also exists and is equal to the double integral I. This is called Fubini's theorem.* This theorem holds even if f has a finite number of discontinuities or an infinite number of discontinuities lying on a finite number of lines $x = C_i$ parallel to the y-axis.

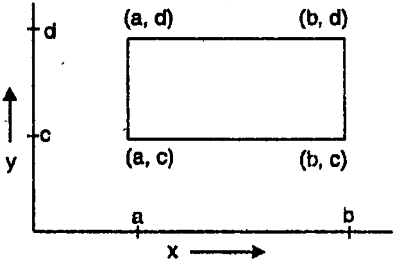


Fig. 2.4.

If a double integral exists, then the two repeated (iterated) integrals cannot exist without being equal. However if the double integral does not exist, the repeated integrals may or may not exist, *i.e.*, the existence of one or of both of the repeated integrals is no guarantee for the existence of the double integral.

2.10 TRIPLE INTEGRALS

Volume of a solid can be obtained by taking volume element $\delta x\, \delta y\, \delta z$ and by triple integral $\iiint dx\, dy\, dz$ under suitable limits. In the general form

$$I = \iiint f(x, y, z)\, dx\, dy\, dz\,. \qquad \text{...(37)}$$

Volume of a solid is generally found by triple integration by taking small element of volume $dx\, dy\, dz$, as $\iiint dx\, dy\, dz$. It may be obtained by a single and double integrals as follows :

Volume (Single Integration) : If a solid is cut by a plane perpendicular to the line chosen (say z-axis) at a distance z from the origin. The area of a plane section will be a function of z, say $f(z)$. The volume of the solid between the planes at the distance z and $z + dz$ will be $f(z)dz$.

$$\therefore \qquad \text{Volume of a solid} = \int_0^h f(z) dz\,. \qquad \text{...(38)}$$

Volume (Double Integration) : If a cylinder is constructed with the cross sectional area $dx\, dy$ on the $z = 0$ plane. If its length is z, then its volume $= z\, dx\, dy$.

$$\therefore \qquad \text{Volume of a solid} = \iint z\, dx\, dy\,. \qquad \text{...(39)}$$

As we have discussed in the previous chapter that *the double integral may be replaced by a triple integral and the line integral by a surface integral by using Gauss's theorem and Stoke's theorem respectively. In the Gauss's theorem, the surface integral is taken over the exterior of surface S, which encloses the volume of that domain or solid. While in the Stoke's theorem the surface integral is taken around the boundary of the curve of the surface.*

$$\iiint_V (f_x + g_y + h_z)\,dx\,dy\,dz = \iint_S (f\,dy\,dz + g\,dz\,dx + h\,dx\,dy)$$

where f_x, g_y and h_z are the partial derivatives of f, g and h with respect to x, y and z respectively.

2.11 CHANGE OF VARIABLES

We can transform the triple integral $\iiint f(x, y, z)\,dx\,dy\,dz$ to another system of variables u, v, w, where x, y, z are given in terms of u, v, w, as follows :

(1) To change the expression $f(x, y, z)$ in terms of $\phi(u, v, w)$

(2) To determine the new limits of u, v, w

(3) To substitute for $dx\,dy\,dz$.

The first two parts are simply the algebraical problems can be solved very easily. For the third part we proceed as :

$$\iiint f(x,y,z)\,dx\,dy\,dz = \int dx \int dy \int f(x,y,z)\,dz\,.$$

z is *first expressed interms of* x, y and w by means of given relations and replace $dz = (\partial z/\partial w)\,dw$. Then we express y interms of v, w and x and replace $dy = (\partial y/\partial v)\,dv$ and lastly use $dx = (\partial x/\partial u)\,du$.

$$\therefore \quad \iiint f(x,y,z)\,dx\,dy\,dz = \iiint \phi(u,v,w)\,\frac{\partial z}{\partial w}\,\frac{\partial y}{\partial v}\,\frac{\partial x}{\partial u}\,dw\,dv\,du\,. \qquad ...(40)$$

Polar Coordinates : The volume in cartesian coordinates (x, y, z) may be transformed into polar coordinates (r, θ, ϕ) by substituting

$x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$.

$$\therefore \quad \iiint dx\,dy\,dz = \iiint r^2 \sin\theta\,dr\,d\theta\,d\phi\,. \qquad ...(41)$$

(Already solved in the first chapter)

2.12 JACOBIANS AND THEIR APPLICATIONS

Jacobians have the remarkable property of behaving like the derivatives of functions of one variable and are the good tools for solving the problems.

Definition : If u_1, u_2... u_n be n differentiable functions of n variables x_1, x_2... x_n, then the determinant

$$\begin{vmatrix} \partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \cdots & \partial u_1/\partial x_n \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \cdots & \partial u_2/\partial x_n \\ \vdots & \vdots & & \vdots \\ \partial u_n/\partial x_1 & \partial u_n/\partial x_2 & \cdots & \partial u_n/\partial x_n \end{vmatrix}$$

is called the Jacobian or the functional determinant of the functions u_1, u_2, ... u_n with respect to x_1, x_2, ... x_n and is denoted by

$$\frac{\partial(u_1,u_2,\dots u_n)}{\partial(x_1,x_2,\dots x_n)} = J\left(\frac{u_1,u_2,\dots u_n}{x_1,x_2,\dots x_n}\right). \qquad ...(42)$$

If the functions u_1, u_2 ... u_n are of the following forms

$u_1 = f_1(x_1)$, $u_2 = f_2(x_1, x_2)$, ..., $u_n = f_n(x_1, x_2, \dots x_n)$, then

$$\frac{\partial(u_1,u_2,\dots u_n)}{\partial(x_1,x_2,\dots x_n)} = \begin{vmatrix} \partial u_1/\partial x_1 & 0 & \cdots & 0 \\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \cdots & 0 \\ \vdots & & & \\ \partial u_n/\partial x_1 & \partial u_n/\partial x_2 & \cdots & \partial u_n/\partial x_n \end{vmatrix}$$

$$= \frac{\partial u_1}{\partial x_1}\cdot\frac{\partial u_2}{\partial x_2}\cdots\frac{\partial u_n}{\partial x_n}\,. \qquad ...(43)$$

This is the case, where the Jacobian reduces to its leading term.

Some Properties—

Theorem 1 : *If $u_1, u_2, \ldots u_n$ are functions of $y_1, y_2, \ldots y_n$ which are themselves functions of $x_1, x_2, \ldots x_n$, then*

$$\frac{\partial(u_1,u_2,\ldots u_n)}{\partial(x_1,x_2,\ldots x_n)} = \frac{\partial(u_1,u_2,\ldots u_n)}{\partial(y_1,y_2,\ldots y_n)}\cdot\frac{\partial(y_1,y_2,\ldots y_n)}{\partial(x_1,x_2,\ldots x_n)} \qquad \ldots(44)$$

For $n = 1$, the theorem reduces to the usual relation

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_1}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_1}.$$

The theorem can be proved by the simple rule of multiplication of determinants combined with the rule for the derivative of a function.

Theorem 2 : If $y_1, y_2, \ldots y_n$ are the functions of $x_1, x_2, \ldots x_n$, then

$$\frac{\partial(y_1,y_2,\ldots y_n)}{\partial(x_1,x_2,\ldots x_n)}\cdot\frac{\partial(x_1,x_2,\ldots x_n)}{\partial(y_1,y_2,\ldots y_n)} = 1. \qquad \ldots(45)$$

Theorem 3 : If $y_1, y_2, \ldots y_n$ are determined as functions of $x_1, x_2, \ldots x_n$ by the equations

$\phi_r(x_1, x_2, \ldots x_n, y_1, y_2, \ldots y_n) = 0$, where $r = 1, 2, \ldots n$,

Then
$$\frac{\partial(\phi_1,\phi_2,\ldots\phi_n)}{\partial(x_1,x_2,\ldots x_n)} = (-1)^n\frac{\partial(\phi_1,\phi_2,\ldots\phi_n)}{\partial(y_1,y_2,\ldots y_n)}\cdot\frac{\partial(y_1,y_2,\ldots y_n)}{\partial(x_1,x_2,\ldots x_n)}. \qquad \ldots(46)$$

Theorem 4 : *(i)* If $y_{m+1}, y_{m+2}, \ldots y_n$ are constant with respect to $x_1, x_2, \ldots x_m$ and *(ii)* if $y_1, y_2, \ldots y_m$ are constant with respect to $x_{m+1}, x_{m+2}, \ldots x_n$, then

$$\frac{\partial(y_1,y_2,\ldots y_m,\ldots y_n)}{\partial(x_1,x_2,\ldots x_m,\ldots x_n)} = \frac{\partial(y_1,y_2,\ldots y_m)}{\partial(x_1,x_2,\ldots x_m)}\cdot\frac{\partial(y_{m+1},y_{m+2},\ldots y_n)}{\partial(x_{m+1},x_{m+2},\ldots x_n)}. \qquad \ldots(47)$$

Theorem 5 : If u, v are functions of l, m, n, and the variables l, m, n are themselves functions of the independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(l,m)}\cdot\frac{\partial(l,m)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(m,n)}\cdot\frac{\partial(m,n)}{\partial(x,y)} + \frac{\partial(u,v)}{\partial(n,l)}\cdot\frac{\partial(n,l)}{\partial(x,y)}. \qquad \ldots(48)$$

Necessary and Sufficient Condition for a Jacobian to Vanish

If $u_1, u_2, \ldots u_n$ be the functions of n independent variables $x_1, x_2, \ldots x_n$. In order that there may exist between these n functions a relation

$$F(u_1, u_2, \ldots u_n) = 0,$$

it is necessary and sufficient that the Jacobian $\frac{\partial(u_1,u_2,\ldots u_n)}{\partial(x_1,x_2,\ldots x_n)}$ should vanish identically.

Thus *(i) If there exist between $u_1, u_2, \ldots u_n$ a relation $F(u_1, u_2, \ldots u_n) = 0$, their Jacobian is zero. It is the necessary condition.*

(ii) If the Jacobian $J(u_1, u_2, \ldots u_n)$ is zero, then there must exist a relation between $u_1, u_2, \ldots u_n$. It is the *sufficient condition.*

EXERCISES

Example 1. *Show that the function f, where*

$$f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2} \quad \text{if } x^2 + y^2 \neq 0$$
$$= 0 \quad \text{if } x = y = 0$$

is differentiable at the origin.

At $x = y = 0$, $f(x, y) = 0$, therefore

$$(\partial f/\partial x)_{0,0} = f_x(0, 0) = (\partial f/\partial y)_{0,0} = f_y(0, 0) = 0.$$

When $x^2 + y^2 \neq 0$,

$$f_x = \frac{\partial f}{\partial x} = \frac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2} \leq \frac{6(x^2+y^2)^{5/2}}{(x^2+y^2)^2} = 6(x^2+y^2)^{1/2}$$

Evidently $\lim\limits_{x,y\to 0} f_x(x,y) = f_x(0,0) = 0.$

Thus f_x is continuous at (0, 0) and f_x (0, 0) exists. Therefore f is differentiable at (0, 0).

Example 2. *Examine the equality of f_{xy} and f_{yx}, where $f(x, y) = x^3y + e^{xy^2}$.*

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}[x^3y + e^{xy^2}] = 3x^2y + y^2e^{xy^2}$$

and

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}[x^3y + e^{xy^2}] = x^3 + 2xy\,e^{xy^2}$$

$\therefore$

$$f_{yx} = \frac{\partial}{\partial y}[f_x] = \frac{\partial}{\partial y}[3x^2y + y^2\,e^{xy^2}] = 3x^2 + 2ye^{xy^2} + 2xy^3\,e^{xy^2}$$

and

$$f_{xy} = \frac{\partial}{\partial x}[f_y] = \frac{\partial}{\partial x}[x^3 + 2xy\,e^{xy^2}] = 3x^2 + 2y\,e^{xy^2} + 2xy^3\,e^{xy^2} = f_{yx}.$$

Example 3. *If $z = x^3 - xy + y^3$, where $x = r\cos\theta$ and $y = r\sin\theta$, find $\partial z/\partial r$ and $\partial z/\partial\theta$.*

$$\frac{\partial z}{\partial x} = 3x^2 - y + 0 \quad\text{and}\quad \frac{\partial z}{\partial y} = 0 - x + 3y^2.$$

and

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial\theta} = -r\sin\theta;\ \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial\theta} = r\cos\theta.$$

$\therefore$

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = (3x^2 - y)\cos\theta + (-x + 3y^2)\sin\theta.$$

and

$$\frac{\partial z}{\partial\theta} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial\theta} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial\theta} = (3x^2 - y)(-r\sin\theta) + (-x + 3y^2)(r\cos\theta)$$

Example 4. *Show that $z = f(x^2y)$, where f is differentiable, satisfies the relation $x(\partial z/\partial x) = 2y(\partial z/\partial y)$.*

Let $x^2y = u$, therefore $z = f(u)$.

Thus

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} = f'(u)\,2xy$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} = f'(u)\,x^2$$

$\therefore$

$$x\frac{\partial z}{\partial x} = f'(u).2x^2y \quad\text{and}\quad 2y\frac{\partial z}{\partial y} = f'(u)\,2x^2y.$$

Hence $x(\partial z/\partial x) = 2y(\partial z/\partial y)$.

Example 5. *Prove that the partial differential equation $\partial^2z/\partial t^2 = c^2\partial^2z/\partial x^2$ reduces to $\partial^2z/\partial u\,\partial v = 0$, where $u = x - ct$ and $v = x + ct$.*

In this problem z is a function of u and v, which are linear functions of the independent variables x and t. Therefore

$$d^2z = \frac{\partial^2z}{\partial u^2}du^2 + 2\frac{\partial^2z}{\partial u\,\partial v}du\,dv + \frac{\partial^2z}{\partial v^2}dv^2 + \frac{\partial z}{\partial u}d^2u + \frac{\partial z}{\partial v}d^2v \qquad \text{...}(i)$$

Since $u = x - ct$ and $v = x + ct$, therefore

$$du = dx - c\,dt,\ dv = dx + c\,dt$$

$\therefore$ $d^2u = 0$ and $d^2v = 0$

Thus equation (i) reduces to

$$d^2z = \frac{\partial^2 z}{\partial u^2}(dx - cdt)^2 + 2\frac{\partial^2 z}{\partial u\,\partial v}(dx - cdt)\,(dx + cdt) + \frac{\partial^2 z}{\partial v^2}(dx + cdt)^2$$

$$= \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right)dx^2 + \left(-2c\frac{\partial^2 z}{\partial u^2} + 2c\frac{\partial^2 z}{\partial v^2}\right)dx\,dt + c^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right)dt^2 \quad ...(ii)$$

Again, since z is a function of independent variables x and t, therefore

$$d^2z = \frac{\partial^2 z}{\partial x^2}dx^2 + 2\frac{\partial^2 z}{\partial x\,\partial t}dx\,dt + \frac{\partial^2 z}{\partial t^2}\cdot dt^2. \quad ...(iii)$$

From equations (*ii*) and (*iii*), we get

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}$$

and

$$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Given that $\partial^2 z/\partial t^2 = c^2\,\partial^2 z/\partial x^2$, therefore

$$c^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right) = c^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\,\partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

or

$$\partial^2 z/\partial u\,\partial v = 0.$$

Example 6. *Find the maxima and minima of the function $f(x, y) = x^3 + y^3 - 3x - 12y + 20$.*

Differentiating w.r. to x and y separately, we get

$$f_x = \partial f/\partial x = 3x^2 - 3 \quad \text{and} \quad f_y = \partial f/\partial y = 3y^2 - 12.$$

$\therefore$ $f_x(x, y) = 0$ for $x^2 = 1$ or $x = \pm 1$

and $f_y(x, y) = 0$ for $3y^2 = 12$ or $y = \pm 2$.

Thus the function has four *stationary points* (1, 2), (–1, 2), (1, –2), (–1, –2).

Differentiating the function f twice partially, we get

$$f_{xx} = \partial^2 f/\partial x^2 = 6x,\ f_{xy} = \partial^2 f/\partial x\,\partial y = 0, \text{ and } f_{yy} = \partial^2 f/\partial y^2 = 6y.$$

Therefore we have

At (1, 2), $f_{xx} = 6$ and $f_{xx}f_{yy} - (f_{xy})^2 = 36 \times 2 - 0 = 72 > 0$

Therefore at point (1, 2), the function has a minima.

At (– 1, 2), $f_{xx} = -6$ and $f_{xx}f_{yy} - (f_{xy})^2 = -72 < 0$

Therefore at point (– 1, 2), the function has neither a maxima nor a minima.

At (1, – 2), $f_{xx} = 6$ and $f_{xx}f_{yy} - (f_{xy})^2 = -72 < 0$.

Therefore at point (1, –2), the function has neither a maxima nor a minima.

At (– 1, –2), $f_{xx} = -6$ and $f_{xx}f_{yy} - (f_{xy})^2 = 72 > 0$.

Therefore at point (–1, –2), the function has maxima.

Example 7. *Show that the function $f(x, y) = y^2 + x^2y + x^4$ has a minima at (0, 0).*

By partial differentiation, we get.

$$f_x = 2xy + 4x^3,\ f_y = 2y + x^2,\ f_{xx} = 2y + 12x^2,\ f_{xy} = 2x,\ f_{yy} = 2.$$

Therefore at origin, we get

$$f_x = 0,\ f_y = 0,\ f_{xx} = 0,\ f_{xy} = 0,\ f_{yy} = 2 \text{ and } f_{xx}f_{yy} - (f_{xy})^2 = 0$$

Thus we see that the function may have extreme value at origin on account of the condition $f_x = 0$ and $f_y = 0$. Since $f_{xx}f_{yy} - (f_{xy})^2$ is zero at origin, therefore further investigation is necessary.

As the function f may be written as

$$f(x, y) = \left(y + \frac{1}{2}x^2\right)^2 + \frac{3}{4}x^4.$$

Therefore $f(0, 0) = 0$ and $f(x, y)$ is always greater than zero for all other values. Hence f has a minimum value at the origin.

Example 8. *Prove that $\partial^2V/\partial x^2 + \partial^2V/\partial y^2$ is invariant for change of rectangular axes :*

In the given problem we are given rectangular axes x and y.

Let us first see the effect of the change of origin. If the origin is shifted so that $x = a + x'$ and $y = b + y'$.

$$\therefore \qquad dx = dx' \quad \text{and} \quad dy = dy'.$$

Therefore all the partial derivatives of V remain the same

$$\therefore \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2} = \text{Invariant.}$$

Let the axes are turned/rotated through an angle α, so that

$$x = x'\cos\alpha - y'\sin\alpha,\ y = x'\sin\alpha + y'\cos\alpha$$

Since the angle α is constant, therefore

$$dx = dx'\cos\alpha - dy'\sin\alpha \text{ and } dy = dx'\sin\alpha + dy'\cos\alpha.$$

$$\therefore \qquad dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = \frac{\partial V}{\partial x}(dx'\cos\alpha - dy'\sin\alpha) + \frac{\partial V}{\partial y}(dx'\sin\alpha + dy'\cos\alpha)$$

$$= \left(\frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\sin\alpha\right)dx' + \left(-\frac{\partial V}{\partial x}\sin\alpha + \frac{\partial V}{\partial y}\cos\alpha\right)dy'.$$

Also $\qquad dV = \dfrac{\partial V}{\partial x'}dx' + \dfrac{\partial V}{\partial y'}dy'$

On comparing, we get

$$\frac{\partial V}{\partial x'} = \frac{\partial V}{\partial x}\cos\alpha + \frac{\partial V}{\partial y}\sin\alpha \text{ and } \frac{\partial V}{\partial y'} = -\frac{\partial V}{\partial x}\sin\alpha + \frac{\partial V}{\partial y}\cos\alpha.$$

We thus get

$$\frac{\partial V}{\partial x} = \frac{\partial V}{\partial x'}\cos\alpha - \frac{\partial V}{\partial y'}\sin\alpha \text{ and } \frac{\partial V}{\partial y} = \frac{\partial V}{\partial x'}\sin\alpha + \frac{\partial V}{\partial y'}\cos\alpha.$$

$$\therefore \qquad \frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\alpha\frac{\partial}{\partial x'} - \sin\alpha\frac{\partial}{\partial y'}\right)\left(\frac{\partial V}{\partial x'}\cos\alpha - \frac{\partial V}{\partial y'}\sin\alpha\right)$$

$$= \frac{\partial^2 V}{\partial x'^2}\cos^2\alpha - 2\frac{\partial^2 V}{\partial x'\partial y'}\sin\alpha\cos\alpha + \frac{\partial^2 V}{\partial y'^2}\sin^2\alpha.$$

and $$\frac{\partial^2 V}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) = \left(\sin\alpha\frac{\partial}{\partial x'} + \cos\alpha\frac{\partial}{\partial y'}\right)\left(\frac{\partial V}{\partial x'}\sin\alpha + \frac{2V}{\partial y'}\cos\alpha\right)$$

$$= \frac{\partial^2 V}{\partial x'^2}\sin^2\alpha + 2\frac{\partial^2 V}{\partial x'\partial y'}\sin\alpha\cos\alpha + \frac{\partial^2 V}{\partial y'^2}\cos^2\alpha.$$

$$\therefore \qquad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial x'^2} + \frac{\partial^2 V}{\partial y'^2}.$$

Thus the expression remains invariant under shifting or rotation of axes.

Example 9. *Convert the expression of example 8 in polar coordinates.*

The cartesian coordinates may be converted into polar coordinates by using the relations $x = r\cos\theta$ and $y = r\sin\theta$. Since V is a function of (x, y) and also of (r, θ).

$$\therefore \qquad \frac{\partial V}{\partial r} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial r} = \cos\theta\frac{\partial V}{\partial x} + \sin\theta\frac{\partial V}{\partial y} \qquad \ldots(i)$$

and $$\frac{\partial V}{\partial \theta} = \frac{\partial V}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial V}{\partial y}\cdot\frac{\partial y}{\partial \theta} = -r\sin\theta\frac{\partial V}{\partial x} + r\cos\theta\frac{\partial V}{\partial y}. \qquad \ldots(ii)$$

On solving, we get

$$\frac{\partial V}{\partial x} = \cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta} \Rightarrow \frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$$

$$\frac{\partial V}{\partial y} = \sin\theta\frac{\partial V}{\partial r} + \frac{\cos\theta}{r}\frac{\partial V}{\partial\theta} \Rightarrow \frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$$

Therefore $$\frac{\partial^2 V}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) = \left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial V}{\partial r} - \frac{\sin\theta}{r}\frac{\partial V}{\partial\theta}\right)$$

$$= \cos^2\theta\frac{\partial^2 V}{\partial r^2} - \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\sin^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial V}{\partial\theta} + \frac{\sin^2\theta}{r}\frac{\partial V}{\partial r}$$

Similarly $$\frac{\partial^2 V}{\partial y^2} = \sin^2\theta\frac{\partial^2 V}{\partial r^2} + \frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 V}{\partial r\partial\theta} + \frac{\cos^2\theta}{r^2}\frac{\partial^2 V}{\partial\theta^2} - \frac{2\cos\theta\sin\theta}{r^2}\frac{\partial V}{\partial\theta} + \frac{\cos^2\theta}{r}\frac{\partial V}{\partial r}$$

$$\therefore \quad \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = \frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial\theta^2} + \frac{1}{r}\frac{\partial V}{\partial r}.$$

Example 10. *Find the value of $\int_c [(x + y^2)dx + (x^2 - y)\,dy]$ taken in the clockwise sense along the closed curve formed by $y^3 = x^2$ and the chord joining the points (0, 0) and (1, 1).*

The equation of the chord joining points (0, 0) and (1, 1) is given by

$$y - 0 = \frac{1-0}{1-0}(x - 0) \quad \text{or} \quad y = x.$$

The curve also passes through points (0, 0) and (1, 1). Therefore the clockwise line integral will be given by

$$\int_{\text{curve}}[(x + y^2)\,dx + (x^2 - y)dy] + \int_{\text{chord}}[(x + y^2)\,dx + (x^2 - y)\,dy]$$

$$= \int_0^1[(x + x^{4/3})dx + (y^3 - y)dy] + \int_1^0[(x + x^2)dx + (y^2 - y)dy]$$

$$= \left[\frac{x^2}{2} + \frac{x^{7/3}}{7/3} + \frac{y^4}{4} - \frac{y^2}{2}\right]_0^1 + \left[\frac{x^2}{2} + \frac{x^3}{3} + \frac{y^3}{3} - \frac{y^2}{2}\right]_1^0$$

$$= \frac{3}{7} + \frac{1}{4} - \frac{1}{3} - \frac{1}{3} = \frac{1}{84}.$$

Example 11. *Show that the change in the order of integration in the double integral $\int_0^1\int_0^1 \frac{x^2 - y^2}{x^2 + y^2}dy\,dx$ is permissible.*

$$\int_0^1 \frac{x^2 - y^2}{x^2 + y^2}dy = \int_0^1\left(\frac{2x^2}{x^2 + y^2} - 1\right)dy = \left[2x\tan^{-1}\frac{y}{x} - y\right]_0^1 = 2x\tan^{-1}\frac{1}{x} - 1$$

$$\therefore \quad \int_0^1 dx\int_0^1\frac{x^2 - y^2}{x^2 + y^2}dy = \int_0^1(2x\tan^{-1}\frac{1}{x} - 1)dx = \left[x^2\tan^{-1}\frac{1}{x}\right]_0^1 - \int_0^1 x^2.\frac{1}{1 + (1/x)^2}.\left(-\frac{1}{x^2}\right)dx - 1$$

$$= \tan^{-1}1 + \int_0^1\left(\frac{x^2 + 1}{1 + x^2} - \frac{1}{1 + x^2}\right)dx - 1 = \tan^{-1}1 + 1 - \tan^{-1}1 - 1 = 0.$$

By interchanging the order of integration, we have

$$\int_0^1 \frac{x^2 - y^2}{x^2 + y^2}dx = \int_0^1\left(1 - \frac{2y^2}{x^2 + y^2}\right)dx = \left[x - 2y\tan^{-1}\frac{x}{y}\right]_0^1 = 1 - 2y\tan^{-1}\frac{1}{y}$$

$$\int_0^1 dy\int_0^1\frac{x^2 - y^2}{x^2 + y^2}dx = \int_0^1\left(1 - 2y\tan^{-1}\frac{1}{y}\right)dy = 0.$$

Thus we see that

$$\int_0^1 dx \int_0^1 \frac{x^2 - y^2}{x^2 + y^2}\, dy = \int_0^1 dy \int_0^1 \frac{x^2 - y^2}{x^2 + y^2}\, dx = 0.$$

or the change in order of integration of a given function is permissible.

Example 12. *Prove that* $\iint_R \sqrt{|\,y - x^2\,|}\, dx\, dy = (3\pi + 8)/6$, *where* $R = [-1, 1;\, 0, 2]$.

The domain R is divided into two parts, as shown in Fig. 2.5. Therefore the surface integral $\iint_R \sqrt{|\,y - x^2\,|}\, dx\, dy =$

$$\int_{-1}^1 dx \int_0^{x^2} \sqrt{(x^2 - y)}\, dy + \int_{-1}^1 dx \int_{x^2}^2 \sqrt{y - x^2}\, dy$$

$$= \frac{2}{3}\int_{-1}^1 x^3 dx + \frac{2}{3}\int_{-1}^1 (2 - x^2)^{3/2} dx = \frac{4}{3}\int_0^1 (2 - x^2)^{3/2}\, dx$$

$$= \frac{16}{3}\int_0^{\pi/4} \cos^4\theta\, d\theta \quad (\text{let } x = \sqrt{2}\,\sin\theta)$$

$$= \frac{4}{3}\int_0^{\pi/4} (1 + \cos 2\theta)^2 d\theta = \frac{4}{3} + \frac{\pi}{2} = \frac{3\pi + 8}{6}.$$

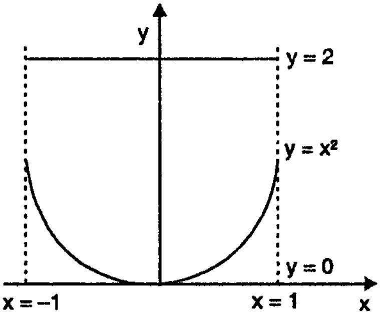


Fig. 2.5. Exampler 12.

Example 13. *Evaluate the surface area of that part of the surface of the cylinder* $x^2 + y^2 = a^2$ *which is cut out by the cylinder* $x^2 + z^2 = a^2$.

The figure shows $\frac{1}{8}$ th of the desired surface.

The equation of the surface has the form

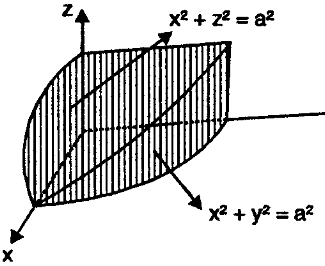


Fig. 2.6. Example 13.

$$y = (a^2 - x^2)^{1/2}$$

$$\therefore \quad \frac{\partial y}{\partial x} = -\frac{x}{(a^2 - x^2)^{1/2}}, \quad \frac{\partial y}{\partial z} = 0.$$

$$\therefore \quad \left[1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2\right]^{1/2} = \frac{a}{(a^2 - x^2)^{1/2}}$$

The domain of integration is a quarter circle $x^2 + z^2 = a^2, x \geq 0, z \geq 0$, on the xz–plane. Thus we have

$$\frac{1}{8}S = \int_0^a dx \int_0^{(a^2 - x^2)^{1/2}} \frac{a}{(a^2 - x^2)^{1/2}}\, dz = a^2$$

$\therefore$ The surface area $S = 8a^2$.

Example 14. *The x and y coordinates of a point on the paraboloid* $2z = x^2/a + y^2/b$ *are expressed in the form* $x = a \tan\theta\cos\phi$, $y = b\tan\theta\sin\phi$, *where* θ *is the angle of inclination of the normal at any point to the axis of z. Show by using Jacobian that the area of the cap of the surface cut off by the curve* $\theta = \lambda$ *is* $2\pi ab\,(\sec^3\lambda - 1)/3$.

The surface area is given by $S = \iint \sqrt{1 + (\partial z/\partial x)^2 + (\partial z/\partial y)^2}\, dx\, dy$

In this problem $\partial z/\partial x = x/a$ and $\partial z/\partial y = y/b$. Thus the integrand may be written as

$$\sqrt{1 + (x/a)^2 + (y/b)^2} = \sqrt{1 + \tan^2\theta\cos^2\phi + \tan^2\theta\sin^2\phi} = \sqrt{1 + \tan^2\theta} = \sec\theta.$$

$$\text{The Jacobian } J = \frac{\partial(x, y)}{\partial(\theta, \phi)} = \begin{vmatrix} \partial x/\partial\theta & \partial x/\partial\phi \\ \partial y/\partial\theta & \partial y/\partial\phi \end{vmatrix} = \frac{\partial x}{\partial\theta}\cdot\frac{\partial y}{\partial\phi} - \frac{\partial x}{\partial\phi}\cdot\frac{\partial y}{\partial\theta}$$

$$= a\sec^2\theta\cos\phi.\, b\tan\theta\cos\phi + a\tan\theta\sin\phi.\, b\sec^2\theta\sin\phi$$

$$= ab\sec^2\theta\tan\theta\,(\cos^2\phi + \sin^2\phi) = ab\sec^2\theta\tan\theta.$$

$$\therefore \qquad S = \int_0^{2\pi} d\phi \int_0^\lambda \sec\theta . ab\sec^2\theta . \tan\theta\, d\theta$$

$$= 2\pi\int_0^\lambda ab\sec^3\theta\tan\theta\, d\theta = \frac{2\pi ab}{3}(\sec^3\lambda - 1)$$

Example 15. *Evaluate the surface integral* $\iint_S (x^4+y^4+z^4)dS$, *where S is the surface* $x^2/a^2+y^2/b^2+z^2/c^2=1$ *and p is the perpendicular from the origin to the tangent plane at the point (x, y, z).*

Symmetry of the integral and the surface S show that its value is 8 times the value of the integral over the portion of the surface in the first octant.

Let $x = a\cos\theta\cos\phi$, $y = b\cos\theta\sin\phi$, $z = c\sin\theta$, where $0 \le \phi \le \pi/2$ and $0 \le \theta \le \pi/2$.

$$\therefore \quad \frac{\partial(x,y)}{\partial(\theta,\phi)} = -ab\cos\theta\sin\theta,\ \frac{\partial(y,z)}{\partial(\theta,\phi)} = -bc\cos^2\theta\cos\phi,\ \frac{\partial(z,x)}{\partial(\theta,\phi)} = -ac\cos^2\theta\sin\phi.$$

and $$\left[\frac{\partial(x,y)}{\partial(\theta,\phi)}\right]^2+\left[\frac{\partial(y,z)}{\partial(\theta,\phi)}\right]^2+\left[\frac{\partial(z,x)}{\partial(\theta,\phi)}\right]^2 = a^2b^2c^2\cos^2\theta\left[\frac{\cos^2\theta\cos^2\phi}{a^2}+\frac{\cos^2\theta\sin^2\phi}{b^2}+\frac{\sin^2\theta}{c^2}\right]$$

and $$p = 1\Big/\left[\frac{\cos^2\theta\cos^2\phi}{a^2}+\frac{\cos^2\theta\sin^2\phi}{b^2}+\frac{\sin^2\theta}{c^2}\right]^{1/2}$$

Hence the surface integral

$$\iint_S p(x^4+y^4+z^4)dS = 8\int_0^{\pi/2}d\theta\int_0^{\pi/2}abc\cos\theta\,[a^4\cos^4\theta\cos^4\phi+b^4\cos^4\theta\sin^4\phi+c^4\sin^4\theta]\,d\phi.$$

$$= 8abc\int_0^{\pi/2}\cos\theta\left[a^4\cos^4\theta\frac{3.1}{4.2}\cdot\frac{\pi}{2}+b^4\cos^4\theta\frac{3.1}{4.2}\frac{\pi}{2}+c^4\sin^4\theta\frac{\pi}{2}\right]d\theta$$

$$= 8abc\frac{\pi}{2}\left[\frac{3}{8}(a^4+b^4).\frac{4.2}{5.3}+\frac{c^4}{5}\right] = \frac{4\pi abc\,(a^4+b^4+c^4)}{5}.$$

Example 16. *Find the volume of the solid bounded by the surface* $z = 1-4x^2-y^2$ *and the plane* $z = 0$.

The solid is a segment of the elliptical paraboloid lying above the xy-plane (*i.e.*, $z = 0$ plane). The paraboloid cuts the xy-plane along the ellipse $4x^2+y^2=1$, which forms the base D of the solid. Thus the solid is bounded by $z = 1-4x^2-y^2$ and below by the ellipse $4x^2+y^2=1$. Moreover the solid being symmetrical, its volume V is four times the volume lying in the first octant.

$$\therefore \quad V = 4\int_0^{1/2}dx\int_0^{\sqrt{1-4x^2}}(1-4x^2-y^2)\,dy = \frac{8}{3}\int_0^{1/2}(1-4x^2)^{3/2}dx$$

$$= \frac{4}{3}\int_0^{\pi/2}\cos^4 t\,dt = \frac{\pi}{4}.$$

Example 17. *Find the Jacobians J for cylindrical and spherical polar coordinates.*

(a) The cylindrical polar coordinates are $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\therefore \quad \frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial x}{\partial\theta} = -r\sin\theta,\ \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial y}{\partial\theta} = r\cos\theta,\ \frac{\partial z}{\partial z} = 1.$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial\theta}\end{vmatrix} = \begin{vmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

(*b*) The spherical polar coordinates (r, θ, ϕ) are related with cartesian coordinates (x, y, z) as :

$$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta.$$

$$\frac{\partial x}{\partial r} = \sin\theta\cos\phi,\ \frac{\partial x}{\partial\theta} = r\cos\theta\cos\phi,\ \frac{\partial x}{\partial\phi} = -r\sin\theta\sin\phi.$$

$$\frac{\partial y}{\partial r} = \sin\theta\sin\phi,\ \frac{\partial y}{\partial\theta} = r\cos\theta\sin\phi,\ \frac{\partial y}{\partial\phi} = r\sin\theta\cos\phi.$$

$$\frac{\partial z}{\partial r} = \cos\theta,\ \frac{\partial z}{\partial\theta} = -r\sin\theta.$$

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos\phi\,(0 + r^2\sin^2\theta\cos\phi) + r\cos\theta\cos\phi\,(r\sin\theta\cos\theta\cos\phi - 0)$$

$$-r\sin\theta\sin\phi\,(-r\sin^2\theta\sin\phi - r\cos^2\theta\sin\phi)$$

$$= r^2\sin^3\theta\cos^2\phi + r^2\sin\theta\cos^2\theta\cos^2\phi + r^2\sin^3\theta\sin^2\phi + r^2\sin\theta\cos^2\theta\sin^2\phi$$

$$= r^2\sin^3\theta + r^2\sin\theta\cos^2\theta = r^2\sin\theta.$$

Example 18. *Evaluate the surface integral* $I = \iint_S (y^2z\,dx\,dy + xz\,dy\,dz + x^2y\,dz\,dx)$, *where S is the outerside of the surface situated in the first octant and formed by the paraboloid of revolution* $z = x^2 + y^2$, *cylinder* $x^2 + y^2 = 1$ *and the coordinate planes.*

Using the Gauss's theorem

$$\iint_S (f\,dx\,dy + g\,dy\,dz + h\,dz\,dx) = \iiint_V (f_x + g_y + h_z)\,dx\,dy\,dz$$

We get

$$I = \iint_S (y^2z\,dx\,dy + xz\,dy\,dz + x^2y\,dz\,dx) = \iiint_V (x^2 + y^2 + z)\,dx\,dy\,dz,$$

where V is the volume enclosed by S.

$$\therefore \quad I = \iint_D dx\,dy \int_0^{x^2+y^2} (x^2 + y^2 + z)\,dz,$$

where D $(x^2 + y^2 \le 1)$ is the projection on the xy-plane of the domain enclosed by S.

$$\therefore \quad I = \frac{3}{2}\iint_D (x^2 + y^2)^2\,dx\,dy$$

For cylinderical coordinates $x = r\cos\theta$, $y = r\sin\theta$

$$I = \frac{3}{2}\iint F(x,y)\,J\,dr\,d\theta.$$

$$= \frac{3}{2}\iint r^4 . r\,dr\,d\theta = \frac{3}{2}\int_0^{\pi/2} d\theta \int_0^1 r^5 dr = \frac{\pi}{8}.$$

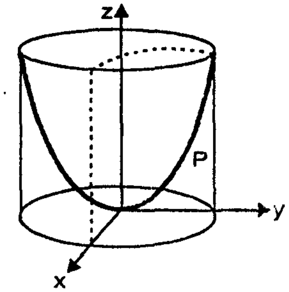


Fig. 2.7. Example 18.

Example 19. *Evaluate the volume integral* $I = \iiint_V \left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right) dx\,dy\,dz$, *taken over the region* $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$.

Let us change the cartesian coordinates to spherical polar coordinates, assuming $x/a = r\sin\theta\cos\phi$, $y/b = r\sin\theta\sin\phi$, $z/c = r\cos\theta$, with

$$0 \le r \le 1, 0 \le \theta \le \pi, 0 \le \phi \le 2\pi.$$

$$\therefore \quad \text{Jacobian } J = \frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = abc\,r^2\sin\theta.$$

Therefore

$$I = \int_0^{2\pi}\int_0^{\pi}\int_0^1 abc\,r^2\sin\theta\,f(x,y,z)\,dr\,d\theta\,d\phi$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi}\sin\theta\,d\theta \int_0^1 (\sqrt{1 - r^2})\,r^2 dr = \pi^2 abc/4.$$

Example 20. *Evaluate the volume integral* $I = \iiint z^2 dx\,dy\,dz$ *taken over the region common to the surfaces* $x^2 + y^2 + z^2 = a^2$ *and* $x^2 + y^2 = ax$.

The region is bounded above and below by the surfaces

$$z = (a^2 - x^2 - y^2)^{1/2} \quad \text{and} \quad z = -(a^2 - x^2 - y^2)^{1/2}$$

and its projection on the $x - y$ plane is the circular domain D given by $x^2 + y^2 \le ax$.

$$\therefore \qquad I = \iiint z^2 dx\, dy\, dz = \iint_D dx\, dy \int_{-(a^2-x^2-y^2)^{1/2}}^{(a^2-x^2-y^2)^{1/2}} z^2 dz = \frac{2}{3}\iint_D (a^2 - x^2 - y^2)^{3/2} dx\, dy.$$

Changing to polar coordinates, the domain D becomes the circle $r = a\cos\theta$

$$I = \frac{2}{3}\iint f(x,y)\, J\, dr\, d\theta = \frac{2}{3}\int_0^{\pi} d\theta \int_0^{a\cos\theta} (a^2 - r^2)^{3/2} r dr.$$

$$= \frac{2}{15} a^5 \int_0^{\pi} (1 - \sin^5\theta) d\theta = \frac{2a^5(15\pi - 16)}{225}$$

$$\left[\text{using } a^2 - r^2 = p^2, -2rdr = 2pdp, \therefore \int_0^{a\cos\theta} (a^2 - r^2)^{3/2} rdr = \int_a^{a\sin\theta} p^4 dp = -\frac{1}{5}[p^5]_a^{a\sin\theta} = a^5(1 - \sin^5\theta)\right]$$

PROBLEMS

1. Prove that the function $f(x, y) = (xy)^{1/2}$ is not differentiable at the point (0, 0), and the partial derivatives f_x and f_y are continuous except at the origin.
2. Verify that $f_{xy} = f_{yx}$ for the function (*a*) $(2x - y)/(x + y)$, (*b*) $x \tan xy$.
3. If $z = e^{xy^2}$, $x = t\cos t$, $y = t\sin t$, calculate $\partial z/\partial t$ at $t = \pi/2$.
4. Show that $z = \log[(x - a)^2 + (y - b)^2]$ satisfies $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ except at (a, b).
5. For a given function $u = e^x \cos y + e^y \sin z$, find all first partial derivatives and verify the equality of u_{xy} and u_{yx}, u_{xz} and u_{zx}; and $u_{yz} = u_{zy}$.
6. If $z = x^3 - xy + y^3$, $x = r\cos\theta$, $y = r\sin\theta$, find $\partial z/\partial r$ and $\partial z/\partial\theta$.
7. Show that $f(x, y, z) = (x + y + z)^3 - 3(x + y + z) - 24xyz + a^3$ has a minima at (1, 1, 1) and a maxima at (–1, –1, –1).
8. Show that the function $f(x, y, z) = 2xyz - 4zx - 2yz + x^2 + y^2 + z^2 - 2x - 4y + 4z$ has five stationary points but has a minimum value only at (1, 2, 0).
9. Show that $\partial^2\theta/\partial x\, \partial y = -(\cos 2\theta)/r^2$.
10. If $u = f(x + 2y) + g(x - 2y)$, show that $4\, \partial^2 u/\partial x^2 = \partial^2 u/dy^2$.
11. If $V = f(x, y)$ and $x = e^u \cos t$, $y = e^u \sin t$, show that $\dfrac{\partial^2 V}{\partial u^2} + \dfrac{\partial^2 V}{\partial t^2} = e^{2u}\left[\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2}\right]$.
12. Find the value of $\int_C xydx$ along the arc of the parabola $x = y^2$ from (1, – 1) to (1, 1). [4/5]
13. Find the value of $\int_0^1\int_0^1 \dfrac{x^2}{1 + y^2} dxdy$. ($\pi/12$)
14. Compute the double integral of the function $f(x, y) = x^2 y e^{xy}$ over $R = [0, 1; 0, 2]$. (2)
15. Evaluate $\iint y\, dx\, dy$ over the part of the plane bounded by the lines $y = x$ and the parabola $y = 4x - x^2$. (54/5)
16. Evaluate $\iint (x^2 + y^2) dx\, dy$ over the domain bounded by $y = x^2$ and $y^2 = x$. (6/35)
17. Show that the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is $2(\pi - 2)a^2$.
18. Evaluate the area of the part of the earth's surface (say spherical of radius R km) contained between the meridians $\phi = 30°$, $\phi = 60°$ and parallels $\theta = 45°$ and $\theta = 60°$. $\pi R^2(\sqrt{3} - \sqrt{2})/12$
19. Show that the surface integral $\iint_S (yz\, dy\, dz + zx\, dz\, dx + xy\, dx\, dy) = 3/8$, where S is the outer surface of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant.
20. Compute the volume V, common to the ellipsoid of revolution $x^2/a^2 + y^2/a^2 + z^2/b^2 = 1$ and the cylinder $x^2 + y^2 - ay = 0$. $\frac{2}{9}a^2 b\,(3\pi - 4)$
21. Show that the volume enclosed by the solid $(x/a)^{2/3} + (y/b)^{2/3} + (z/c)^{2/3} = 1$ is $4\pi abc/35$.
22. Show that the volume enclosed by the paraboloid $x^2 - y^2 = 2az$, the cylinder $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ and the plane $z = 0$ is $a^3/6$.
23. Compute the volume of the solid bounded by the cylinders $z = 4 - y^2$ and $z = y^2 + 2$ and the planes $x = -1$, $x = 2$ (8)
24. A right cone has its vertex in the surface of a sphere and its axis coincident with the diameter of the sphere passing through that point. Find the volume common to the cone and the sphere. $4\pi a^3(1 - \cos^4\alpha)/3$.
25. Show that the volume common to the surfaces $y^2 + z^2 = 4ax$ and $x^2 + y^2 = 2ax$ is $\frac{2}{3}(3\pi + 8)a^3$.

3

ELECTRIC CHARGE & ELECTROSTATIC FIELD

3.1 ELECTRIC CHARGE

The best known interaction is *gravitation*. In certain circumstances, the interactions between separated bodies completely overshadow the gravitational attraction. In these cases we say that these bodies are either magnetic or charged. We can show that a glass rod rubbed with silk will repel the second glass rod rubbed with silk and will attract a hard rubber rod rubbed with fur. Benjamin Franklin, American Physicist, named the kind of charge that appeared on the glass positive and the kind that appeared on the hard rubber negative. He also observed that equal negative and positive charges were obtained at the same time by the rubber on the one hand and by the body rubbed on the other. Thus the like charges repel and unlike charges attract. Now-a-days, it is assumed that the matter contains equal amount of negative (electrons) and positive (protons) electricity. When any substance is rubbed with another, a small amount of one kind of electricity (electrons) is transferred from one to the other body. In this way one body would become positive and other negative.

Charge is quantized : The atomic theory of matter has shown that fluids (*e.g.*, water and air) are not continuous, but are made up of atoms. It is also observed that electron and proton have charges equal in magnitude but opposite in sign. No one has been able to detect a charge smaller in magnitude than the charge of the electron e except the quarks suggested in elementary particles. In addition Milikan's oil drop experiment showed that the magnitudes of all other charges were found to be integer multiples of the magnitude of the charge on the electron. Thus we see that charge exists in discrete packets rather than in continuous amounts and hence is said to be *quantized*. The basic packet, or quantum, of charge has magnitude e. All charged elementary particles, known upto this time carry charges of precisely the same magnitude and have their anti-particles. It shows that the *quantization of charge is a deep and universal law of nature*. There are two kinds of charge quanta. Both have the same magnitude. One is that on the electron and other is that on the proton. The quantization of charge can be explained with the help of higher theories of quantum mechanics.

The quantum of charge e is so small that the graininess of electricity does not show up in most of the experiments. For example in an ordinary 220 volts 60 watt light bulb, 1.7×10^{18} elementary charges (electrons) leave and enter filament every second.

Conservation of Charge : We know that the total charge in an *isolated system* remains constant. By isolated we mean that no matter is allowed to enter or leave the system. The charges can be neither created nor destroyed. This is known as *conservation of charge* and is valid both for large scale events and at the atomic and nuclear level, no exceptions have been found. An interesting example of charge conservation is the *creation of pair* of electron and positron with the high energy photons. Experimentally it was observed that the electric charges of electron and positron were equal in magnitude and opposite in sign. These particles are related to one another as *particle* to *antiparticle.* The reverse process, in which energy appears in the form of two gamma rays when an electron and a positron are bought close to each other, is also observed. This process is known as *annihilation* process. In these processes the net charge is zero both before and after the event so that *charge is conserved.*

Another example of the conservation of charge is found in *radioactive decay.* Few decay processes are :

$$^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}$$

$$^{226}_{88}\mathrm{Ra} \rightarrow {}^{222}_{86}\mathrm{Rn} + {}^{4}_{2}\mathrm{He}.$$

Charge conservation is also valid in *nuclear reactions.* Few examples are :

$$^{44}_{20}\mathrm{Ca} + {}^{1}_{1}\mathrm{H} \rightarrow {}^{44}_{21}\mathrm{Sc} + {}^{1}_{0}\mathrm{n}$$

$$^{4}_{2}\mathrm{He} + {}^{14}_{7}\mathrm{N} \rightarrow {}^{17}_{8}\mathrm{O} + {}^{1}_{1}\mathrm{H}$$

$$^{7}_{3}\mathrm{Li} + {}^{1}_{1}\mathrm{H} \rightarrow 2\,{}^{4}_{2}\mathrm{He}.$$

Here we see that the sum of the atomic numbers (*i.e.*, number of protons in the atom) before the reaction is exactly equal to the sum of the atomic numbers after the reaction. It means that total charge remains constant or *charge is conserved.*

The law of conservation of charge is also true for relativistic motion. In other words we can say that the total electric charge of an isolated system is *relativistically invariant.*

Unit of Charge : One cannot explain the charge in terms of any other known property. In macroscopic charging process the number of electrons involved is very large and we use a unit of charge, the *coulomb.* In nuclear or atomic problems or in microscopic processes, the unit of charge is taken as the charge on the electron (*i.e.*, 1.6021×10^{-19} coulomb), *the electron charge.*

Conductors and Insulators : For the purpose of electrostatic theory all substances can be divided into two main classes : *conductors and insulators.* In conductors electric charges are free to move from one place to another, whereas in insulators they are tightly bound to their respective atoms. In an uncharged body there are an equal number of positive and negative charges.

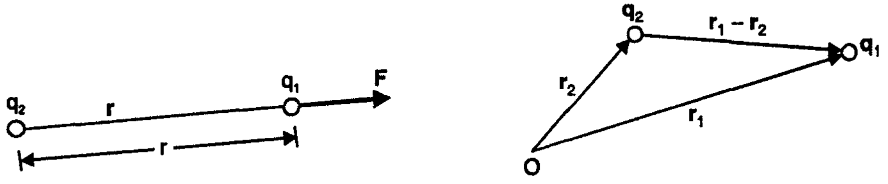
The examples of conductors of electricity are the metals, human body and the earth and that of insulators are glass, hard rubber and plastics. In metals, the free charges are free electrons known as conduction, metallic or free electrons. But in electrolytes, each molecule of electrolyte separates into positively and negatively charged parts, known as ions which can move independently of each other. Although there is no perfect insulator or perfect conductor. The insulating ability of fused quartz is about 10^{25} times as great as that of copper and hence used as perfect insulator. The concepts of perfect insulator and of perfect conductor are useful in electrostatic problems. There are a number of substances that are neither good conductors of electricity nor good insulators. These substances are called *semiconductors.*

Charge and Matter : Every matter is consisted of neutral atoms. The atoms are made up of a dense positively charged nucleus surrounded by the electrons in the orbital motion. The radius of the nucleus is of the order of fermi ($1f = 10^{-15}$ m) and is about 10^{-5} times smaller than the radius of the orbit of outer electron or that of an atom. In the nucleus protons and neutrons are packed by the strong *attractive forces,* known as *nuclear forces.* These forces are much stronger than the electrostatic repulsive forces between the protons. The forces that bind the electrons of an atom to the nucleus, the forces that bind atoms together to form molecules and the forces that bind molecules together to form matter may be described with the help of Coulomb's law and the structure of the matter.

3.2 COULOMB'S LAW

The French colonel, Charles Augustin de Coulomb performed experiments in 1785 to find how the force varies with the distance between two point charges. From his experimental results, *Coulomb* gave a law, can be stated as :

(*a*) The force which two charged bodies 1 and 2 (say), whose dimensions are small compared to their separation (*i.e.,* point charges) exert on one another has a direction of the line joining the charges and is inversely proportional to the square of their separation r, *i.e.,* $\mathbf{F} \propto (1/r^2)\,\hat{r}$, where $\hat{r} = \mathbf{r}/r$ is the unit vector in the direction from charge 2 to charge 1, if the force **F** is acting on charge 1 due to charge 2 [Fig. 3.1 (*a*)].



(*a*) (*b*)

Fig. 3.1. Force between two point charges.

(*b*) This force is proportional to the product of the charges on these charged bodies, *i.e.,* $F \propto q_1 q_2$.

(*c*) This force acts as a repulsive force when q_1 and q_2 have same sign and as an attractive force when two signs are opposite.

(*d*) The force between any two charges is independent of the presence of other charges.

Combining these relations we have $\mathbf{F} \propto \frac{q_1 q_2}{r^2}\,\hat{r}.$ ...(1)

It is better to write $\mathbf{F}_{21}$ instead of **F** as we are concerned with the force on charge 1 due to charge 2. Hence the force $\mathbf{F}_{12}$ on charge 2 due to charge 1 will be $-\mathbf{F}_{21}$.

In Gaussian system of units* the unit of charge is thus defined so that upon a quantity of electricity equal to itself, at a distance 1 cm., it exerts a force of 1 dyne. In this case constant of proportionality for the charges in vacuum or air becomes unity. In SI units the constant of proportionality is usually written in a more complex way as $1/4\pi\varepsilon_0$. Thus we have

$$\mathbf{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r^2}\hat{r}. \quad ...(2)$$

There is no choice about the constant ε_0, *known as the permittivity of free space.* It must have that value which makes the right hand side equal to the left hand side. Thus we have

$$\varepsilon_0 = 8.85418 \times 10^{-12}\ \text{coul}^2/\text{nt.m}^2 \quad \text{or} \quad 1/4\pi\varepsilon_0 = 9 \times 10^9\ \text{nt. m}^2/\text{coul}^2.$$

The equation (2) may be written as**

$$\mathbf{F}_{21} = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r_{21}^{\ 3}}r_{21}, \quad ...(3)$$

where r_{21} is the vector distance to q_1 from q_2. If r_1 and r_2 be the position vectors of the points [Fig. 3.1(*b*)] where charges q_1 and q_2 are placed respectively, then

$$r_{21} = r_1 - r_2. \quad ...(4)$$

If more than two charges are present, the force exerted on an one say q_1, by all the others q_2, q_3, q_4, etc., can be given by adding vectorially all the forces obtained by using Eq. (2).

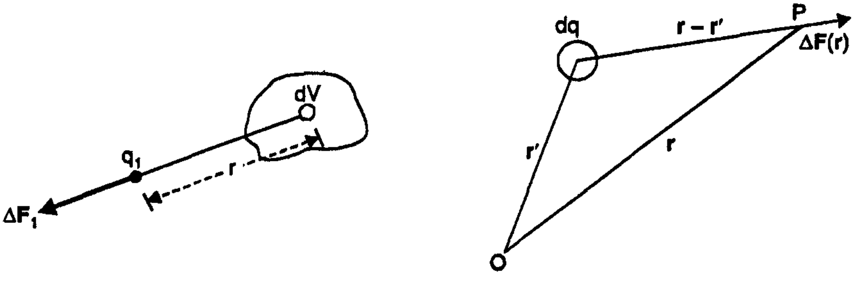
$$\mathbf{F}_1 = \mathbf{F}_{21} + \mathbf{F}_{31} + \mathbf{F}_{41} + ...,$$

where $\mathbf{F}_{21}$ stands for the force exerted on q_1 due to q_2. Thus we have

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_1q_2}{r_{21}^{\ 2}}\hat{r}_{21} + \frac{q_1q_3}{r_{31}^{\ 2}}\hat{r}_{31} + ...\right] = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_1q_i}{r_{i1}^{\ 2}}\hat{r}_{i1}. \quad ...(5)$$

Here the unit vectors $\hat{r}_{21}$ and $\hat{r}_{31}$ have the directions of the lines to q_1 from q_2 and q_3 respectively.

This is just the *superposition principle* for forces.



(*a*) (*b*)

Fig. 3.2. Force due to continuous charge distribution.

In general for a system of n charges, the force on the i^{th} charge is given by

$$\mathbf{F}_i = \frac{q_i}{4\pi\varepsilon_0}\sum_{j\neq i}^{n}\frac{q_j}{\mathbf{r}_{ji}^{\ 3}}\mathbf{r}_{ji} = \frac{q_i}{4\pi\varepsilon_0}\sum_{j\neq i}^{n}\frac{q_j(\mathbf{r}_i - \mathbf{r}_j)}{|\mathbf{r}_i - \mathbf{r}_j|^3} \quad ...(6)$$

---

* In the Gaussian system of units the constant of proportionality in Coulomb's law is arbitrarily chosen equal to unity and the units of charge, current, etc. thereby established. In SI units we start with the definition of the unit of current strength and from this the units of the other quantities such as voltage, charge, etc. were deduced.

** Some authors use $\mathbf{F}_{12}$ for the force on charge 1 due to charge 2 and $\mathbf{r}_{12}$ for the distance to charge 1 from charge 2.

Instead of discrete charge, if the charge is continuously distributed in a region. To calculate the total force on q_1, let us consider the charge as made up of small charge elements and then the forces due to each element dq is added vectorially. As the charge distribution is continuous, hence the summation is replaced by integration.

Thus we have

$$\mathbf{F}_1 = \frac{1}{4\pi\varepsilon_0}\int q_i \frac{dq}{r^2}\hat{r} = \frac{q_1}{4\pi\varepsilon_0}\int \frac{dq}{r^2}\hat{r}, \qquad ...(7)$$

where $\hat{r}$ is a variable unit vector that points from each charge element dq toward the location of charge q_1.

The charge distribution may be described by means of point functions :

(a) Volume Charge Density ρ, which is defined by

$$\rho = \underset{\Delta V \to 0}{\text{limit}} \frac{\Delta q}{\Delta V} \text{ (C/m}^3\text{)},$$

where Δq is the net charge in an infinitesimal volume ΔV.

(b) Surface Charge Density σ, which is defined by

$$\sigma = \underset{\Delta S \to 0}{\text{limit}} \frac{\Delta q}{\Delta S} \text{ (C/m}^2\text{)},$$

where Δq is the net charge on an infinitesimal surface, ΔS. It is also denoted by ρ_s.

If charge is distributed through a volume V and on the surface S that bounds a volume V, then the force on a point charge q located at $\mathbf{r}$ is given by

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\int_V \frac{\rho dV}{r^3}\mathbf{r} + \frac{q}{4\pi\varepsilon_0}\int_S \frac{\sigma dS}{r^3}\mathbf{r}. \qquad ...(8)$$

If variable $\mathbf{r}'$ is used to locate a point within the charge distribution, then we have

$$\mathbf{F}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\int_V \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}')\,dV' + \frac{q}{4\pi\varepsilon_0}\int_S \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(\mathbf{r}')dS' \qquad ...(9)$$

Here $\mathbf{r}'$ plays the role of the source point $\mathbf{r}_j$ in Eq. (6).

Here we see that

(1) The force on q_1 due to q_2, $\mathbf{F}_{21} = -\mathbf{F}_{12}$ the force on q_2 due to q_1.
(2) Like charges (charges of same sign) repel each other while unlike charges attract.
(3) The distance r must be large compared to the linear dimensions of the charged bodies.
(4) The charges must be static or at rest.
(5) The signs must be accurately taken into account.

Coulomb's law applies to point charges, whose spatial dimensions are very small compared with any other length used in the problem. It also applies to the interactions of elementary particles, such as electrons and protons. It is found to hold even for electrostatic repulsion between protons inside the nucleus, however nuclear forces dominate over this repulsion. The spontaneous emission of α-particles, breaking up of the nucleus into two large fragments, presence of more neutrons than the protons in the heavy nuclei are the Coulomb's repulsion effects. We do not know whether this law holds for very large astronomical distances or not.

3.3 ELECTRIC FIELD STRENGTH

The gravitational force may be explained by assuming that every point in space near the earth or planet is associated with a field, known as *gravitational field* of that planet or satellite. Similarly the space surrounding a charged body is associated with a field known as *electric field or electrostatic field* (the field due to static charge).

If a charge q_1 is placed at any point, it sets up an electric field in the space around itself. This field is indicated by the shaded region in Fig. 3.3. If a charge q_2 is placed in the field region of charge q_1, the former will experience a force **F** or $\mathbf{F}_{12}$. In this way *the field plays an intermediary role in the forces between the charges.*

Electric Field Strength : A test charge q_0 (assumed positive for convenience) is placed at any point in the region of any charge, where we want to calculate the electric field strength. If this test charge experiences a force **F**, then the electric *field intensity or strength* **E** at that point is defined as

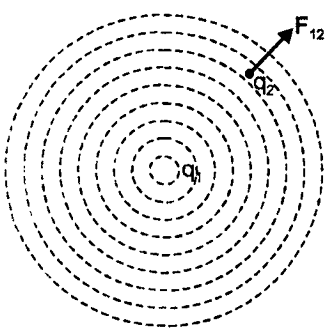


Fig. 3.3. Charge q_2 in the field of charge q_1.

$$\mathbf{E} = \mathbf{F}/q_0. \qquad ...(10)$$

The direction of **E** is the direction of **F** and its unit is Newton/Coulomb.

We must assume a test charge as small as possible so that it will not influence the behavior of the primary charges that are responsible for the field to be determined. Thus Eq. (10) should be replaced by

$$\mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0}. \qquad ...(11)$$

Actually the *test is fictitious.* We merely ask what would be the force on it, if placed at the observation point without affecting the original electric field to be measured. The requirement that the test charge be vanishingly small compared with all sources of the field limits the practical validity of the definition, *i.e.* the definition is more suitable for macroscopic phenomena.

The force experienced by test charge q_0 placed at a distance r from a point charge q may be written from Coulomb's law as $\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{qq_0}{r^2}\hat{r}$,

$$\therefore \qquad \mathbf{E} = \lim_{q_0 \to 0} \frac{\mathbf{F}}{q_0} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = -\frac{q}{4\pi\varepsilon_0}\nabla\left(\frac{1}{r}\right). \qquad ...(12)$$

If we consider coordinates of the points instead of absolute distance between the charges, then Eq. (12) will become

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1(\mathbf{r}-\mathbf{r}_1)}{|\mathbf{r}-\mathbf{r}_1|^3}, \qquad ...(13)$$

where electric field is calculated at point **r** due to a point charge q_1 placed at point $\mathbf{r}_1$ and $|\mathbf{r}-\mathbf{r}_1|$ is the absolute distance between these points.

The resultant electric field at **r** due to many point charges q_i located at $\mathbf{r}_i$, $i = 1, 2, \ldots n$, may be obtained by superposition principle for forces, as

$$\mathbf{E}(\mathbf{r}) = \sum_{i=1}^{n}\frac{q_i(\mathbf{r}-\mathbf{r}_i)}{|\mathbf{r}-\mathbf{r}_i|^3}. \qquad ...(14)$$

If the charges are continuously distributed, the field can be calculated by dividing the charge into infinitesimal elements and integrating the contributions due to all the elements, such as

$$\mathbf{E}(\mathbf{r}) = \int d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r^2}\hat{r}, \qquad ...(15)$$

where $\hat{r}$ is the unit vector pointing from dq toward the point in space at which **E** is being calculated.

For a volume distribution of charge specified by the volume charge density $\rho(\mathbf{r}')$ in the volume V, a surface distribution of charge specified by the surface charge density $\sigma(\mathbf{r}')$ on the surface S and n point charges $q_1, q_2, \ldots q_n$ located at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively, electric field **E** at point **r** is given as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_V \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}|^3}\rho(\mathbf{r}')dV' + \int_S \frac{(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(\mathbf{r}')dS' + \sum_{i=1}^{n}\frac{(\mathbf{r}-\mathbf{r}_i)q_i}{|\mathbf{r}-\mathbf{r}_i|^3}\right] \qquad ...(16)$$

3.4 CALCULATION OF ELECTRIC FIELD STRENGTH

(a) Electric Dipole : If two electric charges, equal in magnitude but opposite in signs coincide at any point, the electric field strength due to these charges in the space around them vanishes. If the charges suffer a small relative displacement, there is an electric field of sensible magnitude in the neighbourhood of these charges. Such a combination of two equal and opposite charges is termed as *electric dipole.*

Let the charges $-q$ and $+q$ be located at points $\mathbf{r}'$ and $\mathbf{r}' + \mathbf{l}$ respectively with respect to the origin O of coordinate system. The electric field at arbitrary point $\mathbf{r}$ may be written as

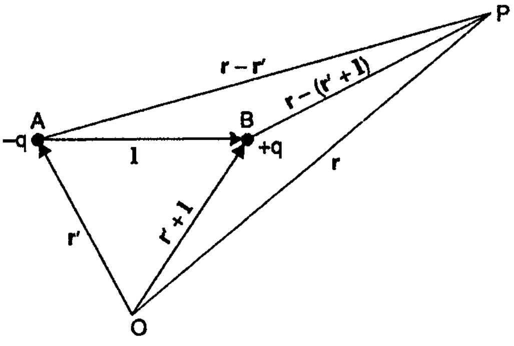


Fig. 3.4. Electric dipole AB.

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}-(\mathbf{r}'+\mathbf{l})}{|\mathbf{r}-(\mathbf{r}'+\mathbf{l})|^3} - \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}. \quad ...(17)$$

This is the general equation for the electric field due to two point charges.

In the case of an electric dipole, the separation $\mathbf{l}$ is small compared with $\mathbf{r} - \mathbf{r}'$. Hence we may expand the denominator of the first term of Eq. (17) as

$$|\mathbf{r}-(\mathbf{r}'+\mathbf{l})|^{-3} = [(\mathbf{r}-\mathbf{r}'-\mathbf{l})^2]^{-3/2} = [|\mathbf{r}-\mathbf{r}'|^2 - 2(\mathbf{r}-\mathbf{r}').\mathbf{l} + l^2]^{-3/2}$$

$$= |\mathbf{r}-\mathbf{r}'|^{-3}\left[1 - \frac{2(\mathbf{r}-\mathbf{r}').\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^2} + \frac{l^2}{|\mathbf{r}-\mathbf{r}'|^2}\right]^{-3/2}$$

Since $l << |\mathbf{r} - \mathbf{r}'|$, therefore the last term can be neglected. By the binomial theorem, we get

$$|\mathbf{r}-(\mathbf{r}'+\mathbf{l})|^{-3} = |\mathbf{r}-\mathbf{r}'|^{-3}\left[1 + \frac{3(\mathbf{r}-\mathbf{r}').\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^2} + ...\right] \quad ...(18)$$

The combination of Eqs. (17) and (18) gives

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{r}-(\mathbf{r}'+\mathbf{l})}{|\mathbf{r}-\mathbf{r}'|^3}\left\{1 + \frac{3(\mathbf{r}-\mathbf{r}')\cdot\mathbf{lr}'}{|\mathbf{r}-\mathbf{r}'|^2} + ...\right\} - \overset{r}{\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}}\right]$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{r}-\mathbf{r}'-\mathbf{l})(\mathbf{r}-\mathbf{r}')\cdot\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^5} - \frac{\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^3} + ...\right] \quad ...(19)$$

The other terms consist of higher powers of $\mathbf{l}$ and thus be neglected, as $\mathbf{l}$ is very small. For $\mathbf{l} = 0$, $\mathbf{E}(\mathbf{r}) = 0$, *i.e.*, the *electric field intensity is zero for a point dipole.* For a limiting case $l \to 0$, $q \to \infty$, the product $q\mathbf{l}$ (= $\mathbf{p}$, the electric dipole moment) is finite.

Thus Eq. (19) may be written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{r}-\mathbf{r}')\cdot\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^5}(\mathbf{r}-\mathbf{r}') - \frac{\mathbf{p}}{|\mathbf{r}-\mathbf{r}'|^3}\right]. \quad ...(20)$$

This is the general expression for the electric field due to an electric dipole. If we transform the point $\mathbf{r}'$ to the origin (or point A coincides 0), then the above expression will become

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3(\mathbf{r}\cdot\mathbf{p})\,\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right] \quad ...(21)$$

If the point P is along the dipole *i.e.*, $\mathbf{r} \parallel \mathbf{p}$, we get

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\frac{3\mathbf{p}}{r^3} - \frac{\mathbf{p}}{r^3}\right] = \frac{1}{4\pi\varepsilon_0}\frac{2\mathbf{p}}{r^3}. \quad ...(22)$$

If the point P is along the perpendicular bisector, *i.e.*, $\mathbf{r} \perp \mathbf{p}$, we get

$$\mathbf{E}(\mathbf{r}) = -\mathbf{p}/4\pi\varepsilon_0 r^3. \qquad ...(23)$$

For the electric field at point P with polar coordinates (r, θ), we may resolve moment $\mathbf{p}$, into two components, one parallel and other perpendicular to $\mathbf{r}$. The field E_1 due to the component $p \cos\theta$ is equal to $2p\cos\theta/4\pi\varepsilon_0 r^3$ and the field E_2 due to the component $p \sin\theta$ is equal to $p \sin\theta/4\pi\varepsilon_0 r^3$ which is perpendicular to the former. Hence magnitude of the resultant field $\mathbf{E}$ is given by

$$E = \sqrt{E_1^{\,2} + E_2^{\,2}} = \frac{p}{4\pi\varepsilon_0 r^3}\sqrt{1 + 3\cos^2\theta}. \qquad ...(24)$$

Its inclination ϕ to OP is given by $\tan\phi = \dfrac{p\sin\theta/4\pi\varepsilon_0 r^3}{2p\cos\theta/4\pi\varepsilon_0 r^3} = \dfrac{1}{2}\tan\theta.$ ...(25)

(*b*) Charged Rod : Let us consider a uniformly charged rod whose linear charge density λ (charge per unit length, also denoted by ρ_l) is constant. We wish to find the electric field at point P due to this rod. *Coulomb's law cannot be applied directly in this case as it applies only to point charges.* However we can imagine the rod to be divided into a large number of small segments, so that each segment will act as a point charge. Coulomb's law can be applied to this segment. If Δx be the length of the segment at a distance x from point P, *which is along the length of this rod* (Fig. 3.5). The electric field due to this segment at point P is given by

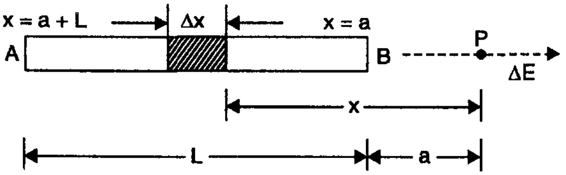


Fig. 3.5. A charged rod.

$$\Delta \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda \Delta x}{x^2}\hat{x}.$$

It is along BP direction. Since the fields due to all segments of the rod are in the same direction as $\Delta\mathbf{E}$, hence

$$E = \Sigma\Delta E = \Sigma\frac{\lambda}{4\pi\varepsilon_0}\frac{\Delta x}{x^2}. \qquad ...(26)$$

If Δx is sufficiently small, we can replace the sum by an integral, thus we get

$$E = \int_{(a+L)}^{a}\frac{\lambda}{4\pi\varepsilon_0}\frac{dx}{x^2} = \frac{\lambda}{4\pi\varepsilon_0}\int_{(a+L)}^{a}\frac{dx}{x^2}$$

$$E = \frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{a+L}\right]. \qquad ...(27)$$

To find the electric field at a point P to one side of the rod [Fig. 3.6], we can not write relation (26), as $\Delta\mathbf{E}$ due to each segment of the rod will point in different directions. The y-component

$$\Delta E_y = \Delta E\cos\theta.$$

As all the y-components are in the same direction, thus

$$E_y = \Sigma\,\Delta\,E\cos\theta = \Sigma\frac{\lambda\Delta x\cos\theta}{4\pi\varepsilon_0} = \int_a^{a+L}\frac{\lambda dx}{4\pi\varepsilon_0}\frac{\cos\theta}{r^2}.$$

Unfortunately, there are three variables (x, r and θ). To convert in one let us use the relation $r^2 = x^2 + b^2$ and $\cos\theta = b/r$.

$$\therefore\quad E_y = \int_a^{a+L}\frac{\lambda b}{4\pi\varepsilon_0}\frac{dx}{(x^2+b^2)^{3/2}} = \frac{\lambda b}{4\pi\varepsilon_0}\left[\frac{1}{b^2}\frac{x}{(x^2+b^2)^{1/2}}\right]_a^{a+L}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{a+L}{[(a+L)^2+b^2]^{1/2}} - \frac{a}{(a^2+b^2)^{1/2}}\right]. \quad ...(28)$$

The x-component of $\mathbf{E}$ can be obtained in the similar way.

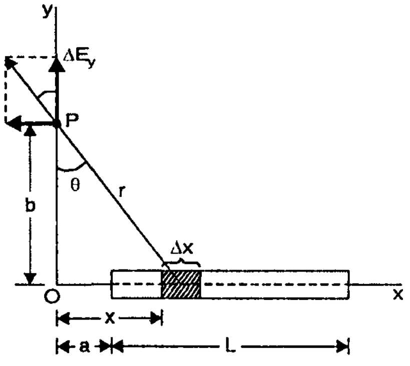


Fig. 3.6. Point P to one side of a charged rod.

$$E_x = \Sigma\Delta E \sin\theta = \int_a^{a+L} \frac{\lambda}{4\pi\varepsilon_0} \frac{x\,dx}{(x^2+b^2)^{3/2}}$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\left[-\frac{1}{[(a+L)^2+b^2]^{1/2}} + \frac{1}{(a^2+b^2)^{1/2}}\right] \qquad \text{...(29)}$$

This relation shows that $E_x = 0$ for $a = -L/2$ *i.e.*, on the points lying on the perpendicular bisector of the rod. For these points, the value of ΔE_x due to the left hand segment cancels the ΔE_x due to the right hand segment at the same distance.

Another interesting property of the result will be obtained when point P is very far from rod, *i.e.*, $b >> L$. In this case

$$E_y = \frac{\lambda}{4\pi\varepsilon_0}\frac{L}{b^2} = \frac{q}{4\pi\varepsilon_0 b^2} \text{ and } E_x = 0. \qquad \text{...(30)}$$

This relation is same as due to a point charge $q(=\lambda L)$. It is expected, as the rod will appear to be a point charge when viewed from far away.

In the special case when the *rod is of infinite length*, the field can be obtained by integrating with in the limits $x = -\infty$ to $x = +\infty$ and the result will be

$$E_y = \lambda/2\pi\varepsilon_0 b \text{ and } E_x = 0. \qquad \text{...(31)}$$

(c) Charged Loop : Consider a circular loop of radius a carrying a uniform charge λ per unit length (also denoted by ρ_l). It is very difficult to compute **E** at any point in the space, however easy at points on its axis. Let us divide the loop in very small segments so that each segment can be assumed as a point charge. The electric field at a point P on the axis of the loop at a distance x from its center due to the charge on the segment of length δl (Fig. 3.7) is given by

$$\Delta E = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\delta l}{r^2}.$$

Its direction is along the line from δl to point P. The contributions to **E** of the various parts of the ring will be in different directions. The y-component ΔE_y cancels with the contribution due to the oppositely placed element of the loop and the net y-component is thus zero. The resultant field **E** will therefore be along x-axis

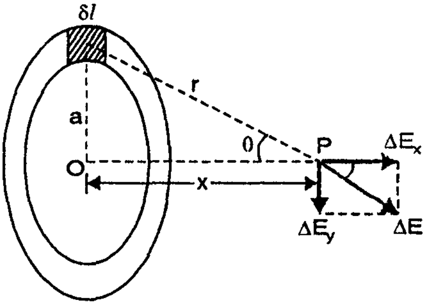


Fig. 3.7. Charged loop.

$$\therefore \quad E = \Sigma\Delta E_x = \int \frac{\lambda}{4\pi\varepsilon_0}\frac{dl}{r^2}\cos\theta.$$

Using relations $r^2 = a^2 + x^2$ and $\cos\theta = x/r$, we get

$$E = \frac{\lambda x}{4\pi\varepsilon_0(a^2+x^2)^{3/2}}\cdot\int dl = \frac{\lambda}{4\pi\varepsilon_0}\frac{x.2\pi a}{(a^2+x^2)^{3/2}}$$

$$= \frac{qx}{4\pi\varepsilon_0(a^2+x^2)^{3/2}}, \qquad \text{...(32)}$$

where q is the total charge on the loop $= 2\pi a\lambda$.

At a very large distance $x >> a$ and we get $E = q/4\pi\varepsilon_0 x^2$, ...(33)

which is same as due to a point charge.

(d) Infinite Plane Distribution of Charge : Let us find the field at a point P, at a height z above an infinite plane distribution of charge of uniform charge density σ. For this, let us consider a charge element ΔS at any point A on the plane with the coordinates x, y, 0 (Fig. 3.8).

Hence field due to this element ΔS at point P is given by

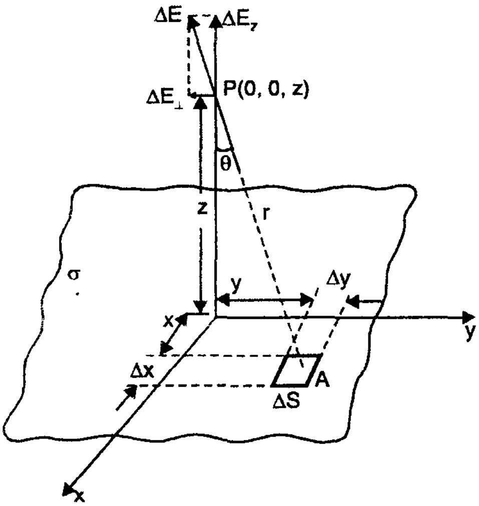


Fig. 3.8. Plane sheet of charge.

$$\Delta E = \frac{1}{4\pi\varepsilon_0}\frac{\sigma \Delta S}{r^2}\hat{r}.$$

$$\therefore \quad \Delta E_z = \frac{\sigma}{4\pi\varepsilon_0}\frac{\Delta x\, \Delta y}{r^2} \times \frac{z}{r}$$

$$= \frac{\sigma z\, \Delta x\, \Delta y}{4\pi\varepsilon_0 (x^2 + y^2 + z^2)^{3/2}}.$$

All the components along the z-direction are added while perpendicular components are cancelled due to the symmetry of the plane of infinite size. Hence the total electric field due to the infinite plane of charge

$$E = \Sigma E_z = \frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} \frac{z dx}{(x^2 + y^2 + z^2)^{3/2}}.$$

$$= \frac{\sigma}{4\pi\varepsilon_0}\int_{-\infty}^{+\infty} dy \frac{2z}{y^2 + z^2} = \frac{\sigma}{2\varepsilon_0}. \qquad \text{...(34)}$$

Thus we see that the field strength at a point outside an infinite plane distribution of charge is independent of the distance z of P from the plane and is in the opposite direction on the other side of the plane and hence suffers a discontinuity at the plane.

The above relation can also be obtained by dividing the plane in circular rings. We leave it as an exercise for the students.

The above problem can be solved by using the cylindrical coordinates. If the plane is assumed in the $z = 0$ plane as above, and the point A on this plane will have coordinates $(r, \phi, 0)$. The point P will have coordinates $(0, \phi, z)$. Thus the electric field at point P due to the charge element at A

$$dE = \frac{\sigma r\, dr\, d\phi}{4\pi\varepsilon_0 (r^2 + z^2)}$$

The vector along AP will be given by

$$\mathbf{R} = (0 - r)\,\hat{r} + (\phi - \phi)\,\hat{\phi} + (z - 0)\hat{z}$$

$$\therefore \quad \text{Unit vector } \hat{\mathbf{R}} = \frac{-r\hat{r} + z\hat{z}}{\sqrt{r^2 + z^2}}$$

$$\therefore \quad d\mathbf{E} = \frac{\sigma r\, dr\, d\phi\, (-r\hat{r} + z\hat{z})}{4\pi\varepsilon_0\, (r^2 + z^2)^{3/2}}$$

The symmetry about the z-axis results in cancellation of the radial components. Therefore

$$\mathbf{E} = \int_0^{2\pi}\int_0^{\infty} \frac{\sigma r z\, dr\, d\phi}{4\pi\varepsilon_0 (r^2 + z^2)^{3/2}}\hat{z} = \frac{\sigma z}{4\pi\varepsilon_0}\int_0^{\infty} \frac{r\, dr}{(r^2 + z^2)^{3/2}} \int_0^{2\pi} d\phi\, \hat{z}$$

$$= \frac{\sigma z}{2\varepsilon_0}\int_0^{\infty} \frac{r dr}{(r^2 + z^2)^{3/2}}\hat{z} = \frac{\sigma z}{2\varepsilon_0}\left[\frac{-1}{(r^2 + z^2)^{1/2}}\right]_0^{\infty}\hat{z} = \frac{\sigma}{2\varepsilon_0}\hat{z}.$$

(*e*) Charged Disc : Let us calculate the field strength on the axis of a disc of radius R, at a distance x from its center O. For this let us divide the disc into annular rings. The field at a point P(Fig. 3.9) due to charge element δs on the annular ring of radius a and of thickness δa and thus surface area δS (= $\delta s\, \delta a$) is given by

$$\Delta \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma \delta S}{r^2}\hat{r}.$$

Its component along the axis of the disc or along the x-direction

$$\Delta E_x = \frac{1}{4\pi\varepsilon_0}\frac{\sigma\delta S}{r^2}\frac{x}{r}.$$

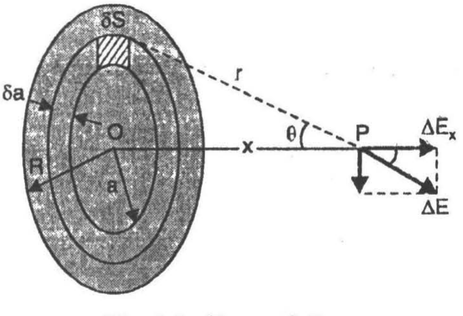


Fig. 3.9. Charged disc.

For reasons of symmetry the total field is along x-direction and is obtained by adding vectorially or by integrating. Assuming that the surface density of charge σ is constant, then

$$E = E_x = \frac{\sigma}{4\pi\varepsilon_0}\int\frac{x}{r^3}da\int ds = \frac{\sigma}{4\pi\varepsilon_0}\int\frac{x}{r^3}da.\,2\pi a$$

$$= \frac{x\sigma}{2\varepsilon_0}\int_0^R\frac{a\,da}{(x^2+a^2)^{3/2}} = \frac{\sigma}{2\varepsilon_0}\left[1-\frac{x}{(x^2+R^2)^{1/2}}\right] \quad ...(35)$$

If the *surface density of charge* σ *is a function of the distance from the centre of the disc, i.e.*, $\sigma = ba$, where b is a constant of proportionality, then

$$E = E_x = \frac{1}{4\pi\varepsilon_0}\int_0^R\frac{x\,b\,a\,da}{(a^2+x^2)^{3/2}}\int ds = \frac{bx}{2\varepsilon_0}\int_0^R\frac{a^2\,da}{(a^2+x^2)^{3/2}}$$

$$= \frac{bx}{2\varepsilon_0}\left[\log_e\frac{R+(R^2+x^2)^{1/2}}{x} - \frac{R}{(R^2+x^2)^{1/2}}\right]. \quad ...(36)$$

3.5 EFFECT OF ELECTRIC FIELD ON A POINT CHARGE

The force $\boldsymbol{F}$ on a charge particle of mass m and charge q due to an electric field of strength $\boldsymbol{E}$ is given by

$$\boldsymbol{F} = q\boldsymbol{E}. \quad ...(37)$$

This force will produce an acceleration in the charged particle, given by

$$\boldsymbol{a} = \boldsymbol{F}/m = (q/m)\boldsymbol{E}. \quad ...(38)$$

This acceleration is of charged particle placed, not of that due to which the electric field $\boldsymbol{E}$ is produced. For example, the earth's gravitational field can not have any effect on the earth, but only on a second body, say a stone, placed in that field.

3.6 LINES OF FORCE

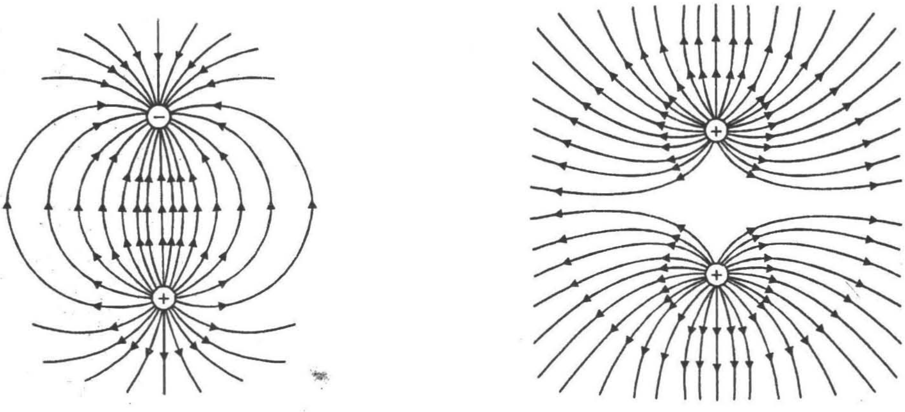
When the vector **E** is known for all points in an electric field the field is completely specified. The direction and magnitude of field at any point indicate how a small charge q_0 would begin to move if put there. By following this direction from point to point curves are obtained. These are termed as *line of force*. Thus *the line of force in an electric field is a curve such that the tangent at any point on it gives the direction of the resultant electric field strength at that point. This is also the path on which a test charge will tend to move, if free to move.*

These lines are imaginary, their existence can be shown by sprinkling saw dust or gypsum salt in the electrostatic field. These particles acquire charges and place themselves along the lines of force. *The properties of electric lines of force are* :

1. No line of force originates or terminates in the space surrounding a charge. Every line of force is a continuous and smooth curve originating from a positive charge and ending on a negative charge.
2. The tangent to a line of force at any point gives the direction of $\boldsymbol{E}$ at that point.
3. They do not pass but leave or end on *a* charged conductor normally when the charges on the conductor are in equilibrium. Suppose the lines of force are not ⊥ to the conductor surface. In this situation the component of electric field || to the surface would cause the electrons to move and would therefore give rise to a current. Since there is no current on the surface, *hence lines of force are always ⊥ to the conductor surface.*

4. Lines of force never intersect. If it happens then we say that at the point of intersection the electric intensity is zero, otherwise it would have to be tangential to two different curves at the same instance which is impossible.
5. The number of lines per unit area crossing a surface at right angle to the field direction at every point is proportional to the electric intensity. Hence the lines of force are closely spaced where the intensity is large and are widely separated where the intensity is small.
6. They repel each other in the case of two like charges.

The direction of lines of force in two cases are shown in Fig. 3.10.



(a) Opposite charges *(b) Similar charges*

Fig. 3.10. Lines of forces due to two point charges.

We define unit field (1 N/C) arbitrarily as corresponding to unit density of lines of force (one line/m^2). To determine the number of lines per unit charge, let us draw a spherical surface of radius r around the charge q at its centre. The electric field E at the surface is given by $E = q/4\pi\varepsilon_0 r^2$. As the density of lines of force is E, the total number of lines of force originating at q and crossing the spherical surface will be $4\pi r^2 E = q/\varepsilon_0$. Hence the *number of lines originating from a unit charge is* $1/\varepsilon_0$.

Electric lines of force are very useful as they give qualitative picture of the electric field distribution. There is a simple connection between the lines of force and inverse square law. We know that the density of lines at r meter from an isolated charge q is $q/4\pi\varepsilon_0 r^2$. As the area of the spherical surface subtended by a bundle of lines originating from the point charge is four times greater for a surface $2r$ meter away than for a surface at r meter distance. Thus the density of lines of force at $2r$ meter is just one quarter that at r meter in agreement with the value from *inverse square law*.

3.7 SOLID ANGLE

Solid angle is the analogue in three dimensions of the usual angle in two dimensions. In two dimensions, the unit angle is the angle subtended by an arc of length equal to its radius and is known as *radian*. In three dimensions, a given area on a sphere subtends a certain solid angle at its center. The unit solid angle is that subtended at the center of a sphere by an area equal to the square of its radius and is known as *steradian*. Since total surface area of the sphere is $4\pi r^2$, hence the total solid angle subtended at its center is 4π steradians.

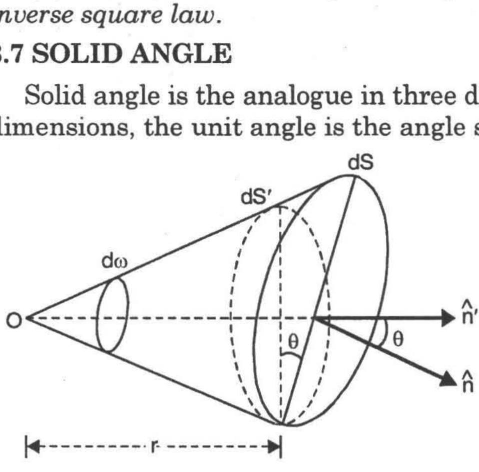


Fig. 3.11. Solid angle.

In Fig. 3.11, dS and dS' subtend the same solid angle. Thus the solid angle subtended by any surface dS at a point O a distance r away is given by

$$d\omega = dS \cos\theta / r^2, \qquad ...(39)$$

where $dS \cos\theta$ (= dS') is the projection of the surface dS perpendicular to the radius vector r from the point O.

3.8 GAUSS'S LAW

Consider a single positive point charge q surrounded by a closed surface of arbitrary shape, as shown in Fig. 3.12. The electric intensity **E** at every point of the surface is directed radially outward from the charge. Let us consider any sufficiently small area dS of the surface for which **E** can be considered to have same magnitude and direction. If θ be the angle between **E** and the outward normal to the surface at any point of this area. The product E_n, the component of **E** normal to the surface, and the area dS is

$$E_n dS = E\cos\theta\, dS$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\cos\theta\, dS.$$

As $dS\cos\theta/r^2$ can be replaced by the term solid angle $d\omega$ subtended at the charge q by the area dS, hence.

$$E_n\, dS = (q/4\pi\varepsilon_0)\, d\omega.$$

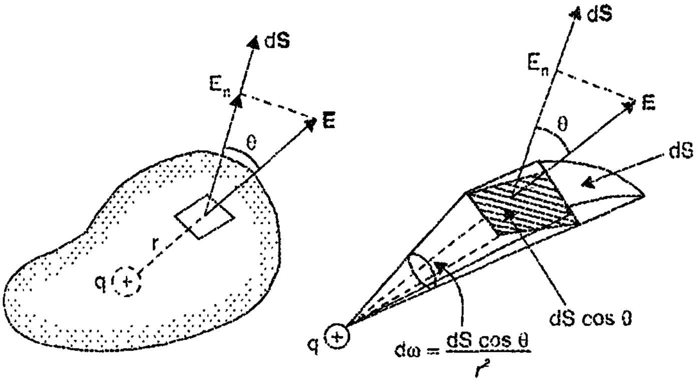


Fig. 3.12. Point charge surrounded by a closed surface.

If the surface is continuous we can integrate both sides of this equation over the entire closed surface and so we have

$$\oint_S E_n dS = \frac{q}{4\pi\varepsilon_0}\oint_S d\omega = \frac{q}{4\pi\varepsilon_0}4\pi = \frac{q}{\varepsilon_0}.$$

[As $\oint d\omega$ = Total solid angle surrounding the charge $q = 4\pi$, regardless of the shape or size of the closed surface.]

As the product $E_n\, dS = E\cos\theta\, dS = \mathbf{E}\cdot d\mathbf{S}$, hence we have

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = q/\varepsilon_0. \qquad ...(40)$$

If the point charge is negative the field **E** is directed radially inward, the angle θ is greater than 90° or E_n is negative. The surface integral is thus negative. If the closed surface does not enclose charge, the flux of **E**, *i.e.*, $\oint_S \mathbf{E}\cdot d\mathbf{S}$ equals zero. Hence the form of Eq. (40) is correct whatever be the sign and value of the charge.

Any arbitrary distribution of charges can be assumed as a system of point charges. We can use Eq. (40) for each point charge and sum over all charges. In this case charge q is equal to Σq. For the continuous distribution of charges q may be written as $\int \rho dV$. Thus

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\sum_{i=1}^{n} q_i = \frac{1}{\varepsilon_0}\int_V \rho dV. \qquad ...(41)$$

This is the equation which expresses Gauss's law : *The surface integral of the normal component of electric field* **E** *over any closed surface in an electrostatic field equals $1/\varepsilon_0$ times the total or net charge contained in a volume enclosed by the surface.*

The left hand side of Eq. (41) is called the flux of **E** across the surface and is represented by Φ. Thus

$$\Phi = \oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_V \rho dV. \qquad ...(42)$$

The term flux is borrowed from hydrodynamics, where a similar integral in which **E** is replaced by the velocity vector **v** gives the net flow of fluid across a surface. The Gauss's law can now be stated as : *the outward flux of electric field across a closed surface equals* $1/\varepsilon_0$ *times the net charge contained in the volume enclosed by the surface. Equation (42) is known as the integral form of the Gauss's law.*

Using the Gauss's divergence theorem

$$\oint_S \mathbf{F}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{F}\, dV.$$

we get

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \int_V \text{div } \mathbf{E}\, dV = \frac{1}{\varepsilon_0}\int_V \rho dV.$$

Since it is true for any arbitrary volume. Thus the integrands must be equal. Therefore

$$\text{div } \mathbf{E} = \rho/\varepsilon_0. \qquad \text{...(43)}$$

This is the *differential form of Gauss's law,* and is the fundamental equation of electrostatics. It states that *the divergence of electric field* **E** *at any point is equal to* $(1/\varepsilon_0)$ *times the charge density at that point.*

In terms of electric displacement density **D** (to be defined later), which is equal to $\varepsilon_0\mathbf{E}$, the Gauss's law can be stated as : *the net outward electric displacement through a closed surface is equal to the net charge contained in the volume enclosed by the surface, i.e.,*

$$\oint_S \mathbf{D}\cdot d\mathbf{S} = \int_V \rho\, dV. \qquad \text{...(44)}$$

$$\int_V \nabla\cdot\mathbf{D}\, dV = \int_V \rho\, dV. \qquad \text{...(45)}$$

or

$$\nabla\cdot\mathbf{D} = \rho, \qquad \text{...(46)}$$

i.e., at every point in a medium the divergence of electric displacement is equal to the charge density at that point. It is the first of the four Maxwell's equations to be derived. The Gauss's law may be called as alternative statement of Coulomb's law. The Gauss's law provides an easy means of finding **E** or **D** for symmetric charge distribution.

Limitation : Gauss's law would be invalid if the inverse square law were not exactly true.

Verification of Gauss's law : Gauss's law may be verified with the help of Fig. 3.13. When the point lies outside the surface, every elementary cone from O cuts the surface an even number of times. Let us draw a cone of solid angle $d\omega$ at point O, intercepting areas dS_1, dS_2, dS_3, dS_4, dS_5 and dS_6 at points A, B, C, D, E and F respectively. The corresponding outward normal over these areas are shown in Fig. 3.13. The angles θ_1, θ_3 and θ_5 are obtuse and θ_2, θ_4 and θ_6 are acute. Thus

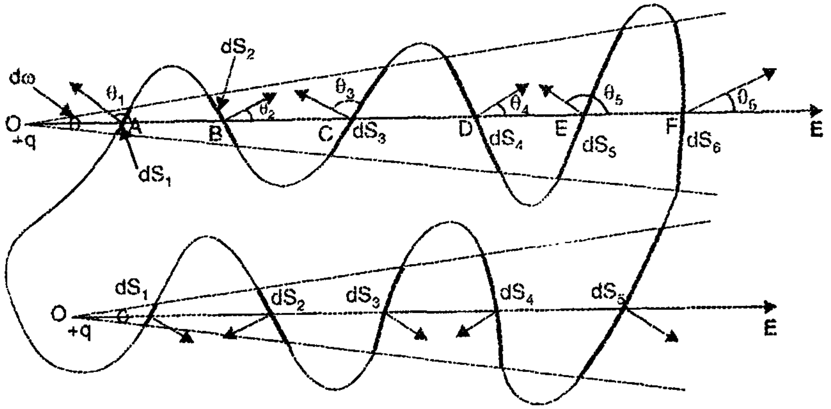


Fig. 3.13. Verification of Gauss's law.

$$d\omega = \frac{dS_1 \cos(\pi - \theta_1)}{r_1^2} = -\frac{dS_1}{r_1^2}\cos\theta_1 = \frac{dS_2}{r_2^2}\cos\theta_2 = -\frac{dS_3}{r_3^2}\cos\theta_3 = \text{etc.}$$

$$\therefore \qquad dS_1 \cos\theta_1 = -r_1^2 d\omega;\ dS_2 \cos\theta_2 = r_2^2\, d\omega;\ dS_3 \cos\theta_3 = -r_3^2\, d\omega;\ \ldots \text{ etc.}$$

Hence the contribution from these six (say) areas to the surface integral is

$$d\Phi = E_1 \cos\theta_1\, dS_1 + E_2 \cos\theta_2\, dS_2 + E_3 \cos\theta_3\, dS_3 + \ldots\ldots$$

$$= \frac{q}{4\pi\varepsilon_0 r_1^2}(-r_1^2 d\omega) + \frac{q}{4\pi\varepsilon_0 r_2^2}(r_2^2 \delta\omega) + \frac{q}{4\pi\varepsilon_0 r_3^2}(-r_3^2 d\omega) + \ldots\ldots$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-d\omega + d\omega - d\omega + d\omega - d\omega + d\omega\right] = 0. \qquad \ldots(47)$$

This is same as required by Gauss's Law. For a point outside the closed surface, the total flux entering the surface is the same as that leaving it.

If the point charge lies inside the surface, the cone of solid angle $d\omega$ will cut the surface an odd number of times (5 in the present case). Thus the net flux will be given by

$$d\Phi = \frac{q}{4\pi\varepsilon_0}(d\omega - d\omega + d\omega - d\omega + d\omega) = \frac{q}{4\pi\varepsilon_0} d\omega.$$

or
$$\Phi = \oint_S \mathbf{E} \cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0}\int d\omega = \frac{q}{\varepsilon_0}.$$

For the negative charge inside the surface, the direction of **E** will be towards O, hence θ_1 will be obtuse instead of acute (in case of the positive charge). Thus net flux will be negative. Therefore we see that the Eq. (40) holds good whether the charge inside the surface is positive, negative or zero.

Graphical Interpretation of Gauss's Law : As the number of electric lines of force per unit area at right angles to their direction is proportional to **E**, the surface integral of **E** over a closed surface is proportional to the total number of lines of force crossing the surface in an outward direction. The total number of lines of force is also proportional to the net charge within the surface.

For closed surface, Φ is positive if the lines of force point outward everywhere (**E** is outward every where, $\theta < 90°$ and $\mathbf{E}\cdot d\mathbf{S}$ will be positive) and negative if they point inward (**E** is inward every where, $\theta > 90°$ and $\mathbf{E}\cdot d\mathbf{S}$ will be negative). Let us consider two equal and opposite point charges and their line of forces as shown in Fig. 3.14. From the statement just given Φ is positive for surface S_1 and is negative for surface S_2. Surface S_4 encloses both charges. The net number of lines of force crossing in an outward direction is zero and the net charge inside the closed surface is also zero. Similarly in surface S_3, number of lines of force in inward direction is same as those in outward direction or the net number of outward lines of force is zero. The charge inside this surface is also zero. In this way it is clear that Gauss's law is correct whatsoever be the charge inside the closed surface.

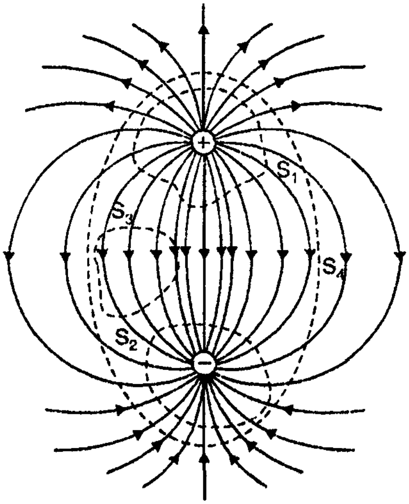


Fig. 3.14. Hypothetical closed surfaces in the electric field.

3.9 APPLICATIONS OF GAUSS'S LAW

As Gauss's law does not provide expression for **E** but provides only for its surface integral. To calculate **E** we must choose a surface in such a way that the surface integral may be replaced by a product of which **E** is a factor. This hypothetical surface is known as *gaussian surface.* Thus the necessary condition for applying Gauss's law for calculating **E** is the uniform and symmetrical charge distribution so that the gaussian surface may also be symmetrical. Let us discuss few simple cases :

(A) Point charge : The electric field due to a point charge is everywhere radial. Its magnitude is the same at all points at the same distance r from the charge. Hence we select gaussian surface (*a closed surface drawn in an electric field*) a spherical surface. It is also clear from Fig. 3.15 that both **E** and $d\mathbf{S}$ (direction of which is always perpendicular to the surface itself) at any point on the gaussian surface are directed radially outward. Hence

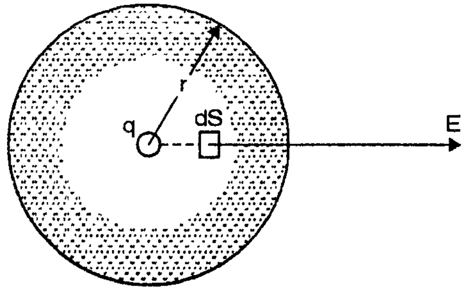


Fig. 3.15. Point charge surrounded by spherical gaussian surface.

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_S E_n dS = ES = E4\pi r^2.$$

But from Gauss's law it should be $1/\varepsilon_0$ times the total charge inside the gaussian surface. Hence

$$4\pi r^2 E = q/\varepsilon_0 \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \qquad ...(48)$$

The force on the point charge q_0 if placed on this surface, *i.e.*, at a distance r from point charge q is then $F = q_0 E = qq_0/4\pi\varepsilon_0\mathbf{r}^2$.

The force is acting in radially outward direction, hence can be represented vectorially as

$$\mathbf{F} = (qq_0/4\pi\varepsilon_0 r^2)\,\hat{r}. \qquad ...(49)$$

It is nothing but Coulomb's law.

(B) Uniformly charged sphere : In the uniformly charged sphere of radius R and charged to q units, the charge density ρ at every point is constant. It is evident that the field due to a spherical charged surface at an external point P has the same symmetry as that of a point charge. Thus we can construct gaussian surface by drawing a concentric sphere of radius OP with centre O. At all points of this sphere the magnitude of electric field is the same and its direction is perpendicular to the surface. Thus

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_S E_n dS = ES = E4\pi r^2.$$

Using Gauss's law, we get

$$4\pi r^2 E = \frac{q}{\varepsilon_0} \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}. \qquad ...(50)$$

This relation is same as that for a point charge. Hence the *electric field at any external point due to a uniformly charged sphere is the same as if the total charge is concentrated at its centre.*

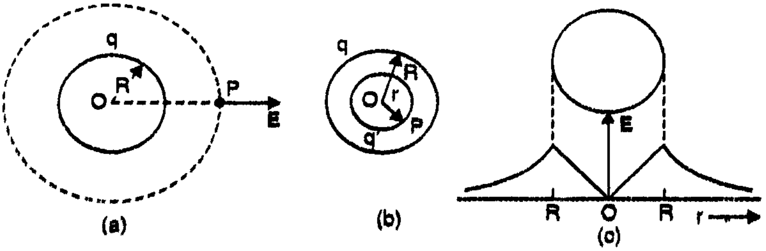


Fig. 3.16. **E** due to a uniformly charged sphere.

At the surface of sphere $r = R$, the electric field

$$E = (1/4\pi\varepsilon_0)q/R^2. \qquad ...(51)$$

For the field inside the charged sphere let us draw gaussian surface of radius $r(<R)$, as shown in Fig. 3.16 (*b*) Thus

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_S E_n dS = 4\pi r^2 E = q'/\varepsilon_0,$$

where q' is that part of charge which is with in the gaussian surface, sphere of radius r. The part of q that lies outside this sphere makes no contribution to **E** at the surface of sphere of radius r. If ρ is the volume density of the charge which is constant, then

$$q' = \frac{4}{3}\pi r^3\rho = \frac{4}{3}\pi r^3\left(\frac{q}{\frac{4}{3}\pi R^3}\right) = \left(\frac{r}{R}\right)^3 q.$$

$$\therefore \qquad 4\pi r^2 E = \frac{q}{\varepsilon_0}\left(\frac{r}{R}\right)^3 \quad \text{or} \quad E = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}. \qquad \text{...(52)}$$

From Eqs. (48), (51) and (52), it is clear that the electric field due to a uniformly charged sphere is maximum at the surface and is zero at the centre [Fig. 3.16(c)]. As the **E** is directed radially outward, the above relations may be written as

$$\mathbf{E}_{\text{outside}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r},\ \mathbf{E}_{\text{surface}} = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}\hat{r},\ \mathbf{E}_{\text{inside}} = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r}.$$

(C) An isolated uniformly charged spherical conductor : In an isolated charged spherical conductor any excess charge on it is distributed uniformly over its outer surface and there is no charge inside it. Hence this problem is same as that of charged spherical shell or hollow sphere. As in the previous case, the field at external points has the same symmetry as that of a point charge so we can construct a gaussian surface of radius $r > R$. The electric intensity calculated, as in the previous case, is given by the relation

$$E = (1/4\pi\varepsilon_0)q/r^2. \qquad \text{...(53)}$$

It is same as due to the uniformly charged sphere or the point charge of same magnitude if placed at the centre.

At the surface $r = R$,

$$E = (1/4\pi\varepsilon_0)q/R^2. \qquad \text{...(54)}$$

If σ be the surface charge density, then $q = 4\pi R^2\sigma$. The electric field at the surface of or very close to the charge spherical conductor is thus given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^2\sigma}{R^2} = \frac{\sigma}{\varepsilon_0}. \qquad \text{...(55)}$$

At all points inside the charged spherical conductor or hollow spherical shell, electric field **E** = 0, as there is no charge inside such a sphere.

(D) Two concentric charged spherical conductors : Consider two concentric spherical conductors or shells, A and B, of radii r_1 and r_2 with charges q_1 and q_2 distributed uniformly over these shells respectively.

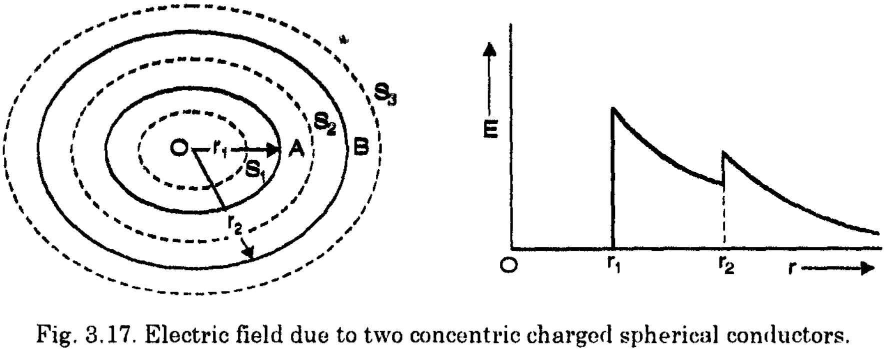


Fig. 3.17. Electric field due to two concentric charged spherical conductors.

As discussed earlier, the electric field **E** is zero inside the inner spherical conductor A of radius r_1, as the gaussian surface, S_1 does not enclose any charge. For a point in between the two conductors A and B, the gaussian surface S_2 encloses the charge on the conductor A only. Thus

$$\oint_{S_2} \mathbf{E} \cdot d\mathbf{S} = E.\ 4\pi r^2 = q_1/\varepsilon_0 \text{ or } E = q_1/4\pi\varepsilon_0 r^2. \qquad ...(56)$$

At $r = r_1$, $E = q_1/4\pi\varepsilon_0 r_1^2$. For $r_1 < r < r_2$, E is given by Eq. (56) and thus decreases with distance. For a point outside the conductor B, the gaussian surface S_3 encloses the total charge $q_1 + q_2$ (*i.e.*, sum of charges on both the conductors). Thus

$$\oint_{S_3} \mathbf{E} \cdot d\mathbf{S} = E 4\pi r^2 = (q_1 + q_2)/\varepsilon_0 \quad \text{or} \quad E = (q_1 + q_2)/4\pi\varepsilon_0 r^2. \qquad ...(57)$$

At $r = r_2$, $E = (q_1 + q_2)/4\pi\varepsilon_0 r_2^2$. For $r_2 < r < \infty$, **E** is given by Eq. (57) and thus decreases with distance. The variation of **E** with **r** is shown in Fig. 3.17(*b*).

(E) Charged cylinder or rod of infinite length : If the cylinder (say a wire) is long and of radius R and we are not too near to either end, then by symmetry the lines of force outside the cylinder are radial and perpendicular to its surface. The electric field has the same magnitude at all points at the same radial distance. Let us draw an imaginary co-axial cylinder (a gaussian surface) of length l through any point at a distance r from the axis of the cylinder as shown in Fig. 3.18(*a*). Lines of force are parallel to the circular caps, hence the flux through these faces is zero and the total flux is due to curved surface only. If λ is the charge per unit length, *i.e.*, linear charge density, the charge with in the gaussian surface will be λl. Thus for the curved surface

$$\oint_S \mathbf{E} \cdot d\mathbf{S} = \oint_S E dS = E 2\pi r l.$$

For Gauss's law $E 2\pi r l = q/\varepsilon_0 = \lambda l/\varepsilon_0$ or $E = \lambda/2\pi\varepsilon_0 r$. ...(58)

If the charge is distributed throughout the volume and charge density is ρ, then

$$2\pi r l\, E = \rho(\pi R^2 l)/\varepsilon_0 \quad \text{or} \quad E = \rho R^2/2\varepsilon_0 r. \qquad ...(59)$$

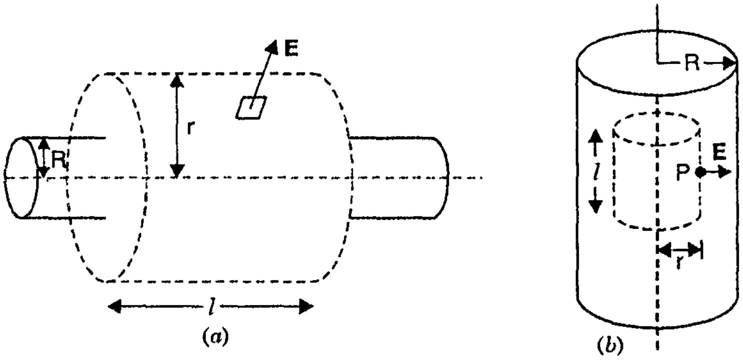


Fig. 3.18. Infinite cylinder or rod of charge.

The electric field inside the charged cylinder will be zero if the charge is on its surface only, as the net charge in the gaussian surface through this point is zero. The case is different when the electric charge is distributed uniformly with in the cylinder of radius R. To find **E** at an inner point P, a distance r apart from the axis of the cylinder, let us draw a gaussian surface passing through P, which is a cylinder of length l and of radius r, Fig. 3.18(*b*). As the flat surfaces do not contribute to the flux and the flux is due to the curved surface for which **E** is ⊥ to the surface or || to the surface vector $d\mathbf{S}$. Hence from Gauss's law we have

$$\oint_S \mathbf{E}.\ d\mathbf{S} = \oint_S E dS = E 2\pi r l = q'/\varepsilon_0.$$

The charge q' inside this gaussian surface $= \pi r^2 l\rho$,

$$\therefore \qquad E 2\pi r l = \pi r^2 l\rho/\varepsilon_0 \quad \text{or} \quad E = r\rho/2\varepsilon_0. \qquad \text{...(60)}$$

Thus the electric intensity at a point inside an infinite uniformly charged cylinder is radially directed and varies as the distance from its axis. At $r = R$, relations (59) and (60) give the same result. The field decreases at points outside.

Here we see that the electric field outside a charged cylinder does not depend upon its radius, therefore we can say that it is same as if the charge on the cylinder was concentrated in a line along its axis. It is right to state that the electric field so calculated is due to the entire charge on the cylinder although only a portion of total charge is used when we apply gauss's law. The existence of the entire charge on the cylinder is taken into account indirectly by considering the symmetry of the problem. If the cylinder is of small length we can not conclude that the field at one end of cylinder is same as at the centre or the field is symmetric, *i.e.*, the lines of force are everywhere not perpendicular to the cylinder.

Charged cylindrical conductor : In an isolated charged cylindrical conductor, there is no charge inside it, the whole of the charge is distributed uniformly over its outer surface. If σ be the surface density of charge, then $q = 2\pi Rl\sigma$ will be the charge with in the gaussian cylindrical surface of length l. Thus from Gauss's law, we get

$$E.2\pi r l = 2\pi R l\sigma/\varepsilon_0 \quad \text{or} \quad E = \sigma R/\varepsilon_0 r. \qquad (r > R) \qquad \text{...(61)}$$

The electric field at or near the curved surface will be

$$E = \sigma/\varepsilon_0. \qquad \text{...(62)}$$

E = 0, inside the conductor as there is no charge inside it.

(F) Infinite plane sheet of charge : Fig. 3.19 shows a portion of a nonconducting flat thin sheet, infinite in size with the constant surface charge density σ. By symmetry, since the sheet is infinite, the field must have the same magnitude and the opposite directions at two points equidistant from the sheet on opposite sides. Let us draw a cylinder AB (gaussian surface) with one end on one side and other end on the other side and of cross sectional area S. No lines of force cross the cylindrical wall, *i.e.*, the component of **E** normal to curved surface is zero. Thus the surface integral of **E**.d**S** over the entire surface of cylinder is reduced to the surface integral due to side surfaces. Hence Gauss's law gives

$$\oint_C \mathbf{E}\cdot d\mathbf{S} + \oint_{S_1} \mathbf{E}\cdot d\mathbf{S} + \oint_{S_2} \mathbf{E}\cdot d\mathbf{S} = q/\varepsilon_0.$$

$$\therefore \quad 0 + ES + ES = \sigma S/\varepsilon_0 \quad \text{or} \quad E = \sigma/2\varepsilon_0. \qquad \text{...(63)}$$

Thus we see that the magnitude of the field is independent of the distance from the sheet. Practically an infinite sheet of charge does not exist. *These results are correct for real charge sheets if points under consideration are not near the edges and the distances from the sheet are small compared to the dimensions of sheet.*

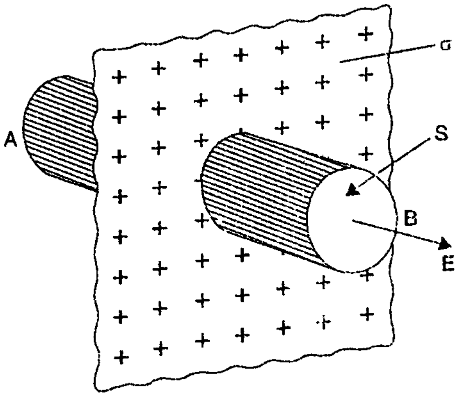


Fig. 3.19. Infinite plane sheet of charge.

(G) Infinite charged conducting plate : When a charge is given to a conducting plate, it distributes itself over the entire outer surface of the plate. The surface density of charge σ is uniform and is the same on both surfaces if plate is of uniform thickness and of infinite size. The electric field at any point thus arises from the superposition of the fields of two sheets of charge, one on each surface of the plate. The field is perpendicular to the plate and is in outward direction if the charge on the plate is positive and uniform. $E = 2.\sigma/2\varepsilon_0 = \sigma/\varepsilon_0$. It may also be obtained by considering a cylinder through any point P, as shown in Fig. 3.20, normal to the plate surface and cross-sectional area S. The other end face of the cylinder lies inside the conductor. This face does not contribute to the flux as the field inside the conductor is zero. Curved surface of the cylinder also does not contribute to the flux as $\mathbf{E}$ is parallel to the curved surface. Thus the only face is the outside end face which contributes to the flux as E is ⊥ to this surface. Hence from Gauss's law

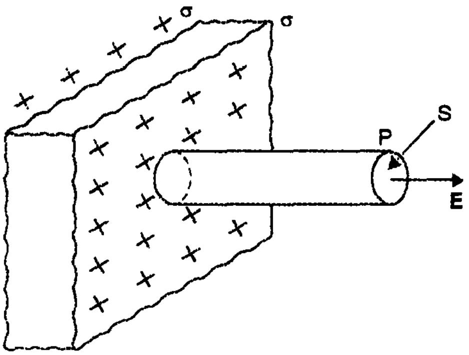


Fig. 3.20. Charged conducting plate.

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = q/\varepsilon_0 \quad \text{or} \quad \oint_S E\,dS = ES = \sigma S/\varepsilon_0,$$

$$\therefore \qquad E = \sigma/\varepsilon_0. \qquad \ldots(64)$$

This result shows that *the field due to a charge conducting plate is twice the field due to a plane sheet of charge with the same surface charge density. It is because the electric flux emerges from both the sides (equally) of the non conducting sheet (or sheet of charge), whereas whole of the flux emerges from the outer surface of the conductor (conducting plate) only, although the total flux in both the cases is same as the surface charge density is same. It has same limitations.*

(H) Electric intensity between oppositely charged parallel plane plates : When two plane parallel infinite conducting plates, separated by a distance, are given equal and opposite charges, the field is uniform in the space between them. The spreading or fringing (edge effect) of the field at the edges of the plates is assumed negligible as the plates are of the infinite size.

Draw a normal cylinder through a point P, between two plates and let the other end of this cylinder lying in the left conducting plate and crossing an area S [Fig. 3.21]. The problem is same as infinite charged conducting plate and the electric field at point P is thus given by $E = \sigma/\varepsilon_0$. To find the electric intensity at an outside point Q, draw the cylinder through the point Q whose second end face lying in the first conductor. It is clear that the surface of the cylinder contributing to the flux is the end face passing through the point Q. Hence from Gauss's law

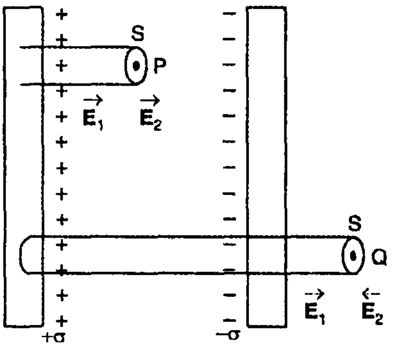


Fig. 3.21. Charged parallel plates.

$$\oint_S \mathbf{E}\,d\mathbf{S} = \oint_S E\,dS = ES = (\sigma S - \sigma S)/\varepsilon_0 = 0, \text{ or } E = 0. \qquad \ldots(65)$$

The electric field at any point due to this system of plates may be written as the superposition of the fields due to infinite plane of sheets of charge. Therefore at point P, the electric field

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 \quad \text{or} \quad E = E_1 + E_2$$

$$\therefore \qquad E = \sigma/2\varepsilon_0 + \sigma/2\varepsilon_0 = \sigma/\varepsilon_0.$$

For point Q, $\mathbf{E}_1$ and $\mathbf{E}_2$ are antiparallel, hence the net field $\mathbf{E} = 0$.

Thus *the electric intensity due to two oppositely charged infinite planes is σ/ε_0 at any point in between the planes and is zero for all external points.*

(I) Coulomb's theorem : Consider a charged conductor of irregular shape. In general, surface charge density will vary from point to point. Let us assume it to be *a* constant, say σ at *a* small surface δS. Let us construct a gaussian surface in the form of a cylinder AB of cross section δS, one end face of which is inside the conductor and other passes through a point P outside the surface close to it, as shown in Fig. 3.22. The end face B does not contribute to flux as **E** is zero everywhere inside the conductor. The curved surface also does not contribute to the flux as **E** is always normal to the charged conductor and hence parallel to the curved surface. Thus the only contribution to the flux is through the end face A which is outside the conductor. Thus from Gauss's law

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_S E dS = E\delta S = \sigma\delta S/\varepsilon_0,$$

or $$E = \sigma/\varepsilon_0. \qquad ...(66)$$

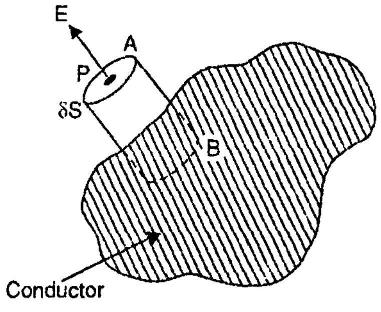


Fig. 3.22. charged conductor of irregular shape.

This agrees with the results already obtained for spherical, cylindrical and plane surfaces and is known as *Coulomb's theorem*. It states that ***the electric intensity at any point close to the charged conductor is*** $1/\varepsilon_0$ ***times the surface charge density on the surface.***

3.10 ELECTRIC FIELD AND CHARGE IN CONDUCTORS

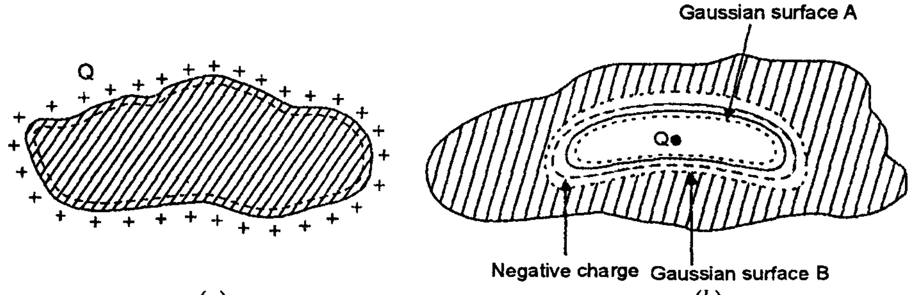
A block of metal is placed into a region of uniform electric field **E**. The field will produce a force on the mobile negatively charged electrons in the metal. The electrons will therefore move towards one surface leaving the net positive charge on the opposite surface of the metal. This charge distribution induced by the external field, generates its own electric field which tends to cancel the former. The free charges continue to readjust their positions until the vector sum of these two fields is exactly zero. Thus we conclude that under electrostatic conditions, the *value of* **E** *at all points within a conductor is zero.* This idea, together with the Gauss's law can be used to prove several interesting facts.

(*i*) Consider a conductor carrying a charge Q and no current is flowing in it [Fig. 3.23 (*a*)]. Now consider a gaussian surface inside the conductor, everywhere on which $\mathbf{E} = 0$. Thus from Gauss's law

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = (1/\varepsilon_0)\,\Sigma q,$$

we get $$\Sigma q = 0, \text{ as } E = 0. \qquad ...(67)$$

Thus the sum of all charges inside the gaussian surface is zero. This surface can be taken just inside the surface of the conductor, hence *under electrostatic conditions, the excess charge on a conductor resides on its outer surface only.*



(*a*) (*b*)

Fig. 3.23. Conductor without and with cavity.

(*ii*) Consider a charge Q suspended in a cavity in a conductor [Fig. 3.23(*b*)]. Let us apply Gauss's law to the gaussian surface A (within the cavity).

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = (1/\varepsilon_0)\Sigma q = Q/\varepsilon_0.$$

Consider a second gaussian surface B, just inside the conductor. $E = 0$ on this gaussian surface. as it is inside the conductor. Hence from Gauss's law

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = (1/\varepsilon_0)\Sigma q, \text{ we get } \Sigma q = 0. \qquad \text{...(68)}$$

This concludes that a charge of $-Q$ must reside on the metal surface of the cavity and thus the sum of this induced charge $-Q$ and the original charge Q is zero. In other words, *a charge Q suspended inside a cavity in a conductor induces an equal and opposite charge –Q on the surface of the cavity.*

(*iii*) The same line of approach can be used to show that the field inside the cavity of a conductor is zero when no charge is suspended in it.

(*iv*) As field inside the conductor is zero, the lines of flux coming from Q cannot penetrate into the metal and must terminate on the surface of the cavity. Since lines of flux terminate on negative charge only, hence an equal negative charge must be induced on the surface of the cavity.

3.11 MECHANICAL FORCE ON THE CHARGED CONDUCTOR

We know that similar charges repel each other, hence the charge on any part of surface of the conductor is repelled by the charge on its remaining part. The surface of the conductor thus experiences a mechanical force as the charge is bound to the surface. The electric intensity at any point P near the conductor surface can be assumed as due to the small part of the surface of area say δS immediately in the neighbourhood of the point under consideration and due to the rest of the surface. Let $\mathbf{E}_1$ and $\mathbf{E}_2$ be the electric intensities due to these parts respectively [Fig. 3.24]. We know from Coulomb's theorem that the total electric field $\mathbf{E}$ (= $\mathbf{E}_1 + \mathbf{E}_2$) has magnitude σ/ε_0 at any point P just outside the conductor and is zero at point Q just inside the conductor,

$$\begin{aligned} \textit{i.e.,} \quad E &= E_1 + E_2 = \sigma/\varepsilon_0 && \text{at point } P \\ \text{and} -E_1 + E_2 &= 0 && \text{at point } Q \\ \therefore \quad E_1 &= E_2 = \sigma/2\varepsilon_0. \end{aligned}$$

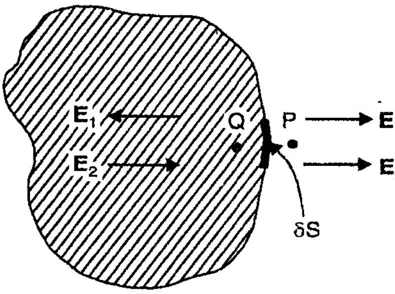


Fig. 3.24. Charged conductor.

Hence the force experienced by the small surface of area δS due to the charge on the rest of the surface is

$$\sigma\delta S E_2 = \sigma\delta S\sigma/2\varepsilon_0 = \sigma^2\delta S/2\varepsilon_0.$$

Thus the mechanical force per unit area on the charged conductor

$$P = \sigma^2/2\varepsilon_0 = (\varepsilon_0 E)^2/2\varepsilon_0 = \varepsilon_0 E^2/2. \qquad \text{...(69)}$$

It is directed along the outdrawn normal. Thus the charged conductor experiences an outward electrostatic pressure. If the medium is of dielectric constant k, the above relation becomes

Outward mechanical force/area = $\frac{1}{2}\varepsilon_0 k E^2$ newton/m^2.

Electrostatic energy in the medium : When a conductor situated in a dielectric is charged, a force acts outward on the surface of the conductor. Imagine the unit area of the charged surface to be moved in the opposite direction through a small distance dx. The work done $dW = \frac{1}{2}\varepsilon_0 k E^2 \times dx$. Since the volume swept out by this imaginary area is dx, hence the work done per unit volume will be $\frac{1}{2}\varepsilon_0 k E^2$. This work is stored as an energy of the medium. Therefore the energy stored up in unit volume of the medium is

$$W = \frac{1}{2}\varepsilon_0 k E^2 \text{ joule/m}^3. \qquad \text{...(70)}$$

Let us consider few applications of the mechanical force.

(*i*) **Electrified soap bubble :** We have proved in mechanics that the excess of pressure inwards on a soap bubble of radius r due to surface tension T is given by $P = 4T/r$.

If the bubble is now given a charge q, then the outward mechanical pressure

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(\frac{q}{4\pi r^2}\right)^2 = \frac{q^2}{32\pi^2\varepsilon_0 r^4},$$

∴ Next excess of pressure $$P = \frac{4T}{r} - \frac{q^2}{32\pi^2\varepsilon_0 r^4} = \frac{4}{r}\left[T - \frac{q^2}{128\pi^2\varepsilon_0 r^3}\right]. \qquad ...(71)$$

Thus we see that the effective value of surface tension of the bubble decreases and therefore the bubble will expand. The increase in radius of bubble can be obtained as follows : We know that $P \propto 1/V$ or $P \propto 1/r^3$ $(= k/r^3)$.

$$\therefore \qquad \frac{dP}{dr} = -\frac{3k}{r^4} = -\frac{3P}{r} \text{ or } dr = -\left(\frac{r}{3P}\right)dP.$$

As change in pressure due to charge q, $dP = (q^2/32\pi^2\varepsilon_0 r^4)$, hence the increase in radius

$$dr = -\frac{r}{3P}\cdot\frac{q^2}{32\pi^2\varepsilon_0 r^4} = -\frac{q^2}{96\varepsilon_0\pi^2 P r^3}. \qquad ...(72)$$

(*ii*) Formation of a cloud on charged particles (Wilson cloud chamber) : We have proved in mechanics that the surface tension helps in the contraction of the liquid drop, the inward pressure is $2T/r$. If charge q is given to the drop, the outward mechanical pressure due to this charge is $q^2/32\pi^2\varepsilon_0 r^4$. Thus if the latter is greater than the former, the drop will tend to grow instead of contracting. On the other hand if r is small and the drop is uncharged, *i.e.*, the former is very large, the tendency to form drops on uncharged particles is extremely small.

Air mixed with saturated vapour of a liquid is contained in a cylinder fitted with a piston. It may be expanded adiabatically, which will result in a fall of temperature. The vapour will become super saturated, it will condense in cloud of water droplets, if nuclei on which the water vapour may condense are provided. If the dust particles are present in the air, they will act as condensation nuclei. In a dust free air, the formation of cloud will be extremely difficult. C. T. R. Wilson discovered that the condensation cloud could be produced even in dust free air provided the air was ionized by an ionizing agent (X-rays, electrons or any nuclear radiations). If the ionizing particle entered the chamber either immediately before or immediately after the expansion, the ions left in its path would act as condensation nuclei. He observed that as super saturation was increased, the negative ions first served as centres of condensation, then both positive and negative ions served as the nuclei of droplets. A close array of fine droplets, called a *cloud track* was thus formed. This track appeared as a white line on a dark background, when a suitable flash light was allowed to fall from the side. This track was photographed by means of a camera. The schematic diagram of Wilson expansion cloud chamber is shown in Fig. 3.25.

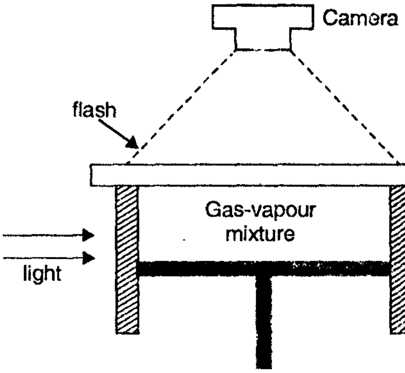


Fig. 3.25. Wilson cloud chamber.

3.12 CAVENDISH'S PROOF OF INVERSE SQUARE LAW

The inverse square law was experimentally observed by Coulomb himself using torsion balance. Cavendish gave the following method for the verification of this law.

Let C be any point inside a uniformly charged hollow sphere (spherical shell). Draw through C a double cone of infinitesimal solid angle $d\omega$ cutting out areas dS_1 and dS_2 of the spherical shell at distances r_1 and r_2 from point C. From the properties of solid angles, we have

$$d\omega = \frac{dS_1 \cos\theta_1}{r_1^2} = \frac{dS_2 \cos\theta_2}{r_2^2} \quad ...(73)$$

Fig. 3.26 shows that the triangle $dS_1 - O - dS_2$ is isoscales, therefore θ_1 and θ_2 are equal, Thus we have

$$dS_1 = d\omega r_1^2/\cos\theta \text{ and } dS_2 = d\omega r_2^2/\cos\theta.$$

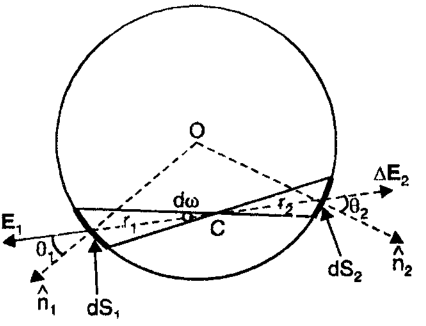


Fig. 3.26. Field inside a hollow conductor.

If σ be the surface density of charge, then charges on these surfaces will be $\sigma\, dS_1$ and $\sigma\, dS_2$. If we assume that the electrostatic force arises as the n^{th} power of the distance, then the resultant intensity at point C due to these charges

$$\Delta E = \frac{\sigma}{4\pi\epsilon_0}\left[\frac{dS_2}{r_2^n}\hat{r}_2 + \frac{dS_1}{r_1^n}\hat{r}_1\right]$$

$$= \frac{\sigma d\omega}{4\pi\epsilon_0 \cos\theta}\left[\frac{1}{r_2^{n-2}} - \frac{1}{r_1^{n-2}}\right]\hat{r}_2 \quad (\because \hat{r}_1 = -\hat{r}_2) \quad ...(74)$$

Since $r_1 > r_2$, as shown in Fig. 3.26, therefore following three cases arise :

(*i*) If $n > 2$, the resultant intensity will be positive, because $1/r_2^{n-2} > 1/r_1^{n-2}$. Hence there will be resultant field at C acting towards left, *i.e.*, directed toward the centre O.

(*ii*) If $n < 2$, the resultant intensity will be negative, hence the resultant field at C will act towards right, *i.e.*, directed away from the centre O.

(*iii*) If $n = 2$, the resultant intensity will be zero.

The remaining surface of the sphere can be divided into pairs of element, hence the electric field at point C due to the whole charge on the shell comes out to be zero for $n = 2$.

Thus we see that the resultant electric intensity at this point C is zero, if the inverse square law holds good. Therefore the *inverse square law is proved if electric field inside the charged shell is found to be zero.* This was experimentally shown by Cavendish. In his experiment a conducting sphere A was supported inside a bigger concentric sphere B. These were insulated from each other. The outer sphere B was first charged and connected with sphere A with the help of a wire II. The connection between these spheres was broken and the residual charge on the inner sphere was tested by electroscope. No charge was found on the sphere A. Thus he proved that the exponent n in the force (*i.e.*, power of r in the relation for electric field) law was two. He repeated his experiment many times and found that n was between 2.02 and 1.98. Cavendish did not publish his results so that almost no body knew about them at that time. Maxwell repeated Cavendish's experiment with more accuracy and obtained values as $2 \pm 5 \times 10^{-5}$. In 1936. Plimpton and Lawton repeated the experiment again and set the limits as $2 \pm 2 \times 10^{-9}$. Thus we see that the inverse square law is correct, but not exactly.

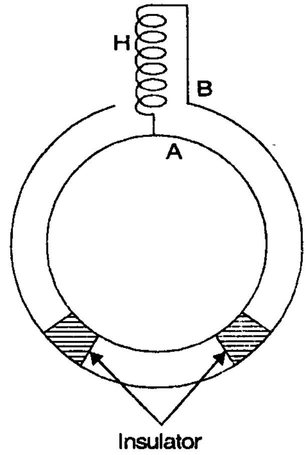


Fig. 3.27. Two spheres insulated with each other.

Electrostatic Shield : We can shield an object from the influence of electrostatic fields by simply enclosing it within a conducting sheath. It need not be a solid conductor, a wire mesh will do. The conducting enclosure shields the interior from external electrostatic fields. It can be seen that the ungrounded metallic shell does not shield the exterior from electric fields due to charges that reside with in the shell, however the grounded metallic shell shields the field due to the interior charge from the external world.

3.13 DETERMINATION OF ELECTRONIC CHARGE

The first attempt to determine the electronic charge was made by J.J. Thomson in 1898. He used a method developed by C.T.R. Wilson and later on modified by H.A. Wilson. The method was based on the principle that the cloud of water droplets could be formed on charged ions under special conditions. This cloud falls down with a certain velocity. Its velocity was observed and the value of '*e*' was determined. The value so obtained was 40% higher than the standard value determined later.

In 1909, R.A. Millikan not only demonstrated the discrete nature of electric charge but gave the correct value of the charge on an individual electron. In this experiment two exactly parallel horizontal metal plates *A* and *B* are insulated from each other and separated by ebonite or glass pieces of thickness about a cm as shown in Fig. 3.28. Potential difference is applied between these plates by connecting a power supply through a reversing switch with the help of which the direction of electric current can be changed or can be set zero. The upper plate *A* has some fine holes, through which few of the droplets of a non-volatile oil from an atomiser are allowed to fall. A majority of these drops becomes charged by friction at the atomiser nozzle during the time of spray. The droplets, illuminated by the light beam, appear as tiny bright stars falling slowly with a terminal velocity governed by their weights, viscous force and by buoyancy force. These droplets are observed through a microscope *M* whose eyepiece carries a micrometer scale at the position of cross wires to measure the distance travelled by the droplets.

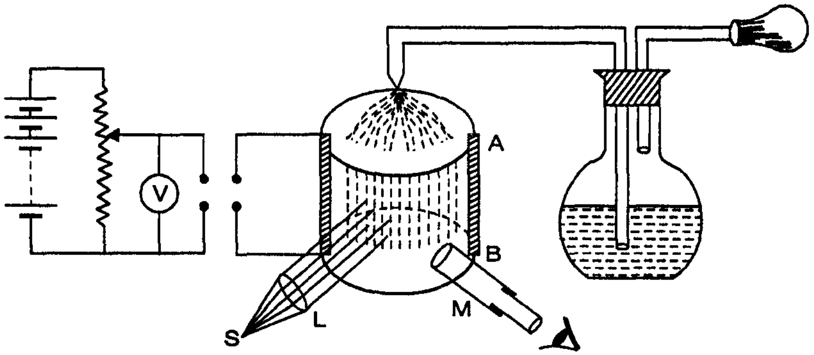


Fig. 3.28. Millikan's apparatus.

When electric field is not applied, a drop falls down under gravity, opposed by the viscous force and buoyancy force and acquires a steady terminal velocity v. If a be the radius of the oil drop, ρ the density of oil and σ the density of air then by Stokes law, we get

Viscous force $6\pi\eta a v$ = Apparent weight of the drop = $\frac{4}{3}\pi a^3\rho g - \frac{4}{3}\pi a^3\sigma g$

or $$a = [9\eta v/2(\rho - \sigma)g]^{1/2}. \qquad ...(75)$$

If the charge on the droplet is n times the electronic charge e, and the plates are maintained at a potential difference V such that E (= V/d) is in downward direction. The force on the negatively charged drop will be in upward direction and having value neE or neV/d. On the other hand the downward velocity of positively charged droplets will increase further and the velocity of uncharged particles remains same. If one of these droplets is selected for observations. Firstly this droplet is allowed to move under the electric field. Field is so chosen that the droplet moves with a constant velocity, known as terminal velocity. The velocity v' is measured with the help of a stop-watch. In equilibrium, downward forces = upward forces, *i.e.*,

$$\frac{4}{3}\pi a^3\rho g = \frac{4}{3}\pi a^3\sigma g + 6\pi\eta v' + neE$$

$$\therefore \qquad neE = \tfrac{4}{3}\pi a^3(\rho - \sigma)g - 6\pi\eta a v' = 6\pi\eta a(v - v')$$

$$\text{or} \qquad ne = 9\sqrt{2}\,\pi\eta^{3/2}(\rho - \sigma)^{-1/2}g^{-1/2}v^{1/2}(v - v')/E$$

$$= 9\pi\left[\frac{2\eta^3}{g(\rho - \sigma)}\right]^{1/2}\frac{v^{1/2}}{E}(v - v'). \qquad \text{...(76)}$$

The value of ne is calculated for each set separately by observing velocities of the droplets in the presence and in the absence of an electric field. These values of charges $n_1 e$, $n_2 e$, ... are divided by the highest common factor of all such values. Millikan measured the charges of some thousands of drops and found that the charge on any one droplet was integral multiple of a basic charge e. He gave the value of e as 1.60207×10^{-19} coulomb.

EXERCISES

Example 1. *Prove that the gravitational force is negligible in comparison to electrostatic force in the hydrogen atom in which the electron and the proton are about 5.3×10^{-11} meter apart.*

From Coulomb's law, the electrostatic force between electron and proton

$$F_e = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$$

$$= (9.0 \times 10^9\ \text{N·m}^2/\text{coul}^2) \times \frac{(1.6023 \times 10^{-19}\text{coul})^2}{(5.3 \times 10^{-11}\text{m})^2}$$

$$= 8.2 \times 10^{-8}\ \text{newton},$$

From gravitational law, the gravitational force

$$F_g = G\frac{m_1 m_2}{r^2}$$

$$= (6.7 \times 10^{-11}\ \text{N. m}^2/\text{kg}^2)\ \frac{(9.1 \times 10^{-31}\text{kg})\,(1.7 \times 10^{-27}\text{kg})}{(5.3 \times 10^{-11}\text{m})^2}$$

$$= 3.69 \times 10^{-47}\ \text{newton},$$

which is about 10^{-39} times smaller than the electrostatic force, thus can be neglected in comparison to the latter in nuclear or atomic problems.

Example 2. *Two identically charged spheres are suspended by strings of equal length. The strings make an angle of 30° with each other. When suspended in a liquid of density 0.8 gm cm^{-3}, the angle remains the same. Calculate the dielectric constant of the liquid. The density of the material of the spheres is 1.6 gm cm^{-3}.*

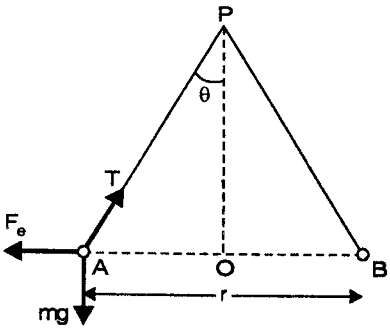


Fig. 3.29. Example 2.

Let two spheres A and B each having a charge q be suspended at the same point P. Each sphere is acted upon by following forces :

(*i*) Its weight mg (= ρVg) acting vertically downward.

(*ii*) Force of repulsion $F_e = q^2/4\pi\varepsilon_0 r^2$.

(*iii*) Tension T of the string along the string.

In equilibrium, moment of these forces about any point (say P) will be zero. Thus we have $mg \times OA = F_e \times PO$ or $F_e = mg \tan\theta$,

$$\therefore \qquad q^2/4\pi\varepsilon_0 r^2 = V\rho g\tan\theta \text{ or } q^2 = 4\pi\varepsilon_0 r^2 V\rho g\tan\theta. \qquad \text{...(}i\text{)}$$

When the spheres are suspended in the liquid of density σ, the upthrust force $F_{up} = V\sigma g$ will also act. Thus at equilibrium, we have

$$F_e' = (mg - F_{up}) \tan\theta.$$

$$\therefore \quad q^2/4\pi\varepsilon_0 k r^2 = V(\rho - \sigma) g \tan\theta$$

$$\text{or} \quad q^2 = 4\pi\varepsilon_0 k r^2\, V(\rho - \sigma) g \tan\theta \qquad ...(ii)$$

From Eqs. (*i*) and (*ii*), we get

$$\rho = k(\rho - \sigma) \text{ or } k = \rho/(\rho - \sigma).$$

$\therefore$ Dielectric constant of the liquid $k = 1.6/(1.6 - 0.8) = 2$.

Example 3. *Three identical small spheres of mass m are suspended by threads of negligible masses and equal length l from a common point. When a charge Q is divided equally between the spheres, these spheres come to equilibrium at the corners of a horizontal lateral triangle whose sides are d. Show that*

$$Q^2 = 12\ \pi\varepsilon_0\, mgd^3 (l^2 - d^2/3)^{-1/2}.$$

Let A, B and C are the charged spheres, each of mass m and charged to $\frac{1}{3}Q$. They are suspended from a common point P with threads each of length l. At equilibrium forces acting at charged sphere B :

(1) mg due to its weight downwards

(2) T due to tension of the string along the string BP.

(3) Electrostatic repulsive forces $\mathbf{F}_{CB}$ and $\mathbf{F}_{AB}$ due to charges at C and A respectively.

Since masses and charges are equal and the triangle formed is in horizontal plane, therefore the point of suspension P must be just above the centre of mass of the system, point O on ΔABC. $\angle ABD = \angle CBD = 30°$.

If θ is the angle of T from the horizontal, *i.e.*, $\angle PBD = \theta$, then

Vertical component $T \sin\theta = mg$

Horizontal component $T\cos\theta = F_{AB}\cos 30° + F_{CB}\cos 30°$

$$= \frac{2.(Q/3)^2 \cos 30°}{4\pi\varepsilon_0 d^2}$$

$$\therefore \quad Q^2 = 12\ \sqrt{3}\ \pi\varepsilon_0 mgd^2 \cot\theta. \qquad ...(i)$$

From ΔBCD, $\quad BD = [\ d^2 - (d/2)^2]^{1/2} = \sqrt{3}\ d/2$

$$\therefore \quad BO = \tfrac{2}{3} BD = d/\sqrt{3}\,.$$

Since $BP = l$ and $\angle BOP = 90°$, hence

$$PO = [\ l^2 - d^2/3]^{1/2}$$

$$\therefore \quad \cot\theta = \frac{BO}{PO} = \frac{d/\sqrt{3}}{(l^2 - d^2/3)^{1/2}} \qquad ...(ii)$$

Therefore $\quad Q^2 = 12\ \pi\varepsilon_0 mgd^3 (l^2 - d^2/3)^{-1/2}$.

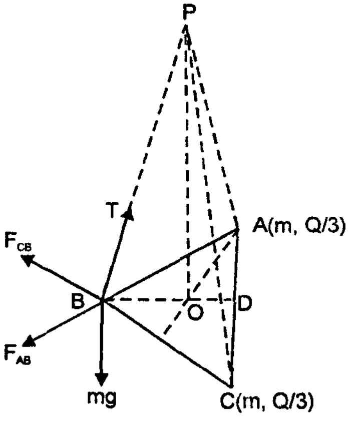


Fig. 3.30. Example 3.

Example 4. *Two positive charges q_1 and q_2 are placed at a distance r apart. A third charge q is placed such that all the three charges are in equilibrium. What is the magnitude, sign and position of charge q. Check whether is it a stable equilibrium.*

Let the charge q be placed at point P at a distance x from charge q_1, as shown in Fig. 3.31. For q to be in equilibrium, the charges must be collinear and the forces on it exerted by q_1 and q_2 charges must be equal and opposite. Since $AB = r$, therefore $PB = r - x$ and

$$F_{q_1q} = \text{Force on } q \text{ by } q_1 = q_1q/4\pi\varepsilon_0x^2$$

$$F_{q_2q} = \text{Force on } q \text{ by } q_2 = q_2q/4\pi\varepsilon_0(r-x)^2.$$

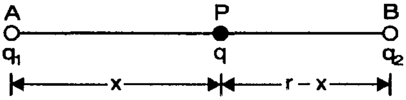


Fig. 3.31. Example 4.

$$\therefore \quad \frac{q_1q}{4\pi\varepsilon_0x^2} = \frac{q_2q}{4\pi\varepsilon_0(r-x)^2} \text{ or } \sqrt{\frac{q_1}{q_2}} = \frac{x}{(r-x)}.$$

It gives
$$AP = x = \frac{(q_1/q_2)^{1/2}r}{1+(q_1/q_2)^{1/2}}.$$

For equilibrium q may be positive or negative. For the system to be in equilibrium, the charges q_1 and q_2 must also be in equilibrium. For q_1 to be in equilibrium

$$\mathbf{F}_{qq_1} = -\mathbf{F}_{q_2q_1} \text{ or } \frac{qq_1}{4\pi\varepsilon_0x^2} = \frac{q_1q_2}{4\pi\varepsilon_0r^2}$$

$$\therefore \quad q = q_2(x/r)^2 = q_1[1+(q_1/q_2)^{1/2}]^{-2}.$$

q must have sign opposite of q_1, *i.e.*, negative. If q be negative, it must experience a force of attraction due to both the charges q_1 and q_2. A slight displacement of q from its equilibrium position will increase the force of attraction due to one charge and decrease the force due to other charge. Hence q will no longer be in equilibrium, while displaced along AB. Thus the original equilibrium is an unstable equilibrium.

If the charge q is displaced perpendicular to AB, the resultant of two forces of attraction will bringe the charge q back to its original position. Thus the equilibrium is stable.

Reverse will be the cases for q positive.

Example 5. *ABCD is a square of 4 cm side, charges of 16 × 10⁻⁹, −16 × 10⁻⁹ and 32 × 10⁻⁹ coulomb are placed at the points A, C and D respectively. Find the intensity of the field at point B.*

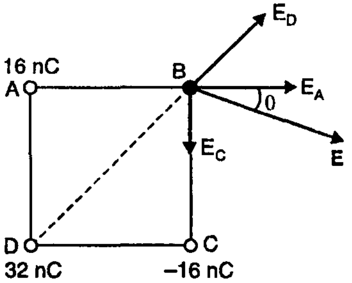


Fig. 3.32. Example 5.

We know the electric field intensity **E** due to a point charge q at a distance r is given by the relation

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r},$$

where $\hat{r}$ is along r increasing.

As **E** is a vector quantity, the total intensity at B is the vector sum of three intensities due to charges at A, C and D, where

$$E_A = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} = 9\times10^9\times\frac{16\times10^{-9}}{(0.04)^2} = 9\times10^4 \text{ along AB}$$

$$E_C = 9\times10^9\times\frac{(-16\times10^{-9})}{(0.04)^2} = 9\times10^4 \text{ along BC,}$$

and
$$E_D = 9\times10^9\times\frac{32\times10^{-9}}{32\times10^{-4}} = 9\times10^4 \text{ along DB.}$$

$$\therefore \text{ Resultant intensity } \mathbf{E} = \mathbf{E}_A+\mathbf{E}_C+\mathbf{E}_D.$$

and
$$E = \sqrt{(E_x^2+E_y^2)} = \sqrt{[(E_A+E_D\cos45°)^2+(E_C-E_D\sin45°)^2]}$$

$$= \sqrt{[E_A^2+E_D^2+E_C^2+2E_D(E_A\cos45°-E_C\sin45°)]}$$

$$= 9\sqrt{3}\times10^4 \text{ newtons/coul.}$$

and $$\tan\theta = \frac{E_y}{E_x} = \frac{E_C - E_D \sin 45°}{E_A + E_D \cos 45°} = \frac{(1 - 1/\sqrt{2})9 \times 10^4}{(1 + 1/\sqrt{2})9 \times 10^4}$$

$$= 0 .1716$$

or $$\theta = 9 .44.$$

Example 6. *An electron is constrained to move along the axis of the ring of charge q and of radius a. Show that the electron can perform oscillations whose frequency is given by*

$$\omega = \sqrt{qe/4\pi\varepsilon_0 m a^3}\ ,$$

- Electric intensity E at a point P on the axis of the ring of radius a at a distance x from its centre, as given in article 3.4(*c*), is given by

$$E = \frac{1}{4\pi\varepsilon_0}\frac{qx}{(a^2 + x^2)^{3/2}}.$$

It is along the axis of the ring.

The force on the electron if placed at point P

$$F = qex/4\pi\varepsilon_0\,(a^2 + x^2)^{3/2} = qex/4\pi\varepsilon_0 a^3 \qquad \text{(for } x << a\text{)}$$

Thus we see that for $x << a$, the force is proportional to the distance from the mean position, *i.e.*, the centre of the ring. It means that the electron executes simple harmonic motion about the centre of the ring. Its frequency is given by the relation

$$\omega = \sqrt{\frac{d^2x/dt^2}{x}} = \sqrt{\frac{F/m}{x}} = \sqrt{\frac{qe}{4\pi\varepsilon_0 m a^3}}.$$

Example 7. *A very long uniformly charged thin straight wire is oriented along the axis of the thin charged wire ring with one of its end coinciding with the centre of the ring. Find the expression for the interaction force between the wire and the ring.*

Consider a ring of radius a with total charge q. The electric field intensity at an axial point P at a distance x from the centre of the ring

$$E = qx/4\pi\varepsilon_0\,(a^2 + x^2)^{3/2}.$$

If λ be the charge per unit length on the straight wire. The force experienced by an element Δx of the wire due to the charge on the ring

$$\Delta F = E\ (\lambda\Delta x) = q\lambda x\Delta x/4\pi\varepsilon_0\,(a^2 + x^2)^{3/2}.$$

The force of attraction between the whole wire and the ring

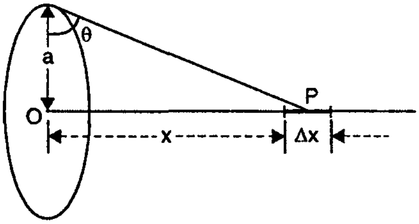


Fig. 3.33. Example 7.

$$F = \Sigma\Delta F = \int_0^\infty \frac{q\lambda x dx}{4\pi\varepsilon_0(a^2 + x^2)^{3/2}}.$$

To solve the integral, put $x = a\tan\theta$ and $dx = a\sec^2\theta\, d\theta$.

$$\therefore \qquad F = \int_0^{\pi/2} \frac{q\lambda}{4\pi\varepsilon_0}\frac{a\tan\theta .\, a\sec^2\theta d\theta}{a^3\sec^3\theta}$$

$$= \frac{q\lambda}{4\pi\varepsilon_0 a}\int_0^{\pi/2} \sin\theta d\theta = \frac{q\lambda}{4\pi\varepsilon_0 a}.$$

Example 8. *A thin conducting ring of radius a has an electric charge q. Find the change in tension of the wire, if a point charge q′ is placed at the centre of the ring.*

The charge per unit length on the ring $\lambda = q/2\pi a$. Consider a small element Δl on the ring. It will experience a force of repulsion due to charge q' placed at the centre of the ring. This outward force is given by

$$\Delta F = \frac{1}{4\pi\varepsilon_0}\frac{q'(\lambda\Delta l)}{a^2} = \frac{1}{4\pi\varepsilon_0}\frac{q\,q'}{2\pi a^3}\Delta l. \qquad ...(i)$$

To balance this force, tension of the wire increases. If T be the increase in tension at the ends of the element. The inward force on the element Δl due to increase in tension

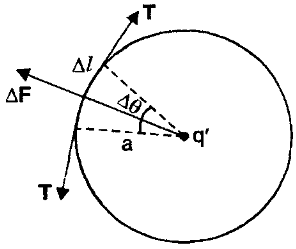


Fig. 3.34. Example 8.

$$\Delta F' = 2\ T\sin(\Delta\theta/2).$$

For a small element Δl, $\Delta\theta$ is very small, therefore

$$\Delta F' = 2\ T(\Delta\theta/2) = T.(\Delta l/a) \qquad ...(ii)$$

For equilibrium the inward force due to increase in tension and the electrostatic force due to charge q' at the centre must be equal. Therefore

$$T\frac{\Delta l}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q\,q'\Delta l}{2\pi a^3} \text{ or } T = \frac{q\,q'}{8\pi^2\varepsilon_0 a^2}.$$

Example 9. *Find the electric field at the centre of a uniformly charged semicircular arc.*

Let the charge per unit length of arc be λ and the radius of the arc be a. If the arc is divided into small segments, so that each will work as a point charge. The electric field at the central point O due to a segment of length ΔL,

$$\Delta E = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\Delta L}{a^2}.$$

Its x-component $\Delta E_x = \Delta E\cos\theta$ will be cancelled by the contribution from a symmetrically placed ΔL on the right half of the arc, hence net component $E_x = \Sigma\Delta E_x = 0$. Thus the y-component contributes to E only.

$$\therefore \qquad E = E_y = \Sigma\Delta E_y = \int\frac{\lambda}{4\pi\varepsilon_0 a^2}.dl\sin\theta.$$

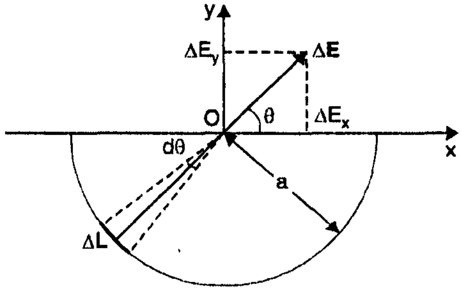


Fig. 3.35. Example 9.

To convert two variables l and θ in one, let us use the relation $dl = a\,d\theta$ (arc = radius × angle subtended by the arc at the centre).

$$E = \frac{\lambda}{4\pi\varepsilon_0 a}\int_0^{\pi}\sin\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 a}\left[-\cos\theta\right]_0^{\pi} = \frac{\lambda}{2\pi\varepsilon_0 a}.$$

Example 10. *Show the variation of electric field due to the positive charge in the gold atom ($Z = 79$).*

Gold atom is spherical and is of radius 1×10^{-10} meter. The positive charge of the atom is $79 \times 1.6023 \times 10^{-19}$ coul. and is distributed uniformly within the nucleus of radius 6.9×10^{-15} meter. The electric intensity outside the nucleus is given by the relation (48), *i.e.*,

$$E = q/4\pi\varepsilon_0 r^2.$$

At the surface of atom $r = 10^{-10}$ meter,

$$E = \frac{9\times10^9\times79\times1.6023\times10^{-19}}{(10^{-10})^2} = 1.14\times10^{13}\text{ nt/coul.}$$

At the nuclear surface $r = R = 6.9 \times 10^{-15}$, and the electric intensity

$$E = \frac{9\times10^9\times79\times1.6023\times10^{-19}}{(6.9\times10^{-15})^2} = 2.39\times10^{21}\text{ nt./coul.}$$

Inside the nucleus the electric intensity is given by

$$E = qr/4\pi\varepsilon_0 R^3.$$

It is zero at the centre ($r = 0$) and increases with r upto the maximum value of 2.39×10^{21} nt./coul. at nuclear surface. Beyond it, decreases inversely as the square of the distance from the centre.

Example 11. *Find the force on a point charge 30 μC at (0, 0, 5) due to a charge of 500 $\pi\mu C$ which is uniformly distributed over the circular disc ($r \le 5m$, $z = 0$ m).*

The charge of 500 π μC is distributed uniformly over the circular disc of radius 5m placed in the x-y plane (as $z = 0$). Therefore the charge density $\sigma = \frac{q}{A} = \frac{500\pi}{\pi . 5^2} = 20\ \mu C/m^2$.

In cylindrical coordinates, $\mathbf{R} = -r\hat{r} + 5\hat{z}$.

Here ϕ does not appear explicitly in this expression, nevertheless both $\mathbf{R}$ and $\hat{\mathbf{R}}$ vary with ϕ through $\hat{r}$.

The differential charge on a differential surface dS of the disc

$$dq = \sigma\ dS = \sigma(rd\phi)\ dr = \sigma r dr d\phi.$$

∴ Force on a point charge at point P due to this differential charge

$$d\mathbf{F} = \frac{Qdq}{4\pi\varepsilon_0 R^3}\mathbf{R}$$

$$= \frac{(30 \times 10^{-6})\,(20 \times 10^{-6})\, rdrd\phi}{(9 \times 10^9)^{-1}(r^2 + 5^2)^{3/2}}(-r\hat{r} + 5\hat{z})$$

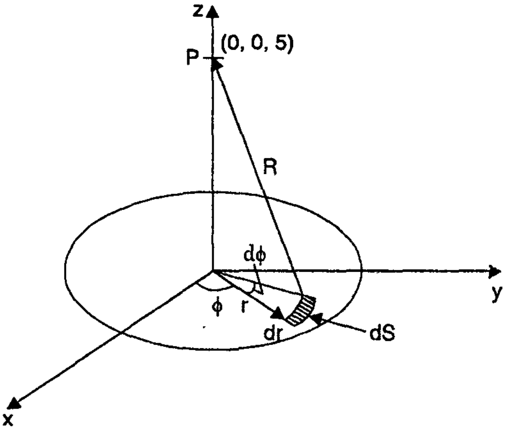


Fig. 3.36. Example 11.

Since the point P is symmetrical about the charge distribution on the disc, therefore the radial components will cancel and the resultant force will be along the z-direction *i.e.*, along $\hat{z}$.

$$\therefore \quad \mathbf{F} = \left[27\int_0^5 \frac{rdr}{(r^2 + 5^2)^{3/2}} \int_0^{2\pi} d\phi\right]\hat{z} = 54\pi\left[\int_0^5 \frac{rdr}{(r^2 + 5^2)^{3/2}}\right]\hat{z}$$

$=9$ $.93\ \hat{z}$ newtons.

Example 12. *Calculate the electric field due to charge uniformly distributed over an infinite plane with charge density σ using the cylindrical coordinates.*

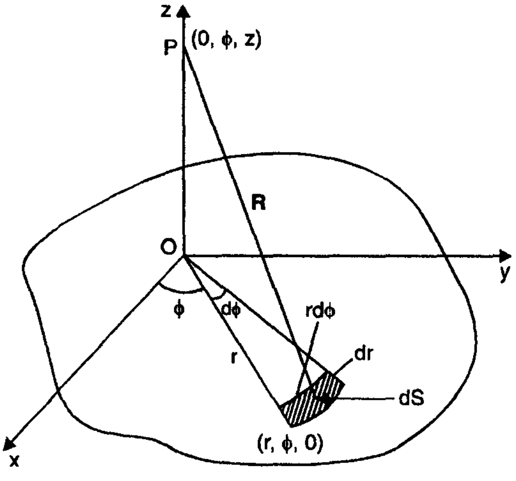


Fig. 3.37. Example 12.

In cylindrical coordinate system, if the charge is in the $z = 0$ plane, as shown in Fig. 3.37.

Let us consider a differential area dS on the plane at point with cylindrical coordinates $(r, \phi, 0)$. The electric field at point $P\ (0, \phi, z)$ is given by

$$d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma dS}{(r^2 + z^2)}\hat{\mathbf{R}}$$

$$\text{Since } \hat{\mathbf{R}} = \frac{-r\hat{r} + z\hat{z}}{(r^2 + z^2)^{1/2}} \text{ and } dS = rdrd\phi$$

$$\therefore \quad d\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r\, dr\, d\phi}{(r^2 + z^2)^{3/2}}(-r\hat{r} + z\hat{z})$$

The symmetry of the charge distribution about the z-axis results in cancellation of the radial components. Therefore

$$\mathbf{E} = \frac{\sigma z}{4\pi\varepsilon_0}\hat{z}\int_0^\infty \frac{r\,dr}{(r^2 + z^2)^{3/2}}\int_0^{2\pi} d\phi = \frac{\sigma z}{2\varepsilon_0}\left[\int_0^\infty \frac{rdr}{(r^2 + z^2)^{3/2}}\right]\hat{z}$$

$$= \frac{\sigma z}{2\varepsilon_0}\left[-\frac{1}{\sqrt{r^2+z^2}}\right]_0^{\infty}\hat{z} = \frac{\sigma}{2\varepsilon_0}\hat{z}.$$

This result is for points above the xy-plane. Below the plane the unit vector changes to $-\hat{z}$. Thus the generalized relation may be written as

$$\mathbf{E} = (\sigma/2\varepsilon_0)\hat{n},$$

where $\hat{n}$ is the unit normal vector.

The above result show that the electric field is every where normal to the plane of the charge and independent of the distance from the plane.

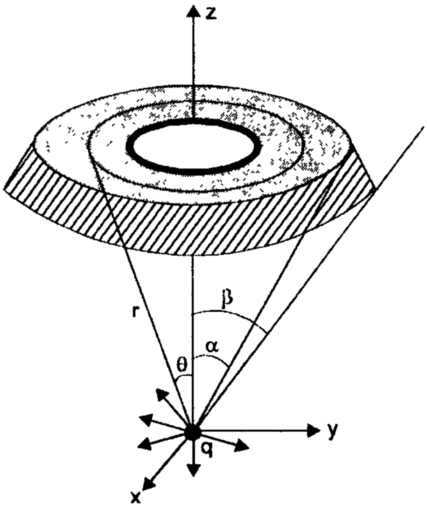


Fig. 3.38. Example 13.

Example 13. *A point charge q is at the origin of a spherical coordinate system. Find the flux which crosses the portion of a spherical shell described by* $\alpha \le \theta \le \beta$. *What will happen if* $\alpha = 0$ *and* $\beta = \pi/2$.

The total flux Φ due to a charge q crosses a complete spherical shell of area $4\pi r^2$ is given by

$$\Phi = \int \mathbf{E}\cdot d\mathbf{S} = \frac{q}{4\pi\varepsilon_0 r^2}.4\pi r^2 = \frac{q}{\varepsilon_0}.$$

The area of the strip between the angles α and β is given by

$$A = \int_0^{2\pi}\int_\alpha^\beta r^2 \sin\theta d\theta\, d\phi$$

$$= 2\ \pi r^2 (\cos\alpha - \cos\beta).$$

Therefore the flux through this strip

$$\Phi' = \frac{A}{4\pi r^2}.\frac{q}{\varepsilon_0} = \frac{q}{2\varepsilon_0}(\cos\alpha - \cos\beta).$$

For $\alpha = 0$ and $\beta = \pi/2$, a hemisphere,

$$\Phi' = \frac{q}{2\varepsilon_0}(1-0) = \frac{q}{2\varepsilon_0}.$$

Example 14. *A charge of uniform density* $\sigma = 0.3\ nC/m^2$ *exists on a plane uniform sheet* $2x - 3y + z = 6\ m$. *Find the electric field on the side of the plane containing the origin.*

Since the charge configuration is a uniform sheet, therefore the electric field $E = \sigma/2\varepsilon_0$ in the direction normal to its surface

$$\therefore\ \mathbf{E} = \frac{\sigma}{2\varepsilon_0}\hat{n} = 0.3\times10^{-9}\times2\pi\times9\times10^{9}\hat{n} = 16.97\hat{n}.$$

The unit normal vector for a plane $ax + by + cz = d$ are

$$\hat{n} = \pm\frac{a\hat{x}+b\hat{y}+c\hat{z}}{\sqrt{a^2+b^2+c^2}}$$

$\therefore$ For a plane $2x - 3y + z = 6$, the unit normal vectors are

$$\hat{n} = \pm\frac{2\hat{x}-3\hat{y}+\hat{z}}{\sqrt{2^2+3^2+1^2}} = \pm\frac{2\hat{x}-3\hat{y}+\hat{z}}{\sqrt{14}}.$$

It is evident from the figure 3.39, that the unit vector on the side of the plane containing the origin is produced by the negative sign. Therefore the electric field at the origin is

$$\mathbf{E} = 16.97\left[\frac{-2\hat{x}+3\hat{y}-\hat{z}}{\sqrt{14}}\right] = 4.54\ (-2\hat{x}+3\hat{y}-\hat{z})\ \text{N/C}$$

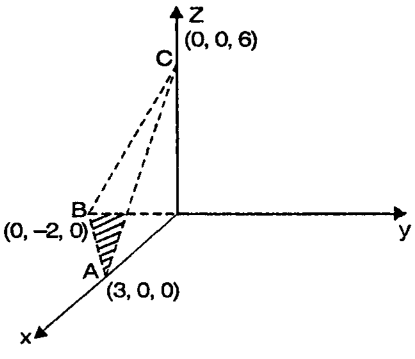


Fig. 3.39. Example 14.

Example 15. *Planes $x = 2$ and $y = -2$ respectively carry charges $10\ nC/m^2$ and $12\ nC/m^2$. If the line $x = 0$, $z = 2$ carries charge $8\pi\ nN/m$, calculate the electric field intensity at $(1, 1, -1)$ due to these three charge distributions.*

Let $\mathbf{E}_1$, $\mathbf{E}_2$ and $\mathbf{E}_3$ are, respectively the contributions to **E** at point $(1, 1, -1)$ due to infinite plane sheet $(x = 2)$, infinite plane sheet $(y = -2)$ and infinite line $(x = 0, z = 2)$, as shown in Fig. 3.40.

We know that for an infinite sheet of charge $\mathbf{E} = (\sigma/2\varepsilon_0)\,\hat{n}$, where $\hat{n}$ is a unit vector normal to the sheet. The electric field is normal to the sheet and is independent of the distance between the sheet and the point of observation P. Therefore

$$\mathbf{E}_1 = \frac{\sigma_1}{2\varepsilon_0}(-\hat{x}) = -\frac{10\times10^{-9}\hat{x}}{2/36\pi\times10^{9}} = -180\pi\,\hat{x}.$$

$$\mathbf{E}_2 = \frac{\sigma_2}{2\varepsilon_0}(\hat{y}) = \frac{12\times10^{-9}\hat{y}}{2/36\pi\times10^{9}} = 216\pi\,\hat{y}.$$

$$\mathbf{E}_3 = \frac{\rho_L}{2\pi\varepsilon_0 r}\hat{r} = \frac{8\pi\times10^{-9}}{(18\times10^{9})^{-1}}\frac{\hat{r}}{r}, = 144\pi\frac{\hat{r}}{r}.$$

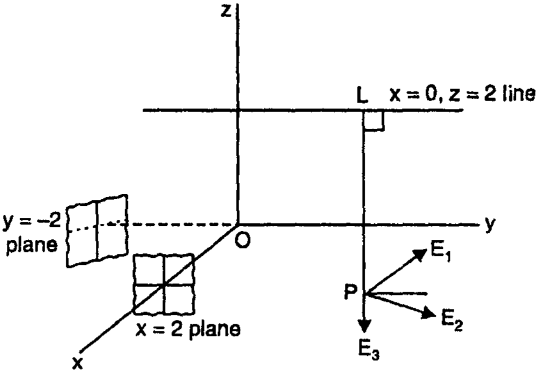


Fig. 3.40. Example 15.

where $\hat{r}$ is the unit vector along LP, perpendicular to the line of charge from the point P and r is the distance LP. If we consider plane $y = 1$ on which $\mathbf{E}_3$ lies. Thus

$$\mathbf{r} = \hat{x} - 3\hat{z}$$

$$\therefore \quad r = |\,\mathbf{r}\,| = [1^2 + 3^2]^{1/2} = \sqrt{10} \quad \text{and} \quad \hat{r} = \frac{\hat{x} - 3\hat{z}}{\sqrt{10}}$$

$$\therefore \quad \mathbf{E}_3 = 144\,\pi\,(\hat{x} - 3\hat{z})/10 = 14.4\pi\hat{x} - 43.2\pi\hat{z}.$$

Therefore the total field

$$\begin{aligned}\mathbf{E} &= -180\pi\hat{x} + 210\pi\hat{y} + 14.4\pi\hat{x} - 43.2\pi\hat{z}\\ &= -165.6\pi\hat{x} + 210\pi\hat{y} - 43.2\pi\hat{z}.\end{aligned}$$

Example 16. *If the electric field is given by $\mathbf{E} = 8\hat{x} + 4\hat{y} + 3\hat{z}$, calculate the electric flux through a surface of area 100 units lying in the x-y plane.*

In this problem $\hat{x}$, $\hat{y}$ and $\hat{z}$ are the unit vectors along x, y and z axes respectively, generally taken as **i**, **j** and **k**.

Since the electric flux is defined as $\Phi = \mathbf{E.S}$. The vector surface area in the x-y plane will be in the direction of the out drawn normal, *i.e.*, in the z-direction. Therefore

$$\mathbf{S} = 100\hat{z}\ (= 100\,\mathbf{k})$$

$$\therefore \quad \Phi = \mathbf{E}\cdot\mathbf{S} = (8\hat{x} + 4\hat{y} + 3\hat{z}).100\hat{z} = 300 \text{ units.}$$

Example 17. *Calculate electric flux for a cube of side a as shown in Fig. 3.41, where $E_x = bx^{1/2}$, $E_y = E_z = 0$, $a = 10$ cm and $b = 800$ nt coul-$m^{1/2}$.*

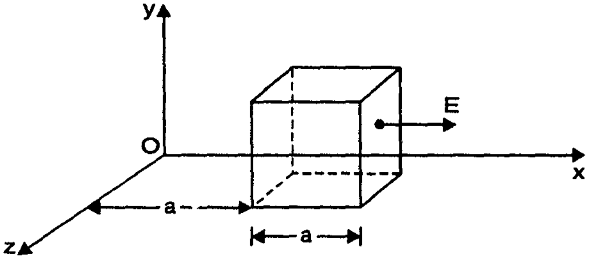


Fig. 3.41. Example 17.

Since the electric field is acting only in x-direction and its y and z components are zero. The net electric flux is thus given as

$$\begin{aligned}\Phi &= \Phi_{\text{right surface}} - \Phi_{\text{left surface}}\\ &= \int_R \mathbf{E}\cdot d\mathbf{S} - \int_L \mathbf{E}\cdot d\mathbf{S} = (E_x a^2)_{\text{right}} - (E_x a^2)_{\text{left}}\\ &= a^2[E_{2a} - E_a] = a^2\,[b\,(2a)^{1/2} - ba^{1/2}]\\ &= b\ a^{5/2}\left(\sqrt{2} - 1\right)\\ &= 800 \times (0.1)^{5/2}\left(\sqrt{2} - 1\right) = 1.05 \text{ mks units.}\end{aligned}$$

Example 18. *Calculate the flux of the electric field strength through each of the faces of a closed cube of length 2a, if a charge q is placed (a) at its centre and (b) at one of its vertices.*

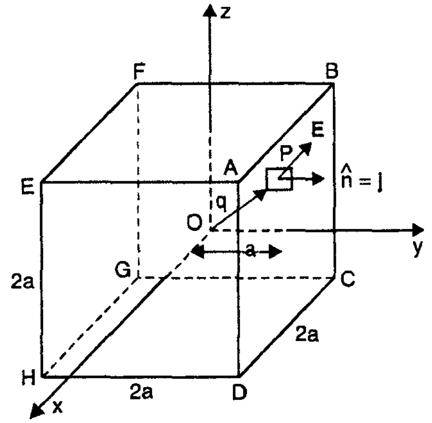


Fig. 3.42. Example 18.

Consider a cube with sides $2a$. A charge q is placed at its centre O, assumed to be the origin of the cartesian coordinates, whose axes lie parallel to the edges of the cube.

The field strength due to charge q on an infinitesimal element $dxdz$ at point of the surface $ABCD$ will be given by

$$\mathbf{E} = \frac{q}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3} = \frac{q}{4\pi\varepsilon_0 r^3}(x\mathbf{i} + a\mathbf{j} + z\mathbf{k}).$$

The flux through the surface $dxdz$

$$\Delta\Phi = \mathbf{E}\cdot\hat{n}dA = \mathbf{E}\cdot\mathbf{j}dxdz$$
$$= qa\,dxdz\ /4\pi\varepsilon_0(x^2 + a^2 + z^2)^{3/2}.$$

The flux through the surface $ABCD$

$$\Phi = \frac{qa}{4\pi\varepsilon_0}\iint_{-a}^{a}\frac{dxdz}{(x^2 + a^2 + z^2)^{3/2}} = \frac{qa}{\pi\varepsilon_0}\int_0^a dx\int_0^a\frac{dz}{(z^2 + x^2 + a^2)^{3/2}}$$

$$= \frac{qa}{\pi\varepsilon_0}\int_0^a dx\left[\frac{z}{(x^2 + a^2)\sqrt{z^2 + x^2 + a^2)}}\right]_0^a = \frac{q\,a^2}{\pi\varepsilon_0}\int_0^a\frac{dx}{(x^2 + a^2)\sqrt{x^2 + 2a^2}}$$

$$= \frac{qa^2}{\pi\varepsilon_0}\left[\frac{1}{a^2}\tan^{-1}\frac{x}{\sqrt{x^2 + 2a^2}}\right]_0^a = \frac{q}{\pi\varepsilon_0}\tan^{-1}\frac{1}{\sqrt{3}}$$

$$= q\ /6\varepsilon_0.$$

The same result can be obtained much more easily by making use of Gauss's law as follows :

Symmetry of six faces of a cube about its centre ensures that the flux Φ of $\mathbf{E}$ through each of the faces will be same when charge q is placed at the centre.

$$\therefore \qquad \Phi = 6\Phi_S = q/\varepsilon_0 \quad \text{or} \quad \Phi_S = q/6\varepsilon_0.$$

In the second case, when q is at one of the vertices, the flux through each of the three faces meeting at this vertex will be zero, as $\mathbf{E}$ is parallel to these surfaces. The flux through other three faces will be same, say Φ_S. We know that one-eighth of the flux originating from charge q passes through this cube, as the faces of the cube meeting at the charge include one-eighth of the spherical surface drawn with charge q as centre.

$$\therefore \qquad 3\Phi_S = \frac{1}{8}\frac{q}{\varepsilon_0}, \text{ or } \Phi_S = \frac{1}{24}\cdot\frac{q}{\varepsilon_0}.$$

Example 19. *A dielectric cylinder of radius a is infinitely long. It is non-uniformly charged such that volume charge density* ρ *varies directly as the distance from the axis of the cylinder. Calculate the electric intensity due to it, if* ρ *is zero at the axis and is* ρ_s *on the surface.*

To calculate net charge within the charged cylinder, let us divide it in large number of co-axial thin cylindrical shells. Consider a co-axial shell of radii x and $x + \delta x$ and of length l. If the thickness δx is very small so that the charge density may be assumed as constant at each point on it. Thus the charge on this shell $\delta q = 2\pi x l \delta x \rho$.

As ρ varies directly as the distance x, hence ρ may be written as cx, where c is a constant given by the relation $ca = \rho_s$ or $c = \rho_s/a$.

The intensity at any point P may be calculated as given in article 3.9(*E*). The field E at the point at a distance r outside the cylinder is given by

$$E2\pi rl = \frac{q}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int dq.$$

Charge inside the gaussian surface is the charge on the cylinder of length l, *i.e.*, the integral of dq must be over the whole cylinder.

$$\therefore \qquad E = \frac{1}{2\pi rl\varepsilon_0}\int_0^a 2\pi lcx^2dx = \frac{ca^3}{3r\varepsilon_0} = \frac{\rho_s a^2}{3r\varepsilon_0}.$$

At points inside the cylinder, E is given by

$$E2\pi rl = \frac{q'}{\varepsilon_0} = \frac{1}{\varepsilon_0}\int dq = \frac{1}{\varepsilon_0}\int_0^r 2\pi lcx^2dx,$$

as the charge inside the gaussian surface is the charge on the cylinder of radius r and of length l. Therefore

$$E = cr^2/3\varepsilon_0 = \rho_s r^2/3a\varepsilon_0.$$

At the points on the surface of the cylinder, $E = \rho_s a/3\varepsilon_0$.

In all the cases the field is in radially outward direction.

Example 20. *Calculate the charge density at (1, $\pi/4$, 3) and the total charge enclosed by the cylinder of radius 1 m with $-2 \le z \le 2$ m. Given that $D = z\,\rho\cos^2\phi\;\hat{z}$ C/m^2.*

From Gauss's law, we have $\nabla\cdot\mathbf{D} = \rho$

$$\rho_V = \partial D_z/\partial z = \rho\cos^2\phi.$$

$$\text{At } (1, \pi/4, 3),\ \rho_V = 1.\cos^2\pi/4 = 0.5 \text{ C/m}^2.$$

Total volume charge $Q = \int_V \rho_V dV = \int_V \rho\cos^2\phi\,(\rho\, d\rho\, d\phi\, dz)$

$$= \int_{-2}^{2} dz \int_{\phi=0}^{2\pi}\cos^2\phi\, d\phi \int_{\rho=0}^{1}\rho^2 d\rho = \frac{4\pi}{3}\text{ coul.}$$

Example 21. *Calculate the electric field intensity due to a spherical charge distribution, given by*

$$\rho = \rho_0(1 - r/a),\ \textit{when } r \le a \textit{ and } \rho = 0 \textit{ when } r > a.$$

Find the value of r for which E is maximum.

Let us consider a thin spherical shell of radii x and $x + dx$. The volume of this shell $\delta V = 4\pi x^2 dx$. To calculate electric field intensity at a point outside the spherical charge distribution let us assume a gaussian spherical surface passing through the point which is at a distance r from the centre. Using Gauss's law, we get

$$\int \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int dq = \frac{1}{\varepsilon_0}\int_0^a \rho_0\left(1 - \frac{x}{a}\right)4\pi x^2 dx.$$

Here integration is between $x = 0$ to $x = a$, as whole of the charge is within the gaussian surface.

$$\therefore \qquad \oint E dS = 4\pi r^2 E = \pi\rho_0 a^3/3\varepsilon_0$$

$$\text{or} \qquad E = \rho_0 a^3/12\varepsilon_0 r^2.$$

Field inside the charge distribution is given by

$$\int_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int dq = \frac{1}{\varepsilon_0}\int_0^r \rho_0\left(1 - \frac{x}{a}\right)4\pi x^2 dx.$$

Here integration is between $x = 0$ to $x = r$, as the charge inside the gaussian surface is the charge within the sphere of radius r

$$\therefore \quad \oint E dS = 4\pi r^2 E = \frac{4\pi\rho_0}{\varepsilon_0}\left[\frac{r^3}{3} - \frac{r^4}{4a}\right]$$

or
$$E = \frac{\rho_0}{\varepsilon_0}\left[\frac{r}{3} - \frac{r^2}{4a}\right].$$

Electric field E will be maximum, when

$$\frac{dE}{dr} = 0 \quad \text{or} \quad \frac{d}{dr}\left[\frac{\rho_0}{\varepsilon_0}\left(\frac{r}{3} - \frac{r^2}{4a}\right)\right] = 0.$$

$$\therefore \quad r = 2a/3.$$

Example 22. *If a heavy atom can be pictured as a spherical nucleus with charge +q and radius a embeded in a much larger sphere of negative charge, the electrons. This negative charge is distributed uniformly throughout this sphere of radius b. Find the electric field intensity outside the nucleus.*

As the atom is always neutral therefore the total charge inside the gaussian surface passing through any point *outside the atom* is zero, hence by Gauss's law

$$\int \mathbf{E} \cdot d\mathbf{S} = (1/\varepsilon_0)\Sigma q = 0.$$

As surface area $d\mathbf{S}$ is always || to the radius vector, hence $\mathbf{E}$ cannot be ⊥ to $d\mathbf{S}$ and thus zero.

At a point inside the atom but outside the nucleus and a distance r apart from the centre of the nucleus, the total charge with in the gaussian sphere is

$$\Sigma q = q - \left(\tfrac{4}{3}\pi r^3\right)\rho.$$

$$\therefore \quad \int \mathbf{E} \cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\left[q - \frac{4}{3}\pi r^3 \cdot \left(\frac{q}{\frac{4}{3}\pi b^3}\right)\right]$$

$$E 4\pi r^2 = \frac{q}{\varepsilon_0}\left[1 - \left(\frac{r}{b}\right)^3\right]$$

or
$$E = \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - \left(\frac{r}{b}\right)^3\right]$$

The first term is due to the nucleus, while the second term represents the cancelling field due to electrons. At point very close to nucleus the second term is negligible, the nucleus and innermost electrons of an atom are therefore not greatly affected by the outer electrons of the atom.

Example 23. *Calculate the flux of* **E** *through (i) each of the bases, (ii) the curved surface of a right cylindrical closed surface of radius a and length l, due to the charge q situated at its geometrical centre.*

To find the electric flux through the base of the cylinder, let us divide the base (say lower) into large number of concentric rings, as shown in Fig. 3.43. The electric field at any point P of an annular ring between radii r and $r + \delta r$ is

$$E = q/4\pi\varepsilon_0(r^2 + l^2/4) \text{ along } \mathbf{OP}.$$

The flux through a very small element of this annular ring is $E \cos\theta$ × area of the element. As $E \cos\theta$ is same for all elements on this annular ring, hence the flux through this annular ring

$$\delta\Phi = E\ \cos\theta.2\pi r\delta r$$

or $$\delta\Phi = \frac{q.2\pi r\delta r\cos\theta}{4\pi\varepsilon_0(r^2+l^2/4)} = \frac{ql\ r\delta r}{4\varepsilon_0(r^2+l^2/4)^{3/2}}$$

∴ Total flux through the base of cylinder

$$\Phi = \int_0^a \frac{qlrdr}{4\varepsilon_0(r^2+l^2/4)^{3/2}} = \frac{q}{2\varepsilon_0}\left[1-\frac{l/2}{(a^2+l^2/4)^{1/2}}\right].$$

By symmetry the flux through the above base will also be same. As the total flux originating from a point charge is always q/ε_0, hence by Gauss's law

$$\Phi_{curved} + 2\Phi_{bases} = q/\varepsilon_0$$

or $$\Phi_{curved} = \frac{q}{\varepsilon_0} - \frac{q}{\varepsilon_0}\left[1-\frac{l/2}{(a^2+l^2/4)^{1/2}}\right] = \frac{ql}{2\varepsilon_0(a^2+l^2/4)^{1/2}}.$$

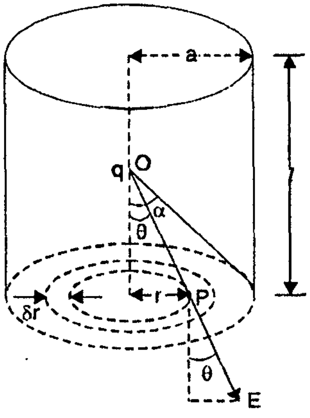


Fig. 3.43. Example 23.

Example 24. *Two plane metal plates are placed parallel to each other, one carries a surface charge density +σ and the other −2σ. Find the charge densities on the two surfaces of the third plane metal plate placed at the centre. The central plate is parallel to the other plates and is assumed connected to the earth.*

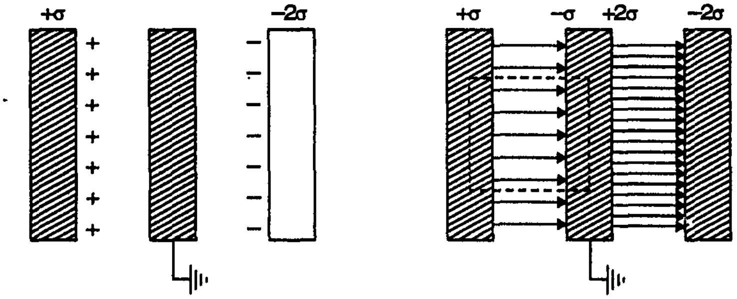


Fig. 3.44. Example 24.

The lines of force coming from the plate on the left must end on the left side of the central plate. Thus the charge density must be −σ. Similarly the charge density on the right surface of the central plate must be +2σ, as it is due to the lines of force going towards the right plate having charge density −2σ. Thus the central plate has a net charge +σ.

The same result can be obtained by using Gauss's law. Consider a gaussian surface shown by the dotted lines in Fig. 3.44. Thus by Gauss's law

$$\int \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\Sigma q.$$

As the net flux coming out of the gaussian surface is zero, hence Σq must be zero. This can be true only if the charge on the left side of the central plate is equal and opposite to the charge on the left plate. Similarly we can show that the charge on the right side will be equal and opposite to the charge on the right plate.

Example 25. *If the electric field near the earth's surface be 300 volts/meter directed downwards, what is the surface density of charge on the earth's surface ?*

The electric field just outside a charged conductor of surface charge density σ is given by $E = \sigma/\varepsilon_0$.

In this problem the electric field near the earth's surface, which may be treated as a charged conductor, is given as 300 volts/meter. Since the permittivity of the free space $\varepsilon_0 = 8.85 \times 10^{-12}$ coul^2/nt–m^2. Thus the surface density of charge on the earth's surface

$$\sigma = \varepsilon_0 E = 8.85 \times 10^{-12} \times 300 = 2.66 \times 10^{-9}\ \text{coul/m}^2.$$

Example 26. *A gold foil weighing 50 mg per sq cm is placed on a horizontal charged plate. What would be the density of charge so that the foil may just rise ?*

Mechanical force per unit area of a charged plate = $\sigma^2/2\varepsilon_0$.

In this problem outward mechanical force is balanced by the force due to the weight of gold foil. Hence at equilibrium

$$\sigma^2/2\varepsilon_0 = mg, \text{ or } \sigma^2 = 2\varepsilon_0 mg,$$

where σ is the surface charge density and m is the mass per unit area of the foil.

$\therefore \qquad \sigma = 2 \times 8.85 \times 10^{-12} \times 50 \times 10^3 \times 9.81$

or $\qquad \sigma = 9.313 \times 10^{-6}$ coul/m^2.

Example 27. *Two uniform infinite sheets of electric charge densities +σ and −σ intersect at right angles. Find the magnitude and direction of the electric field everywhere and sketch the lines of* **E**.

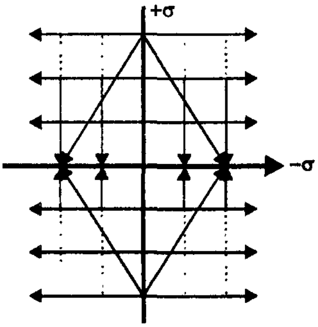


Fig. 3.45. Example 27.

The electric field due to the infinite sheet of charge density σ at any point in the space is given by $E = \sigma/2\varepsilon_0$, It is perpendicular to the surface of the sheet in the outward direction (thin line)

The electric field due to the infinite sheet of charge density $-\sigma$ at any point in the space is also equal to $\sigma/2\varepsilon_0$ but in inward direction perpendicular to the surface (dotted lines).

The resultant at any point can be obtained by the superposition of these two fields at that point

$$\therefore \qquad E = \sqrt{(\sigma/2\varepsilon_0)^2 + (\sigma/2\varepsilon_0)^2} = \sqrt{2}\,\sigma/2\varepsilon_0.$$

The resultant field will make an angle of 45° with the surface of the sheet, as shown by the thick line in the Fig. 3.45.

Example 28. *In the electrostatic separation of solids, small particles of quark and phosphate in a florida phosphate ore, are separated by applying a uniform electric field, when the ore is allowed to fall freely. Assuming zero initial displacement and velocity, determine the separation between the particles after falling 80 cm. Given $q/m = 9\ \mu C/kg$ for both positively and negatively charged particles and $E = 400\ kV/m$.*

The positively and negatively charged particles are separated by the horizontal electric field E

$\therefore \qquad q\mathrm{E} = m\ (d^2x/dt^2)\,\hat{x}$

or $\qquad d^2x/dt^2 = (q/m)\mathrm{E}$

On integrating twice, we get

$$x = (q/2m)\,Et^2 + C_1 t + C_2$$

where C_1 and C_2 are constants of integration.

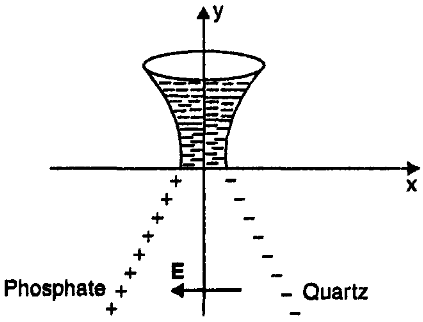


Fig. 3.46. Example 27.

Similarly the vertical force is the gravitational force due to the weight of the particle.

$\therefore \qquad m d^2y/dt^2 = -mg \quad \text{or} \quad d^2y/dt^2 = -g$

On integrating twice we get

$$y = -(g/2)t^2 + C_3 t + C_4.$$

At the initial stage displacement and velocity are zero, therefore

$$x\,(t = 0) = 0, \text{ which gives } C_2 = 0;\ y(t = 0) = 0 \Rightarrow C_4 = 0$$

$$x'(t=0) = 0 \Rightarrow C_1 = 0,\; y'(t=0) = 0 \Rightarrow C_3 = 0.$$

$$x = (qE/2\text{m})t^2 \text{ and } y = -(g/2)t^2$$

or
$$x = \frac{qE}{mg}y = \frac{9\times10^{-6}\times400\times10^3}{9.8}\times0.8 = 0.294 \text{ m.}$$

∴ The separation between the particles = $2x = 0.588$ m = 58.8 cm.

Example 29. *An electric charge Q is uniformly distributed over the surface of a sphere of radius a. Show that the force on a small charge element dq is radial and outward and is given by dF = (1/2) E dq, where E is the electric field at the surface of the sphere.*

The surface charge density $\sigma = Q/4\pi a^2$

and the electric field at the surface of the sphere $E = Q/4\pi\varepsilon_0 a^2$.

Consider a small charge element of area ds on the surface of the sphere. The charge dq on this element will be σds. This charge will produce an electric field E_{1p} at point P inside the sphere close to an element ds, which is approximately that due to a uniformly charged infinite plate, such as

$$\mathbf{E}_{1p} = -(\sigma/2\varepsilon_0)\,\hat{n},$$

where $\hat{n}$ is the unit vector normal to ds in the outward direction.

The net electric field is zero inside the sphere. Hence, if $\mathbf{E}_{2p}$ be the electric field at point P due to the rest of charges on the sphere, then

$$\mathbf{E}_p = \mathbf{E}_{1p} + \mathbf{E}_{2p} = 0 \text{ or } \mathbf{E}_{2p} = (\sigma/2\varepsilon_0)\,\hat{n}$$

∴
$$\mathbf{E}_{2p} = \frac{\sigma}{2\varepsilon_0} = \frac{Q}{8\pi\varepsilon_0 a^2} = \frac{1}{2}E.$$

As point P is close to the ds, $\mathbf{E}_{2p}$ may be considered as the field strength at ds due to the charges of the spherical surface. Hence the force acting on ds is

$$dF = dq\,\mathbf{E}_{2p} = \tfrac{1}{2}E\,dq.$$

Example 30. *Derive an expression for the electric charge required to expand the bubble to twice its dimensions.*

Let P_1 be the internal pressure inside the bubble of radius r and surface tension T. If P be the atmospheric pressure, then $P_1 = P + 4T/r$.

When the bubble after charging expands, the radius becomes $2r$, the volume becomes 8 times that initial volume. Therefore by Boyle's law the pressure inside the bubble reduces to $P_1/8$. This decrease in pressure is due to the mechanical outward pressure due to the charge q given to the bubble.

∴ Total outward pressure = total inward pressure

$$\frac{1}{8}\left(P + \frac{4T}{r}\right) + \frac{q^2}{32\varepsilon_0\pi^2(2r)^4} = P + \frac{4T}{2r}$$

$$q^2 = 64\,\pi^2\varepsilon_0 r^3\,[7Pr + 12T] \quad \text{or} \quad q = 8\pi\sqrt{\varepsilon_0 r^3(7Pr + 12T)}.$$

Example 31. *An insulated soap bubble 10 cm in radius is charged with 20 stat-coulomb. Find the increase in radius due to the charge. Given the atmospheric pressure = 10^5 nt/m^2.*

From Boyle's law PV = const = k

Pressure P of a sphere of radius $R = \dfrac{k}{\left(\frac{4}{3}\pi R^3\right)} = K/R^3$.

Differentiation gives $dP/dR = -3K/R^4 = -3P/R$ or $dR = -(R/3P)\,dP$.

The change in pressure due to the charge q is given by

$$-dP = -\sigma^2/2\varepsilon_0 = -(q/4\pi R^2)^2/2\varepsilon_0.$$

$$\therefore \qquad dR = -\frac{R}{3P}dP = \frac{R}{3P}\cdot\frac{q}{32\pi^2\varepsilon_0 R^4} = \frac{q^2}{96\pi^2\varepsilon_0 PR^3}.$$

Given $q = 20$ stat coul $= 20$ e. s. u. of charge $= 20 \times 3.36 \times 10^{-10}$ coul.

Thus the substitution of numerical values gives $dR = 5.3 \times 10^{-9}$ cm.

Example 32. *In Millikan's experiment an oil drop of radius 10^{-4} cm. remains suspended between the plates which are 1 cm apart. If the drop has charge of 5e over it, calculate the potential difference between the plates. The density of oil may be taken as 1.5 gm./cc.*

Since the drop remains suspended between the plates, the viscous force on the drop is zero and the weight mg of the drop is balanced by the force due to electric field between the plates. If V be the potential difference between the plates, then

$$qE = mg \quad \text{or} \quad qV/d = \tfrac{4}{3}\,\pi r^3(\rho - \sigma)\,g.$$

In this problem $q = 5e = 5 \times 1.6 \times 10^{-19}$ coul, $d = 1$ cm $= 0.01$ *m*, $r = 10^{-4}$ cm $= 10^{-6}$ m, $\rho = 1.5 \times 10^3$ kg/m^3. Assuming the density of air as negligible in comparison to the oil density, we get

$$V = \frac{0.01}{5 \times 1.6023 \times 10^{-19}} \times \frac{4}{3} \times 3.14 \times (10^{-6})^3 \times 1.5 \times 10^3 \times 9.81$$

$$= 770 \text{ volts.}$$

ORAL QUESTIONS

1. Given an insulated rod, how could you determine if it was charged ? How could you find the sign of the charge on it ?
2. The quantum of charge is 1.6023×10^{-19} coul, is there any single quantum of mass ?
3. Give some examples of physical quantity to be (*a*) quantized or (*b*) conserved.
4. Whether the constant of proportionality in Coulomb's law is a measured value or a calculated value ?
5. A stone is placed in the gravitational field of earth. Can we also say that the earth lies in the gravitational field of the stone ?
6. Electric lines of force never cross, why ?
7. Two point charges are placed at a certain distance apart. The electric field strength is zero at one point between them. What can you conclude about the charges ?
8. What is the electric flux through a surface encloses an electric dipole ?
9. A charge q is placed inside an enclosure. Compare the total flux coming out of the walls of the enclosure, if it is spherical, cubic, rectangular, parallelopiped or hemisphere.
10. Show that no work is done in moving the test charge from point to point on the surface of the metal.
11. Plane $z = 10$ m carries charge 20 nC/m^2. The electric intensity at the origin is ————.
12. The electric field at a point inside a charged hollow spherical conductor is ————.
13. Is it correct that the electric flux is a vector quantity ?
14. The electric field due to infinite plane sheet of charge is : (*a*) independent of distance or (*b*) varies inversely as the square of the distance ?
15. The electric field inside a charged non conducting sphere : (*a*) is zero everywhere; (*b*) increases linearly with distance from the centre or (*c*) is constant everywhere.
16. The surface charge density is represented by (*a*) λ, (*b*) σ or (*c*) ρ.
17. An ebonite rod rubbed with fur is brought near a glass rod rubbed with silk, then (*a*) two will attract each other, (*b*) repel each other, (*c*) nothing with happen.

18. For detecting and testing of small electric charges, we use an (*a*) electrophorous, (*b*) ammeter; (*c*) electroscope; (*d*) voltmeter.
19. The lighting rods are made of (*a*) metals, (*b*) insulators, (*c*) semiconductors.
20. The filament used in the electric bulb is made of (*a*) tungston, (*b*) nichrome, (*c*) copper, (*d*) aluminium.
21. Define electric field strength. Give its unit.
22. What is the difference between a uniform and non-uniform electric field ? Give examples.
23. Define flux of a field vector.
24. When very dry hair is combed with vulcanite comb, a crackling sound is heard. This is caused by (*a*) friction, (*b*) electric sparks, (*c*) viscosity.
25. During the thunderstorm, you are travelling in a car. To avoid yourself from the lightning you will have to (*a*) get out of the car and lay down on the ground, (*b*) take shelter under a nearby tree, (*c*) remain in the car.
26. A man is placed with an electrical measuring instrument inside a large closed metal sphere. What will the man observe as (*a*) charge is placed on the sphere, or (*b*) a large object is brought close to the sphere ?
27. As you penetrate a uniform sphere of charge, find the effect on E while moving towards the centre.
28. A spherical rubber balloon carries a charge that is uniformly distributed over its surface. How does *E* vary for points (*a*) inside, (*b*) on the surface and (*c*) outside the balloon, as the balloon is blown up ?
29. In Millikan's apparatus, how can you find the sign of charge on the droplets from the atomiser.

Ans. 11. $-360\pi\ \mathbf{a}_z$ V/M; **12.** Zero; **13.** No; **14.** (*a*); **15.** (*b*); **16.** (*b*); **17.** (*a*), **18.** (*a*), **19.** (*a*), **20.** (*a*), **24.** (*b*); **25.** (*c*).

PROBLEMS

1. In the Bohr model of atomic hydrogen an electron of mass 9.11×10^{-31} kg revolves, about a nucleus consisting one proton, in a circular orbit of radius 5.29×10^{-11} m. If mass of proton is 1.67×10^{-27} kg, calculate the radial acceleration and angular velocity of the electron.
2. If two equally charged balls of identical masses of 0.20 gm are suspended from 50 cm long strings. Calculate the value of each charge, if the strings make an angle of 37° to the vertical.

   (3.2×10^{-7} coul)
3. Two charges *Q* and *Q* are placed at the diagonally opposite corners of a square, while charge *q* and *q* are placed at the remaining corners. If the resultant force on one charge *Q* is zero, find the relation between *Q* and *q*. ($Q = 2\sqrt{2}q$)
4. A pith ball covered with tin foil having a mass of *m* kg hangs by a fine silk thread *l* meter long in an electric field *E*. When the ball is given electric charge of *q* coulomb, it stands out *d* metre from the vertical line. Show that the strength of the electric field is given by

$$E = mgd/q\sqrt{l^2 - d^2} \text{ newton/coulomb.}$$

5. Three particles each of mass 1 gm and carrying a charge *q* are suspended from a common point by insulated massless strings each 100 cm. long. If the particles are in equilibrium and are located at the corners of an equilateral triangle of side length 3 cm, calculate the charge *q* on each particle. (3.16×10^{-9} *C*)
6. Two similar helium filled spherical balloons tied to a 5 gm weight with strings and each carrying an electric charge *q* float in equilibrium as shown in Fig. 3.46. Find (*i*) *q* and (*ii*) the volume of each baloon. Given density of air = 0.00129 gm cm^{-3} and density of helium inside the balloon = 0.0002 gm cm^{-3}. (5.55×10^{-7}*C*, 2294 *cc*)
7. A pendulum bob of mass 80 mg and carrying a charge of 2×10^{-8} C is at rest in a horizontal uniform electric field of 2×10^{4} Vm^{-1}. Find the tension in the thread of the pendulum and the angle it makes with the vertical. (8.8×10^{-4} *N*, 27°3′)

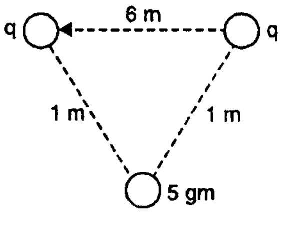


Fig. 3.46. Problem 6.

8. A simple pendulum consists of a small sphere of mass m suspended by a thread of length l. The sphere carries a positive charge q. The pendulum is placed in a uniform electric field E directed vertically upwards. With what period will the pendulum oscillate if the electrostatic force acting on the sphere is less than the gravitational force ? $2\pi[l/(g - qE/m)]^{1/2}$

9. A thin circular ring of radius 20 cm is charged with a uniform charge density of λ coul/m. A small section of 1 cm length is removed from the ring. Find the electric field intensity at the centre of the ring. ($2.25\ \lambda \times 10^9$ *N/C*)

10. Prove that the electric field intensity due to a uniformly charged ring is maximum at a distance $1/\sqrt{2}$ times its radius from the centre on its axis.

11. A circular ring of radius a carries a charge which varies as $\lambda = \lambda_0 \sin 0$. Find the electric field at the centre of the ring. $(\lambda_0/4a\varepsilon_0)$

12. Charge lies in the $z = -3m$ plane in the form of a square sheet defined by $-2 \le x \le 2$m, $-2 \le y \le 2$m with charge density $\rho_s = \sigma = 2(x^2 + y^2 + 9)^{3/2}$ nC/m². Find the $\dot{E}$ at the origin. (864k *N/C*)

13. Two parallel infinite wires carry uniform charges of λ_1 and λ_2 coul/m. If the separation of the wires is b, find the force on unit length of one as a result of the other. $(\lambda_1 \lambda_2/2\pi\varepsilon_0 b)$

14. A uniform line charge with $\lambda = 50$ μC/m lies along the x-axis. Calculate the flux per unit length crosses the portion of the $z = -3$ m plane bounded by $y = \pm 2$ m. (6.36 μC/m, $9.36/\varepsilon_0$ μ web/m)

15. An infinitely long wire is stretched horizontally 4 meters above the surface of the earth. It has a charge of microcoulomb per cm of its length, calculate the electric field at a point on earth vertically below the wire. (4.5×10^5 *N/C*)

16. An α-particle, approaching the surface of a gold nucleus, is at a distance equal to the nuclear radius (6.9×10^{-15} m) from the surface. If the α-particle is assumed as point charge and of mass 6.7×10^{-27} kg, calculate the force and acceleration of the α-particle.
(1.9×10^2 *N*, 2.8×10^{28} *m/sec*²)

17. A 10 MeV α-particle moves head on towards a stationary nucleus having 80 protons. Calculate the distance of its closest approach. (1.44×10^{-15} *m*)

18. Five thousand lines of force enter a certain volume of space and three thousand lines emerge from it. What is the total charge in coulombs within the volume ? (-1.77×10^{-8} *coul*)

19. Calculate the electric field intensity at a point (*i*) *inside and* (*ii*) outside the spherically symmetric charge distribution of radius a. The charge volume density varies as $\rho(r) = \rho_0(1 - r^3/a^3)^{1/2}$ inside and is zero outside the distribution.
$$\frac{2a^3\rho_0}{9\varepsilon_0 r^2}\left[1-\left(1-\frac{r^3}{a^3}\right)^{3/2}\right], \frac{2a^3\rho_0}{9\varepsilon_0{}^2}.$$

20. Electric charge of uniform density ρ per unit volume is distributed within a spherical shell of inner and outer radii 'a' and 'b' respectively. Calculate the electric field at a point distant r from the centre such that (*i*) $a < r < b$ and (*ii*) $r < b$.
$$\frac{\rho}{3\varepsilon_0}\left[\frac{r^3 - a^3}{r^2}\right], \frac{\rho}{3\varepsilon_0}\left[\frac{b^3 - a^3}{r^2}\right]$$

21. If a uniform sphere of charge q, radius R and charge density ρ has a narrow straight tunnel along its diameter, and a point charge $-q'$ is placed at the entrance to the tunnel and released. Show that its motion is simple harmonic. Find the period of this motion and the maximum velocity acquired by the charge $-q$. Given the mass of the charge q' as m.
$2\pi[3\varepsilon_0\, m/q'\rho]^{1/2}$, $R[q'\rho/3\varepsilon_0 m]^{1/2}$.

22. The inner surface of a non-conducting hemi-spherical bowl of radius a has uniformly spread charge of surface density σ over it. Find the electric field at the centre of the flat surface of the bowl.
($\sigma/4\varepsilon_0$ *N/C*)

23. Two concentric thin metallic spherical shells of radii R_1 and R_2, where $R_1 < R_2$, bear charges q_1 and q_2 coulombs respectively. Using Gauss Theorem, show that

(*a*) The electric field intensity at radius $r < R_1$ is zero.

(*b*) The electric field intensity at radius *r* between R_1 and R_2 is $q_1/4\pi\varepsilon_0 r^2$

(*c*) The electric field intensity at radius $r < R_2$ is $(q_1 + q_2)/4\pi\varepsilon_0 r^2$.

24. A spherical shell of outer and inner radii *a* and *b* respectively is concentric to a metal sphere (of radius *c*) inside the shell. Find the net charge on the outer surface of the spherical shell, if the electric field at a point *P* outside the shell at a distance 40 cm from the centre is 200 N/C. (3.55×10^{-9} *coul*)

25. A cylinder of radius *b* is uniformly charged with a volume charge density ρ coul/m². Find expressions for the electric field as a function of *r* for inside and outside the cylinder, if the charge density varies as $\rho = \rho_0 r^3$ within the cylinder. ($\rho_0 r^4/5\varepsilon_0$, $\rho_0 b^5/5\varepsilon_0 r$)

26. A long conducting cylinder of length *l* carrying a total charge $+q$ is surrounded by a conducting cylindrical shell of total charge $-2q$. Calculate E (*i*) at point outside the shell and (*ii*) in the region between the cylinders. ($q/2\pi\varepsilon_0$ *lr radially inward*, $q/2\pi\varepsilon_0$ *lr radially outward*)

27. A small sphere whose mass is 1×10^{-4} kg carries a charge of 3×10^{-10} coul and is attached to one end of a silk fibre 5 cm long. The other end of the fibre is attached to a large vertical conducting plate which has a surface charge of 25×10^{-6} C/m³. Find the angle which the fiber makes with the vertical. (41°)

28. Three infinite metal plates *A, B, C* are arranged parallel to each other. Plate *B* carries a uniform positive charge having a charge density σ on both the sides. Plate *A* and *C* carry unknown charges, but the arrangement produces zero electric field in the region left to plate *A* and right to plate *C*, *i.e.*, outside this arrangement. Find (*i*) the electric intensity at points between *A* and *B*. (*ii*) charges on the two surfaces of plate *A*. ($\sigma_{\text{right}} = \sigma$, $\sigma_{\text{left}} = 0$)

29. The surface density of charge on a conductor, in dry air, is 2.67×10^{-13} coul cm^{-2}. What is the force per unit area due to this charge. (4.03×10^{-7} *newton/m*²)

30. Find the greatest charge which can be carried by a metallic sphere of 10 cm diameter, if the dielectric strength of air is 2×10^4 volts/cm. (0.55 *micro coul*)

31. A bubble of radius *a* is formed from a soap solution having a surface tension *T*. If the bubble is raised to a potential *V* by being touched with a wire from a static machine, show that the radius of the bubble increases to *r*, given by

$P((r^3 - a^3) + 4T(r^2 - a^2) - \frac{1}{2}\varepsilon_0 V^2 r = 0$, where *P* is the atmospheric pressure.

32. An oil droplet of mass 3×10^{-11} gm and of radius 2×10^{-4} cm carries 10 excess electrons. What is its terminal velocity (*a*) when falling in a region in which there is no electric field, (*b*) when falling in an electric field 3×10^5 N/C, directed downward ? Given the viscosity of air = 1.80×10^{-7} N sec./m². (4.34×10^{-4} *m/sec.*, 2.74×10^{-4} *m/sec. upward*)

4

ELECTROSTATIC POTENTIAL

4.1 LINE INTEGRAL OF ELECTRIC FIELD INTENSITY

In mechanics, the line integral plays a very fundamental role in problems involving work. We know that if a force **F** acts on a body while the body suffers a displacement $d\mathbf{l}$, the work done

$$dW = \mathbf{F} \cdot d\mathbf{l} = F \cos \theta dl, \qquad \text{...(1)}$$

where θ is the angle between **F** and $d\mathbf{l}$. For a path of finite length,

$$W = \int_A^B \mathbf{F} \cdot d\mathbf{l} \qquad \text{...(2)}$$

Component forms in the various coordinate systems of the differential displacement vector are as follows :

$$d\mathbf{l} = dx\,\hat{x} + dy\,\hat{y} + dz\,\hat{z} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} \quad \text{(cartesian)}$$

$$= d\rho\hat{\rho} + r\,d\phi\hat{\phi} + dz\hat{z} \quad \text{(cylindrical)}$$

$$= dr\,\hat{r} + r\,d\theta\hat{\theta} + r\sin\theta\,d\phi\hat{\phi} \quad \text{(spherical)}$$

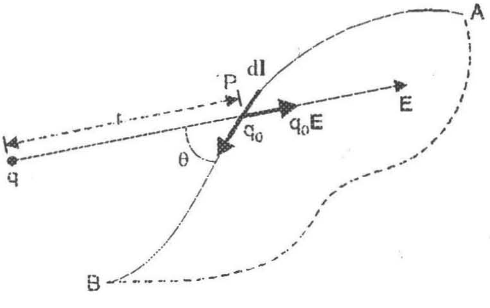


Fig. 4.1. Work done on a test charge.

We know that the earth's gravitational field, a vector field, is a *conservative field, i.e., the potential energy of a system is independent of the path taken and is defined by its positions only.* If we prove that the electrostatic field, *an electric field due to stationary charges,* is a conservative field, then we can say that the use of work and energy is as natural in *electrostatics* as in *mechanics.*

To show that the electrostatic field is a conservative field, let us consider the field due to a point charge q. If **E** be the electric field at any point P due to the point charge q, the force on the test charge q_0 (its presence should not affect the electric field **E**) will be $q_0\mathbf{E}$. Hence the external force $\mathbf{F} = -q_0\mathbf{E}$ is required to displace the test charge q_0 (Fig. 4.1).

External work necessary to move the test charge through a differential distance $d\mathbf{l}$ making an angle θ with the force $\mathbf{F}$ is $dW = \mathbf{F}\cdot d\mathbf{l}$. Hence the total external work to move the test charge q_0 from A to B against the electric field due to charge q is given by

$$W_{AB} = \int_A^B \mathbf{F}\cdot d\mathbf{l} = -q_0\int_A^B \mathbf{E}\cdot d\mathbf{l}.$$

In Fig. 4.1, $\mathbf{E}$ points to the right and $d\mathbf{l}$, which is always in the direction of motion, points to the left. Therefore

$$\mathbf{E}\cdot d\mathbf{l} = E\,dl\cos(180-\theta) = -E dl\cos\theta.$$

As we move a distance $d\mathbf{l}$ along the motion, we are moving in the direction of decreasing r, because r is measured from q which is assumed as origin, therefore $dl\cos\theta = -dr$ and

$$W_{AB} = -q_0\int_A^B \mathbf{E}\cdot d\mathbf{l} = -q_0\int_A^B E\,dr. \qquad ...(3)$$

If r_A and r_B be the radial distances of points A and B respectively from the charge q, then

$$W_{AB} = -q_0\int_{r_A}^{r_B}\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}dr = \frac{q_0 q}{4\pi\varepsilon_0}\left[\frac{1}{r_B}-\frac{1}{r_A}\right] \qquad ...(4)$$

It depends only on the radial positions of points A and B and is independent of path taken by the test charge between these points. The work done along a closed path (A to B along the continuous line + B to A along the dotted line) is thus zero. Therefore the field of a point charge is *conservative*. As the work done is a scalar quantity, total work done in moving a charge q_0 from point A to point B in an electric field due to a distribution of charges is just the sum of the work required to move this test charge when charges are considered individually. Thus the field due to arbitrary distribution of charge is also a *conservative field.*

An equivalent statement of the conservative nature of an electrostatic field is

$$\oint \mathbf{E}\cdot d\mathbf{l} = 0, \qquad ...(5)$$

or

$$\int_A^B \mathbf{E}\cdot d\mathbf{l} = -\int_B^A \mathbf{E}\cdot d\mathbf{l}. \qquad ...(6)$$

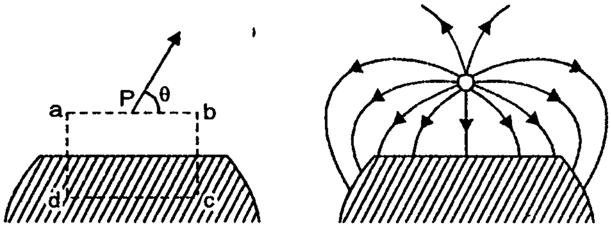
There are following three cases in which the methods of integral calculus are not necessary to evaluate the line integral of $\mathbf{E}$:

(*i*) If $\mathbf{E}$ is parallel and has the same magnitude at all points of the path of length $\mathbf{l}$, then E = constant and the integral = El.

(*ii*) If $\mathbf{E}$ is perpendicular to the path at all points on it, then the integral is zero, as $\mathbf{E}\cdot d\mathbf{l} = Edl\cos 90° = 0$.

(*iii*) If $\mathbf{E}$ is zero at all points of the path, the integral is zero.

To prove that the electric field lines are always perpendicular to the surface of the conductor under electrostatic conditions, let us consider a portion of the surface of the conductor shown in Fig. 4.2 (*a*). The line integral around the path *a b c d*



(*a*) (*b*)

Fig. 4.2. Electric field near a conductor.

$$\oint \mathbf{E}\cdot d\mathbf{l} = \int_a^b \mathbf{E}\cdot d\mathbf{l} + \int_b^c \mathbf{E}\cdot d\mathbf{l} + \int_c^d \mathbf{E}\cdot d\mathbf{l} + \int_d^a \mathbf{E}\cdot d\mathbf{l} = 0.$$

We know that the electric field inside the charged conductor is zero, hence the integral along *cd* is zero. The sides *ad* and *bc* are so short that they make no contribution to the line integral. Hence the integral around the closed path

$$\oint \mathbf{E}\cdot d\mathbf{l} = \int_a^b \mathbf{E}\cdot d\mathbf{l} = 0. \qquad ...(7)$$

This is only possible when cos θ = 0 or θ = 90° (or –90° if the surface has –ve charge, as **E** cannot be zero. Thus we can state that *electric lines of force meet the surface of a conductor at right angles provided the charges in the conductor are at rest as shown in Fig. 4.2 (b).*

4.2 ELECTROSTATIC POTENTIAL ENERGY AND POTENTIAL

We have seen that the electrostatic field is conservative. We generally associate the conservative force with the potential energy. Thus the external work to move a test charge q_0 from A to B against the electric field **E** is equal to the difference in potential energy of the charge q_0 at point B and at point A, in an electric field **E**, *i.e.*,

$$(P.E.)_B - (P.E.)_A = -q_0\int_A^B \mathbf{E}\cdot d\mathbf{l}.$$

The potential energy is a scalar quantity. It has the unit of work or energy. In SI units its unit is *joule.*

The difference in potential energy per unit charge

$$\underset{q_0 \to 0}{\text{Limit}} \frac{(P.E.)_B - (P.E.)_A}{q_0} = -\int_A^B \mathbf{E}\cdot d\mathbf{l}.$$

This quantity is known as the *potential difference* (*pd* or voltage) between points A and B and is generally denoted by V_{AB} or $V_B - V_A$. As the potential energy and charge are both scalars, the ratio of these, the potential difference, is hence the *scalar quantity.* Its unit in SI-system of units is joule/coulomb. This unit is called volt (*V*), named in honour of the Italian Scientist *Alessandro Volta,* and accepted as a practical unit of potential difference.

Thus the potential difference between two points A and B in an electric field **E** is given as

$$V_{AB} = V_B - V_A = -\int_A^B \mathbf{E}\cdot d\mathbf{l}. \qquad ...(8)$$

The potential difference between two points A and B in an electric field due to a point charge q is thus given as

$$V_B - V_A = -\int_A^B \mathbf{E}\cdot d\mathbf{l} = -\int_A^B (-Edl\cos\theta) = -\int_A^B Edr$$

$$= -\int_{r_A}^{r_B} \frac{q}{4\pi\varepsilon_0 r^2}dr = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]. \qquad ...(9)$$

In determining V_{AB}, A is the initial point while B is the final point. If V_{AB} is negative, there is a loss in potential energy in moving a test charge q_0 from A to B, or the work is being done by the field. If V_{AB} is positive, there is a gain in potential energy. Relation (9) also shows that V_{AB} is independent of the path taken, only depends on the initial and final positions.

Usually point A is chosen to be *at infinity for the reference point* for zero potential. This gives us

$$V_B = \frac{1}{4\pi\varepsilon_0}\frac{q}{r_B} = \frac{W}{q_0}, \qquad ...(10)$$

i.e., the potential V at a given point in an electric field is defined as the $1/q_0$ *times the work that an external agent must do in moving the test charge* q_0 *from infinity to this point against that electric field. Potential at any point thus obtained is called absolute potential.*

The potential is assumed to be zero at infinity as the height is assumed zero at sea level in mechanics. This reference point can be chosen at any other value. In many circuit problems the earth is taken as a reference of potential and assigned the zero value. Potential V near an isolated positive charge is positive because positive work is required to push a positive test charge in from infinity. Similarly, the potential V is negative due to an isolated negative charge as the positive test charge is attracted by the negative charge.

If the point charge q is located at point, whose position vector is $\mathbf{r}'$, the potential at a point $\mathbf{r}$ is given by

$$V(\mathbf{r}) = q/4\pi\varepsilon_0 \,|\, \mathbf{r} - \mathbf{r}' \,|.$$

For a distribution of charge, the superposition principle gives the expression for a potential at $\mathbf{r}$ due to n point charges $q_1, q_2, \ldots q_n$ located at points with position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots \mathbf{r}_n$, as

$$V(\mathbf{r}) = \frac{q_1}{4\pi\varepsilon_0 \,|\, \mathbf{r} - \mathbf{r}_1 \,|} + \frac{q_2}{4\pi\varepsilon_0 \,|\, \mathbf{r} - \mathbf{r}_2 \,|} + \ldots \frac{q_n}{4\pi\varepsilon_0 \,|\, \mathbf{r} - \mathbf{r}_n \,|}$$

$$= \frac{1}{4\pi\varepsilon_0} \sum \frac{q_i}{|\, \mathbf{r} - \mathbf{r}_i \,|} \qquad \text{(point charges)} \qquad \ldots(11)$$

For continuous charge distributions, we replace q_i with charge elements λdl, σdS or ρdv and summation becomes an integration. Thus the potential at $\mathbf{r}$ becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\lambda(\mathbf{r}')\, dl'}{|\, \mathbf{r} - \mathbf{r}' \,|} \qquad \text{(line charge)}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_S \frac{\sigma(\mathbf{r}')\, dS'}{|\, \mathbf{r} - \mathbf{r}' \,|} \qquad \text{(surface charge)}$$

$$= \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho(\mathbf{r}')\, dv'}{|\, \mathbf{r} - \mathbf{r} \,|} \qquad \text{(volume charge)} \qquad \ldots(12)$$

Here λ, σ and ρ are respectively linear, surface and volume charge densities at $\mathbf{r}'$.

4.3 EQUIPOTENTIAL SURFACE

In any region in which $\mathbf{E} = 0$ at all points, such as the region very far from all charges or the *interior of a charged conductor* in which all the charges are at rest, the line integral of $\mathbf{E}$ is zero along any path (*i.e.*, $\oint \mathbf{E} \cdot \mathbf{dl} = 0$). It means that the potential difference between any two points inside the charged conductor is zero, *i.e.*, *all points in the conductor are at the same potential and the interior of a charged conductor is an equipotential volume.*

If we consider an arc of a circle, at the center of which a point charge is placed, the electric field $\mathbf{E}$ will be $\perp$ to this arc and the work done in moving the test charge along this circle is zero as $\oint \mathbf{E} \cdot \mathbf{dl} = 0$. Hence each point on this circle is at the same potential and this circle constitutes *equipotential line.*

If the surface is in such a way that it is everywhere at right angles to an electric field (surface perpendicular to electric lines of forces) then any path lying in this surface is at right angles to the field and the line integral along this line is zero. Hence *all the points on this surface are at same potential and the surface is known as an equipotential surface.*

Equipotential surface in electrostatics act as wavefronts in optics. The wavefront is the locus of points which are in the same phase. The wavefronts are the planes perpendicular to the direction of rays, similarly the equipotential surfaces are the planes perpendicular to the lines of force. The equipotential surface is plane, spherical, or cylindrical corresponding to the charges at infinity (or infinite plane sheet of charge), point charge (or spherical charge) or the line charge (cylinder of charge) respectively, similar to the wavefronts due to source at infinity, point source or slit source, as shown in Fig. 4.3.

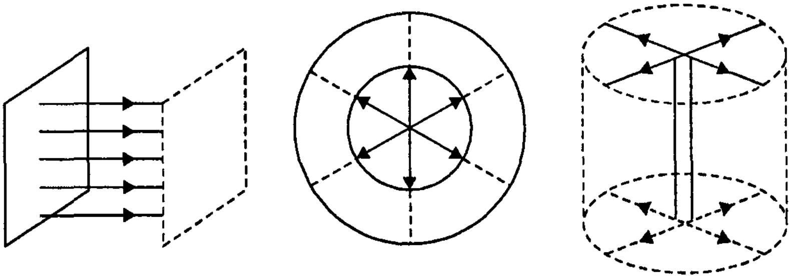

(a) (b) (c)

Fig. 4.3. Equipotential surfaces (dotted).

4.4 POTENTIAL GRADIENT

We have determined how to calculate electrostatic potential V from an electric field **E**. Let us now consider how to calculate **E** if V is known. If we consider two points A and B at a distance Δl apart from each other in an electric field **E**, set up by any arrangement of charges. The work done by the external agency exerting the force **F** ($= -q_0$ **E**) in moving a test charge q_0 along the path $\Delta \mathbf{l}$ is,

$$\Delta W = \mathbf{F} \cdot \Delta \mathbf{l} = -q_0 \mathbf{E} \cdot \Delta \mathbf{l}.$$

$$\therefore \qquad \Delta W / q_0 = \Delta V = -\mathbf{E} \cdot \Delta \mathbf{l} = -E \Delta l \cos (\pi - \theta) = E \Delta l \cos \theta.$$

As the component $E \cos \theta$ is the component of **E** in the direction of $-\mathbf{l}$, hence $-E \cos \theta$, which we call E_l, is the component of **E** along **l**. Thus in the differential limit the above equation becomes

$$dV = - E_l dl \text{ or } E_l = -dV/dl, \qquad \text{...(13)}$$

Therefore the component of electric intensity in any direction **l** is equal to the negative rate of change of potential with distance in the direction of $d\mathbf{l}$. Its unit is thus *volt/meter*. There will be one direction **l** for which dV/dl is a maximum. E_l will also be maximum for this direction. This maximum value of E_l will in fact be E itself. Thus

$$E = (E_l)_{\max} = -(dV/dl)_{\max}. \qquad \text{...(14)}$$

The maximum rate of change of potential with distance at a given point is known as the potential gradient at that point. The direction of **l** for which dV/dl has its value maximum is always at right angles to the equipotential surface and is along **E**.

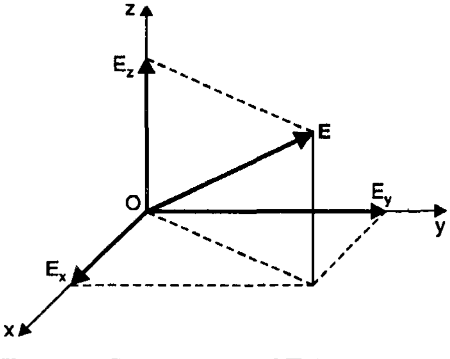


Fig. 4.4. Components of E in cartesian coordinates.

We can find the components of E at any point in the directions of x, y and z-axes with the help of the potential gradients due to this field at the point along these axes. Thus

$$E_x = -\partial V / \partial x, \; E_y = -\partial V / \partial y \;\text{ and }\; E_z = -\partial V / \partial z. \qquad \text{...(15)}$$

In the case of polar co-ordinates, the components are radial along radius vector and azimuthal perpendicular to the radius vector. These can be calculated as

$$E_r = -\left(\frac{\partial V}{\partial r}\right)_\theta \quad \text{and} \quad E_\theta = \frac{1}{r}\left(\frac{\partial V}{\partial \theta}\right)_r \qquad \text{...(16)}$$

We know that $\mathbf{E} = E_x \mathbf{i} + E_y \mathbf{j} + E_z \mathbf{k}$.

$$\therefore \qquad \mathbf{E} = -\frac{\partial V}{\partial x}\mathbf{i} - \frac{\partial V}{\partial y}\mathbf{j} - \frac{\partial V}{\partial z}\mathbf{k} = -\left[\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right] V$$

$$= -\nabla V \equiv -\text{grad } V. \qquad \text{...(17)}$$

$\mathbf{E} = -\nabla V = 0$ inside the conductor implies that the potential is the same everywhere in the conductor.

The relation (17) is very useful in many problems in which it is easier to calculate V rather that **E**. From vector calculus, we thus get

$$\text{curl}\,\mathbf{E} = \nabla \times \mathbf{E} = \nabla \times (-\nabla V) = 0. \qquad \text{...(18)}$$

Thus we see that the electrostatic field is curl free, or an *irrotational field.*

Electrostatic field due to a group of charges as a gradient of a scalar : The electric field at a point r due to a group of point charges is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\,\mathbf{r} - \mathbf{r}_i\,|^3}, \qquad \text{...(19)}$$

$$\therefore \qquad \text{curl}\,\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i \;\text{curl}\left[\frac{\mathbf{r} - \mathbf{r}_i}{|\,\mathbf{r} - \mathbf{r}_i\,|^3}\right].$$

Using the vector identity curl (φ**A**) = φ curl **A** + (grad φ) × **A**, we get

$$\text{curl}\,\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i \left[\frac{1}{|\,\mathbf{r} - \mathbf{r}_i\,|^3}\nabla \times (\mathbf{r} - \mathbf{r}_i) + \nabla\left(\frac{1}{|\,\mathbf{r} - \mathbf{r}_i\,|^3}\right) \times (\mathbf{r} - \mathbf{r}_i)\right]$$

Since $\nabla \times (\mathbf{r} - \mathbf{r}_i) = 0$ and $\nabla\,|\,\mathbf{r} - \mathbf{r}_i\,|^{-3} = -3(\mathbf{r} - \mathbf{r}_i)/|\,\mathbf{r} - \mathbf{r}_i\,|^{-5}$.

$$\therefore \qquad \text{curl}\,\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i \left[0 - \frac{3(\mathbf{r} - \mathbf{r}_i)}{|\,\mathbf{r} - \mathbf{r}_i\,|^5} \times (\mathbf{r} - \mathbf{r}_i)\right]$$

$$= 0 \qquad [\because (\mathbf{r} - \mathbf{r}_i) \times (\mathbf{r} - \mathbf{r}_i) = 0]$$

Since the curl of a gradient of a scalar function is always zero, hence $\mathbf{E}(r) = -\,\text{grad}\,\varphi(r)$. Thus the electrostatic field is irrotational and may be expressed as a gradient of a scalar function φ, which is *known as electrostatic potential.* Since $\nabla\,|\,\mathbf{r} - \mathbf{r}_i\,|^{-1} = -\,|\,\mathbf{r} - \mathbf{r}_i\,|^{-3}\,(\mathbf{r} - \mathbf{r}_i)$. Therefore Eq. (19) may be written as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n} q_i \left[-\nabla\left(\frac{1}{|\,\mathbf{r} - \mathbf{r}_i\,|}\right)\right] = -\nabla\left[\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\,\mathbf{r} - \mathbf{r}_i\,|}\right] = -\nabla\varphi(\mathbf{r}),$$

$$\therefore \qquad \varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\frac{q_i}{|\,\mathbf{r} - \mathbf{r}_i\,|}. \qquad \text{...(20)}$$

It is also represented by $V(r)$.

If the charge distribution is assumed as continuous, then Eq. (20) becomes

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}. \qquad \text{...(21)}$$

The electrostatic potential at a point **r** due to a group of charges together with a continuous charge distribution with *a* volume charge density ρ (**r**′) in a volume v and surface charge density σ (**r**′) on the surface S that bounds a volume v is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\sum_{i=1}^{n}\frac{q_i}{|\,\mathbf{r} - \mathbf{r}_i\,|} + \int_V \frac{\rho(\mathbf{r}')dv'}{|\,\mathbf{r} - \mathbf{r}'\,|} + \int_S \frac{\sigma(\mathbf{r}')dS'}{|\,\mathbf{r} - \mathbf{r}'\,|}\right] \qquad \text{...(22)}$$

4.5 CALCULATION FOR ELECTROSTATIC POTENTIAL

(a) Potential due to a charged sphere : Let us consider a *conducting sphere* of radius R, having a charge q, uniformly distributed over its surface, so that the surface density of charge $\sigma = q/4\pi R^2$.

Cut the charged sphere by two closed parallel planes *AB* and *CD*, so as to form an annular charged ring *ACDB*. From Fig. 4.5. it is clear that the area of this annular ring

$$= 2\pi\,(R \sin\theta)\,(R\,d\theta) = 2\pi\,R^2 \sin\theta\,d\theta.$$

$\therefore$ Charge on this ring $\Delta q = 2\pi R^2 \sin\theta\, d\theta \times \sigma = \frac{1}{2}\, q \sin\theta\, d\theta$.

As the ring is of small width, we can assume each point of this ring as equidistant from any point on its axis. Hence the potential at any point P at a distance r from the centre of the sphere and a distance x from the ring is given by

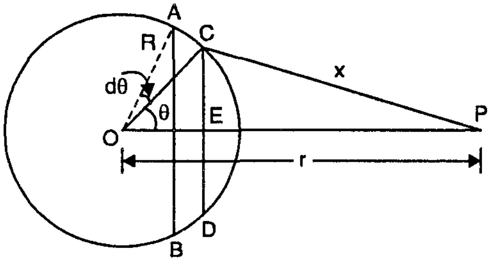


Fig. 4.5. A charged sphere.

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{\Delta q}{x}, = \frac{1}{4\pi\varepsilon_0}\frac{q \sin\theta\, d\theta}{2x}.$$

Here x and θ both are variables, let us convert them in one only. From $\Delta\, COP$, $x^2 = R^2 + r^2 - 2\,Rr\cos\theta$. Therefore

$2x dx = 2Rr \sin\theta\, d\theta$ or $\sin\theta\, d\theta = x\, dx/Rr$.

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{q dx}{2Rr}.$$

Hence the potential at this point P (outside the sphere)

$$V_0 = \frac{1}{4\pi\varepsilon_0}\int_{r-R}^{r+R}\frac{q}{2Rr}dx = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}. \qquad \text{...(23)}$$

Thus the potential at any point outside the uniformly charged sphere is the same as if whole of the charge be concentrated at the center of the sphere.

Potential at the point P on the sphere

$$V_s = \frac{1}{4\pi\varepsilon_0}\int_{R}^{+R}\frac{q}{2R^2}dx = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}. \qquad \text{...(24)}$$

Potential at the point P inside the sphere

$$V_i = \frac{1}{4\pi\varepsilon_0}\int_{R-r}^{R+r}\frac{q}{2Rr}dx = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}. \qquad \text{...(25)}$$

Thus the potential at any point inside the charged sphere is the same as on the surface of the sphere and the charged conductor acts as an equipotential volume.

We know that $E = -dV/dr$, hence electric intensity inside the charged sphere

$$E_i = -\frac{d}{dr}\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{R}\right] = 0 \qquad \text{...(26)}$$

Electric intensity outside the charged sphere

$$E_0 = -\frac{d}{dr}\left[\frac{1}{4\pi\varepsilon_0}\frac{q}{r}\right] = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2} \qquad \text{...(27)}$$

It will be along the direction of **r**. Hence

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\hat{r}}{r^2}. \qquad \text{...(28)}$$

At the surface $r = R$, we thus get

$$E_s = \frac{1}{4\pi\varepsilon_0}\frac{q}{R^2}. \qquad \text{...(29)}$$

Thus we see that electric intensity is zero inside the charged sphere, maximum at the boundary and decreases outside the sphere, as shown in Fig. 4.6.

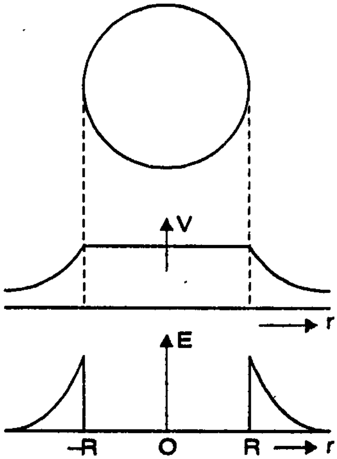


Fig. 4.6. Variation of V and E with r due to charged sphere.

Non-conducting sphere : If the given sphere is non-conducting and the charge is assumed to have been spread uniformly throughout the volume of the sphere. In such a case it is easy to calculate electric field by using Gauss's law and then to find the potential with the help of the relation (8). The electric intensity outside the sphere ($r > R$) is given as

$$\mathbf{E}_0 = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}.$$

Therefore the potential (V) outside the surface of the sphere

$$V_0 = -\int_\infty^r \mathbf{E}\cdot d\mathbf{r} = -\int_\infty^r \frac{q}{4\pi\varepsilon_0 r^2}\hat{r}\cdot d\mathbf{r} = -\int_\infty^r \frac{q\,dr}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r}. \qquad ...(30)$$

At the surface of the sphere (*i.e.*, $r = R$), we get

$$V_S = q\ /4\pi\varepsilon_0 R. \qquad ...(31)$$

The electric intensity inside the sphere $\mathbf{E}_i = \frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\hat{r}.$

The potential inside the surface of the sphere

$$\begin{aligned} V_i &= -\int_\infty^r \mathbf{E}\cdot d\mathbf{r} = -\int_\infty^R \mathbf{E}_0\cdot d\mathbf{r} - \int_R^r \mathbf{E}_i\cdot d\mathbf{r} \\ &= -\int_\infty^R \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}\cdot d\mathbf{r} - \int_R^r \frac{1}{4\pi\varepsilon_0}\frac{q\,r}{R^3}\hat{r}\cdot d\mathbf{r} \\ &= \frac{q}{4\pi\varepsilon_0 R} - \frac{q}{8\pi\varepsilon_0 R^3}[r^2 - R^2] \end{aligned}$$

$$\therefore \qquad V_i(r) = \frac{q}{4\pi\varepsilon_0}\left[\frac{3}{2R} - \frac{r^2}{2R^3}\right]. \qquad ...(32)$$

(*b*) Potential due to an electric dipole : Let us consider dipole AB, two equal charges q of opposite signs and separated by a small distance $2a$, as shown in Fig. 4.7.

We want to calculate an electric potential at a point $P(r, \theta)$, which is not too close to the dipole. Using the expression for the potential due to point charges, we get

$$V_P = \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{PB} - \frac{q}{PA}\right].$$

Assuming that $r >> a$ we can write

$$PB = r - a\cos\theta \text{ and } PA = r + a\cos\theta.$$

$$\begin{aligned} \therefore \qquad V_P &= \frac{1}{4\pi\varepsilon_0}\left[\frac{q}{r - a\cos\theta} - \frac{q}{r + a\cos\theta}\right] \\ &= \frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{r^2 - a^2\cos^2\theta} \end{aligned}$$

As $r >> a$, hence $a^2\cos^2\theta$ can be neglected in comparison to r^2.

$$\therefore\ V_P = \frac{1}{4\pi\varepsilon_0}\frac{2qa\cos\theta}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2} = \frac{\mathbf{p.r}}{4\pi\varepsilon_0 r^3}. \qquad ...(33)$$

$$= -\frac{1}{4\pi\varepsilon_0}\mathbf{p}\cdot\nabla\left(\frac{1}{r}\right) = -\frac{1}{4\pi\varepsilon_0}\mathbf{p}\cdot\text{grad}\left(\frac{1}{r}\right), \qquad ...(34)$$

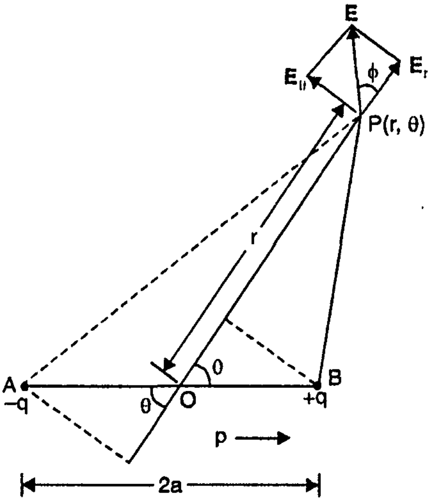


Fig. 4.7. An electric dipole.

where **p** is the dipole moment of the electric dipole. It is directed from the negative charge to positive charge and having magnitude $2qa$.

Relation (34) shows that the potential V is zero at all the points of which $\theta = 90°$, *i.e., no work is required to bring a test charge in from infinity along the perpendicular bisector of the dipole.*

In a general case, shown in Fig. 3.4, let $\mathbf{r}'$ and $\mathbf{r}' + \mathbf{l}$ be the position vectors of the charges $-q$ and $+q$ respectively with respect to the origin O. l is the distance between the charges or the length of the dipole. The electric potential at an arbitrary point P with position vector $\mathbf{r}$ may be written as

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'-\mathbf{l}|} - \frac{q}{4\pi\varepsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r}'|}. \qquad ...(35)$$

Since $\mathbf{l}$ *is small compared with* $\mathbf{r}-\mathbf{r}'$, hence we may expand the denominator of the first term of Eq. (35) as

$$|\mathbf{r}-\mathbf{r}'-\mathbf{l}|^{-1} = |(\mathbf{r}-\mathbf{r}'-\mathbf{l})^2|^{-1/2} = [|\mathbf{r}-\mathbf{r}'|^2 - 2(\mathbf{r}-\mathbf{r}').\mathbf{l} + l^2]^{-1/2}$$

$$= |\mathbf{r}-\mathbf{r}'|^{-1}\left[1 - \frac{2(\mathbf{r}-\mathbf{r}').\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^2} + \frac{l^2}{|\mathbf{r}-\mathbf{r}'|^2}\right]^{-1/2}$$

Since $l \ll |\mathbf{r}-\mathbf{r}'|$, therefore the last term may be neglected. By the Binomial theorem, we thus have

$$|\mathbf{r}-\mathbf{r}'-\mathbf{l}|^{-1} = |\mathbf{r}-\mathbf{r}'|^{-1}\left[1 + \frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^2} + \ldots\right]. \qquad ...(36)$$

The combination of Eqs (35) and (36) gives

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0}\cdot\frac{(\mathbf{r}-\mathbf{r}')\cdot\mathbf{l}}{|\mathbf{r}-\mathbf{r}'|^3}$$

Here higher order terms are neglected as they are the higher power terms of l. For $l \to 0$ and $q \to \infty$, we have $q\mathbf{l} = \mathbf{p}$, the dipole moment of the electric dipole.

$$\therefore \qquad V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3} = -\frac{1}{4\pi\varepsilon_0}\mathbf{p}\cdot\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \qquad ...(37)$$

If we transform the point $\mathbf{r}'$ to the origin, then we have the same result as obtained earlier, Eq. (33).

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}.\mathbf{r}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}, \qquad ...(33)$$

where θ is the angle between $\mathbf{p}$ and $\mathbf{r}$.

We cannot obtain $\mathbf{E}$ from the gradient of V at once. The components of $\mathbf{E}$ in two perpendicular directions are E_r (the radial component along the direction of r) and E_θ (the transverse or azimuthal component perpendicular to the direction of r), given by the relations

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left[\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right] = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}$$

and

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}\right] = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}.$$

$$\therefore \qquad E = \sqrt{\left(E_r^2 + E_\theta^2\right)} = \frac{1}{4\pi\varepsilon_0}\frac{p}{r^3}\sqrt{\left(1 + 3\cos^2\theta\right)}. \qquad ...(38)$$

The angle ϕ which the vector $\mathbf{E}$ makes with the radius vector $\mathbf{r}$ is thus given by

$$\phi = \tan^{-1}(E_\theta/E_r) = \tan^{-1}\left(\tfrac{1}{2}\tan\theta\right). \qquad ...(39)$$

If $\theta = 0°$, then electric field intensity $\mathbf{E}$ is in the direction of $\mathbf{p}$ and having magnitude $(1/4\pi\varepsilon_0)\,2p/r^3$. If $\theta = 90°$, then electric field intensity $\mathbf{E}$ is in the direction of $-\mathbf{p}$ and having magnitude $(1/4\pi\varepsilon_0)p/r^3$.

(c) Potential due to an infinite line charge : Let us consider an infinite line charge of constant linear charge density λ, Fig. (4.8). We have proved that the electric field strength at any point P in the space at a distance y from the line is in the direction perpendicular to the direction of this charge and is given by the relation

$$E = \lambda/2\pi\varepsilon_0 y. \qquad \text{...(40)}$$

The electric potential at this point is thus given by

$$V_y = -\int_\infty^y \mathbf{E}\cdot\mathbf{dy} = -\int_\infty^y \frac{\lambda}{2\pi\varepsilon_0 y}dy = \frac{\lambda}{2\pi\varepsilon_0}\left[\log_e y\right]_y^\infty$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\log_e \infty - \log_e y\right] = \infty \qquad \text{...(41)}$$

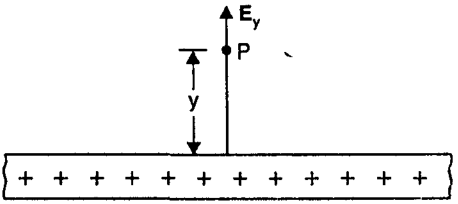


Fig. 4.8. Infinite line charge.

Thus the electric potential due to an infinite line charge is infinity everywhere. This is correct. In practical problems we are interested in potential difference between positions with finite separation and never in absolute potentials. In this case our choice of infinity for the point of zero potential was arbitrary. We may, however, set $V = 0$ at some arbitrary distance y_0 as

$$V_y = \frac{\lambda}{2\pi\varepsilon_0}\left[\log_e y\right]_y^{y_0} = \frac{\lambda}{2\pi\varepsilon_0}\log_e y_0 - \frac{\lambda}{2\pi\varepsilon_0}\log_e y = -\frac{\lambda}{2\pi\varepsilon_0}\log_e y + C. \qquad \text{...(42)}$$

Here $(\lambda/2\pi\varepsilon_0)\log_e y_0$ has been replaced by a constant C, since y_0 is constant. Relation (42) may be used for the calculation of an electric field strength E at any point due to infinite line charge.

In the previous chapter we have proved that an electric field due to a charged cylinder of infinite length at any point outside to it is equal to that due to infinite line charge. Proceeding similarly we may therefore obtain the *similar expression for the potential due to the charged cylinder of infinite length.*

(*d*) Potential due to an uniformly charged disc : Let us find the electric potential at any point P on the axis of a uniformly charged disc of radius R having surface charged density σ, Fig. 4.9.

Let us divide this disc into large number of flat circular strips and consider one strip of radius x and of width Δx. Since Δx is very small, each point of this strip can be assumed to be at equal distance from the point P. Potential at point P due to this circular strip is

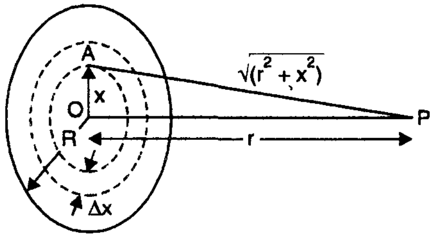


Fig. 4.9. A charged disc.

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{\Delta q}{AP} = \frac{1}{4\pi\varepsilon_0}\frac{2\pi x\Delta x\sigma}{\sqrt{(r^2+x^2)}}. \qquad \text{...(43)}$$

Thus the potential due to the charge on the whole disc

$$V = \int dV = \frac{\sigma}{2\varepsilon_0}\int_0^R \frac{x\,dx}{\sqrt{(r^2+x^2)}}$$

$$= \frac{\sigma}{2\varepsilon_0}\left[\sqrt{(R^2+r^2)} - r\right] \quad r > 0. \qquad \text{...(44)}$$

The potential on the axis point P along the negative direction of r is given by

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+r^2} + r\right] \qquad r < 0. \qquad \text{...(45)}$$

At the central point O of the disc, the potential

$$V = \sigma R/2\varepsilon_0. \qquad \text{...(46)}$$

Using Binomial theorem for $r >> R$, relation (44) may be written

$$V = \frac{\sigma}{2\varepsilon_0}\left[r + \frac{R^2}{2r} - r\right] = \frac{\sigma R^2}{4\varepsilon_0 r} = \frac{\pi R^2\sigma}{4\pi\varepsilon_0 r} = \frac{q}{4\pi\varepsilon_0 r}, \qquad \text{...(47)}$$

where q is the charge on the whole disc.

This is the relation as obtained due to a point charge. Thus at far away points the distribution of charge becomes insignificant. It is very difficult to calculate the potential at the points other than on the axis. However *potential on the edge of the disc can be calculated as under :*

To calculate the potential at point P on the edge (Fig. 4.10), let us divide the disc in large number of rings with P as centre. The potential due to one segment between radii r and $r + \delta r$ is given as

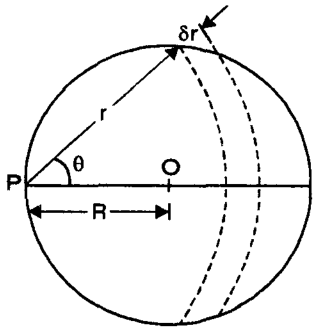


Fig. 4.10. A charged disc.

$$\Delta V = 2\ r\theta\delta r\sigma/4\pi\varepsilon_0 r.$$

∴ Total potential due to the entire disc

$$V = \frac{\sigma}{2\pi\varepsilon_0}\int_0^{2R}\theta dr.$$

As $r = 2R\cos\theta$ and $dr = -2R\sin\theta d\theta$,

$$\therefore \qquad V = -\frac{\sigma}{2\pi\varepsilon_0}\int_{\pi/2}^{0}\theta 2R\sin\theta d\theta$$

$$= \frac{\sigma R}{\pi\varepsilon_0}\int_0^{\pi/2}\theta\sin\theta d\theta = \frac{\sigma R}{\pi\varepsilon_0}. \qquad \text{...(47)}$$

Comparing with Eq. (46) we see that the potential at the centre of the disc is greater than the potential at the edge. Thus a *uniformly charged disc is not an equipotential surface* and thus we may expect the electric field component in the plane of the disc in radially outward direction. The case will be different if the disc is conducting. As under static conditions a conductor behaves as an equipotential surface, hence the charge on the conductor can not be distributed uniformly. *A conducting disc must therefore have a continuous decrease in charge density from the edge to the centre, maximum at the edge and minimum at the centre.*

The **electric field** at the axial point P may be obtained by the Eq. (44), as

$$E(r_+) = -\frac{\partial V}{\partial r} = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{r}{(R^2 + r^2)^{1/2}}\right] \quad r > 0. \qquad \text{...(49)}$$

For points on the left side of the disc, *i.e.*, $r < 0$, we get

$$E(r_-) = \frac{\sigma}{2\varepsilon_0}\left[1 + \frac{r}{(R^2 + r^2)^{1/2}}\right] \qquad r < 0. \qquad \text{...(50)}$$

At the centre of the disc, *i.e.*, $r = 0$, we get

$$E_0 = \sigma/2\varepsilon_0. \qquad \text{...(51)}$$

For very large distances $r >> R$, Eq. (49) gives

$$E = \frac{\sigma}{2\varepsilon_0}\left[1 - \frac{r}{r(1 + R^2/r^2)^{1/2}}\right] = \frac{\sigma}{2\varepsilon_0}\left[1 - \left(1 + \frac{R^2}{r^2}\right)^{-1/2}\right]$$

$$= \frac{\sigma R^2}{4\varepsilon_0 r^2} = \frac{\pi R^2\sigma}{4\pi\varepsilon_0 r^2} = \frac{q}{4\pi\varepsilon_0 r^2}, \qquad \text{...(52)}$$

where q be the total charge on the disc. Thus the *disc acts as a point source at very large distances.* Eqs. (49) and (50) show that the disc appears as infinite sheet of charge if viewed from points very closed to it.

4.6 THE DIPOLE APPROXIMATION FOR AN ARBITRARY DISTRIBUTION OF CHARGES

Let us calculate the potential at a distance large compared with the size of the set of charges. Let the charge q_i be situated at a distance r_i from an origin O chosen somewhere in the middle of the group of charges. The potential at a point P at a large distance r from the origin due to whole charge is given by

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{|\mathbf{r} - \mathbf{r}_i|}, \qquad \text{...(53)}$$

where $|\mathbf{r} - \mathbf{r}_i|$ is the distance from the charge q_i to the point P.

As the point P is at a large distance, hence $|\mathbf{r}-\mathbf{r}_i| \simeq r$. Thus

$$V = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_i q_i = \frac{Q}{4\pi\varepsilon_0 r}, \qquad \dots(54)$$

where Q is the total charge. Thus we see that the potential at a large distance due to a set of charges (pump of charge) is same as if the whole charge was concentrated at one point, the origin.

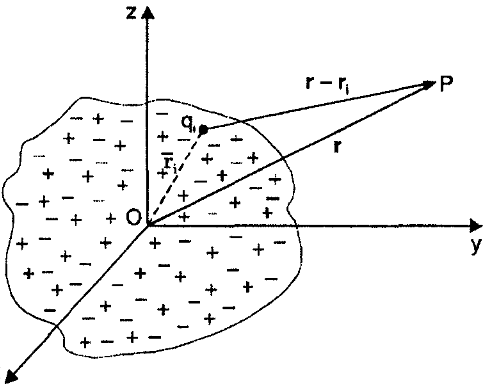


Fig. 4.11. An arbitrary distribution of charges.

Let us calculate the effect of the separate charges, taking most accurate expression for $|\mathbf{r}-\mathbf{r}_i|$. For the point P at a large distance, we may assume

$$|\mathbf{r}-\mathbf{r}_i| = r - \mathbf{r}_i.\hat{r},$$

where $\hat{r}$ is the unit vector in the direction of $\mathbf{r}$. Since $r_i \ll r$, the above relation may be written to our approximation as

$$\frac{1}{|\mathbf{r}-\mathbf{r}_i|} = \frac{1}{r}\left[1-\frac{\mathbf{r}_i\cdot\hat{r}}{r}\right]^{-1} = \frac{1}{r}\left[1+\frac{\mathbf{r}_i\cdot\hat{r}}{r}+\dots\right].$$

Hence
$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r}\left(1+\frac{\mathbf{r}_i\cdot\hat{r}}{r}+\dots\right) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_i q_i + \frac{1}{4\pi\varepsilon_0}\sum_i q_i\frac{\mathbf{r}_i\cdot\hat{r}}{r^2}+\dots$$

or
$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{r}+\sum_i q_i\frac{\mathbf{r}_i\cdot\hat{r}}{r^2}+\dots\right) \qquad \dots(55)$$

Here the dots indicate the terms of higher order in r_i/r and are the successive terms in the expansion of $1/|\mathbf{r}-\mathbf{r}_i|$. The first term is same as Eq. (54). It is the potential at very large distance, as the higher order terms are negligible and is known as *monopole* contribution. For molecules such as *HCl*, CO, CH_3Cl, $HCCl_3$, etc., the total charge is zero and the second term is dominant as the higher order terms are again negligible. If we define $\Sigma q_i\mathbf{r}_i = \mathbf{p}$, then we get

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}. \qquad \dots(56)$$

The quantity $\mathbf{p}$ is called the dipole moment of the distribution of charge and the potential V, the *dipole potential*. Dipole potential decreases as $1/r^2$ and varies as $\cos\theta$. Dipole fields are very important for the studies of molecular and atomic properties. For many molecules, such as CO_2, the dipole moment vanishes because of the symmetry of molecules, for them we take next term, which varies as $1/r^3$ and is called *quadrupole potential.*

4.7 MULTIPOLE EXPANSION FOR THE CONTINUOUS CHARGE DISTRIBUTION

We know that a point charge is known as *electric monopole*. If a monopole is displaced through a very small distance and the second monopole of equal and opposite charge is placed at the original position of the first monopole, an *electric dipole* is produced. Similarly an *electric quadrupole* is produced by displacing a dipole through a very small distance and placing the another dipole of same magnitude and opposite sign at the original position of the former dipole. The *octupole* may similarly be obtained by two quadrupoles. On continuing the process multipoles of higher order can be obtained. When all the point charges of the multipole lie on the straight line, the multipole is called a *linear* or an *axial multipole.*

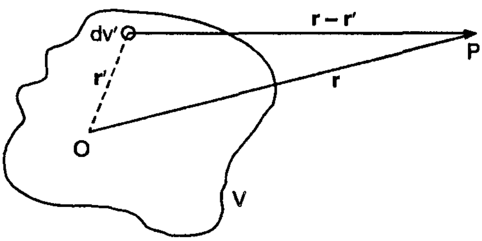


Fig. 4.12. Continuous charge distribution.

Let us find the electrostatic potential due to the continuous charge distribution at a point *P*, at a large distance *r* from the origin *O*. Consider an volume element dv' at $\mathbf{r}'$ in the charge distribution. If $\rho(\mathbf{r}')$ be the volume charge density at this point, the electrostatic potential at point *P* due to this element is given as

$$\Delta V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\rho(\mathbf{r}')\,dv'}{|\mathbf{r}-\mathbf{r}'|}.$$

The potential due to entire charge distribution

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')\,dv'}{|\mathbf{r}-\mathbf{r}'|}. \qquad \text{...(57)}$$

Since $r' << r$, the quantity $|\mathbf{r}-\mathbf{r}'|$ may be expanded as :

$$|\mathbf{r}-\mathbf{r}'|^{-1} = (r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2)^{-1/2} = \frac{1}{r}\left[1 - \frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right]^{-1/2}$$

$$= \frac{1}{r}\left[1 - \frac{1}{2}\left(\frac{-2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right) + \frac{3}{8}\left(\frac{-2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right)^2 + \dots\right]$$

Since $r' << r$, $(r'/r)^2$ may therefore be neglected in comparison to $2\mathbf{r}\cdot\mathbf{r}'/r^2$. On neglecting higher powers of r', we get

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = \left[\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{1}{2}\left(\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right) + \dots\right]$$

Thus Eq. (57) may be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int\left[\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{1}{2}\left(\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right) + \dots\right]\rho(\mathbf{r}')\,dv'. \qquad \text{...(58)}$$

The third term in the bracket on the right side may be written as

$$\frac{1}{2}\left(\frac{3(\mathbf{r}.\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right) = \frac{1}{2r^5}\left\{3(\mathbf{r}\cdot\mathbf{r}')^2 - r'^2r^2\right\} = \frac{1}{2r^5}\left\{3\left(\sum_{i=1}^{3} x_i x_i'\right)^2 - r'^2\sum_{j=1}^{3} x_j^2\right\}$$

$$= \frac{1}{2r^5}\left\{3\left(\sum_{i=1}^{3} x_i x_i' \sum_{j=1}^{3} x_j x_j'\right) - r'^2\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \delta_{ij}\right\}$$

$$= \frac{1}{2r^5}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j\,(3x_i'x_j' - r'^2\delta_{ij}),$$

where δ_{ij} is Kronecker delta defined as $\delta_{ij} = 1$ $\quad i = j$

$= 0 \quad i \neq j$

Thus (58) may be written as

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\int_V \rho(\mathbf{r}')\,dv' + \frac{1}{4\pi\varepsilon_0}\frac{\mathbf{r}}{r^3}\cdot\int \mathbf{r}'\rho(\mathbf{r}')\,dv.$$

$$+ \frac{1}{4\pi\varepsilon_0}\frac{1}{2r^5}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j \int_V (3x_i'x_j' - r'^2\delta_{ij})\,\rho(\mathbf{r}')\,dv' + \dots$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{r} + \frac{1}{4\pi\varepsilon_0}\frac{(\mathbf{r}\cdot\mathbf{p})}{r^3} + \frac{1}{4\pi\varepsilon_0}\frac{1}{r^5}\sum_{i=1}^{3}\sum_{j=1}^{3} x_i x_j.\,Q_{ij} + \dots \qquad \text{...(59)}$$

where q is the total charge or monopole moment, p the dipole moment of the charge distribution and Q_{ij} the quadrupole moment tensor, given by the relations

$$q = \int_V \rho(\mathbf{r}')\,dv',\ \mathbf{p} = \int_V \mathbf{r}'\rho(\mathbf{r}')\ dv' \text{ and and } Q_{ij} = \frac{1}{2}\int_V (3x_i'x_j' - r'^2\delta_{ij})\rho\,(\mathbf{r}')\,dv'. \qquad \text{...(60)}$$

Thus we see that the potential at an external point due to a charge distribution may be expressed as the sum of the potentials due to monopole, dipole, quadrupole, octupole, etc. The quadrupole moment Q may also be defined as

$$Q = \frac{1}{2}\int_V r'^2(3\cos^2\theta - 1)\,\rho(\mathbf{r}')\,dv'. \qquad \text{...(61)}$$

The above discussion indicates that an arbitrary charge distribution is equivalent to a combination of a point charge (monopole), a dipole, a quadrupole, an octupole and so forth as shown in Fig. 4.13.

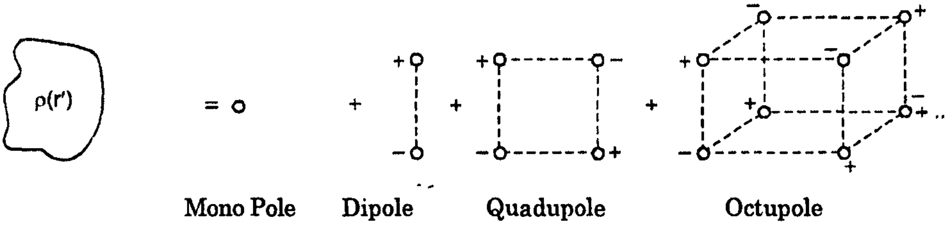


Fig. 4.13. Charge distribution equivalent to its various multipoles.

4.8 ELECTROSTATIC ENERGY OF DISTRIBUTED CHARGES

We know that when we raise a stone from the earth's surface the work we do against gravitational attraction is stored in the form of potential energy in the system (earth + stone). This energy converts in the form of kinetic energy when stone falls and finally transferred into heat energy in the system, earth + stone.

The potential due to the charge q_1 at a point P at a distance r is given by

$$V = q_1/4\pi\varepsilon_0 r.$$

If a charge q_2 is moved from infinity to this point, then from the definition of potential the work required is given by

$$W = Vq_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}.$$

This work is stored in the form of electrostatic energy U of the system $q_1 + q_2$ similar to the gravitational potential energy in the system earth + stone.

$$\therefore \qquad U = W = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2}{r}. \qquad \text{...(62)}$$

If $\mathbf{r}_1$ and $\mathbf{r}_2$ be the position vectors of points A and B respectively, we have $\mathbf{r}_{21} = \mathbf{r}_2 - \mathbf{r}_1$. If a third charge q_3 is at point C with position vector $\mathbf{r}_3$, the total electrostatic energy of the system of three charges will be given as

$$U = \frac{q_2q_1}{4\pi\varepsilon_0 r_{21}} + \frac{q_3q_1}{4\pi\varepsilon_0 r_{31}} + \frac{q_3q_2}{4\pi\varepsilon_0 r_{32}}. \qquad \text{...(63)}$$

For the calculation of potential energy of the system containing large number of particles, the procedure is to compute the potential energy for every pair of charges separately and to add these algebraically. Hence for n charges

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{all\ pairs}\frac{q_iq_j}{r_{ij}} = \frac{1}{2}\,\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{q_iq_j}{r_{ij}} \quad (i \neq j) \qquad \text{...(64)}$$

In the expression $\frac{1}{2}$ is introduced as each pair is counted twice here. This relation can be written in different forms by noting that the final value of the potential V at the j^{th} point charge is

$$V_j = \sum_{i=1}^{n}\frac{q_i}{4\pi\varepsilon_0 r_{ij}} \quad (i \neq j)$$

$$\therefore \text{ Electrostatic energy of the system} = \frac{1}{2}\sum_{j=1}^{n} q_jV_j. \qquad \text{...(65)}$$

For a continuous distribution of charge, we may write $\rho(\mathbf{r})dv$ in place of q_i and $\rho(\mathbf{r}')dv'$ in place of q_j. Thus energy necessary to form a continuous charge distribution is

$$U = \frac{1}{8\pi\varepsilon_0}\int_v \frac{\rho(\mathbf{r})\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv\,dv' = \frac{1}{2}\int_v \rho(\mathbf{r})V(\mathbf{r})\,dv. \qquad ...(66)$$

This formula may be written in another form, using the vector identity

$$\nabla.(V\mathbf{E}) = V\nabla.\mathbf{E} + \mathbf{E}.\nabla V$$

$$\therefore \qquad U = \frac{1}{2}\int_v \rho V\,dv = \frac{1}{2}\varepsilon_0\int_v V\nabla\cdot\mathbf{E}\,dv = \frac{1}{2}\varepsilon_0\int_v[\nabla\cdot(V\mathbf{E})\ @\ \mathbf{E}\cdot\nabla V]\,dv$$

$$= \frac{1}{2}\varepsilon_0\int_v V\mathbf{E}.d\mathbf{S} + \frac{1}{2}\varepsilon_0\int_v \mathbf{E}.\mathbf{E}.dv.$$

If the surface S is allowed to approach infinity, the surface integral vanishes. Thus we have

$$U = \frac{1}{2}\varepsilon_0\int_v E^2 dv\,. \qquad ...(67)$$

Thus there exists an energy density u, every where in space, given by

$$u = \tfrac{1}{2}\varepsilon_0 E^2. \qquad ...(68)$$

If the charge increment is added not only to a volume element dv, but also to a surface element dS. We may write $dq = \rho\ dv$ and $dq = \sigma dS$. Thus the electrostatic energy of an arbitrary charge distribution is given by

$$U = \frac{1}{2}\int_V \rho(\mathbf{r})\,V(\mathbf{r})\,dv + \frac{1}{2}\int_S \sigma(\mathbf{r})\,V(\mathbf{r})\,dS, \qquad ...(69)$$

where the potential $V(\mathbf{r})$ is given by

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_V \frac{\rho(\mathbf{r}')\,dv'}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\varepsilon_0}\int_S \frac{\sigma(\mathbf{r}')\,dS'}{|\mathbf{r}-\mathbf{r}'|} \qquad ...(70)$$

Electrostatic Energy of a Charged Sphere : Let us imagine that we assemble the sphere by building up a succession of thin spherical layers of infinitesimal thickness. It q_r is the charge of the sphere when it has been built up to the radius r, the work done in bringing a further charge δq from infinity to the layer of thickness δr is

$$\Delta W = \Delta U = q_r\delta q/4\pi\varepsilon_0 r.$$

If ρ be the charge density in the sphere, then $q_r = \frac{4}{3}\pi r^3\rho$ and $\delta q = \frac{4}{3}\pi 3r^2\rho\delta r = 4\pi r^2\rho\delta r$,

$$\therefore \qquad \Delta U = 4\pi\rho^2 r^4\delta r/3\varepsilon_0.$$

Total energy required to assemble the sphere of radius R is

$$U = \int_0^R dU = \frac{4\pi\rho^2R^5}{15\varepsilon_0} = \frac{3}{5}\frac{q^2}{4\pi\varepsilon_0 R}, \qquad ...(71)$$

where q is the total charge on the sphere of radius R.

Electrostatic Energy in Nuclei : The nucleons that hold the stable nuclei are held together by strong attractive forces and *the energy must be supplied to the nucleus to separate it into the individual constituents.* This energy is called the binding energy of the nucleus. The binding energy may be regarded as being made up of a number of terms, each of which represents some general characteristic of nuclei. The only charge dependent term in the relation for binding energy is the *Coulomb repulsion* between the protons, all other terms are based on the fact that the *p-p*, *n-n* and *n-p forces are identical and are the short range forces.*

The electrostatic energy is simply the work done against electrostatic forces in assembling the Z-protons in a sphere of radius R. The problem is same as that of the electrostatic energy of a charged sphere discussed earlier. The total charge in the nucleus q = charge on Z protons = Ze. Thus Eq. (71) gives

$$U = \frac{3}{5}\frac{Z^2e^2}{4\pi\varepsilon_0 R}. \qquad ...(72)$$

This relation is not correct for small number of protons (Z small). In such cases Z^2 must be replaced by $Z(Z-1)$.

$$\therefore \qquad U = \frac{3}{5}\frac{Z(Z-1)e^2}{4\pi\varepsilon_0 R}. \qquad ...(73)$$

This relation may be used for calculating the nuclear radius as given below. Consider a positron emission

$${}_Z^AX \rightarrow {}_{Z-1}{}^AY + {}_{+1}e + \nu \qquad ...(74)$$

The difference in the electrostatic energies between these nuclei is given by

$$\Delta U = \frac{3}{5}\frac{e^2}{4\pi\varepsilon_0 R}[Z^2 - (Z-1)^2] = \frac{3}{5}\frac{e^2\,(2Z-1)}{4\pi\varepsilon_0 R}. \qquad ...(75)$$

The maximum positron energy, when the daughter nucleus is formed in the ground state, is given by

$$E_{\beta+} = \frac{3}{5}\frac{e^2(2Z-1)}{4\pi\varepsilon_0 R} - m_0c^2 - (m_n - m_p)\,c^2,$$

where m_0c^2 is the rest mass energy of the positron and $(m_n - m_p)c^2$ is the energy absorbed in transforming a proton of mass m_p into a neutron of mass m_n.

Let us use a relation $R = R_0A^{1/3}$ and $2Z - 1 \doteq A$, we get

$$E_\beta = \frac{3}{5}\frac{A^{2/3}e^2}{4\pi\varepsilon_0 R_0} - 0.51\,MeV - 1.29\,MeV. \qquad ...(76)$$

The slope of the curve between E_β and $A^{2/3}$ gives us $R_0 \doteq 1.2 \times 10^{-15}$ m.

4.9 THE ELECTRON VOLT

When an electron or proton falls through a potential difference of $V_B - V_A$ volts, it acquires a kinetic energy

$$K.E. = e(V_B - V_A) = (1.6023 \times 10^{-19})\,(V_B - V_A) \text{ joule}. \qquad ...(77)$$

This amount is very small and is multiple of 1.6023×10^{-19}, hence we introduce a new unit of energy, known as *electron volt* (eV). It is the energy in joules divided by the magnitude of the quantum of charge 1.6023×10^{-19} coulomb.

$$\therefore \qquad 1eV = 1.6023 \times 10^{-19} \text{ joule}. \qquad ...(78)$$

The utility of this unit is obvious, as charge in atomic and nucleus cases is generally taken in terms of the electronic charge (*i.e.*, Ze) and the potential difference in volts. If an α-particle, having charge $2e$, falls through a potential difference of $100V$, then it will acquire energy $200eV$. Its million and billion times quantities are respectively abbreviated as MeV and BeV.

4.10 HIGH VOLTAGE BREAKDOWN

Let us consider a big sphere A of radius a connected with a small sphere B of radius b by a wire as shown in Fig. 4.14. The wire will have little influence on the fields outside but will keep the spheres at the same potential. If the bigger ball carries a charge Q and the smaller ball carries a charge q, then their potentials can be written as

$$V_A = \frac{1}{4\pi\varepsilon_0}\frac{Q}{a}, \text{ and } V_B = \frac{1}{4\pi\varepsilon_0}\frac{q}{b},$$

As $V_A = V_B$, hence $Q/a = q/b$.

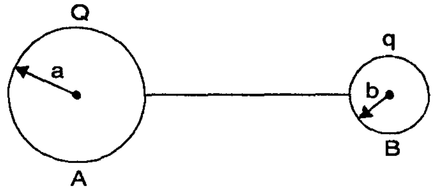


Fig. 4.14. Two charged spheres connected by a wire.

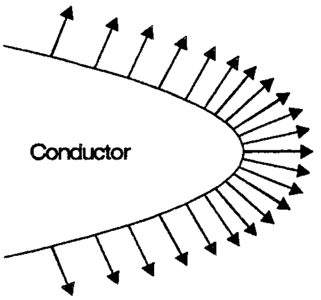


Fig. 4.15. A pointed conductor.

The ratio of the electric fields at the surfaces

$$\frac{E_A}{E_B} = \frac{Q/4\pi\varepsilon_0 a^2}{q/4\pi\varepsilon_0 b^2} = \frac{Qb^2}{qa^2} = \frac{b}{a}. \qquad ...(79)$$

Therefore the field is higher at the surface of the small sphere.

The same can be explained with a conductor having a very sharp end, Fig. 4.15. Since the charges try to spread out on the surface of the conductor and the tip of the sharp end is far away from most of the surface. Some of the charges on the conductor get pushed to the sharp end. A relatively small charge on the sharp end can thus produce a large surface charge density, hence *a* high field just outside of it.

Due to this very high field at the sharp ends of conductors, the air near these points *breaksdown.* In this process the loose charge (electron or ion) present somewhere in the air is accelerated by this field and picks up enough speed before it hits another atom to be able to knock an electron off that atom. In this way more and more ions are produced. Their motion constitutes a discharge or spark. *Hence for no spark in air while charging an object, we must require the smooth surface.*

4.11 ELECTROSTATIC GENERATOR

Let us consider large hollow metal sphere B with an original positive charge q_B, indicated by +ve signs with circles around them and having inner radius r_B (Fig. 4.16). If it has small opening so that a small metal sphere A of radius r_A with a positive charge q_A can be placed inside it. When sphere A is at the centre of B and the effect due to opening in B is very small and can be neglected, the positive charge on sphere A will induce negative charge on the inner surface of B and the electric field between these two charges is symmetric and radial. From Gauss's law, the field between spheres A and B is due to the charge on A only and at a distance r from the centre of sphere A *is given by*

$$E = \frac{1}{4\pi\varepsilon_0}\frac{q_A}{r^2},$$

$\therefore$ Potential difference between spheres A and B is

$$V_A - V_B = \frac{q_A}{4\pi\varepsilon_0}\left(\frac{1}{r_A} - \frac{1}{r_B}\right). \qquad ...(80)$$

Thus the *pd* is independent of the original charge q_B on the sphere B. For q_A positive, the $V_A > V_B$, or the inner sphere A will always be at higher potential then the outer hollow sphere B.

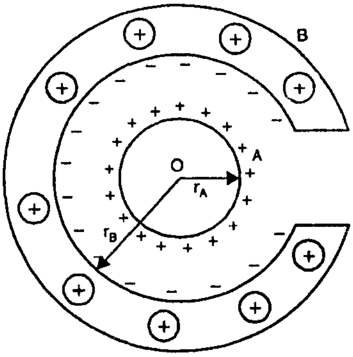


Fig. 4.16. Two charged spheres.

If the spheres are connected by a fine wire (outer surface of A with the inner surface of B), the charge will flow to the outer sphere B as it was on the lower potential, till the potential difference $V_A - V_B = 0$ or from above relation until q_A becomes zero. This leads to the conclusion that all the charge q_A of the sphere A is transferred to the hollow sphere B, irrespective of the initial value of charge and potential of the hollow sphere B. An electrostatic generator works on this principle to produce very high voltages.

Van de Graaff Generator : It was invented by R.J. Van de Graaff in the United States in 1931, for producing high voltages in the million volts region. Its design is based on the principle that *if a charged conductor is brought into internal contact with a hollow conductor, all of its charge transfers to the hollow conductor no matter how high the potential of the latter may be.*

In consists of a belt, made of paper, silk, rayon or other flexible non-conducting material, which is run by means of a motor at high speed over two pulleys (Fig. 4.17). The positive pole of a dc power supply at a few kilovolts is connected to a pointed comb like conductor. Because of the large electrostatic field in the air near the spray point, a corona breakdown is produced. The positive and negative ions are formed and a spray of positive ions is repelled from the points. These positive ions attach themselves to the surface of a moving belt (actually the negatively charged electrons pass from the belt to the points, thus leaving the belt with a positive charge) and are carried mechanically by the belt to the terminal where they are removed by another corona discharge to another set of needle points. This set of needle points is connected with the internal surface of the hollow hemispherical conductor mounted on insulating supports. The upper pulley is within this large metal hemisphere. The charge is thus transferred to this hemisphere and increases its potential.

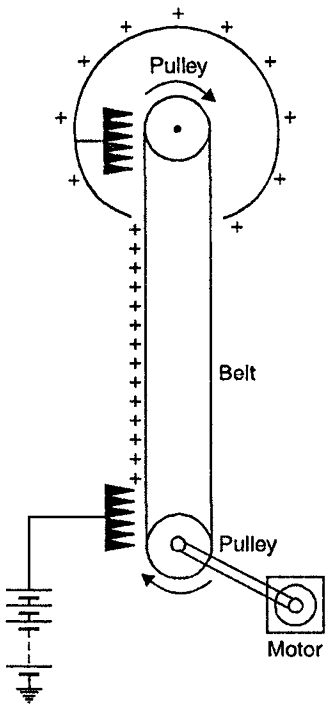


Fig. 4.17. Van de Graaff Generator.

The potential V of the sphere at any moment is equal to Q/C, where *Q is the charge on the sphere at that instant and C* its capacity. The charge transfer process is repeated and the potential of the hollow sphere rises until the rate of loss of charge by leakage equals the rate at which charge is introduced. The leakage can be minimized and the maximum potential is limited by electrical discharge of the surrounding atmosphere. The electric field at the surface of the hollow hemisphere of radius R is V/R, where V is the potential of the hemisphere. Experimentally it is observed that the sparking field or the *dielectric strength* in dry air at atmospheric pressure is about 3 mega volts/m. Thus theoretically the maximum voltage attainable is 3 mega volts for a sphere of radius one meter. In practice, the steady operating potential of the high voltage terminal in machines operating in atmospheric air is about one-third of the theoretical values for isolated spheres.

The maximum electric field that can exist outside of the sphere without breakdown goes to larger values if the pressure of the gas is increased, or even better, if the space is completely evacuated. Satisfactory vacuum is not easy. If highly pressed air alone is used a definite fire hazard will exist by virtue of the compressed oxygen. Modern machines often use high pressure nitrogen. The addition of one or two per cent of an electro-negative gases such as freon ($CCl_2\,F_2$) or sulphur hexafluoride increases the electrical breakdown strength by a factor of perhaps two.

4.12 LINEAR ACCELERATOR

The idea of accelerating positive ions with the help of an alternating radio frequency field was first suggested by G. Ising in 1924. On this principal the first working Linear accelerator was constructed by R. Wideroe and then modified by Sloan and Lawrence. The linear accelerator consists of a number of cylindrical electrodes of increasing lengths arranged in a straight line. Alternate electrodes are connected together, the electrodes numbered 1, 3, 5, ... etc. being joined to one terminal and the remaining 2, 4, 6 ... etc. to the second terminal of the high frequency electrical oscillator (Fig. 4.18). Thus the polarity of the electrode changes in every half period.

Suppose positive ions from the ion source move from left to the right along the common axis of the cylindrical electrodes. The ions receive no acceleration while passing through the electrode, as the potential is uniform inside the electrode. If the first electrode is at positive potential and the second is at negative, the positively charged particles will be accelerated in the gap between these electrodes. The positive ions then enter the second electrode and travel through it at a constant but higher speed. The length of the second electrode is such that just as the ions reach the gap between it and the third electrode, the potential of these electrodes are reversed.

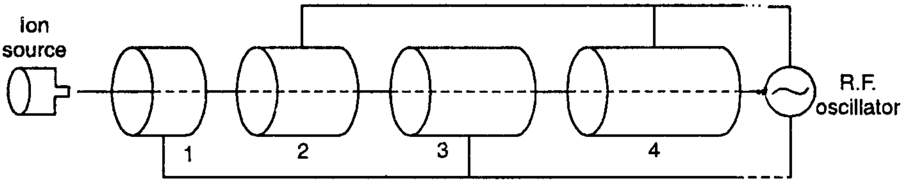


Fig. 4.18. Linear accelerator.

Now the second electrode becomes positive and the third negative. Thus the positive ions are further accelerated in the gap. Thus the ions are accelerated while passing through the each gap. The energy acquired by the ion of charge q coulomb while passing through a potential difference of V volts will be qV. The velocity of the ion of mass M in the n^{th} electrode is given by

$$\tfrac{1}{2}Mv_n^2 = (n-1)qV + K, \qquad \text{...(81)}$$

where K is the kinetic energy of the ion entering the first cylinder. The length of the n^{th} cylinder is given by

$$L_n = \frac{v_n}{2f} = \frac{1}{2f}\left[\frac{2[(n-1)qV + K]}{M}\right]^{1/2}, \qquad \text{...(82)}$$

where f is the frequency of the applied field.

EXERCISES

Example 1. *Find potential difference between the centre and one corner C of a square, if on the other three corners A, B and D, charges –5, +2 and + 3 μC respectively be placed. Given that the length of each side of the square is $3\sqrt{2}$ meter.*

Potential at the centre of square

$$V_0 = \frac{1}{4\pi\varepsilon_0}\sum \frac{q_i}{r_i} = \frac{1}{4\pi\varepsilon_0}\left[\frac{-5\times10^{-6}}{3} + \frac{2\times10^{-6}}{3} + \frac{3\times10^{-6}}{3}\right] = 0.$$

Thus the central point O is at the same electrical level as infinity and no work will be needed to bring a test charge from infinity up to this point.

Similarly potential at the corner C of the square

$$V_C = \frac{1}{4\pi\varepsilon_0}\left[\frac{-5\times10^{-6}}{6} + \frac{2\times10^{-6}}{3\sqrt{2}} + \frac{3\times10^{-6}}{3\sqrt{2}}\right]$$

$$= 9\times10^9 \times (5\sqrt{2} - 5)\times 10^{-6}/6 = 3106.6 \text{ volts.}$$

$$\therefore \qquad V_C - V_O = 3106.6 \text{ volts.}$$

Example 2. *Determine the law of variation of potential with distance in an electric field produced between two parallel infinite planes, bearing equal and opposite charges.*

In chapter 3, it has been established that the electric field **E** is the same at any point between such plates. The potential difference between plate A and an arbitrary point at distance x from the plate A is given by

$$V_{Ax} = Ex = V_A - V_x.$$

If the potential of the negatively charged plate B is assumed zero and the separation of the plates is d, then we have

$$V_A = V_{AB} = Ed.$$

$\therefore$ Potential at point at a distance x from the plate A is

$$V_x = V_{AB} - V_{AB}\,(x/d) = V_{AB}\,(1 - x/d).$$

Thus we see that the potential difference between two infinite planes bearing charges of opposite sign and equal charge density varies with distance according to linear law and vanishes on moving from positive to negative plate.

Example 3. *Find out the points where (i) $V = 0$ and (ii) $E = 0$ in the following figure :*

(*i*) Let P be the point on the axis, either on the left or the right of charge $+q$, at a distance r. Therefore the potential due to both the charges at this point if assumed on the left of q, will be

$$V_P = \frac{q}{4\pi\varepsilon_0 r} + \frac{-3q}{4\pi\varepsilon_0(1+r)} = 0.$$

∴ $r = 0.5$ m or -0.25 m.

Thus the potential will be zero at point P on the axis which is either 0.5 m on the left or 0.25m on the right of charge $+q$.

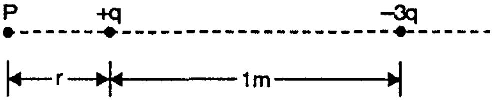


Fig. 4.19. Example 3.

(*ii*) The charges are of opposite signs. The electric field will be in the same direction at every point in between these charges. The total field can be zero only on the left of charge $+q$. Hence

$$E_P = \frac{q}{4\pi\varepsilon_0 r^2} + \frac{(-3q)}{4\pi\varepsilon_0(1+r)^2} = 0 \text{ or } \frac{1}{r^2} - \frac{3}{(1+r)^2} = 0.$$

∴ $r = (1 \pm \sqrt{3})/2.$

Since the negative sign is not permissible, the only point on the axis at which total electric field E will be zero is $(1 + \sqrt{3})/2$ metre on the left of charge $+q$.

Example 4. *The potential of a certain charge configuration is expressed by $V = 2x + 3xy + y^2$ volts. Find the electric intensity at point (5, 2). What acceleration does an electron experience in the x-direction ? Distances are in metre.*

We know that the electric intensity at any point is defined as the negative gradient of a potential at that point.

$$\therefore \quad E = -\nabla V = -\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)(2x + 3xy + y^2)$$

$$= -\mathbf{i}\frac{\partial}{\partial x}(2x + 3xy + y^2) - \mathbf{j}\frac{\partial}{\partial y}(2x + 3xy + y^2) - \mathbf{k}\frac{\partial}{\partial z}(2x + 3xy + y^2)$$

$$= -\mathbf{i}(2 + 3y) - \mathbf{j}(3x + 2y).$$

∴ **E** at point (5, 2) = $-8\mathbf{i} - 19\mathbf{j}$.

Force on the electron of charge e is given by

$$\mathbf{F} = (-e)\mathbf{E} = 8 \times 1.6023 \times 10^{-19}\,\mathbf{i} - 19 \times 1.6023 \times 10^{-19}\,\mathbf{j}.$$

Acceleration experienced by the electron in the x-direction will be due to the x-component of force.

$$\therefore \quad a_x = F_x/m = 8 \times 1.6023 \times 10^{-19}/9.1 \times 10^{-31} = 1.41 \times 10^{12} \text{ m/sec}^2.$$

Example 5. *Two drops of water, each with a charge of 3×10^{-9} coulomb and having a surface potential of 500 volts form a single drop. Calculate the surface potential of the new drop so formed.*

Let r be the radius of the each drop, the surface potential of the drop is given by the relation

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{r} \text{ or } r = \frac{1}{4\pi\varepsilon_0}\frac{q}{V}.$$

Substituting the numerical values

$$r = 9 \times 10^9 \times 3 \times 10^{-9} \times (1/500) = 0.054 \text{ m}.$$

The volume of the each drop is $\frac{4}{3}\pi\,(0.054)^3$ m^3. The volume of the combined drop will thus be $\frac{8}{3}\pi\,(0.054)^3$. The radius of this drop R is given as

$$\frac{4}{3}\pi R^3 = \frac{8}{3}\pi\,(0.054)^3 \text{ or } R = 2^{1/3} \times 0.054 = 0.068 \text{ m.}$$

and the total charge $q' = 2q = 6 \times 10^9$ coul. Therefore the potential of this new drop is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q'}{R} = 9 \times 10^9 \times \frac{6 \times 10^{-9}}{0.068} = 793.7 \text{ volts.}$$

Example 6: *In the Bohr model of the hydrogen atom, the electron rotates in a circle of radius 0.53 A° about the nucleus. Calculate the speed of the electron in the orbit and the amount of energy needed to tear the electron loose from the nucleus of the atom.*

Electron rotates in an orbit due to centripetal force provided by the Coulomb attraction between it and the nucleus. In hydrogen nucleus we have one proton having charge equal to the electronic charge. Therefore

$$\frac{mv^2}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}.$$

After substituting values (m of electron $= 9.1 \times 10^{-31}$ kg., e on electron $= 1.6023 \times 10^{-19}$ C, r of the orbit $= 0.53 \times 10^{-10}$ metre and $1/4\pi\varepsilon_0 = 9 \times 10^9$ Nm^2/C^2), we get $V = 2.18 \times 10^6$ m/sec.

To find the energy needed to tear the electron loose, we must know the work required to carry the electron from its orbit out to infinity, which will be the product of electron charge and the potential at the surface of the orbit.

$$\therefore \qquad W = eV = e\,\frac{1}{4\pi\varepsilon_0}\frac{e}{r} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}.$$

As charge on the electron is negative, hence is attracted by the positively charged nucleus, thus a negative work will be done in bringing this electron from infinity to surface. Therefore the work done in tearing the electron loose (*i.e.*, out to infinity) from the nucleus will be positive. Substituting numerical values, we get

$$W = 9 \times 10^9 \times (1.6023 \times 10^{-19})^2/0.53 \times 10^{-10} = 4.36 \times 10^{-18} \text{ joule.}$$

Actually, all this amount of work is not required, as electron has its kinetic energy

$$E = \frac{1}{2}mV^2 = 2.16 \times 10^{-18} \text{ joule.}$$

$\therefore$ External energy needed to tear the electron loose

$$= W - E = 2.20 \times 10^{-18} \text{ joule} = 13.73 \text{ eV.}$$

This is known as the *ionization energy* of the hydrogen atom.

Example 7. *A proton, which is at a distance 10^{-14} m from the centre of a mercury nucleus is released from nucleus. Calculate the K.E. and its speed when it gets far away from the nucleus.*

Mercury nucleus has a charge $80 \times 1.6023 \times 10^{-19}$ C (as its atomic number is 80), proton will be repelled and caused to move radially toward infinity by this positive charge. Potential of the proton at its initial position is

$$V = \frac{q}{4\pi\varepsilon_0 r} = \frac{(9 \times 10^9)\,(80 \times 1.6023 \times 10^{-19})}{10^{-14}} = 11.5 \times 10^6 \text{ volts.}$$

This is the work done when we bring this proton from infinity to this point against the electric field due to the +vely charged nucleus. When this proton moves away, this work is released as K.E. and is acquired by the proton.

$$\therefore \qquad K.E. = 11.5 \times 10^6 \text{ electron volts} = 11.5 \text{ MeV.}$$

As mass of proton $m = 1.67 \times 10^{-27}$ kg and 1 $MeV = 1.6 \times 10^{-13}$ joule, hence the proton velocity.

$$v = \sqrt{(2\,K \cdot E/m)} = 4.69 \times 10^7 \text{ m/sec.}$$

As this velocity is very large, we must use relativistic relation

$$K.E. = m_0c^2\left[\frac{1}{\sqrt{(1-v^2/c^2)}} - 1\right]$$

Substituting numerical values, the relativistic velocity v may be calculated.

Example 8. *Two fixed, equal positive charges, each of magnitude 5×10^{-5}C are located at points A and B separated by a distance of 6m. An equal and opposite charge moves towards them along the line COD, the perpendicular bisector of the line AB (Fig. 4.20). The moving charge, when it reaches to point C at a distance of 4m from O, has a K·E. of 4J. Calculate the distance of the farthest point D at which the negative charge will reach before returning towards C.*

The potential energy of the charge $-q$ at point C due to the charges at A and B is given by

$$P.E. = \frac{1}{4\pi\varepsilon_0}\left[\frac{q(-q)}{AC} + \frac{q(-q)}{BC}\right] = -9 \times 10^9 \times 2 \times \frac{(5 \times 10^{-5})^2}{5}$$

$$= -\ 9 \text{ joules.} \qquad [\because\ AC = BC = (3^2 + 4^2)^{1/2} = 5]$$

The $K.E.$ at $C = 4J$ (given), therefore the total energy $= 4 - 9 = -5\ J$.

The negative charge $-q$ will reach the farthest point D with $K.E. = 0$, such that its total energy is again $-5J$. Therefore $P.E.$ at point D must be $-5J$.

$$\therefore \quad -5 = -\frac{1}{4\pi\varepsilon_0}\left[\frac{q^2}{AD} + \frac{q^2}{BD}\right] = \frac{-2q^2}{4\pi\varepsilon_0 AD}$$

$$= -\frac{9 \times 10^9 \times 2 \times (5 \times 10^{-5})^2}{AD}$$

or $\quad AD = 9$ m

$$\therefore \quad OD = [AD^2 - OA^2]^{1/2} = (9^2 - 3^2)^{1/2} = 6\sqrt{2} \text{ m.}$$

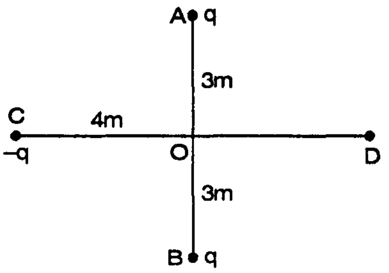


Fig. 4.20. Example 8.

Example 9. *Three concentric spherical metallic shells A, B and C of radii a, b and c (a < b < c) have surface charge densities σ, –σ and σ respectively.*

(i) Find the potential of the three spherical shells A, B and C. (ii) If the shells A and C are at the same potential, obtain the relation between the radii a, b and c.

We know that the potential at any point inside a spherical charged conductor or spherical shell is the same as on the surface of the shell. Therefore potential of A

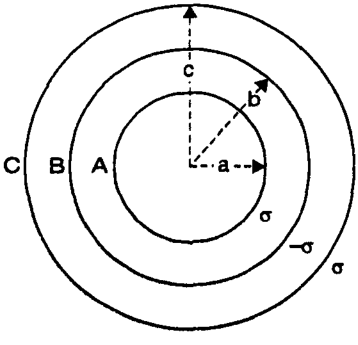


Fig. 4.21. Example 9.

$$V_A = \text{Potential due to } \sigma \text{ on } A + \text{due to } (-\sigma) \text{ on } B + \text{due to } \sigma \text{ on } C$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi a^2\sigma}{a}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{-4\pi b^2\sigma}{b}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi c^2\sigma}{c}\right)$$

$$= \frac{\sigma}{\varepsilon_0}\ (a - b + c).$$

Similarly the potential of B

$$V_B = \text{Potential due to } \sigma \text{ on } A + \text{due to } -\sigma \text{ on } B + \text{due to } \sigma \text{ on } C$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi a^2\sigma}{b}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{-4\pi b^2\sigma}{b}\right) + \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi c^2\sigma}{c}\right)$$

$$= \frac{\sigma}{\varepsilon_0}\left[\frac{a^2}{b} - b + c\right].$$

Potential of C

$$V_C = \text{Potential due to } \sigma \text{ on } A + \text{due to } -\sigma \text{ on } B + \text{due to } \sigma \text{ on } C$$

$$= \frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi a^2\sigma}{c}\right)+\frac{1}{4\pi\varepsilon_0}\left(\frac{-4\pi b^2\sigma}{c}\right)+\frac{1}{4\pi\varepsilon_0}\left(\frac{4\pi c^2\sigma}{c}\right)$$

$$= \frac{\sigma}{\varepsilon_0}\left[\frac{a^2}{c}-\frac{b^2}{c}+c\right].$$

(*ii*) Given that $V_A = V_C$, therefore

$$\frac{\sigma}{\varepsilon_0}(a-b+c) = \frac{\sigma}{\varepsilon_0}\left(\frac{a^2}{c}-\frac{b^2}{c}+c\right) \text{ or } a-b=\frac{a^2-b^2}{c}$$

$$\therefore \qquad c = a+b \text{ as } a \neq b.$$

Example 10. *Calculate the potential due to a uniformly charged ring at a point on its axis and hence deduce an expression for electric intensity.*

Consider the circular ring with charge λ per unit length. To find the potential at a point P (Fig. 3.7) on the axis at a distance x, divide the ring into small elements each of length Δl. Each point of this element is at the same distance r from point P, hence potential due to the charge on this element

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\Delta l}{r}.$$

Therefore the potential due to the whole ring

$$V = \frac{1}{4\pi\varepsilon_0}\Sigma\frac{\lambda\Delta l}{r} = \frac{1}{4\pi\varepsilon_0}\int\frac{\lambda dl}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda}{r}\int dl.$$

As charge density λ is same and each point on the ring is at the same distance from point P hence we have taken λ and r outside the integral. The integral is extended around the ring and the sum of the elements of length Δl around the ring is $2\pi a$.

$$\therefore \qquad V = \frac{\lambda}{2\varepsilon_0}\frac{a}{r} = \frac{\lambda}{2\varepsilon_0}\frac{a}{(a^2+x^2)}$$

This relation is true for all values of x.
The electric field along the x-axis

$$E_x = -\frac{\partial V}{\partial x} = \frac{\lambda a x}{2\varepsilon_0(a^2+x^2)^{3/2}}.$$

Example 11. *Calculate the potential due to a thin charged rod of length L at the point along and ⊥ to its length.*

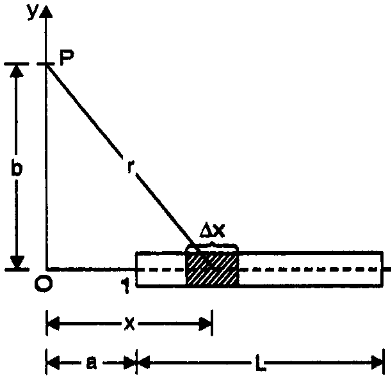


Fig. 4.22. Example 11.

Suppose the thin rod has a uniform charge of charge density λ per unit length. To find potential at point P (Fig. 4.22), let us divide the rod into a series of small elements, each may be assumed as point charge. The potential at this point P due to one element of length Δx is given by

$$\Delta V = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\Delta x}{r}.$$

x Total potential due to whole rod $V = \Sigma\Delta V = \Sigma\lambda\Delta x/4\pi\varepsilon_0 r$.

Assuming $\Delta x \to 0$, summation can be replaced by an integration.

$$\therefore \qquad V = \int_a^{a+L}\frac{\lambda dx}{4\pi\varepsilon_0(b^2+x^2)^{1/2}} = \frac{\lambda}{4\pi\varepsilon_0}\int_a^{a+L}\frac{dx}{(b^2+x^2)^{1/2}}$$

or $$V = \frac{\lambda}{4\pi\varepsilon_0}\left[\log_e\left\{x+\left(b^2+x^2\right)^{1/2}\right\}\right]_a^{a+L} = \frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{a+L+\left\{b^2+(a+L)^2\right\}^{1/2}}{a+\left(b^2+a^2\right)^{1/2}}\right].$$

Let us restrict this result to following limiting cases :

(*i*) *$b = 0$ and $L << a$ or at a far distance on its axis*

$$V = \frac{\lambda}{4\pi\varepsilon_0}\log_e\frac{2(a+L)}{2a} = \frac{\lambda}{4\pi\varepsilon_0}\log_e\left(1+\frac{L}{a}\right).$$

As $\log_e(1+z) = z - z^2/2 + z^3/3 - \ldots = z$ if z is less than 1, hence

$$V = \frac{\lambda}{4\pi\varepsilon_0}\frac{L}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q}{a},$$

where $\lambda L = q$, the total charge on the rod.

The relation is same as due to a point charge. Thus at a large distance charged rod behaves as a point charge.

(*ii*) *$a = 0$, $L << b$ or at the points on the line ⊥ to axis*

$$V = \frac{\lambda}{4\pi\varepsilon_0}\log_e\left[\frac{L}{b}+\left(1+\frac{L^2}{b^2}\right)^{1/2}\right] = \frac{\lambda}{4\pi\varepsilon_0}\log_e\left(1+\frac{L}{b}\right)$$

$$= \frac{\lambda}{4\pi\varepsilon_0}\frac{L}{b} = \frac{q}{4\pi\varepsilon_0 b},$$

which is again as due to a point charge.

Example 12. *Determine the potential distribution about long parallel wires having equal and opposite charge distributions along their lengths.*

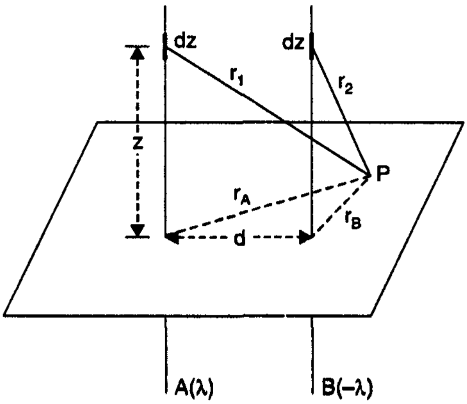


Fig. 4.23. Example 12.

Let A and B be two wires (length $2L$) with linear charge densities λ and $-\lambda$ respectively. The potential at any point P due to small elements dz at a distance z from the plane of symmetry (Fig. 4.23) is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\left(\frac{\lambda}{r_1}-\frac{\lambda}{r_2}\right)dz$$

$$\therefore \quad V = \frac{\lambda}{4\pi\varepsilon_0}\int_{-L}^{L}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)dz.$$

If the point P lies on the plane of symmetry and is at the distance r_A and r_B from wires A and B respectively. Then $r_1 = (z^2+r_A{}^2)^{1/2}$ and $r_2 = (z^2+r_B{}^2)^{1/2}$.

$$\therefore \quad V = \frac{\lambda}{2\pi\varepsilon_0}\int_0^L\left[\frac{1}{(z^2+r_A{}^2)^{1/2}}-\frac{1}{(z^2+r_B{}^2)^{1/2}}\right]dz$$

or $$V = \frac{\lambda}{2\pi\varepsilon_0}\left[\log_e\left(z+\sqrt{z^2+r_A{}^2}\right)-\log_e\left(z+\sqrt{z^2+r_B{}^2}\right)\right]_0^L$$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\log_e\frac{L+\sqrt{L^2+r_A{}^2}}{L+\sqrt{L^2+r_B{}^2}}-\log_e\frac{r_A}{r_B}\right]$$

As L approaches infinity, the first term in the bracket approaches zero ($r_A << L$ and $r_B << L$). Therefore for infinite parallel wires with equal and opposite uniform charges

$$V = -\frac{\lambda}{2\pi\varepsilon_0}\log_e\frac{r_A}{r_B}.$$

At the point in between the wires on the plane of symmetry, $r_A = r_B$ and we have $V = 0$. $V =$ const, if r_A/r_B is also constant. One may prove that the locus of a constant potential will be the family of circles. Because of invariance in the z-direction, the equipotential surfaces will be cylinders.

Example 13. *A dielectric cylinder of radius 'a' is of infinite length and is non-uniformly charged, such that the charge volume density* ρ *varies inside directly as the distance from its axis, with* $\rho = 0$ *at the axis and* ρ_0 *at the surface. Calculate the p.d. between the axis and the surface.*

To solve this problem, let us divide the charged cylinder into large number of co-axial cylindrical shells. Consider a cylindrical shell of radii x and $x + \delta x$, the electric intensity E must be calculated first as given in example 13 of chapter 3.

We know that $\rho = cx$, $\rho_0 = ca$, E inside the cylinder $= \rho_0 r^2/3a\varepsilon_0$ and E out side the cylinder $= \rho a^2/3r\varepsilon_0$. We also know that the potential difference between two points, where E is the electric field intensity, is given by

$$V_A - V_B = -\int_B^A \mathbf{E}\cdot d\mathbf{r}.$$

Hence the p.d. between the axis and the surface of the cylinder

$$V_{axis} - V_{surface} = -\int_a^0 \frac{\rho_0 r^2}{3a\varepsilon_0}dr = \frac{\rho_0}{3a\varepsilon_0}\int_0^a r^2\,dr$$

$$= \rho_0 a^2/9\varepsilon_0.$$

Example 14. *Calculate the electrostatic potential at the centre, inside, outside and on the surface of a charged sphere having charge distribution as*

$$\rho = \rho_0(1 - r/a) \textit{ when } r \le a \textit{ and } \rho = 0 \textit{ when } r > a.$$

To find potential due to a spherical distribution of charge, the sphere is divided in large number of spherical shells. Let us imagine a spherical shell of radii r and $r + \delta r$. The potential at the centre due to the charge δq over this is given by

$$\Delta V_c = \delta q/4\pi\varepsilon_0 r = 4\pi r^2\delta r\rho/4\pi\varepsilon_0 r = \rho r\delta r/\varepsilon_0.$$

$\therefore$ V_C due to entire spherical charge $= \int_0^a \frac{r}{\varepsilon_0}\rho_0\left(1 - \frac{r}{a}\right)dr$

$$= \frac{\rho_0}{\varepsilon_0}\left[\frac{r^2}{2} - \frac{r^3}{3a}\right]_0^a = \frac{\rho_0 a^2}{6\varepsilon_0}.$$

Potential V_0 outside this spherical charge distribution $= q/4\pi\varepsilon_0 r_0$.

$$\therefore \quad V_0 = \frac{1}{4\pi\varepsilon_0 r_0}\int dq = \frac{1}{4\pi\varepsilon_0 r_0}\int_0^a 4\pi r^2\rho_0\left(1 - \frac{r}{a}\right)dr$$

$$V_0 = \frac{\rho_0}{\varepsilon_0 r_0}\left[\frac{r^3}{3} - \frac{r^4}{4a}\right]_0^a = \frac{\rho_0 a^3}{12\varepsilon_0 r_0}.$$

Potential V_i inside the charge distribution is equal to V due to the portion of sphere between radii 0 and r_i and due to the portion of sphere between radii r_i and a. Hence

$$V_i = \frac{1}{4\pi\varepsilon_0 r_i}\int dq + \int dV$$

$$= \frac{1}{4\pi\varepsilon_0 r_i}\int_0^{r_i} 4\pi r^2\rho_0\left(1 - \frac{r}{a}\right)dr + \int_{r_i}^a \frac{4\pi r^2}{4\pi\varepsilon_0 r}\rho_0\left(1 - \frac{r}{a}\right)dr$$

$$= \frac{\rho_0}{\varepsilon_0 r_i}\left[\frac{r^3}{3} - \frac{r^4}{4a}\right]_0^{r_i} + \frac{\rho_0}{\varepsilon_0}\left[\frac{r^2}{2} - \frac{r^3}{3a}\right]_{r_i}^a = \frac{\rho_0}{6\varepsilon_0}\left(a^2 - r_i^2 + \frac{r_i^3}{2a}\right).$$

Example 15. *An electric quadrupole is formed by a charge $-2e$ at the origin and charges $+e$ at the points $(\pm a, 0)$. Show that the electrostatic potential at a large distance r ($>> a$) is approximately given by the expression $V = ea^2 (3\cos^2\theta - 1)/4\pi\varepsilon_0 r^3$, where θ is the angle between the direction of r and the axis of x. Find an expression for electric field at this point.*

Applying relation for potential due to a point charge to Fig. 4.24. Assuming charge $+e$ at A, $-2e$ at O and $+e$ at B, we get total potential at point P as

$$V = \Sigma\ V_i = \frac{1}{4\pi\varepsilon_0}\left[\frac{e}{PB} - \frac{2e}{PO} + \frac{e}{PA}\right].$$

As $OP = r$ and $BO = OA = a$, and $\angle POA = \theta$, hence from the geometry of the figure

$$AP = |\ \mathbf{r} - \mathbf{a}\ | = (r^2 + a^2 - 2\mathbf{r}\cdot\mathbf{a})^{1/2}$$

and $$BP = |\ \mathbf{r} + \mathbf{a}\ | = (r^2 + a^2 + 2\mathbf{r}\cdot\mathbf{a})^{1/2}.$$

Thus we have

$$V(\mathbf{r}, \theta) = \frac{e}{4\pi\varepsilon_0}\left[\frac{1}{(r^2 + a^2 + 2\mathbf{r}.\mathbf{a})^{1/2}} - \frac{2}{r} + \frac{1}{(r^2 + a^2 - 2\mathbf{r}.\mathbf{a})^{1/2}}\right]$$

$$= \frac{e}{4\pi\varepsilon_0 r}\left[\left(1 + \frac{2\mathbf{r}.\mathbf{a}}{r^2} + \frac{a^2}{r^2}\right)^{-1/2} - 2 + \left(1 - \frac{2\mathbf{r}.\mathbf{a}}{r^2} + \frac{a^2}{r^2}\right)^{-1/2}\right].$$

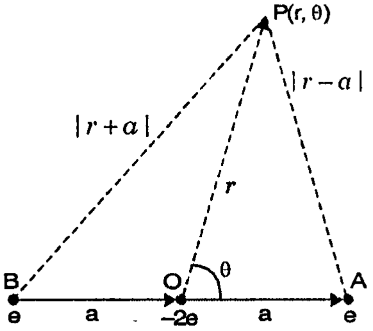


Fig. 4.24. Example 15.

Using the relation $\mathbf{r}.\mathbf{a} = ra\cos\theta$ and expanding by Binomial theorem, we get

$$V(\mathbf{r}, \theta) = \frac{e}{4\pi\varepsilon_0 r}\left[\left(1 - \frac{a}{r}\cos\theta + \frac{a^2(3\cos^2\theta - 1)}{2r^2} + \ldots\right) - 2 + \left(1 + \frac{a}{r}\cos\theta + \frac{a^2(3\cos^2\theta - 1)}{2r^2} + \ldots\right)\right]$$

For $r >> a$, one may neglect higher order terms.

$$\therefore \qquad V(\mathbf{r}, \theta) = \frac{e}{4\pi\varepsilon_0 r}\frac{a^2}{r^2}(3\cos^2\theta - 1) = \frac{ea^2(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3}.$$

Thus we see that the *potential due to a quadrupole varies inversely as the cube of the distance, provided $r >> a$.*

To obtain $\mathbf{E}(\mathbf{r})$ from $V(\mathbf{r})$, we first calculate $\mathbf{E}$ in two perpendicular directions $\mathbf{r}$ and θ, as

$$E_r = -\frac{\partial V}{\partial r} = -\frac{\partial}{\partial r}\left[\frac{ea^2(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3}\right] = \frac{3ea^2(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^4}$$

$$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{ea^2(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3}\right] = \frac{6ea^2\cos\theta\sin\theta}{4\pi\varepsilon_0 r^4}$$

$$\therefore \qquad E = \sqrt{E_r^{\ 2} + E_\theta^{\ 2}} = \frac{3ea^2}{4\pi\varepsilon_0 r^4}\sqrt{(3\cos^2\theta - 1)^2 + 4\cos^2\theta\sin^2\theta}$$

$$= \frac{3ea^2}{4\pi\varepsilon_0 r^4}\sqrt{5\cos^4\theta - 2\cos^2\theta + 1}$$

The angle ϕ which the vector $\mathbf{E}$ makes with the radius vector $\mathbf{r}$ is given by

$$\phi = \tan^{-1}\left(\frac{E_\theta}{E_r}\right) = \tan^{-1}\left(\frac{\sin 2\theta}{3\cos^2\theta - 1}\right).$$

Example 16. *The NaCl crystal consists of Na^+ and Cl^- ions. These are arranged at the corners of cubes stacked together in such a way that the ions in nearest neighbours to any ion have opposite charges. If the side of the cube is R and the electronic charge is $-e$, calculate the potential energy or the electrostatic energy (i) per ion, and (ii) per gm of NaCl crystal.*

If the crystal is not of very small size, the number of ions on its surface will be negligible compared to the total number of ions inside the crystal. *NaCl* crystal is made up of large number of cubes. One cube is shown in Fig. 4.25. The potential energy of the ions inside the crystal will be nearly same as due to all the ions. Let us now consider an ion *A*, say Na^+, in the interior of the crystal. When we bring it from infinity to position, work is done. To find the electrostatic energy associated with this ion we should consider all other ions inside the crystal. It has six nearest Cl^-, each at a distance *R* from it, the electrostatic energy of this ion due to these Cl^- is $-6e^2/4\pi\varepsilon_0 R$ (there are six pairs, each consists two equal and opposite charges *e* and $-e$ separated by a distance *R*). The next nearest neighbours are the twelve Na^+, each are at the diagonally opposite corners and at a distance $R\sqrt{2}$ from ion *A*. The energy contributed by these 12 ions is $+12\ (e^2/4\sqrt{2}\ \pi\varepsilon_0 R)$, the next nearest neighbours are eight Cl^-, each at a distance $R\sqrt{3}$ from ion *A*, thus contributing $-8(e^2/4\sqrt{3}\ \pi\varepsilon_0 R)$ energy. The next are six Na^+, each at a distance $R\sqrt{4} = 2R$ and contributing $6(e^2/8\pi\varepsilon_0 R)$ energy and so on. Proceeding in this way and adding all the terms, we get the total potential energy of ion *A*.

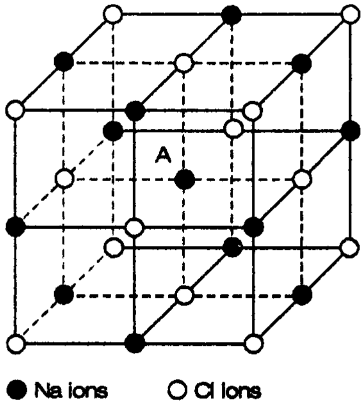


Fig. 4.25. Example 16.

$$U = \frac{1}{4\pi\varepsilon_0}\left[-\frac{6e^2}{R}+\frac{12e^2}{R\sqrt{2}}-\frac{8e^2}{R\sqrt{3}}+\frac{6e^2}{2R}\ldots\right]$$

In this infinite series the latter positive and negative terms cancel each other. Actual addition of large number of terms gives

$$U = -1.7476e^2/4\pi\varepsilon_0 R.$$

It is the energy per ion of an ionic crystal. In general the factor 1.7476 is replaced by α, known as *Madelung Constant,* which depends upon the crystal structure.

Total number of ions in 1 gm of NaCl

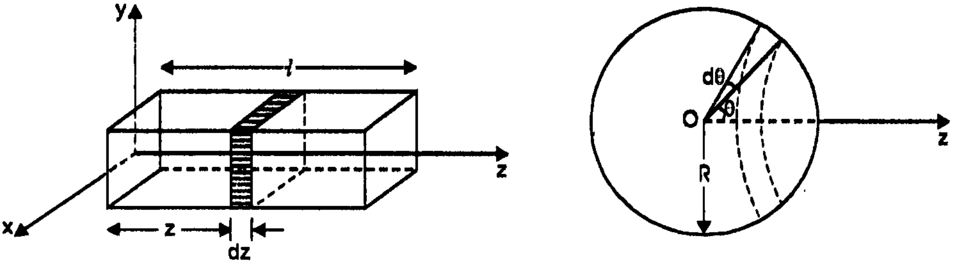
$$= \frac{2\times\text{Avogadro's number}}{\text{Molecular weight of NaCl}} = \frac{2\times 6.023\times 10^{23}}{58.5}$$

∴ The electrostatic potential energy per gm of NaCl crystal

$$= \frac{1}{2}\times\frac{2\times 6.023\times 10^{23}}{58.5}\times\left(-1.7476\frac{e^2}{4\pi\varepsilon_0 R}\right) = \frac{-1.80\times 10^{22}e^2}{4\pi\varepsilon_0 R}.$$

Here factor $\frac{1}{2}$ is introduced as each ion-pair is counted twice while summing up the energy per ion for a large number of ions. The –ve sign in the result shows that this much amount of energy is required to remove one ion from the crystal. The result thus obtained is not exactly correct, however it can be obtained by quantum mechanics.

Example 17. *Calculate the dipole moments of the following charge distributions : (a) A rectangular rod of charge with a charge density* $\rho = \alpha\ (z - l/2)$ *for* $0 \le z \le l$, *and (b) A sphere with an angular charge distribution having surface charge density given by* $\sigma = \sigma_0 \cos 0$.



(*a*) (*b*)

Fig. 4.26. Example 17.

(*a*) Let A be the area of cross section of the rod. The rod is of length l and placed along the z-direction. Its dipole moment is given by

$$\mathbf{p} = \int z dq \hat{z} = \int_0^l z(\rho A dz)\,\hat{z} = \hat{z}\alpha A \int_0^l z(z - l/2)\,dz$$

$$= \frac{1}{12}\alpha A l^3 \hat{z}.$$

It is independent of our choice of origin, as the total charge in the rod $\int dq = 0$.

(*b*) Let R be the radius of the sphere. Let us consider a ring at a distance z from the centre and of width dz. As clear from the Fig. 4.26 (*b*), radius of the ring $r = R \sin\theta$ and the thickness $dz = Rd\theta$. Thus the charge dq on this ring $= 2\pi(R\sin\theta)\,(Rd\theta)\sigma$. Therefore the dipole moment of the sphere

$$p = \int z dq \hat{z} = \int_0^\pi (R\cos\theta)\,(2\pi R^2 \sin\theta d\theta\, \sigma_0 \cos\theta)\hat{z}$$

$$= \hat{z}2\pi R^3\sigma_0 \int_0^\pi \cos^2\theta \sin\theta\, d\theta = \hat{z}\frac{4}{3}\pi R^3\sigma_0 = \hat{z}\sigma_0\, v,$$

where v is the volume of the sphere. Again as $\int dq = 0$, therefore the dipole moment is independent of the choice of the origin.

Example 18. *Calculate the resultant dipole moment of a system of charges, shown in Fig. 4.27. Hence determine the electric potential and intensity at a point quite far away from the charges.*

The monopole moment A_0 for this system of charges is zero as net charge is zero. To find dipole moment, it is easy to find its x-and y-components due to individual charges.

As each side is of length a, hence by the property of hexagonal each corner of the hexagon is at the same distance a *from* its centre.

Hence (x, y) co-ordinates of these corners are respectively $(-a, 0)$, $(-a/2, a\sqrt{3}/2)$, $(a/2, a\sqrt{3}/2)$, $(a, 0)$, $(a/2, -a\sqrt{3}/2)$ and $(-a/2, -a\sqrt{3}/2)$.

∴ Component of dipole moment along x-axis for all charges

$$p_x = \int_v x\rho dv = \int x dq$$

$$= 2q(-a) - 3q(-a/2) + 3q(a/2) - 2q(a) + 3q(a/2) - 3q(-a/2) = 2q\,a.$$

Similarly its y-component $p_y = \int_v y\rho dv$ for all charges.

$$p_y = 2q(0) - 3q\,(a\sqrt{3}/2) + 3q\,(a\sqrt{3}/2) - 2q(0) + 3q\,(-a\sqrt{3}/2) - 3q\,(-a\sqrt{3}/2) = 0$$

∴ Total dipole moment $\mathbf{p} = p_x\mathbf{i} + p_y\mathbf{j} = 2qa\mathbf{i}$.

Hence its component along $OP = 2qa\cos\theta$.

∴ $V_p = \mathbf{p.r}\ /4\pi\varepsilon_0 r^3 = 2qa\cos\theta/4\pi\varepsilon_0 r^2$.

Thus the electric field components are

$$E_r = -\partial V/\partial r = 4qa\cos\theta/4\pi\varepsilon_0 r^3 = 2p\cos\theta/4\pi\varepsilon_0 r^3.$$

and $$E_\theta = -(1/r)\,\partial V/\partial\theta = 2qa\sin\theta/4\pi\varepsilon_0 r^3 = p\sin\theta/4\pi\varepsilon_0 r^3.$$

∴ $$E = (p/4\pi\varepsilon_0 r^3)\,(1 + 3\cos^2\theta)^{1/2}$$

and the direction of this resultant with OP is given by

$$\phi = \tan^{-1}(E_\theta/E_r) = \tan^{-1}\left(\frac{1}{2}\tan\theta\right).$$

Example 19. *If four charges each of value q are placed at points (±a, 0) and (0, ± a) and a fifth charge −4q is at the origin. Calculate the potential and electric intensity at a point far away from the system of charges.*

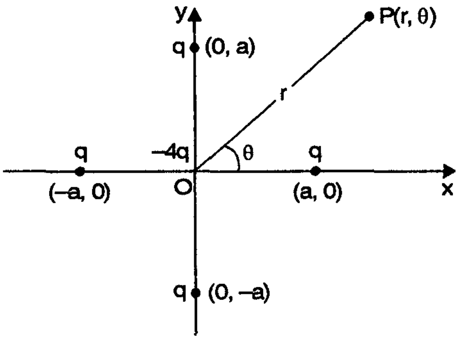


Fig. 4.28. Example 19.

Let us now consider a point P at a distance r from the origin. It is clear that for all charges the net value of $\int_v \rho dv$ is zero, hence monopole moment is zero. Similarly for all charges the net value of $\int_v \rho r dv$ is zero, hence dipole moment is also zero. The potential at point P, with polar coordinates (r, θ) is thus due to quadrupole moment of the charge distribution, where

$$Q = \int_v \frac{1}{2}\rho\,(3\cos^2\theta - 1)r^2 dv \text{ for all charges.}$$

As point P is very far from origin, the angle θ can be assumed for charges on the axis of x and $90° - \theta$ for the charges on the y-axis.

Contribution to Q of charge $-4q$ at point $(0, 0) = 0$

Contribution to Q of charge $+q$ at point $(+a, 0) = \frac{1}{2}qa^2(3\cos^2\theta - 1)$

Contribution to Q of charge $+q$ at point $(-a, 0) = \frac{1}{2}qa^2(3\cos^2\theta - 1)$

Contribution to Q of charge $+q$ at point $(0, +a) = \frac{1}{2}qa^2(3\sin^2\theta - 1)$

Contribution to Q of charge $+q$ at point $(0, -a) = \frac{1}{2}qa^2(3\sin^2\theta - 1)$

$$\therefore \qquad Q = qa^2[3\cos^2\theta - 1 + 3\sin^2\theta - 1] = qa^2.$$

Hence the potential at $P = V(\mathbf{r}, \theta) = Q/4\pi\varepsilon_0 r^3 = qa^2/4\pi\varepsilon_0 r^3$.

Potential is spherically symmetric as it does not depend on θ, hence the electric intensity at point P will be in radially outward direction.

$$\therefore \qquad \mathbf{E} = -\frac{\partial V}{\partial r}\hat{r} = \frac{3qa^2}{4\pi\varepsilon_0 r^4}\hat{r}.$$

Example 20. *A static charge is distributed in a spherical shell of inner radius R_1 and outer radius R_2. The electric charge density is given by $\rho = a + br$, where r is the distance from the centre. Find expression for the electric potential and energy density for $r < R_1$.*

Since P is a function of r only, therefore we can take a concentric spherical surface of radius r as the Gaussian surface and the electric field strength/intensity $\mathbf{E}$ may be obtained by using Gauss's law as :

$$\int_S \mathbf{E}d\mathbf{S} = \frac{1}{\varepsilon_0}\int \rho(r)\,dv = \frac{1}{\varepsilon_0}\int (a + br)\,4\pi r^2 dr$$

For $0 < r < R_1$, $\mathbf{E}_1 = 0$ as there is no charge within this region.

For $R_1 < r < R_2$, we get $4\pi r^2 E_2 = \frac{4\pi}{\varepsilon_0}\int_{R_1}^{r}(a + br')\,r'^2 dr'$

or $$\mathbf{E}_2 = \frac{1}{\varepsilon_0 r^2}\left[\frac{a}{3}(r^3 - R_1^3) + \frac{b}{4}(r^4 - R_1^4)\right]\hat{r}.$$

For $r > R_2$, we get $4\pi r^2 E_3 = \frac{4\pi}{\varepsilon_0}\int_{R_1}^{R_2}(a + br')\,r'^2 dr'$

or $$\mathbf{E}_3 = \frac{1}{\varepsilon_0 r^2}\left[\frac{a}{3}(R_2^3 - R_1^3) + \frac{b}{4}(R_2^4 - R_1^4)\right]\hat{r}.$$

The potential is assumed to be zero at $r \to \infty$, therefore the potential for $r < R_1$ is given by

$$V(r) = \int_\infty^r \mathbf{E}\cdot d\mathbf{r} = -\left[\int_\infty^{R_2}\mathbf{E}_3\cdot d\mathbf{r} + \int_{R_2}^{R_1}\mathbf{E}_2\cdot d\mathbf{r} + \int_{R_1}^{r}\mathbf{E}_1\cdot d\mathbf{r}\right]$$

$$= -\int_{\infty}^{R_2} \frac{1}{\varepsilon_0}\left[\frac{a}{3}\left(R_2^3 - R_1^3\right) + \frac{b}{4}\left(R_2^4 - R_1^4\right)\right]\frac{1}{r^2}\hat{r}.d\mathbf{r} - \int_{R_1}^{R_1} \frac{1}{\varepsilon_0}\left[\frac{a}{3}\left(r^3 - R_1^3\right) + \frac{b}{4}\left(r^4 - R_1^4\right)\right]$$

$$\frac{1}{r^2}\hat{r}\cdot d\mathbf{r} - \int_{R_1}^{r} 0\cdot dr$$

$$= \frac{1}{\varepsilon_0}\left[\frac{a}{2}(R_2^2 - R_1^2) + \frac{b}{3}(R_2^3 - R_1^3)\right]$$

As $\mathbf{E}_1 = 0$ for $r < R_1$, the energy density for $r < R_1$ is

$$W = \tfrac{1}{2}\varepsilon_0 E_1^2 = 0.$$

Example 21. *A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the centre. Find the field* **E** *and the potential ϕ at the centre of the hollow sphere.*

Consider an arbitrary point P of the hollow region at a distance **r** from the centre of the sphere. If O' is the centre of this hollow region, and let $O'P = r'$. Therefore $\mathbf{r}' = \mathbf{r} - \mathbf{a}$.

In the absence of the hollow region the electric field at point P would be

$$\mathbf{E}_1 = \frac{q}{4\pi\varepsilon_0 r^2}\hat{r} = \frac{\frac{4}{3}\pi r^3\rho}{4\pi\varepsilon_0 r^3}\mathbf{r} = \frac{1}{3}\frac{\rho}{\varepsilon_0}\mathbf{r}.$$

If only the spherical hollow region has charge density ρ, the electric field at point P would be

$$\mathbf{E}_2 = (\rho/3\varepsilon_0)\mathbf{r}'.$$

Hence the superposition theorem gives the electric field at P which is given by

$$\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}(\mathbf{r} - \mathbf{r}') = \frac{\rho}{3\varepsilon_0}\mathbf{a}.$$

Thus the field inside the hollow region is uniform and is the same at the point O' also.

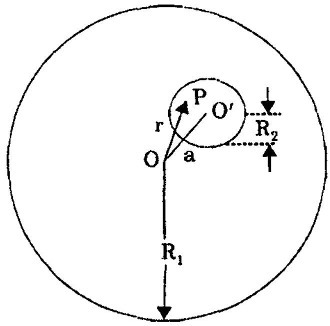


Fig. 4.29. Example 21.

We can find the electric fields inside and outside the sphere of radius R with a uniform charge density ρ, as

$$\mathbf{E}(\mathbf{r}) = \rho\mathbf{r}/3\varepsilon_0 \qquad r < R$$

$$= \frac{\rho R^3}{3\varepsilon_0 r^3}\mathbf{r} \qquad r > R.$$

The potential at any arbitrary point inside the sphere is

$$V(\mathbf{r}) = -\int_{\infty}^{r}\mathbf{E}.d\mathbf{r} = +\int_{r}^{R}\mathbf{E}.d\mathbf{r} + \int_{R}^{\infty}\mathbf{E}.d\mathbf{r}$$

$$= \frac{\rho}{6\varepsilon_0}(R^2 - r^2) + \frac{\rho R^3}{3\varepsilon_0}\cdot\frac{1}{R} = \frac{\rho}{6\varepsilon_0}(3R^2 - r^2)$$

As calculated the electric field, for our present problem we have

$$V \text{ at point } O' = V_1 - V_2 = \frac{\rho}{6\varepsilon_0}(3R_1^{\,2} - a^2) - \frac{\rho}{6\varepsilon_0}(3R_2^{\,2} - 0)$$

$$= \frac{\rho}{6\varepsilon_0}\left[3(R_1^{\,2} - R_2^{\,2}) - a^2\right].$$

Example 22. *Two large parallel metal plates are separated by a distance of 0.50 cm. If a 100 volts battery is connected across them, calculate the field between the plates and the charge surface density on them.*

As **E** is constant between the parallel large plates, the work done in carrying a unit charge from one plate to the other is (force/unit charge) × (distance) = *Ed*. Thus by the definition of potential

Potential difference between the plates $V_B - V_A = Ed$.

$$\therefore \quad 100 \text{ volts} = E \times 0.5 \times 10^{-2} \text{ metre.}$$

or
$$E = 2 \times 10^4 \text{ V/m (or N/C)}.$$

To find the surface charge density on the positive plate, let us consider a Gaussian box with one end in this surface and other opposite end in between these plates. As no lines of flux emerge from the end in the metal plate and sides of the box, the number of lines of flux passing through the opposite end face of the box is *ES*.

Thus by Gauss's law

$$\int \mathbf{E}\cdot d\mathbf{S} = q/\varepsilon_0 \text{ or } ES = \sigma S/\varepsilon_0.$$

$$\therefore \quad \sigma = \varepsilon_0 E = (8.85 \times 10^{-12} \text{ C}^2/\text{Nm}^2) \times (2 \times 10^4 \text{ N/C}) = 1.77 \times 10^{-7} \text{ C/m}^2.$$

Example 23. *The inside of a grounded spherical metal shell with inner radius R_1 and outer radius R_2, is filled with space charge of uniform charge density ρ. Find the electrostatic energy of the system. Calculate the potential at the centre.*

From Gauss's law

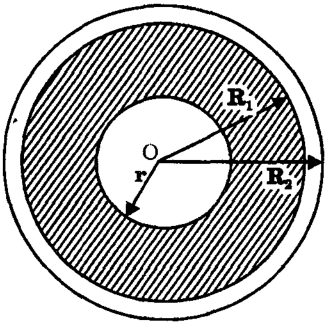


Fig. 4.30. Example 23.

$$\int \mathbf{E}\cdot d\mathbf{S} = q/\varepsilon_0 \quad \text{or} \quad E\cdot 4\pi r^2 = \tfrac{4}{3}\pi r^3\rho/\varepsilon_0$$

$$\therefore \quad \mathbf{E} = (r\rho/3\varepsilon_0)\,\hat{r}.$$

As the shell is grounded $V(R_1) = 0$ and $E = 0$ for $r > R_2$, Thus

$$V(r) = \int_r^{R_1} \mathbf{E}\cdot d\mathbf{r} = \int_r^{R_1}\left(\frac{\rho}{3\varepsilon_0}\right) r\,dr = \frac{\rho}{6\varepsilon_0}\left[R_1^{\,2} - r^2\right]$$

At the centre, $r = 0$ and the potential

$$V_0 = \rho R_1^2/6\varepsilon_0.$$

The electrostatic energy is thus given by

$$W = \int \frac{1}{2}\rho V d\tau = \frac{1}{2}\int_0^{R_1} \rho.\frac{\rho}{6\varepsilon_0}(R_1^{\,2} - r^2).4\pi r^2 dr$$
$$= 2\pi\rho^2 R_1^5/45\varepsilon_0.$$

Example 24. *A line charge of very large length along the z-axis, has a uniform charge density $\lambda(=\rho_l) = 0.5 \times 10^{-9}$ C/m. Find V_{AB}, where A is (2m, π/2, 0) and B is (4m, π, 5m).*

Since the field due to the line charge is completely in the radial direction, the dot product **E**.*d***l** results in *Edr*.

$$\therefore \quad V_{AB} = -\int_B^A \mathbf{E}\cdot d\mathbf{l} = -\int_B^A E\,dr = -\frac{\lambda}{2\pi\varepsilon_0}\int_B^A \frac{1}{r}dr$$
$$= -0.5 \times 10^{-9} \times 2 \times 9 \times 10^9 \left[\log r\right]_{4\text{m}}^{2\text{m}} = -9\left(\log_e r\right)_4^2 = 6.24 \text{ V}.$$

Example 25. *Eight negative charges each of value-e, are placed at the corners of a cube of sides x cm, at the centre to which a positive charge of +2e is placed. Calculate the electrostatic energy of the system.*

The given assembly of charges consists of (*i*) twelve pairs each with $-e$ and $-e$ charges separated by a distance $(AB = BC = \ldots) = x$ and contributing an electrostatic energy

$$U_1 = 12 \times \frac{(-e)(-e)}{4\pi\varepsilon_0 x} = \frac{12e^2}{4\pi\varepsilon_0 x}.$$

(*ii*) twelve pairs each with charge $-e$ and $-e$ separated by a diagonal distance ($AF = BE = ...$) $= x\sqrt{2}$ and contributing an electrostatic energy

$$U_2 = 12 \times \frac{(-e)(-e)}{4\pi\varepsilon_0 x\sqrt{2}} = \frac{12e^2}{4\pi\varepsilon_0 x\sqrt{2}},$$

(*iii*) eight pairs each with charge $-e$ and $2e$ separated by a distance ($OA = OB = ...$) $= x\sqrt{3}/2$ and contributing an electrostatic energy

$$U_3 = 8 \times \frac{(-e)2e}{4\pi\varepsilon_0 x\sqrt{3}/2} = -\frac{16e^2}{4\pi\varepsilon_0 x\sqrt{3}/2},$$

(*iv*) four pairs each with charge $-e$ and $-e$ separated by a distance ($AG = BH = CE = DF$) $= x\sqrt{3}$ and contributing an electrostatic energy

$$U_4 = 4 \times \frac{(-e)(-e)}{4\pi\varepsilon_0 x\sqrt{3}} = \frac{4e^2}{4\pi\varepsilon_0 x\sqrt{3}}.$$

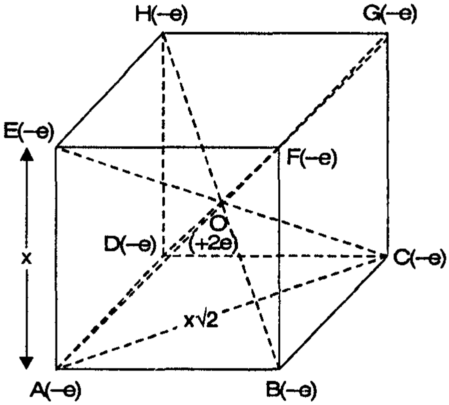


Fig. 4.31. Example 25.

Therefore the total electrostatic potential energy of the given system or the work done in forming such a system of charges

$$U = \frac{12e^2}{4\pi\varepsilon_0 x} + \frac{12e^2}{4\pi\varepsilon_0 x\sqrt{2}} - \frac{16e^2}{4\pi\varepsilon_0 x\sqrt{3}/2} + \frac{4e^2}{4\pi\varepsilon_0 x\sqrt{3}}$$

$$= \frac{4e^2}{4\pi\varepsilon_0 x}\left[3 + \frac{3}{\sqrt{2}} - \frac{8}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right] = 3.89 \times 10^{10}\frac{e^2}{x} \text{ joules.}$$

Example 26. *A conducting sphere of radius R carries a charge q. (a) Calculate the total energy stored in the surrounding space. (b) What is the radius of the spherical surface if half of the stored energy lies within it ?*

At any point P at a distance r from the centre of the sphere ($r > R$), the electric intensity E is given by

$$E = q/4\pi\varepsilon_0 r^2.$$

$\therefore$ Energy density at this point $= \frac{1}{2}\varepsilon_0 E^2 = q^2/32\pi^2\varepsilon_0 r^4$.

The energy which lies in a spherical shell between the radii r and $r + dr$ is

$$dU = (4\pi r^2\, dr).\,(q^2/32\,\pi^2\varepsilon_0 r^4).$$

$$\therefore \quad \text{Total energy stored } U = \int dU = \frac{q^2}{8\pi\varepsilon_0}\int_R^\infty \frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0 R}.$$

In the second part of the question

$$\frac{1}{2}U = \frac{q^2}{8\pi\varepsilon_0}\int_R^{R_0}\frac{dr}{r^2} = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{R} - \frac{1}{R_0}\right]$$

Using value of U, we get $R_0 = 2R$. Thus the radius of the spherical surface in which the half of the energy is stored will be $2R$.

Example 27. *Find the work done in moving a point charge $q = 4\ \mu C$ from the origin to $(2m, \pi/4, \pi/2)$ spherical coordinates, in the field*

$$\mathbf{E} = 5e^{-r/4}\,\hat{r} + (10/r \sin\theta)\,\hat{\phi}\ (V/m).$$

The work done in moving a point charge in an electric field **E** is given by

$$W = -q\int \mathbf{E}\cdot d\mathbf{l}.$$

In the spherical coordinates, the segment

$$dl = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}.$$

$$\therefore \quad W = -4\times10^{-6}\int\left[5e^{-r/4}\hat{r} + (10/r\sin\theta)\,\hat{\phi}\right]\cdot\left[dr\,\hat{r} + rd\theta\,\hat{\theta} + r\sin\theta d\phi\,\hat{\phi}\right]$$

$$= \left(-4\times10^{-6}\right)\left[\int_0^2 5e^{-r/4}dr + \int_0^{\pi/2}10d\phi\right] = 94.3\ \mu\text{J}.$$

Example 28. *The electrostatic potential at a point P due to an idealized dipole layer of dipole moment per unit area p_e on surface S is given by*

$$\varphi_p = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{p}_e\cdot\mathbf{r}}{r^3}dS,$$

where **r** *is the vector from the surface element to the point P.*

(a) If the dipole layer is of infinite extent and lying in the x-y plane of uniform moment density $p_e = p_e\,\hat{z}$. Determine whether ϕ is discontinuous across the layer?

(b) If a point charge +q is located at the centre of a spherical surface of radius a. On this surface there is a uniform dipole layer of dipole moment density p_e and surface charge density σ. Find p_e and σ so that the potential inside the surface will be just that of charge q, while zero outside the surface.

Ans. (*a*) By symmetry the electrostatic potential at point *P* is only dependent on the *z*-coordinate. If the point *P* is on the *z*-axis, using cylindrical coordinates (r, θ, z), then the potential at point *P* is

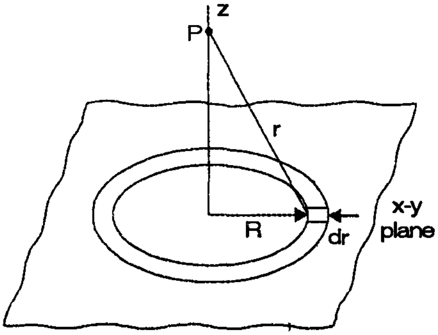


Fig. 4.32. Example 28.

$$\varphi_p = \frac{1}{4\pi\varepsilon_0}\int\frac{\mathbf{p}_e\cdot\mathbf{r}}{r^3}dS = \frac{1}{4\pi\varepsilon_0}\int\frac{p_e\cdot z}{r^3}dS.$$

Since $r^2 = R^2 + z^2$ and $dS = 2\pi R dr$, therefore

$$\varphi_p = \frac{2\pi p_e z}{4\pi\varepsilon_0}\int_0^\infty\frac{Rdr}{(R^2+z^2)^{3/2}} = \begin{bmatrix} p_e/2\epsilon_0 & z>0 \\ -p_e/2\epsilon_0 & z<0. \end{bmatrix}$$

Hence the electrostatic potential is discontinuous across the $x - y$ plane for which $z = 0$. The discontinuity is given by

$$\Delta\varphi = p_e/2\varepsilon_0 - (-p_e/2\varepsilon_0) = p_e/\varepsilon_0.$$

(*b*) In this problem, given that $\varphi = 0$ for $r > a$. Consequently $\mathbf{E} = 0$ for $r > a$. Using Gauss's law

$$\oint\mathbf{E}\cdot d\mathbf{S} = Q/\varepsilon_0, \ = Q/\varepsilon_0,$$

we get $Q = 4\pi a^2\sigma + q = 0$ or $\sigma = -q/4\pi a^2$.

If the potential at infinity is zero, then the potential outside the spherical surface will be zero everywhere. But the potential inside the sphere is $\varphi = q/4\pi\varepsilon_0 r$. For $r = a$, $\varphi = q/4\pi\varepsilon_0 a$. Therefore the discontinuity at the spherical surface is

$$\Delta\varphi = -q/4\pi\varepsilon_0 a.$$

We than have by comparing result of (*a*),

$$p_e/\varepsilon_0 = -q/4\pi\varepsilon_0 a \quad \text{or} \quad p_e = -q/4\pi a \text{ along } \mathbf{r}.$$

Example 29. *An isolated soap bubble of radius 1 cm is at a potential of 100 volts. If it collapses to a drop of radius 1 mm, what is the change of its electrostatic energy ?*

If the soap bubble of radius *r* carries a charge *q*, its potential is given by

$$V = q/4\pi\varepsilon_0 r.$$

Its electrostatic energy $\quad W = \frac{1}{2}\,qV = q^2/8\pi\varepsilon_0 r.$

As the radius changes from r_1 to r_2, the change of electrostatic energy

$$\Delta W = \frac{q^2}{8\pi\varepsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right].$$

The charge on the bubble of radius 1 cm and at a potential of 100 volts is

$$q = 4\pi\varepsilon_0 r_1 V = 4\pi\varepsilon_0 \times 10^{-2} \times 100 = 4\pi\varepsilon_0$$

$$\therefore \qquad \Delta W = \frac{(4\pi\varepsilon_0)^2}{8\pi\varepsilon_0}\left[10^3 - 10^2\right] = 2\pi\varepsilon_0 \times 900 = 5 \times 10^{-8} \text{ joule.}$$

Example 30. *A charge q is placed on the surface of an originally uncharged soap bubble of radius r_0. The radius increases to the value r due to the mutual repulsion of the charged surface. Show that* $q = [\frac{32}{3}\pi^2\varepsilon_0 P r_0 r\ (r^3 + r_0 r + r_0^2)^{1/2}]$, *where P is the pressure of the atmosphere.*

When the bubble of radius x is given a charge q, the outward mechanical pressure

$$= \frac{\sigma^2}{2\varepsilon_0} = \frac{1}{2\varepsilon_0}\left(\frac{q}{4\pi x^2}\right)^2 = \frac{q^2}{32\pi^2\varepsilon_0 x^4}.$$

Work done in expanding the bubble $= \int_{r_0}^{r} \frac{q^2}{32\pi^2\varepsilon_0 x^4} . 4\pi x^2 dx$

$$= \frac{q^2}{8\pi\varepsilon_0}\left(\frac{1}{r_0} - \frac{1}{r}\right) = \frac{q^2(r - r_0)}{8\pi\varepsilon_0 r r_0}.$$

This is the loss of potential energy as the work is done by the bubble. Work done in expansion against the atmospheric pressure P

$$= \int_{r_0}^{r} P . 4\pi x^2 dx = 4\pi P(r^3 - r_0^3)/3.$$

From the principle of conservation of energy, we get

$$q^2(r - r_0)/8\pi\varepsilon_0 r r_0 = 4\pi P(r^3 - r_0^3)/3$$

or

$$q = \left[\frac{32}{3}\pi^2\varepsilon_0 P\, r r_0(r^2 + r r_0 + r_0^2)\right]^{1/2}.$$

Example 31. *In the Vande Graaff generator, a surface charge density 10^{-7} C/m² is transported at a velocity of 3m/sec through a moving belt of width 10 cm. Calculate the charge collected in 10 sec.*

If σ is the surface charge density, v the speed of belt, and w its width, the current

$$I = \sigma v w.$$

$\therefore$ Total charge q collected in time t sec $= \sigma v w t$

$$= 10^{-7} \times 3 \times 0.1 \times 10 = 300 \text{ nC}.$$

Example 32. *A linear accelerator for the acceleration of protons to 45.3 MeV is designed so that, between any pair of accelerating gaps, the protons spend one complete radio frequency cycle inside a drift tube. The rf-frequency used is 200 Mc/sec. What is the length of the final drift tube ?*

Length of the final drift tube $L_f = \frac{v_f}{v} = \frac{1}{v}\sqrt{\frac{2E}{M}}$

$$= \frac{(2 \times 45.3 \times 1.6023 \times 10^{-13})^{1/2}}{200 \times 10^6\ (1.6724 \times 10^{-27})^{1/2}}$$

$$= 46.6 \text{ cm}.$$

ORAL QUESTIONS

1. Does it mean that no work is done in carrying a positive test charge from one point to the other on the equipotential surface ? Does it mean that no force will have to be exerted to carry the test charge from one point to the other on it ?
2. What can you say about E, (*i*) in the region where V is constant, (*ii*) at a point where V is zero ?
3. What is the optical analog of equipotential surface ?
4. What can be said about E, if it is defined as a gradient of scalar field ?
5. Does the potential of a positively charged insulated conductor have to be positive ?
6. A metal sphere suspended by a silk thread is placed in a uniform external electric field **E**. What is the magnitude of the electric field inside the sphere if (*i*) it is uncharged, (*ii*) charged ?
7. The emf of a battery can be measured accurately by (*a*) ammeter, (*b*) voltmeter, (*c*) transformer; (*d*) potentiometer.
8. How is electrostatic shielding achieved ?
9. What do you mean by saying that the electrostatic field is conservative ?
10. Electron volt is a unit of (*a*) charge, (*b*) current; or (*c*) energy.
11. Is it true that the electric field may be expressed as a negative gradient of electrostatic potential.
12. The electrostatic potential inside a charged, non conducting sphere (*a*) increases with distance from the centre; (*b*) decreases with distance from the constant; (*c*) is constant; (*d*) is zero.
13. N small drops of the same size are charged to V volt each. If they coalesce to form a single large drop then its potential will be (*a*) V/N, (*b*) NV; (*c*) $VN^{2/3}$, (*d*) $VN^{1/3}$.
14. An electron traveling in a region of electrostatic potential V_1 passes into a region of higher potential V_2, then the change in its *K.E.* is proportional to (*a*) $V_2 - V_1$, (*b*) $(V_2 - V_1)^{1/2}$, (*c*) $(V_2 - V_1)^2$.
15. Two metallic spheres of radii r_1 and r_2, carrying charges q_1 and q_2 respectively are connected by a thin wire. The potential of the spheres will be ...
16. Two identical conducting balls having positive charges q_1 and q_2 are separated by a distance r. If they are made to touch each other and then separated to the same distance, the force between them will (*a*) increase, (*b*) decrease; (*c*) remain unchanged, (*d*) zero.
17. Which of the following is the equipotential surface : (*a*) uniformly charged disc, (*b*) charged spherical conductor, (*c*) circle with a point charge at the centre, (*d*) spherical surface with a point charge at the centre ?
18. How are *MeV* and the atomic mass unit (*mu* or *u*) related ?
19. What is the energy associated with a doubly charged helium atom when passes through a potential difference of 1,000 volts ?
20. Can we obtain infinite potential from Van de Graff generator ? If not why ?

Ans. 1. Yes; **7.** (*d*); **10.** (*c*); **11.** (yes); **12.** (*b*); **13.** (*c*); **14.** (*a*); **15.** $(1/4\pi\varepsilon_0)(q_1 + q_2)/(r_1 + r_2)$; **16.** (*a*); **17.** (*d*); **18.** $1u = 931$ MeV; **19.** 2 keV.

PROBLEMS

1. Three charges q, $2q$ and $8q$ are placed along a 9 cm long straight line so that the potential energy of the system is minimum. Find the positions of the charges and determine the strength of the electric field at the position of q. [$2q$ at 0 cm, q at 3 cm, $8q$ at 9 cm; zero]
2. A potential difference of 100 V is applied between two parallel plates. Calculate the speed of a proton released from plate B just before it hits plate A. (1.4×10^5 m/sec)
3. A charge q is placed at each of four corners $(\pm a, 0, 0)$, $(0, \pm a, 0)$ of a square. Show that the potential at a point (x, y, z) near the origin is

$$V = \frac{q}{4\pi\varepsilon_0 a}\left[4 + \left(\frac{x^2 + y^2 - 2z^2}{a^2}\right) + \ldots\right].$$

4. A particle of charge $3 \times 10^{-9} C$ is situated in a uniform electric field directed to the left. In moving to the right a distance of 5 cm the work of an applied force is 6×10^{-5} Joule and the change in K.E. of the particle is 4.5×10^{-5} Joule. Calculate the magnitude of the electric field intensity and the work of the electrical force. (10^5 N/C, -15×10^{-6} joule)

5. An α-particle is accelerated through a p.d of 10^6 volts in an electrostatic generator. Find the kinetic energy acquired by it. What K.E would a proton acquire under these same circumstances? Which particle would acquire the greater speed, if both were started from rest ?
(3.2×10^{-13} joule, 1.6×10^{-13} Joule, proton)

6. An infinite number of charges each equal to q are placed along the x-axis at $x = 1, 2, 4, 8, \ldots\infty$. Find the potential and electric field at the point $x = 0$, due to this set of charges. Given that the consecutive charges have opposite signs. ($2q/3$, $4q/5$)

7. Four equal charges each of value $+q$ are symmetrically arranged along the circumference of a circle of radius 'a' and a charge $-7q$ is placed at the centre. Calculate the electric field at a far distant point at a distance r from this system. ($3q/4\pi\varepsilon_0 r^2$)

8. Calculate the electric potential and hence the electric intensity at a point P at a distance r ($>> a$) from the origin due to a system of charges, $-2q$ at the origin, $+q$ at point $(a, 0)$ and $+q$ at a point $(0, a)$. $qa\,(\cos\theta + \sin\theta)/4\pi\varepsilon_0 r^2$, $qa\,\sqrt{(5 + 6\sin\theta\cos\theta)}\,/\,4\pi\varepsilon_0 r^2$

9. Six equal positive charges q are kept at six corners of a regular hexagon of sides 'a', If a charge $-6q$ is placed at the centre, calculate the potential and hence electric intensity at a large distance r. ($3qa^2/8\pi\varepsilon_0 r^3$, $9qa^2/8\pi\varepsilon_0 r^4$)

10. The electric potential at any point in the $x - y$ plane is given by
$$\varphi = 5x(x^2 + y^2)^{1/2} + y\,(x^2 + y^2)^{-1/2}.$$
Find the cartesian components of the electric field intensity at that point.

11. Given point charges $+q$ at $(a/2, 0, 0)$ and $-q$ at $(-a/2, 0, 0)$ in empty space, show that for a point in xy plane at distance r from the origin the limiting form of electric field components when $r >> a$ is as below.
$$E_x = \frac{aq}{4\pi\varepsilon_0 r^2}\left(\frac{3x^2}{r^2} - 1\right), E_y = \frac{aq}{4\pi\varepsilon_0 r^2}\frac{3y}{r^2}.$$

12. Two fixed charges $-2q$ and q are located at the points with co-ordinates $(-3a, 0)$ and $(3a, 0)$ respectively in the $x - y$ plane. Show that all the points in the $x - y$ plane where the electric potential due to the two charges is zero, lie on a circle. Find its radius and location of the centre. If a particle of charge q starts from rest at the centre of the circle, show by a short qualitative argument that the particle eventually crosses the circle. Find its speed when it does so.
$$\left[4a, (5a, 0), q/\sqrt{4\pi\varepsilon_0 ma}\right]$$

13. The electric field in the atmosphere at the earth's surface is approximately 200 volts/metre, directed downward. At 1400 metre above the earth's surface the electric field in the atmosphere is only 20 volts/m, again directed downward. Calculate the average charge density in the earth's atmosphere below 1400 metre. (1.1×10^{-12} coulomb/m^2)

14. **A soap bubble 10 cm in radius with a wall thickness of 3.3×10^{-6} cm and charged to a potential of 100 volts breaks and falls as a spherical drop. Show that the potential of the drop is 10 kV.**

15. **An α-particle ($m = 4 \times 1.67 \times 10^{-27}$ kg, $q = 2 \times 1.6 \times 10^{-19}$ C) is emitted from a radon nucleus ($81 \times 1.6 \times 10^{-19} C$) with an energy 4.79 MeV. Assuming it is released at rest near the outer edge of the nucleus and then lies out into space, calculate the radius of the radium nucleus. (5.3×10^{-14} metre)**

16. **A rod of length L carries a charge $\lambda = kx^3$ C/m. Find the potential at the point $x = 0$, if the rod is along positive x-axis with left end at $x = b$. Also calculate the electric field.**
$3k \times 10^9\,[(L + b)^3 - b^3]$, $-9k \times 10^9\,[(L + b)^2 - b^2]$.

17. Two identical thin rings, each of radius R are coaxially placed a distance R apart. If Q_1 and Q_2 are the charges uniformly spread respectively on the two rings. Calculate the work done is moving a charge q from the centre of one ring to that of the other. $[q(Q_1 - Q_2)\,(\sqrt{2} - 1)/4\sqrt{2}\,\pi\varepsilon_0 R]$

18. A certain very long cylinder of radius b has a charge distribution $\rho = kr^3$, where r is the distance from the cylinder axis. Find the difference in potential between the surface of the cylinder and its axis. ($kb^5/25\ \varepsilon_0$)
19. Find the potential at an arbitrary point inside a uniformly charged sphere and prove that the potential has an extreme value at the centre of the sphere. ($\rho_0/2\varepsilon_0$) ($R^2 - r^2/3$)
20. A spherical charge distribution has a volume density $\rho = A/r$, where r is the distance from the centre of the distribution, inside this distribution and is zero outside to it. Calculate the potential at an arbitrary point. (A/ε_0) ($R - r/2$) for $r \le R$ and $AR^2/2\varepsilon_0 r$ for $r >> R$.
21. Positive charge is distributed with uniform density ρ through out the volume of a sphere. Show that the electric field at a distance r from the centre of the sphere is $E = \rho r/3\varepsilon_0$. Show that if a particle of mass m and charge $-q$ is placed in the sphere, it will execute simple harmonic vibration back and forth through the centre with a period $2\pi\ (2\varepsilon_0\ m/\rho q)^{1/2}$.
22. A charge q is distributed over two concentric hollow spheres of radii r and R ($> r$) such that the surface densities are equal. Find the potential at the common centre. [$q(r + R)/4\pi\varepsilon_0(R^2 + r^2)$]
23. A static electric charge is distributed in a spherical shell of inner radius R_1 and outer radius R_2. The electric charge density is given by $\rho = a + br$, where r is the distance from the centre and zero everywhere else. Find an expression for the electric field everywhere in terms of r. Also find expressions for the electric potential and energy density for $r < R_1$. Given the potential to be 0 at $r \to \infty$.

$$E = 0 \text{ for } r < R;\ \mathrm{E} = \frac{1}{\varepsilon_0 r^3}\left[\frac{a}{3}(r^3 - R_1{}^3) + \frac{b}{4}(r^4 - R_1{}^4)\right]\mathrm{r} \text{ for } R_1 < r < R_2;$$

$$\mathrm{E} = \frac{1}{\varepsilon_0 r^3}\left[\frac{a}{3}(R_2{}^3 - R_1{}^3) + \frac{b}{4}(R_2{}^4 - R_1{}^4)\right]\mathrm{r} \text{ for } r > R_2,$$

$$V(r) = \frac{1}{\varepsilon_0}\left[\frac{a}{3}(R_2{}^2 - R_1{}^2) + \frac{b}{4}(R_2{}^3 - R_1{}^3)\right] \text{ and } W = 0.$$

24. If ${}^{238}_{92}\mathrm{U}$ nucleus splits in two equal fragments. Calculate the (*i*) repulsive force acting on each fragment and (*ii*) the mutual electric potential energy of the two fragments. Given that the radius of the initially spherical ${}^{238}\mathrm{U}$ nucleus is 8.0×10^{-15} metre and the fragments are equal in size and charge, spherical and just touching.
25. Show that the potential energy of two coplaner dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ at a distance r apart is $(p_1 p_2/4\pi\varepsilon_0 r^3)$ $(\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2)$, where θ_1 and θ_2 are the angles made by $\mathbf{p}_1$ and $\mathbf{p}_2$ respectively with the line joining their centres.
26. A spherical conducting shell of radius r, central at the origin has a potential field,

$$V = V_0 \text{ at } r \le a, \quad \text{and} \quad V = V_0 a/r \text{ at } r > a.$$

Find an expression for the stored energy due to this potential. ($2\pi\varepsilon_0 V_0{}^2 a$)

27. Two coaxial cylinders form the plates of a capacitor. The radius of the outer cylinder is fixed. Show that if the radius of the inner cylinder is $1/e$ times the radius of the outer cylinder, the field strength at the surface of the inner cylinder is least for a given potential difference between them.
28. Two protons in a nucleus of ${}^{238}\mathrm{U}$ are 6.0×10^{-15} meter apart. What is their mutual electric potential energy ? The charge on a proton is 1.6×10^{-19} coul. (3.84×10^{-14} Joule.)
29. A sphere of radii 1 cm is charged to a potential of 3000 volts. Calculate the outward pull per unit area.
30. Given the potential $V = (10/r^2) \sin\theta \cos\phi$, find the work done in moving a 10 μC charge from point A (1, 30°, 120°) to B (4, 90°, 60°). (28.125 μJ)
31. If $V = \rho^2 z \sin\phi$, calculate the energy with in the region defined by $1 < \rho < 4$, $-2 < z < 2$ and $0 < \phi < \pi/3$. (6.612 nJ)
32. In a Van de Graff generator, the belt of width 10 cm, moving with a velocity 10m/sec carries charge 0.6 $\mu C/\mathrm{m}^2$. Find the potential difference between the dome and the base, if the leakage paths have resistance $10^{14}\Omega$. (60 MV)
33. In example 32, of a linear accelerator problem, calculate the K.E. of the protons injected into the accelerator if the first drift tube is of 5.35 cm length. If the peak accelerating potential is 1.49 MV, calculate the total length of the accelerator. (600 keV, 9.4 m)

5

CAPACITORS AND ELECTROMETERS

5.1 CAPACITANCE

We know that the potential of a charged conducting sphere of radius R, assuming the sphere as completely isolated, is given by

$$V'_+ = \frac{1}{4\pi\varepsilon_0}\frac{q}{R}, \qquad \text{...(1)}$$

where q is the charge on the sphere.

Let us now imagine a second sphere of radius R, carrying a negative charge $-q$ and situated at a large distance (>> R) from the first sphere so that one can consider each sphere as electrically isolated. The potential of this sphere is given by

$$V'_- = -\frac{1}{4\pi\varepsilon_0}\frac{q}{R}. \qquad \text{...(2)}$$

$\therefore$ Potential difference between these spheres

$$V' = V'_+ - V'_- = \frac{1}{4\pi\varepsilon_0}\frac{2q}{R} \qquad \text{...(3)}$$

or

$$q = (2\pi\varepsilon_0 R)\, V' = C'V'. \qquad \text{...(4)}$$

This relation shows that the potential difference is proportional to the charge on each sphere. The constant of proportionality depends upon R and is known as *capacitance* of the system of two spheres. It is denoted by the symbol C'.

If these spheres are brought close together, the presence of each will now spoil the spherical symmetry of the lines of force starting from each and the above relations do not hold good. In the presence of the negatively charged sphere the potential of the positively charged sphere will decrease from V'_+ to some lower value V_+. Similarly the potential of the negatively charged sphere will be raised from V'_- to a higher value V_-. Thus the potential difference between two spheres is reduced considerably, although the charges on the spheres have not changed. The capacitance of the system of two spheres defined as $C = q/V$ is hence increased.

In this way we see that the capacitance of the system of two conductors can be increased by bringing the conductors close together.

The capacitance of a single isolated spherical conductor, such as a sphere of radius R can be obtained by assuming the second spherical conductor carrying an equal and opposite charge and of very large (infinite) radius or placed at infinite distance. As the potential of the infinitely distant sphere is zero, hence the capacitance of a single isolated conductor is given by

$$C = q/V = 4\pi\varepsilon_0 R. \qquad ...(5)$$

As $C = q$ when $V = 1$, hence the capacity of a conductor may be defined as the charge required to raise its potential by unity. The SI unit of capacitance is coulomb/volt, The term *farad* is used to represent it, after the name of Michael Faraday. The submultiples of the farad, microfarad (10^{-6} farad = $1\mu F$) and the micro-micro or pico farad (10^{-12} farad = $1\ \mu\mu F = 1pF$) are commonly used in practice as the farad is a large quantity of capacitance.

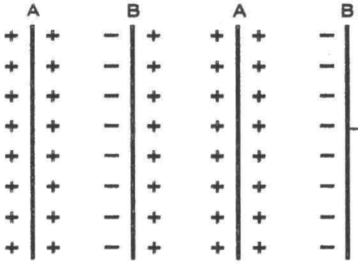


Fig. 5.1. Capacitor.

When a conductor B is brought near a charged conductor A, inductive displacement takes place, the near side of B exhibiting a negative charge and the far side a positive charge. This negative charge tends to lower the potential of A and the positive charge on B tends to raise it. These opposite charges nearly counteract each other, but the negative charge has little greater influence as it is nearer to the conductor A, thus lowers the potential of A. If the conductor B is now earthed, the positive charge on B disappears and negative charge does not go to earth as it is attracted by the +ve charge on the conductor A. Hence the potential of conductor A is much lowered due to the presence of –ve charge on the conductor B. Thus we see that *the potential of an insulated charged conductor can be considerably decreased and hence its capacity can be considerably increased when an earthed conductor is brought near to it.*

The arrangement in which one conductor is charged and other is earthed is name as *capacitor* or *condenser*. The capacitance of a capacitor depends on the geometry of each plate, their separation and the medium in which these plates are placed. In the definition of capacitance charge q is not the net charge, which is zero but the magnitude of the charge on either plate.

Charged capacitor is analogous to the rigid container of volume V containing n molecules of an ideal gas. We know that for a perfect gas

$$PV = nRT \text{ or } V = nRT/P = n/(P/RT). \qquad ...(6)$$

Its comparison with Eq. (5) shows that the capacity C of a capacitor is analogous to the volume V of the container, the charge q on the capacitor to the number of gas molecules in the container and the electrical breakdown limit to the breakdown of the walls. ***Capacitors are very useful to Physicists and Engineers.*** **For example :**

(*i*) Capacitor may be used to establish desired electric *field*. The behaviour of a dielectric material be studied by placing it in an electric field produced by a capacitor.

(*ii*) Capacitor can be used for storing *energy*, as it can confine strong electric fields to small volume. Energy accumulated in a large capacitor is used to accelerate the electrons by discharging the capacitor in a much shorter time as in *electron synchrotrons*.

(*iii*) Capacitors are very useful to reduce voltage fluctuations in electronic power supplies, to transmit signals, to detect electromagnetic oscillations at radio frequencies and to other important electronic circuits.

5.2 CALCULATION OF CAPACITANCES

The method for the calculation of capacitance involves integration of the electric field between two conductors or the plates which are just equipotential surfaces to obtain the potential difference. Thus the capacitance of a capacitor

$$C = \frac{q}{V} = \frac{q}{-\int \mathbf{E}\cdot d\mathbf{l}}. \qquad ...(7)$$

Let us now consider different cases :

(*a*) Spherical conductor : Let us now consider an isolated spherical conductor of radius R. Capacitance of such a spherical conductor

$$C = \frac{q}{V} = \frac{q}{q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R. \qquad ...(8)$$

(*b*) Capacity of a spherical conductor enclosed by an earthed concentric spherical shell : Let the radii of the inner sphere and the outer spherical shell be a and b respectively. If a charge $+q$ be given to the inner sphere, a charge $-q$ will be induced on the inner surface of the outer sphere. We know that the intensity at any point inside a hollow charged conductor is zero, thus the intensity inside the region between two spheres will be that due to the charge on inner sphere only. Intensity at any point P, distant r from the centre of the inner sphere can be obtained by using Gauss's law. Considering a gaussian spherical surface through this point P, we have

$$\int \mathbf{E}\cdot d\mathbf{S} = (1/\varepsilon_0)\Sigma q,$$

$$\therefore \quad E.4\pi r^2 = q/\varepsilon_0, \text{ or } E = q/4\pi\varepsilon_0 r^2.$$

This relation indicates that the field between two concentric spheres is independent of the charge on the outer sphere.

$\therefore$ Potential difference between the spheres

$$V = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = \frac{-q}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2}. = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right].$$

Hence the capacitance of this system

$$C = \frac{q}{V} = 4\pi\varepsilon_0 \frac{ab}{b-a}. \qquad ...(9)$$

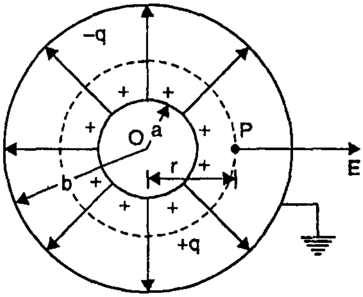


Fig. 5.2. Spherical capacitor.

Thus we see that if $b = \infty$, $C = 4\pi\varepsilon_0 a$, which corresponds to that of an isolated sphere, *i.e.*, the charged sphere may be regarded as a capacitor in which the outer surface has been removed to infinity.

(c) Capacity of an earthed sphere enclosed by a concentric spherical shell : Let a and b be the radii of an earthed sphere and the hollow spherical shell respectively. If the shell is thick, let us assume that its outer radius is c. Suppose the total charge given to the outer shell B is q and the induced charge on the inner sphere A is $-q_1$. The charge $+q_1$ will be on the inner surface of the outer sphere and the remaining charge q_2 $(= q - q_1)$ will remain on the outer surface of the outer sphere. In this way system works as the combination of two capacitors : (*i*) that between sphere A and the inner surface of shell B and (*ii*) that between outer surface of shell B and the earth. Capacitance of the former is $4\pi\varepsilon_0 ab/(b-a)$ while that of the latter is $4\pi\varepsilon_0 c$.

$$\therefore \text{ Capacity of the system } C = 4\pi\varepsilon_0\left[\frac{ab}{b-a} + c\right].$$

This relation may also be derived as follows : The potential is zero at infinity and also that of the inner sphere, the potential of the outer shell is thus given by

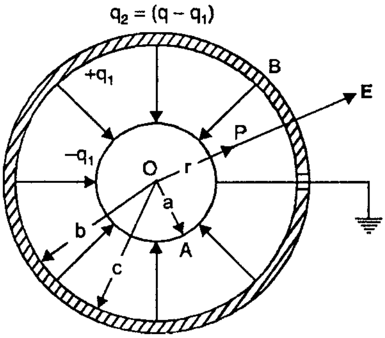


Fig. 5.3. Earthed sphere surrounded by spherical shells.

$$V = -\int_{\infty}^{c} \mathbf{E}\cdot d\mathbf{r} = -\int_{a}^{b} \mathbf{E}\cdot d\mathbf{r}$$

$$= -\int_{\infty}^{c} \frac{1}{4\pi\varepsilon_0} \frac{q_2}{r^2} dr = \int_{a}^{b} \frac{1}{4\pi\varepsilon_0} \frac{q_1}{r^2} dr$$

or
$$\frac{1}{4\pi\varepsilon_0} \frac{q_2}{c} = \frac{q_1}{4\pi\varepsilon_0} \left[\frac{1}{a} - \frac{1}{b}\right].$$

$\therefore$ $q_2 = 4\pi\varepsilon_0 c\, V$ and $q_1 = 4\pi\varepsilon_0\, V ab/(b-a)$,

Hence $q = q_1 + q_2 = 4\pi\varepsilon_0 V[c + ab/(b-a)]$.

$\therefore$ Capacity $C = \dfrac{q}{V} = 4\pi\varepsilon_0 \left[c + \dfrac{ab}{b-a}\right]$. ...(10)

(*d*) Capacity of a cylindrical capacitor : When a metallic cylinder of radius a is placed coaxially inside an earthed hollow metallic cylinder of large radius b, we get cylindrical capacitor. A submarine cable is an example of such a capacitor in which the inner conductor consists of copper cable and the sea water is the outer earthed cylinder.

If a charge q is given to the inner cylinder, induced charge $-q$ will reach the inner surface of the outer cylinder. Assume that the capacitor is of very large length ($l >> b$) so that the lines of force are radial and the fringing of lines of force at the end can be ignored.

To find the electric field between the cylinders, let us consider a gaussian cylinder of radius r ($> a$ and $< b$) and length l'. From Gauss's law, we have

$$\int \mathbf{E}\cdot d\mathbf{S} = (1/\varepsilon_0)\, \Sigma q.$$

As there is no flux through the ends of the cylinder ($\mathbf{E} \perp d\mathbf{S}$) and **E** is constant on and $\perp$ to the gaussian curved surface, hence

$$E.2\pi r l' = (1/\varepsilon_0)\, \lambda l' \text{ or } E = \lambda/2\pi\varepsilon_0 r,$$

where λ is the charge per unit length on the inner cylinder. Therefore, the potential difference between the cylinders

$$V = -\int_{b}^{a} \mathbf{E}\cdot d\mathbf{r} = -\int_{b}^{a} \mathbf{E}\cdot d\mathbf{r} = -\int_{b}^{a} \frac{\lambda}{2\pi\varepsilon_0 r} dr$$

$$= \frac{\lambda}{2\pi\varepsilon_0} \log_e \left(\frac{b}{a}\right).$$

If the capacitor is of length l, its capacity

$$C_l = \frac{q}{V} = \frac{l\lambda}{(\lambda/2\pi\varepsilon_0)\log_e(b/a)} = \frac{2\pi\varepsilon_0 l}{\log_e(b/a)}. \quad ...(11)$$

$\therefore$ Capacity per unit length $C = 2\pi\varepsilon_0/\log_e(b/a)$. ...(12)

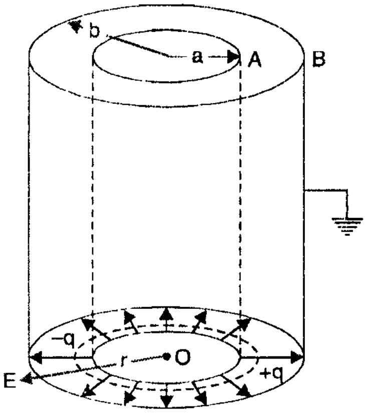


Fig. 5.4. Cylindrical capacitor.

(c) Capacity of a parallel plate capacitor : Two metallic parallel plates of any shape but of same size and separated by a small distance constitute parallel plate capacitor. If charge q is given to the plate P, it will induce a charge $-q$ to the upper surface of the earthed plate Q. Let A be the area of each plate. If the separation d is assumed to be very small compared with the plate dimensions, so that the fringing effect at the edges of the plate may be neglected. Thus electric lines of force starting from plate P and ending at the Q plate are parallel to each other and perpendicular to the plates.

We know from § 3.9 (*H*) that the intensity at a point between the parallel plates is σ/ε_0, where σ is the surface density of charge. Thus the potential of plate P is given by

$$V = -\int_Q^P \mathbf{E}\cdot d\mathbf{r} = \int_0^d \frac{\sigma}{\varepsilon_0}\,dr = \frac{\sigma d}{\varepsilon_0}.$$

Here the distance is measured from the lower plate Q in upward direction, and $\mathbf{E}$ and dr are opposite to each other.

$$\therefore \quad \text{Capacity } C = \frac{q}{V} = \frac{\sigma A}{\sigma d/\varepsilon_0} = \frac{\varepsilon_0 A}{d}. \qquad ...(13)$$

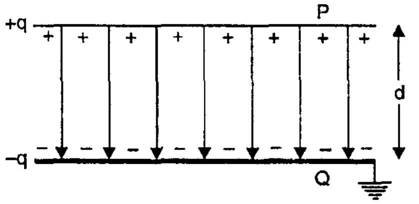


Fig. 5.5. Parallel plate capacitor.

Instead of two plates, if there are n similar plates at equal distances from each other and the alternate plates are connected together, the capacitance of the arrangement is given by $C = (n-1)\varepsilon_0 A/d$.

From relations (8), (9), (10), (12) and (13), it is clear that relations for capacitance depend only on geometrical factors. These relations suggest that ε_0, the constant obtained in Coulomb's law may be measured by building a capacitor of accurately known dimensions and determining its capacitance experimentally by measuring charge q and potential V.

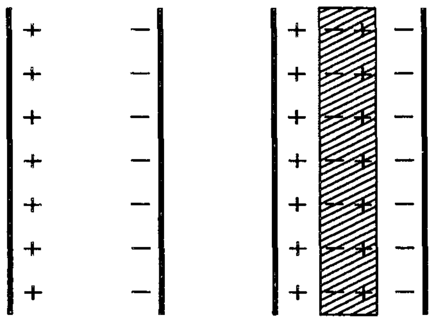
5.3 EFFECT OF A DIELECTRIC

Relations obtained earlier for the capacitances of different shapes and sizes hold only when the plates are in a vacuum or air. Michael Faraday, in 1837, took two identical capacitors and placed a dielectric in one and in other an air at normal pressure. When both capacitors were charged to the same potential difference, he found experimentally that the charge on the one containing the dielectric was greater than that on the other, the ratio was a constant for a particular dielectric. From the relation $C = q/V$, it follows that,

$$\frac{\text{Capacitance with the dielectric } C_d}{\text{Capacitance without the dielectric } C} = a \text{ constant } k.$$

It is a dimensionless quantity as it is just a ratio. It is therefore called *dielectric constant*. Its value is 1 for vacuum and ∞ for metals.

The above experiment may be repeated by inserting a sheet of dielectric between the plates of a capacitor. It will be observed that the potential difference decreases ($V_d/V = 1/k$) as we insert the dielectric and it returns to its original value when the dielectric is removed, showing that the original charges on the plates were not affected by insertion of the dielectric. This reduction in potential difference implies a reduction in the electric intensity, which further implies a reduction in the net charge per unit area. As there is no leakage of charge, the reduction in the net charge is possible as a result of the induced charges appearing on the two surfaces of the dielectric, as shown in Fig. 5.6 (*b*).



(*a*) Without dielectric (*b*) With dielectric

Fig. 5.6. Parallel plate capacitors.

The dielectrics are often used to permit a higher potential difference to be applied between the plates. Thus in the presence of the dielectric between the plates, the capacitances are thus given as :

$C_d = k\,C = k\varepsilon_0 A/d$ for parallel plate capacitor,

$= 2\,\pi k\varepsilon_0 l/\log_e(b/a)$ for cylindrical capacitor of length l,

$= 4\,\pi\varepsilon_0 kR$ for spherical conductor.

In general we can write the capacitance of any capacitor as

$$C_d = k\,\varepsilon_0 L, \qquad ...(14)$$

where L depends on the geometry and has the dimensions of a length.

5.4 ENERGY STORED BY A CHARGED CAPACITOR

A charged capacitor stores an electric potential energy which is equal to the work required to charge it. If the charging is done by a battery, electrical energy is stored at the expense of chemical energy of battery. It is identical to the potential energy stored in a compressed spring or the gravitational potential energy stored in the earth-moon system. This energy can be recovered if the capacitor is allowed to discharge.

For the calculation of the energy stored in a charged capacitor, let q be the charge on the capacitor and V its potential at any instant t. The additional work needed to bring dq charge on the capacitor against the electric field due to the charge q already present will be

$$dW = Vdq = (q/C)dq.$$

$\therefore$ Total work to charge a capacitor to a charge q_0

$$W = \int dW = \int_0^{q_0}\left(\frac{q}{C}\right)dq = \frac{q_0^2}{2C}. \qquad ...(15)$$

$\therefore$ Energy stored by a charged capacitor

$$U = W = \tfrac{1}{2}q_0^2/C = \tfrac{1}{2}CV^2. \qquad ...(16)$$

If q_0 is in coulombs, C in farads and V in volts, U will be in *joules*.

The relation (15) shows that the charged capacitor is the electrical analog of a stretched spring whose elastic potential energy is $\frac{1}{2}kx^2$. The charge q_0 is analogous to the elongation x and $1/C$, *i.e.*, the reciprocal of capacitance to the force constant k.

We may consider that the work done in charging the capacitor produces an electric field. To determine the relationship between stored energy and electric field, let us choose a parallel plate capacitor of area A and plate separation d. It is assumed that the electric field is uniform in the region between two plates and the fringing effect is negligible. Energy stored by such a capacitor is given by

$$U = \frac{1}{2}\frac{q^2}{C} = \frac{1}{2}\frac{(\sigma A)^2}{\varepsilon_0 A/d} = \frac{1}{2}\frac{\sigma^2 Ad}{\varepsilon_0}, \qquad ...(17)$$

or energy stored per unit volume $u = \sigma^2/2\varepsilon_0$. ...(18)

But we know that the electric field **E** due to a parallel plate conductor with a surface charge density σ is σ/ε_0. Thus we get

$$U/\text{volume} = \sigma^2/2\varepsilon_0 = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\varepsilon_0 \mathbf{E}\cdot\mathbf{E}. \qquad ...(19)$$

If the capacitor is filled with a medium of *dielectric constant k*, the value of capacitance $C = k\varepsilon_0 A/d$.

$\therefore$ Energy stored per unit volume $u = \frac{1}{2}k\varepsilon_0 E^2 = \frac{1}{2}k\varepsilon_0 \mathbf{E}\cdot\mathbf{E}$. ....(20)

This relation shows that the energy stored per unit volume depends on E^2. If E is the electric field in a space of volume dv, then the total energy stored in an electrostatic field is given by

$$U = \tfrac{1}{2}\varepsilon_0 k\int E^2 dv. \qquad ...(21)$$

This energy is known as *self energy*. Self energy does not change when we change the relative position of the charges and is not included when we calculate the energy stored in an assembly of charges. The above relations are derived for the special case of a parallel plate capacitor.

Let us consider **two isolated charges,** which are far apart. The self energy of each charge is given by Eq. (21). The total stored energy, when the charges are brought close enough together so that there is an appreciable overlap of the field ($\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$), is thus given by

$$\begin{aligned} U &= \tfrac{1}{2}\varepsilon_0\int(\mathbf{E}_1+\mathbf{E}_2)\cdot(\mathbf{E}_1+\mathbf{E}_2)\,dv \\ &= \tfrac{1}{2}\varepsilon_0\int E_1^2\,dv + \tfrac{1}{2}\varepsilon_0\int E_2^2 dv + \tfrac{1}{2}\int 2\mathbf{E}_1\cdot\mathbf{E}_2 dv. \end{aligned} \qquad ...(22)$$

The first two terms are the self energies of the two separate charges and the third term is the energy resulting from the overlapping of the two fields and is equal to the *potential energy of one charge in the field of the other.*

If the *potential is kept constant* by connecting the capacitor with a battery, the expression for the energy stored by the capacitor of capacitance C can be obtained as under : If the distance between the plates is decreased, the capacity increases and the charge is accumulated and energy is thus stored. Work done by the battery when a charge dq increases the capacitance by dC, is given by

$$dW = Vdq = V^2 dC.$$

∴ Total work done by the battery

$$W = V^2 \int_{C_1}^{C_2} dC = V^2[C_2 - C_1].$$

If $C_1 = 0$, *i.e.*, the plates are initially at infinite distance apart and $C_2 = C$, then

$$W = V^2 C = qV \text{ joules.} \qquad ...(23)$$

5.5 FORCE BETWEEN CAPACITOR PLATES

Let us consider a capacitor consisted of two parallel plates P and Q. Plate P is given a charge $+q$ and plate Q is earthed. Lines of force start from plate P and end at plate Q. If the plates are assumed to be of infinite size and the end effects (curved lines at the ends) are neglected, one can assume the electric intensity of the field between these plates as equivalent to the field of an infinite charged conducting plate. There are two possible cases :

(*a*) When the charge remains the same : The force of attraction per unit area between two plates of a capacitor is equal to the outward electrical force per unit area on the surface of plane P and is given by

$$F = \sigma^2/2\varepsilon_0 = \tfrac{1}{2}\varepsilon_0 E^2 \text{ Newton/meter}^2. \qquad ...(24)$$

(*b*) When the potential difference between the plates remains constant : The potential is applied by connecting a battery across the plates of a capacitor, hence the expression for the force of attraction should be expressed in terms of potential difference between the plates. Since

$$Q = CV = (\varepsilon_0 A/d)\, V = \sigma A,$$

or charge per unit area $\sigma = \varepsilon_0\, V/d$, therefore

$$F = \frac{1}{2\varepsilon_0}\left(\frac{\varepsilon_0 V}{d}\right)^2 = \frac{\varepsilon_0}{2}\frac{V^2}{d^2}. \qquad ...(25)$$

This relation is used in an attracted disc electrometer. The expression for the force can also be derived from the expression for the energy stored. If **F** is the attractive force between the plates, the external force $-$**F** is required to move the plates apart, the external work required to increase the plate separation by an amount d**x** will be $-$**F**·d**x**. Hence the increase in internal energy $dU = -$**F**·d**x**. As the energy stored for the parallel plate capacitor of separation x

$$U = \tfrac{1}{2} q^2/C = q^2 x/2\varepsilon_0 A.$$

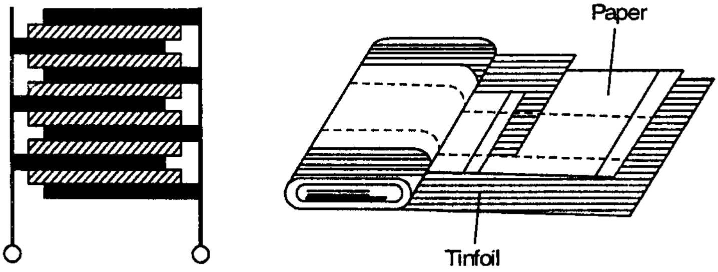
$$\therefore \qquad F = -\frac{dU}{dx} = -\frac{d}{dx}\left[\frac{q^2 x}{2\varepsilon_0 A}\right] = -\frac{q^2}{2\varepsilon_0 A}. \qquad ...(26)$$

As q^2 is positive and A is $+ve$, hence F is negative which indicates that **F** is in the negative **x** direction, giving *attraction* between the plates. In vector form

$$\mathbf{F} = -(q^2/2\varepsilon_0 A)\,\hat{x}. \qquad ...(27)$$

5.6 FORM OF CAPACITORS

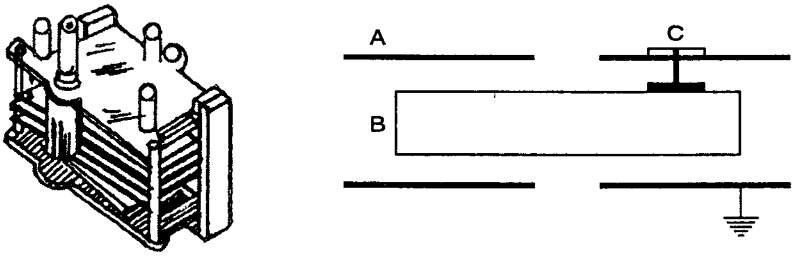
Mica and Paper Capacitors : The capacitors most commonly used are of the parallel plate type. Parallel plate capacitors consist of two metallic foils with *mica or paraffined paper as a* *dielectric between them.* In the former case capacity can be increased by connecting alternating plates to the same terminal (*i.e.*, connecting capacitors in parallel) and in the latter case it is done by taking two strips of metal foil (tin) insulated by the dielectric (paraffined paper) and rolled up in the form of a cylinder for compactness. The paraffinwaxed paper is a cheaper dielectric and can easily by rolled, but the power absorbed in such a capacitor is large in *ac* circuits. Since mica absorbs little power, hence mica capacitors are generally used in *ac circuits, e.g.*, in radio and electronic sets. Apart from the capacitance, an almost equally important factor is the *capacitor working voltage,* which depends on the thickness and dielectric strength of the dielectric. Because of energy dissipated in the capacitor, the working voltage decreases as the frequency of the source is increased, roughly as $1/f^{1/2}$. Mica capacitors are having capacitance range from 0.0001 to 0.05 μF at working voltages up to 500 volts and paper capacitors from 0.01 to 10 μF at working voltages up to 1000 volts. Due to this reason the thickness of dielectric is very small in the latter case.



(*a*) (*b*)

Fig. 5.7. Capacitors : (*a*) mica, (*b*) paper.

Electrolytic Capacitors : Electrolytic capacitors are used when a very high capacitance is required. This type of capacitor consists of two plates of aluminium as electrodes and the extremely thin film of aluminium oxide (10^{-6} cm thick) as dielectric. This film is formed on the positive aluminium electrode by the electrolysis of a solution of aluminium borate. In the *wet* type, the anode is in the form of a cylinder, which is immersed in the electrolyte contained in a metal can, which works as a cathode. In the *dry* type, both plates consist of long strips of aluminium foil, on one of which aluminium oxide is deposited electrically and is separated by cotton or paper guaze soaked in the electrolyte and then rolled up in a cylindrical form. Oxide film on aluminium has a low resistance to current flow in one direction and high resistance in other direction. Thus *electrolytic capacitors can only be used in unidirectional supplies,* such that the potential of the oxide plate is always +ve relative to the other plate. These are widely used in radio-sets as a *smoothing capacitors.* As a capacitance varies with temperature, a stable capacitor must be used in radio-sets, where accurate tuning is required. Silver mica or silver ceramic capacitor is more reliable for this purpose. It consists of a mica sheet with a silver deposit on both sides or a ceramic tube with a silver deposit on inner and outer surfaces. The silver deposits will work as electrodes.



(*a*) (*b*)

Fig. 5.8. Variable capacitors.

Variable Capacitor : If a continuous variation of the capacitance is required, a variable air capacitor of the parallel plates or of the coaxial cylinders is used. The former consists of two sets of plates usually semicircular, one set is fixed and the other can be rotated [Fig. 5.8. (*a*)]. These plates are insulated and parallel and are of aluminium or brass generally. When the movable part is rotated, its plates move in the gaps between the plates of the fixed part and the overlapping area between the two sets changes and thus changes the capacitance of the capacitor. The variable air capacitor of this type is used in ac bridges and in radio-sets.

The latter form consists of two coaxial cylinders of same diameter, with a small gap between them. A smaller cylinder B is caused to slide in or out of the cylinder A by a support which slides upon C, so that the length of B lying with in A can be varied and measured. Cylinder A is insulated and B and C are earthed, Fig. 5.8. (*b*).

The change in capacitance can be calculated by knowing the distance through which cylinder B moves in or out. If a scale of length is attached with a slider cylinder and the absolute capacitance is determined experimentally at one position, the capacitor can be calibrated.

Guard Ring Capacitor : In the derivation of the expression for the capacitance of a parallel plate capacitor, we have assumed the field between the plates as uniform and fringing effect at the edges as negligible. But in actual practice the field is not uniform at the edges and the expression $C = \varepsilon_0 A/d$ is only approximate.

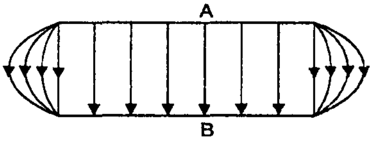


Fig. 5.9. Parallel plate capacitor.

To overcome the ends effect, Lord Kelvin devised a modified form of parallel plate capacitor by using a ring surrounding the plate A. The outer radius of the ring matches that of the bottom earthed plate B and its inner radius is slightly greater than that of the plate A. The ring is in the plane of plate A, leaving a small ring shaped slot for insulation. The fringing field is now shifted to the outside edge of the ring, leaving the uniform field over the whole area of the plate A as shown in Fig. 5.10. It is why the ring is known as *gaurd ring*. Kelvin found that the effective area of the plate = Area of the plate $A + \frac{1}{2}$ Area of the gap between the ring and the plate A. A similar arrangement can be made in the cylindrical capacitor. Two coaxial cylinders constitute a cylindrical capacitor.

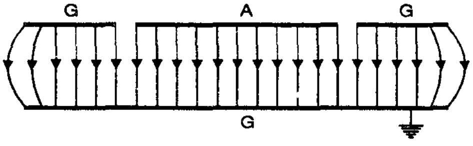


Fig. 5.10. Gaurd ring capacitor.

5.7 KELVIN'S ABSOLUTE OR ATTRACTED DISC ELECTROMETER

For the measurements of potential differences the electromagnetic voltmeters are less accurate as they require some electric current for their working. The instruments devised to measure electrostatic potentials correctly are named as *electrometers*. The simplest type of an electrometer is *gold leaf electroscope*. On account of its low sensitiveness and the uncertainty in the value of its readings, more advanced instruments, *absolute electrometer* and the *quadrant electrometer*, are used.

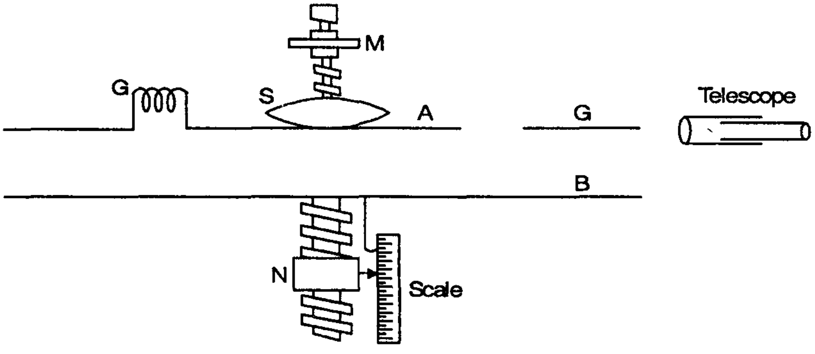


Fig. 5.11. Attracted disc electrometer.

The principle of absolute electrometer depends on the measurement of the force of attraction between the plates of a parallel plate capacitor, across of which the unknown potential difference is applied, in terms of the known weights. It is called absolute electrometer as the potential difference is measured in terms of force of attraction and dimensions of the capacitor which may be expressed in terms of the fundamental units of mass, length and time. It is known as *Kelvin's absolute electrometer as* it was first devised by Lord Kelvin using a guard ring air capacitor.

In this instrument, the disc A is surrounded by a guard ring GG with a small air gap. Both are in the same plane (horizontal) and are connected together by a light flexible wire and hence are always charged to the same potential. The disc A is horizontally suspended by a delicate spring S, connected to the rod which can be raised or lowered by a micrometer screw M. A fine cross is attached to the disc A to see by means of a telescope whether the disc A is in the plane of guard ring GG or not. Another disc B *of radius equal to the outer radius of the guard ring is kept parallel to the disc* A and can be raised or lowered by means of another micrometer screw N. The distance moved by the disc B can be accurately read on a graduated scale, similar to that used in spherometer.

Let V_A and V_B be the potentials of discs A and B respectively, which are at a distance d apart. As the use of guard ring produces a uniform electric field **E** between the discs A and B, hence

$$\int_B^A dV = -\int_B^A \mathbf{E}\cdot d\mathbf{l} = \int_B^A E\,dl = E\left[l\right]_0^d = Ed.$$

$$V_A - V_B = Ed \text{ or } E = (V_A - V_B)/d.$$

We know that the force of attraction exerted by this electric field is given by the relation

$$F = \tfrac{1}{2}\varepsilon_0 E^2 A = \tfrac{1}{2}\varepsilon_0 (V_A - V_B)^2 A/d^2, \text{ or } V_A - V_B = d\sqrt{2F/\varepsilon_0 A}. \qquad \text{...(28)}$$

To use the electrometer all the plates A, B and GG are earthed and the plate A is made in the plane of the guard ring GG with the help of the screw M. A small weight 'm' is then placed on the plate A, which depresses it below the plane of GG. The plate is raised with the help of the screw M, so that it is again in the plane of GG. The weight 'm' is then removed. The spring S pulls the plate A above GG. At this time force 'mg' is required to bring the plate A in the plane of GG. All the earth connections are now cut off.

To measure the unknown potential difference, its positive terminal is connected with the plate A and the negative terminal to the plate B. The plate B is so adjusted that the plate A *comes in the plane of GG*. In this procedure the *main difficulty is the measurement of d, the distance between plates A and B.*

For this we use an alternate procedure given below : The plate A and guard ring GG are now kept at constant high potential by connecting any one with the positive terminal of an electrical machine or a battery whose negative terminal is earthed. The plate B is then connected to the first of the points, the potential difference between which we wish to measure and its position is adjusted by means of screw N until the plate A is in the plane of GG. At this time the force of attraction due to the electrostatic field is equal to mg. If V_1 is the potential of the first point, V_0 that applied to plate A and d_1 the separation of two plates, then

$$V_0 - V_1 = d_1\sqrt{2mg/\varepsilon_0 A}\ .$$

Now plate B is connected to the second point and its position is adjusted again until the plate A is in the plane of GG. If d_2 is the distance between plates A and B and V_2 the potential of the second point, then

$$V_0 - V_2 = d_2\sqrt{2mg/\varepsilon_0 A}$$

$$\therefore \quad V_2 - V_1 = (d_1 - d_2)\sqrt{2mg/\varepsilon_0 A}. \qquad \text{...(29)}$$

The difference $(d_1 - d_2)$ is the difference of the two micrometer readings say x_1 and x_2, hence

$$V_2 - V_1 = (x_1 - x_2)\sqrt{2mg/\varepsilon_0 A}. \qquad \text{...(30)}$$

The potential difference is measured in terms of the difference of micrometer readings, the mass m and the effective area of the plate A, which is equal to the area of plate $A \pm \frac{1}{2}$ the area of the gap.

Determination of Dielectric Constant : Kelvin's attracted disc electrometer may be used for the measurement of dielectric constant of solid substances. The substance is taken in the form of a slab of area equal to the area of the each plate. It is introduced between the plates of an electrometer.

Capacity with dielectric slab of thickness t and of dielectric constant $k = A\varepsilon_0/[d - t\,(1 - 1/k)$.

Thus we see that the effect of introducing the slab is the same as would produced by diminishing d by the amount $t(1 - 1/k)$ without introducing the slab. Hence to bring the capacity to the same value the d should be increased by this amount when the slab is in. Calling this displacement as x, we get

$$x = t(1 - 1/k) \text{ or } k = t/(t - x). \qquad \text{...(31)}$$

Here thickness t is measured by bringing the plates in contact with and without the slab in between these two. Thus k is calculated.

The *main advantage* of this electrometer over the others is that it does not require any known potential difference for the calibration. It measures the unknown potential difference in terms of distance, force and area. It is comparatively less sensitive than the other forms of electrometers. Since the resistance of a voltmeter has a finite value, it always draws some current from the circuit in which it is connected and therefore shows less value of the potential difference. On the other hand, the attracted disc electrometer is an air capacitor having infinite resistance and hence measures the actual potential difference. *Another use over voltmeter* is that it can be used for the determination of the dielectric constant of a material which can be taken in the form of a slab as voltmeter cannot be used for this purpose.

The *main disadvantage* over the voltmeter is its cumbersome arrangement. It is not handy, like a voltmeter. It is not sensitive for measuring or comparing small potential differences and cannot be used for measuring ac-potential differences.

5.8 DOLEZALEK QUADRANT ELECTROMETER

In 1860, Lord Kelvin invented a more sensitive electrometer known as quadrant electrometer. It was later modified by Dolezalek and is therefore known as *Dolezalek Quadrant Electrometer*. It is very sensitive and is used in number of ways.

It *consists* essentially of four hollow brass quadrants so as to form a shallow circular box with short air gaps between the quadrants. The diagonally opposite quadrants are connected in pairs and are well insulated by being supported on amber pillars. A light paddle, made of thin paper coated with silver or of aluminium is suspended by a phosphor bronze strip. It is of a double sector shape and is frequently called the needle from analogy with the galvanometer needle. In the better type of instruments quartz fibre made conducting by dipping in a solution of calcium chloride is used as suspension. The upper end of the suspension is hooked on to the torsion head which can be rotated as well as raised or lowered and then clamped in any position, so that the needle is free to swing in a horizontal plane without touching the quadrants. The deflections of the needle are measured with the help of a lamp and scale arrangement. reflection taking place from a tiny mirror attached to the suspension fibre. The whole instrument is enclosed in a metal box which is earthed to protect it from stray electric field and the disturbance of air draughts. This box also provides window for the light beam. Three levelling screws are fitted to the base.

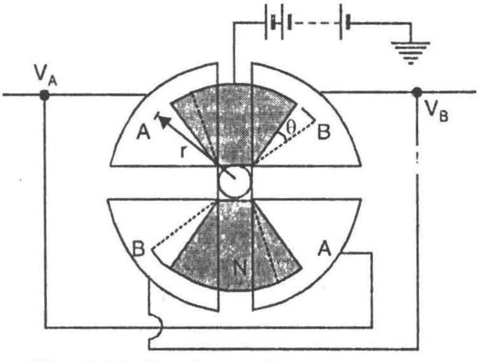


Fig. 5.13. Quadrant electrometer.

The needle is charged to a higher potential and the unknown potential difference is applied between the two pairs of quadrants. Since the capacity of each pair of quadrants is the same, the charge on *BB* is less than on *AA* and therefore the force of repulsion between *BB* and the needle is less than between *AA* and the needle. As a consequence the needle is *deflected towards the pair at lower potential BB* by a couple equal to the difference between the couples due to charges on *BB* and *AA*.

The *complete theory* is somewhat difficult, but a sufficiently accurate result is obtained by an approximate method, *an energy evaluation method.* This method is based on the assumption that *one-half of the paddle is considered to be a sector of a circle of radius r and as the quadrants are edge to edge, the direct capacity between them is assumed to be negligible in comparison of the capacity between quadrants and needle.* Let θ be the angle of deflection when pairs *AA* and *BB and the needle N* are at potentials V_1, V_2 and V_0 respectively. The area of the needle transferred from the *AA* quadrants to the *BB* quadrants is $2\pi r^2\theta/2\pi = r^2\theta$. As there are two faces to the needle the effective area transferred is $2r^2\theta$. Thus the capacity of the capacitor formed by quadrant *B* and the needle *N* is increased and of quadrant *A* and the needle *N* is decreased by the same amount equal to $\varepsilon_0 2r^2\theta/d$, where *d* is the thickness of air space between needle and quadrants. As the potentials remain constant, *A-N* capacitor losses charge $2\varepsilon_0 r^2\theta\,(V_0 - V_1)/d$ whilst the other pair *B–N* gains charge $2\varepsilon_0 r^2\theta(V_0 - V_2)/d$. From the definition of potential difference it follows that the work done by the source of supply in providing this charge to *BN* is $2\varepsilon_0 r^2\theta(V_0 - V_2)^2/d$ and the work done on the source of supply during the loss of charge from *A-N* capacitor is $2\varepsilon_0 r^2\theta(V_0 - V_1)^2/d$. Thus the total energy supplied to the electrometer from the source of supply is given by

$$W = \frac{2\varepsilon_0 r^2\theta}{d}\left[(V_0 - V_2)^2 - (V_0 - V_1)^2\right]$$

$$= \frac{4\varepsilon_0 r^2\theta}{d}\left[V_1 - V_2\right]\left[V_0 - \frac{V_1 + V_2}{2}\right]. \qquad \text{...(32)}$$

This energy is used up partly in increasing the electrostatic potential energy of the capacitors formed by the two pairs of quadrants and the balance in twisting the suspension fibre. As the electrostatic potential energy of a capacitor is $\frac{1}{2}CV^2$, the net increase in electrostatic potential energy of the capacitor system is given by

$$\frac{1}{2}\,\frac{2\varepsilon_0 r^2\theta}{d}(V_0 - V_2)^2 - \frac{1}{2}\,\frac{2\varepsilon_0 r^2\theta}{d}(V_0 - V_1)^2 = \frac{W}{2}. \qquad \text{...(33)}$$

The difference, which is half the total energy supplied from the supply, is used in twisting the fibre. It is proportional to θ, hence the deflecting couple is constant and is equal to $(2\varepsilon_0 r^2/d)\,(V_1 - V_2)\,[(V_0 - \frac{1}{2}\,(V_1 + V_2)]$. If *c* be the torsion couple per unit twist of the suspension fibre, then the restoring couple for deflection θ is $c\theta$. For equilibrium the two couples are equal, hence

$$c\theta = (2\varepsilon_0 r^2/d)\,(V_1 - V_2)\,[V_0 - \tfrac{1}{2}\,(V_1 + V_2)].$$

or

$$\theta = (2\varepsilon_0 r^2/cd)\,(V_1 - V_2)\,[V_0 - \tfrac{1}{2}\,(V_1 + V_2)]$$

$$= K\,(V_1 - V_2)\,[V_0 - \tfrac{1}{2}\,(V_1 + V_2)], \qquad \text{...(34)}$$

where $K = 2\varepsilon_0 r^2/cd$ is a constant for the instrument.

There are two ways of using the quadrant electrometer as a potential measurer :

(*i*) *Heterostatic use :* Here the needle is charged to a very large potential as compared to V_1 and V_2. Thus $V_0 >> V_1$ or V_2 and

$$\theta = K(V_1 - V_2)V_0 = K'\,[V_1 - V_2], \text{ where } K' = KV_0. \qquad \text{...(35)}$$

Hence in heterostatic use, $\theta \propto (V_1 - V_2)$, the potential difference to be measured.

(*ii*) *Idiostatic use :* Here the needle is connected to one pair of quadrants say *AA*. The unknown potential difference is applied across *BB* and the common contact of *N* and *AA*. Thus $V_0 = V_1$ and

$$\theta = K(V_1 - V_2)\,(V_1 - \tfrac{1}{2}\ (V_1 + V_2)] = \tfrac{1}{2}\ K(V_1 - V_2)^2$$
$$= K''\,(V_1 - V_2)^2, \qquad \text{...(36)}$$

where $K'' = \frac{1}{2}\ K.$

Hence in idiostatic use $\theta \propto (V_1 - V_2)^2$.

Relative merits of Heterostatic use and Idiostatic use : We have $K'/K'' = 2V_0$. Hence $K' >> K''$ and therefore the heterostatic use is more accurate than the idiostatic use, as a small potential difference produces a much larger deflection in the former case. Another great advantage of the heterostatic use is that the deflection is proportional to the potential difference. When the instrument is used idiostatically the deflection is proportional to the square of the potential difference and independent of the direction and hence the instrument may be used for the measurement of *alternating potential differences.* For a given deflection, the value of $V_1 - V_2$ will be greater in idiostatic use than in heterostatic use, the range of the instrument is therefore considerably extended. In idiostatic use the needle is connected to the pair of quadrants and thus a high potential source to be connected with the needle is not required. Thus we see that to *measure small potentials, the electrometer is used heterostatically and to measure high potentials and alternating potentials it is used idiostatically.*

Adjustment : The electrometer must be mounted on a very rigid support such as a stone block and must be *adjusted* so that the quadrants are quite horizontal and the needle hangs symmetrically between them. The quadrants are first made horizontal with the spirit level and by opening the case the needle is adjusted by eye in such a way that it swings freely between the top and bottom of the hollow quadrant space and symmetrical between *AA* and *BB* quadrants. The final adjustment is done by the spot of light obtained on the scale, as given under :

AA and *BB* pairs of quadrants are earthed and the needle is charged to a high potential. If the spot of light on scale changes, it means that the needle is not symmetrical between the quadrants. The levelling screws are adjusted until the position of the spot is the same whether the needle is charged or not. As we know that equal and opposite values of $V_1 - V_2$ should give equal and opposite deflections, the electrometer should be adjusted, after connecting needle to about 100 volts, *AA* pair to the earth and the *BB* pair of quadrants to the +ve terminal of the battery, negative terminal of which is earthed, until equal deflections on the both sides of the zero are obtained when the battery terminals are reversed. Both these processes are repeated for the *proper adjustment.*

A small potential difference is applied between the two pairs of quadrants and the potential of the needle is gradually increased. If a graph is plotted between the deflection θ and the potential of the needle, a curve so obtained will be parabolic. The voltage of the needle corresponding to the maximum deflection gives the condition for maximum sensitivity. The needle is thus charged to this potential and the potential difference between the pairs of quadrants is now varied and the deflections are obtained corresponding to known potential differences and a graph is plotted between the potential difference and the deflection. The curve so obtained will be straight line and is known as calibration curve. With the help of this curve the unknown potential difference can be obtained by observing the deflection corresponding to that potential difference.

Quadrant electrometer is not an *absolute* instrument. In precise measurements it is usually used as a null instrument. Used heterostatically it can measure 1/3000 volt of potential. Its sensitivity can be increased by taking special precautions. Its capacitance is 80 *pf.*

Uses of a Quadrant Electrometer : (1) *Measurement of small (steady) potential difference :* Steady small potential can be measured by using the quadrant electrometer *heterostatically. i.e.,* by connecting the needle to the +ve terminal of a high potential battery, the negative terminal of which is earthed and by applying the unknown potential difference across two pairs of quadrants. The deflection in this case is proportional to the unknown potential difference.

(2) *Measurement of alternating potential difference :* In this case the electrometer is used *idiostatically, i.e.,* the needle is connected to one pair of quadrants and the unknown potential difference is applied across two pairs of quadrants. The deflection is proportional to the square of the potential difference.

(3) *Capacitance measurement :* To compare the capacitances of two capacitors C_1 and C_2, connect the apparatus as shown in Fig. 5.13. Here the electrometer is connected heterostatically by connecting the needle to high tension battery, one pair to earth and other pair to battery. Close key K_1, thus charging capacitor C_1 and the quadrant electrometer to a certain potential V_1. If θ_1 is the corresponding deflection indicated by the electrometer, then charge $q = (C + C_1)V_1$, where C is the capacity of the electrometer. Now open key K_1 and close key K_2, thus allowing capacitor C_1 and the electrometer to share the charges with capacitor C_2. If θ_2 is the new deflection then

$$q = (C + C_1 + C_2)V_2$$

$$\therefore \quad q = (C + C_1)V_1 = (C + C_1 + C_2)V_2,$$

$$\text{or} \quad \frac{C_1 + C}{C_2} = \frac{V_2}{V_1 - V_2} = \frac{\theta_2}{\theta_1 - \theta_2}.$$

(As $V_1 \propto \theta_1$ and $V_2 \propto \theta_2$)

If C, the capacity of the electrometer is small and is negligible compared to C_1 and C_2, then

$$C_1/C_2 = \theta_2/(\theta_1 - \theta_2). \quad \text{...(37)}$$

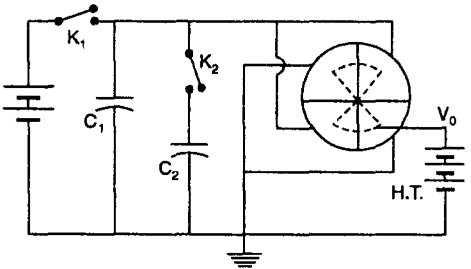


Fig. 5.13. Capacitance measurement.

In this way two capacitances C_1 and C_2 can be compared very easily. If one *capacitance is known, other can be calculated with the* help of this equation.

The capacitance of the quadrant electrometer may be determined as follows : A standard capacitor, say guard ring air capacitor of capacitance C_1 is removed from the circuit 5.13. The electrometer is first charged by closing key K_1 and opening key K_2 and the deflection θ_1 is obtained. Then key K_1 is opened and the charge is shared with the standard capacitor C_2 by closing key K_2 and the deflection θ_2 is also obtained. Hence,

$$q = C V_1 = (C + C_2)V_2,$$

$$\text{or} \quad C = C_2 \frac{V_2}{V_1 - V_2} = C_2 \frac{\theta_2}{\theta_1 - \theta_2}. \quad \text{...(38)}$$

Thus the capacitance of the electrometer C can be calculated.

(4) *Measurement of dielectric constant : dc* methods of determining the dielectric constant give the values 10% greater than the values obtained by *ac* methods. It is due to dielectric absorption phenomenon.

(i) Hopkinson's method for solids : In the Hopkinson's circuit, Fig. 5.14, *AB* is a guard-ring capacitor, *C* a variable capacitor consisting of two coaxial cylindrical conductors (inner is capable of sliding inside the outer one), *Q.E.* the quadrant electrometer, B_1 and B_2 the two batteries with common terminals earthed and the ends at potentials $+V$ and $-V$.

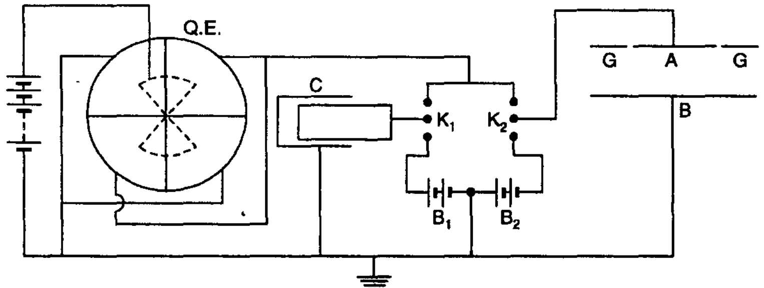


Fig. 5.14. Hopkinson's method for solids.

By pressing keys K_1 and K_2, capacitors are charged to equal and opposite potentials. If the capacitances of these capacitors are equal, then their charges will be equal and opposite. The net charge given to quadrant electrometer is zero and the needle is thus undeflected. Thus by adjusting variable capacitor C, the position of no deflection in the electrometer is obtained. The position of plate B is noted on the scale and the variable capacitor C is kept fixed. The substance of which dielectric constant is to be measured is taken in the form of a slab and is placed on the plate B. Now this plate B is lowered until there is no deflection in the electrometer needle while pressing keys K_1 and K_2. The new position of plate B on the scale is thus noted.

If d is the distance between the plates in the first part and x is the distance by which the plate B is lowered to obtain a null deflection in the second part (when a slab of thickness t and of area equal to that of the guard-ring capacitor is placed), then

$$\frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 A}{[(d-t+x)+t/k]}$$

$$\therefore \quad d-t+x+t/k = d \quad \text{or } k = t/(t-x). \qquad \text{...(39)}$$

The thickness of the dielectric slab can be determined from a micrometer screw.

(ii) Hopkinson's method for determination of k of liquids : In this method the guard ring capacitor AB in Fig. 5.14 is replaced by a double walled cylindrical capacitor in which an insulated metallic cylinder A is hanged [Fig. 5.15]. A is connected with K_2 and the variable capacitor C is adjusted for no deflection as in the previous experiment. If a and b be the inner and outer radii of the cylindrical capacitor C and l_1 be the length of overlap of inner and outer cylinders. The capacity of the capacitor C will be given by

$$C = 2\pi\varepsilon_0 l_1/\log_e(b/a). \qquad \text{...(40)}$$

The liquid of which dielectric constant is to be measured is filled between PQ and the position of C is again adjusted for zero deflection. If l_2 be the length of overlap, then the capacity C' will be given by

$$C' = 2\pi\varepsilon_0 l_2/\log_e(b/a). \qquad \text{...(41)}$$

$$\therefore \quad k = \frac{\text{capacitance of } AB \text{ with liquid}}{\text{capacitance of } AB \text{ without liquid}} = \frac{C'}{C}.$$

$$= l_2/l_1. \qquad \text{...(42)}$$

Thus the dielectric constant k can be determined easily.

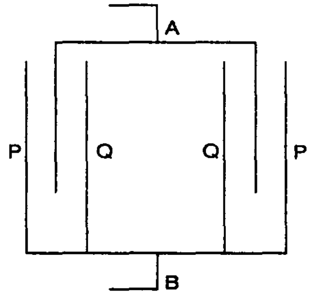


Fig. 5.15. Cylindrical capacitor.

(iii) Boltzmann's method for gases : The circuit used by Boltzmann, shown in Fig. 5.16, consists of two identical metallic plates A and B forming a parallel plate capacitor. These are enclosed in a sealed and earthed metallic chamber C, which can be exhausted or filled with a gas. The plate A is connected to a high tension battery B_1 of n cells each of voltage V *through* a key K_1. Plate B is connected to one pair of quadrants, the other pair of which is earthed. A key K_2 is provided to connect both pairs of quadrants to earth.

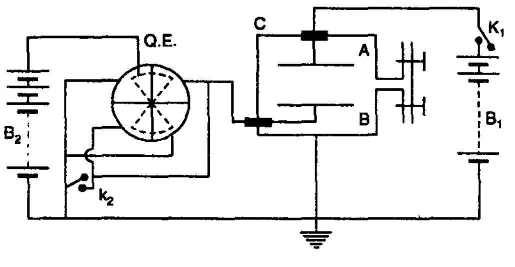


Fig. 5.16. Boltzmann's method for gases.

The chamber C is first exhausted, K_1 and K_2 are closed, thus charging the plate A to a potential nV. There will be no deflection in the electrometer as both the pairs of quadrants are earthed. Keys K_1 and K_2 are opened and the experimental gas is introduced in the chamber C. The capacity of the capacitor formed by A and B plates is increased k times and consequently the potential of plate A falls to nV/k. The key K_1 is again closed and the plate A is charged to potential nV. The increase in potential $nV - nV/k = nV\,(1 - 1/k)$ requires additional charge from the battery B_1. This additional charge on A induces an equal and opposite charge on plate B, which will go to the electrometer and will produce deflection in the needle. Let θ be this deflection, then,

$$\theta \propto nV(1 - 1/k), \text{ or } \theta = KnV(1 - 1/k). \qquad ...(43)$$

If one more cell is added to the battery B_1, the potential of plate A will become $nV + V = (n + 1)V$, and the increase in potential of A by V will also increase the potential of plate B by the same amount. Let θ' be the deflection in needle due to this potential change, then

$$\theta' \propto V \text{ or } \theta' = KV$$

$$\therefore \qquad \theta = n\theta'\,(1 - 1/k) \text{ or } k = n\theta'/(n\theta' - \theta). \qquad ...(44)$$

Thus the dielectric constant k of the gas may be calculated.

(5) *Measurement of ionization currents :* Quadrant electrometer can be used for measuring ionization current as it is too small to be detected by a galvanometer. An ionization chamber, consisting of a metallic box containing two plates insulated and separated from each other, is connected in series with a resistance ($\sim 10^{10}$) ohms and a battery (30 – 200 volts), as shown in Fig. 5.17. When the air between plates A and B is exposed to X-rays, a current flows in this circuit and a potential difference is created between ends of the resistor. Thus potential difference is applied across one pair of quadrants of electrometer, the other pair of which is earthed and the needle is connected to a high tension battery. The steady deflection of the needle is recorded. The electrometer is calibrated by connecting a standard cadmium cell (1.018 volts) across the pair of quadrants with a two-way key. If θ_1 and θ_2 be the deflections in the former and latter cases respectively, then we have

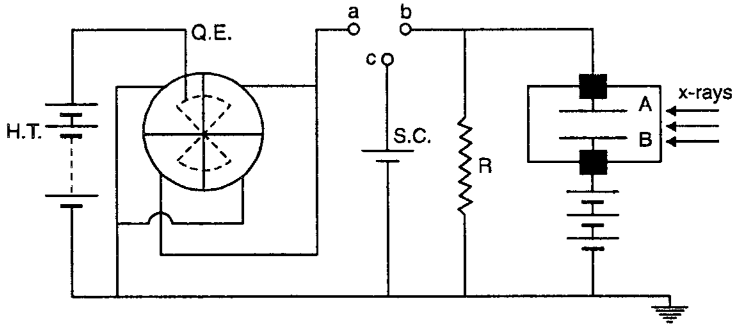


Fig. 5.17. Measurement of ionization current.

$$IR \propto \theta_1 \text{ and } 1.018 \propto \theta_2, \quad \text{or} \quad I = 1.018\theta_1/R\theta_2. \qquad ...(45)$$

It is in amperes if R is measured in ohms.

Modifications of Dolezalek Electrometer

(*a*) *Compton electrometer :* It is the modification of Dolezalek quadrant electrometer and was given by A.H. and K.T. Compton. In this instrument the electrostatic control is varied by tilting the paddle slightly about its long axis and by moving one of the quadrants a little above or below the horizontal plane of the other three.

This method of control gives nearly constant sensitivity over a large range of deflection and increases its sensitivity 70 times greater. It can be quickly adjusted. Its capacity is about 10 *pF*.

(*b*) *Lindemann electrometer :* This instrument, designed in 1924, works on the principle of quadrant electrometer. It is very compact, has a stable zero, a constant sensitivity, very small capacitance (2 *pF*), does not require levelling and is very robust. For its construction and working the readers are advised to consult *Electricity and Magnetism by Fewkes and Yarwood and A Degree Physics Part V, Electricity and Magnetism by C. J. Smith.*

(c) *Electrostatic voltmeter :* This instrument is used to measure potential difference and is based on the electrostatic principles. It consists of two fixed parallel brass plates each of the shape of a double quadrant and are placed vertically, (Fig. 5.18). An aluminium needle is pivoted at the centre so that it can swing about a horizontal axis in a vertical plane mid way between the fixed brass plates. The upper part of the needle carries a pointer which moves over the vertical scale of the instrument and the lower part carries adjustable weights.

The controlling torque may be varied by adjusting these weights and the range of the instrument may thus be altered. The needle is earthed and the potential to be measured is applied to the fixed plates. The scale is first standarized by using known voltages. It is graduated in volts.

The instrument is gravity controlled, as the electrostatic attraction due to the potential difference applied between fixed plates is opposed by the particular weight attached to the lower end of the needle and the latter comes at rest when these two opposing influences are equal and balance each other. With different patterns a total range of about 1,000 volts to 20,000 volts can be obtained.

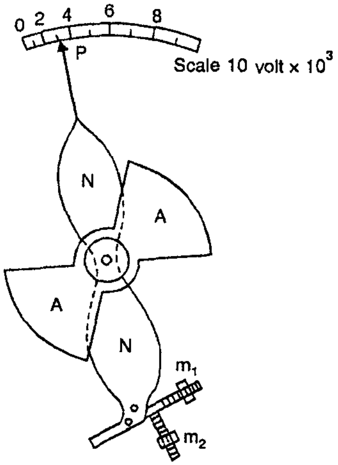


Fig. 5.18. Electrostatic electrometer.

By using a vertical suspension to avoid pivot friction and by increasing the number of moving needles (or vanes) and of fixed brass plates, it may be used to measure much lower potential differences (30 to 120 volts). Higher ranges up to about 2,000 volts can be obtained by modifying the suspension or the vanes.

A great advantage of the electrostatic type is that it absorbs no power and takes only a very small capacity current. However it cannot be used for small potential differences and is costly. The scale is crowded in its lower portion.

A multiplier is used to increase the range of a low-reading electrostatic voltmeter. This consists of a high resistance with tappings at intermediate points. The potential to be measured is connected across the whole resistance and the voltmeter across part of it. The drawback of this multiplier is that the power loss is not zero and the current is not very small.

EXERCISES

Example 1. *A 100 μF capacitor is charged to a potential of 50 volts and the charging battery is then disconnected. This is then connected in parallel with second capacitor. If the potential difference drops to 35 volts, calculate the capacitance of the second capacitor.*

The charge on the first capacitor $q_1 = C_1 V = 100 \times 10^{-6} \times 50$ coul. When this is connected in parallel to another capacitor of capacitance C_2, the capacitance of the combination becomes $C_1 + C_2$, while the total charge will remain q_1. Therefore the potential difference across the combination will be $V_c = q_1/(C_1 + C_2)$.

In the present problem $V_c = 35$ volts. Therefore

$$35 = \frac{5 \times 10^{-3}}{100 \times 10^{-6} + C_2}$$

or $$C_2 = 42.86\ \mu F.$$

Example 2. *Find the equivalent capacitance of the combination in Fig. 5.19. Given that* $C_2 = 20\mu F$ *and that the other capacitances are all* $5\mu F$.

To solve the problem, let q_1, q_2, q_3, q_4, and q_5 be the charges on the left plates of the capacitors C_1, C_2, C_3, C_4 and C_5 respectively as shown in Fig. 5.20. It is easy to write.

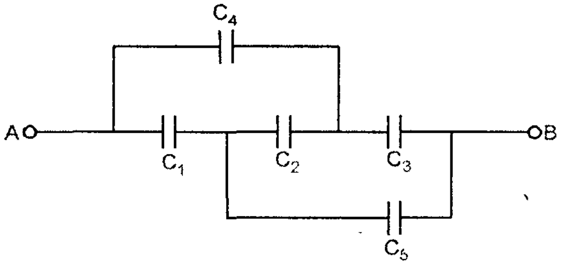


Fig. 5.19. Example 2.

$$V_{AB} = V_{AP} + V_{PB} = V_{AQ} + V_{QB}$$

or $\dfrac{q_1}{C_1} + \dfrac{q_5}{C_5} = \dfrac{q_4}{C_4} + \dfrac{q_3}{C_3}$.

Since $C_2 = 20\,\mu F$ and $C_1 = C_3 = C_4 = C_5 = 5\mu F$

$\therefore\ q_1 + q_5 = q_4 + q_3$. ...(*i*)

Total charge at the intermediate junctions P and Q must be zero, therefore

$-q_1 + q_2 + q_5 = 0$ or $q_1 = q_2 + q_5$

$-q_4 - q_2 + q_3 = 0$ or $q_3 = q_2 + q_4$

Thus equation (*i*) becomes

$q_2 + 2q_5 = q_2 + 2q_4$ or $q_5 = q_4$.

$\therefore$ Equivalent capacitance C = Net charge $(q_1 + q_4)/V_{AB}$

or $C = \dfrac{q_1 + q_4}{(q_1 + q_5)/5\mu F} = 5\mu F.$

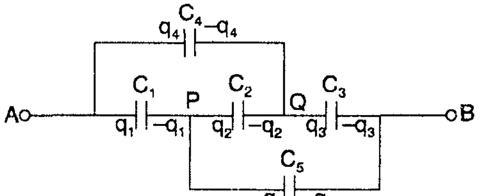


Fig. 5.20. Example 2.

Example 3. *A conductor is charged from an electrophorous by repeated contacts with a plate which after each contact is re-charged with a quantity q of electricity from the electrophorous. If e is the charge of the conductor after the first operation, calculate the ultimate charge.*

On the first contact, the charge taken up by the conductor of capacity C_1 is e and the charge left on the plate of capacity C_2 is $q - e$. Since the potential for either is the same, hence

$$(q - e)/C_2 = e/C_1, \text{ or } e = qC_1/(C_1 + C_2).$$

In the second contact, let e' be the charge taken up by the conductor, then $e + e'$ becomes the total charge over the conductor and the charge left over the plate is $q - e'$. Hence we have

$$(q - e')/C_2 = (e + e')/C_1 \text{ or } e' = qC_1^2/(C_1 + C_2)^2 = e^2/q.$$

Similarly the charge on the conductor on the third contact is e^3/q^2 and so on. Hence the ultimate charge

$$Q = e + e^2/q + e^3/q^2 + ... = eq/(q - e).$$

Example 4. *Three concentric thin spherical shells are of radii a, b, c (a < b < c). The first and third are connected by a fine wire through a small hole in the second and the second is connected to earth through a small hole in the third. Show that the capacity of the capacitor so formed is*

$$4\pi\varepsilon_0\,[ab/(b - a) + c^2/(c - b)].$$

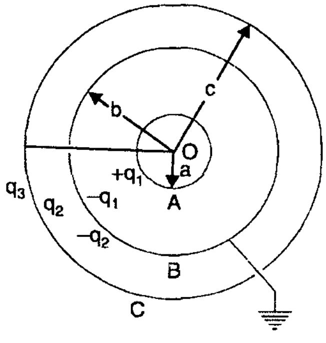


Fig. 5.21. Example 4.

If a charge q is given to the outermost sphere C, as it is connected with the outer surface of innermost sphere A, some of the charge will go to the outer surface of the sphere A, some will go to the inner surface and the rest will remain on the outer surface of this sphere C. The charge q_1 on the inner sphere will induce a charge $-q_1$ on the inner surface of the second sphere B and will induce a charge $+q_1$ on the outer surface which will go to earth. Charge q_2 on the inner surface of the sphere C will also induce a charge $-q_2$ on the outer surface of the sphere B [Fig. 5.21]. In this way three capacitors are formed, one by the outer surface of A and inner surface of B, second by the outer surface of B and inner surface of C and the third by the outer surface of C and the earth. The *pd* between the spheres A and B.

$$V_A - V_B = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = -\frac{q_1}{4\pi\varepsilon_0}\int_b^a \frac{dr}{r^2} = \frac{q_1}{4\pi\varepsilon_0}\left[\frac{1}{a}-\frac{1}{b}\right]$$

$\therefore$ Capacitance $C_1 = \dfrac{q_1}{V_A - V_B} = 4\pi\varepsilon_0 \dfrac{ab}{b-a}.$

Similarly capacitance $C_2 = \dfrac{q_2}{V_B - V_C} = 4\pi\varepsilon_0 \dfrac{bc}{c-b},$

and capacitance $C_3 = 4\pi\varepsilon_0 c.$

$\therefore$ Combined capacity of the capacitor so formed

$$C_A = C_1 + C_2 + C_3$$

$$= 4\pi\varepsilon_0\left[\frac{ab}{b-a} + \frac{bc}{c-b} + c\right] = 4\pi\varepsilon_0\left[\frac{ab}{b-a} + \frac{c^2}{c-b}\right].$$

Example 5. *A one core lead sheathed cable has a conductor core of 0.5 cm diameter and the lead sheath has inside diameter 1.5 cm. The insulating material is rubber. At what voltage will the insulation break down ? Given that rubber has a dielectric strength 400 kV/cm.*

The capacity per unit length of the cable (cylindrical)

$$C = 2\pi\varepsilon/\log_e (r_2/r_1).$$

$\therefore$ Charge per unit length on the cable $\lambda = 2\pi\varepsilon V/\log_e(r_2/r_1)$.

If the voltage applied is of such a value that the insulation is just to break, the intensity of field at the inner surface of sheath will be the same as the dielectric strength, *i.e.*,

Field intensity = $\lambda/2\pi\varepsilon r_2$ = Dielectric strength

$$\therefore \quad \frac{2\pi\varepsilon V}{\log_e(r_2/r_1)}\cdot\frac{1}{2\pi\varepsilon r_2} = \frac{V}{r_2\log_e(r_2/r_1)} = 400 \times 10^3 \text{ volts/cm}$$

$$V/0.75 \times 10^{-2}\log_e(3) = 4 \times 10^7 \text{ volts/m}$$

or $$V = 4 \times 10^7 \times 0.75 \times 10^{-2}\log_e 3 = 3.3 \times 10^5 \text{ volts.}$$

Example 6. *Prove that the capacitance of two very long, thin parallel wires in air is given by $C = \pi\varepsilon_0/\log_e (d/a)$, where a is the radius of each wire and d is the distance between the centers of wires.*

Let A and B are two very long parallel thin wires, separated by a distance d and are in the direction perpendicular to the plane of the paper (Fig. 5.22). Let us consider any point P in the space around the wires, such that $PA = r_1$, $PB = r_2$ and $\angle PAB = \theta$. Let the charges per unit length on A and B are $+\lambda$ and $-\lambda$ respectively. As we know that the electric intensity at any point at distance r due to the long charged cylinder is given by

$$E = \lambda/2\pi\varepsilon_0 r.$$

$$\therefore \quad V = -\int \mathbf{E}\cdot d\mathbf{r} = -\int E\,dr = -\int(\lambda/2\pi\varepsilon_0 r)\,dr.$$

Hence the potential at point P due to wire A of radius R

$$= -\frac{\lambda}{2\pi\varepsilon_0}\int_R^{r_1}\frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_0}(\log_e R - \log_e r_1),$$

where R is the distance of the point of zero potential, say earth.

Similarly, the potential at point P due to wire B

$$= -\frac{-\lambda}{2\pi\varepsilon_0}\int_R^{r_2}\frac{dr}{r} = \frac{\lambda}{2\pi\varepsilon_0}(\log_e r_2 - \log_e R).$$

Hence the total potential at P due to both wires = $(\lambda/2\pi\varepsilon_0)\log_e(r_2/r_1)$

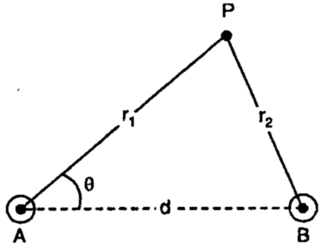


Fig. 5.22. Example 6.

$\therefore$ The potential V_A at wire $A = (\lambda/2\pi\varepsilon_0)\log_e (d-a)/a$

and the potential V_B at wire $B = (\lambda/2\pi\varepsilon_0)\log_e a/(d-a)$.

$\therefore$ Potential difference between the wires

$$V_A - V_B = \frac{\lambda}{2\pi\varepsilon_0}\left[\log_e\left(\frac{d-a}{a}\right) - \log_e\left(\frac{a}{d-a}\right)\right] = \frac{\lambda}{\pi\varepsilon_0}\log_e\frac{d-a}{a}.$$

Thus the capacitance per unit length $= \lambda/(V_A - V_B)$

$$= \frac{\pi\varepsilon_0}{\log_e[(d-a)/a]} = \frac{\pi\varepsilon_0}{\log_e(d/a-1)}$$

$$= \pi\ \varepsilon_0/\log_e (d/a). \qquad \text{(as } d \gg a\text{)}$$

This equation is applied to two parallel telegraph wires which are far removed from the earth.

If the single thin wire of telegraph is at a height h above the earth, the potential difference V between the wire and earth will be half of $V_A - V_B$, i.e., $(\lambda/2\pi\varepsilon_0)\log_e (d/a)$. A single wire will have an electrical image (will be discussed in chapter 6) at a distance h below the earth level, thus the distance between two wires (one wire and other its image) d = 2h.

$\therefore$ *Potential difference* $V = (\lambda/2\pi\varepsilon_0)\log_e (2h/a)$

Therefore the capacitance per unit length $= (\lambda/V) = 2\pi\varepsilon_0/\log_e (2h/a)$.

Example 7. *The plates of parallel plate capacitor are not quite parallel, the separation at one edge being d + a and at the opposite one being d – a. Neglecting edge effects, calculate the capacitance of this capacitor, assuming d >> a.*

Let l and b be the length and breadth of each plate. If θ is the angle which plate AB makes with plate CD such that $\theta l = 2a$.

Let us divide this capacitor in large number of differential capacitors as shown in Fig. 5.23. Consider a capacitor at a distance x from the middle point of the plate and of length dx. The capacitance of this differential capacitor

$$\Delta C = \varepsilon_0 b\, dx/(d + x\theta).$$

As these capacitors are parallel, hence total capacitance

$$C = \Sigma\ \Delta C = \Sigma \frac{\varepsilon_0 b}{d + x\theta}dx = \varepsilon_0 b \int_{-\frac{1}{2}l}^{+\frac{1}{2}l} \frac{dx}{d + x\theta}$$

$$= \frac{\varepsilon_0 b}{\theta}\left[\log_e (d + x\theta)\right]_{-\frac{1}{2}l}^{+\frac{1}{2}l} = \frac{\varepsilon_0 b}{\theta}\log_e \frac{d + \frac{1}{2}\theta l}{d - \frac{1}{2}\theta l}.$$

$$\text{As} \log_e\left(\frac{1+y}{1-y}\right) = 2y\left(1 + \frac{y^2}{3}\right), \theta l = 2a, bl = A,$$

$$\therefore \quad C = \frac{\varepsilon_0 A}{d}\left(1 + \frac{1}{3}\frac{a^2}{d^2}\right).$$

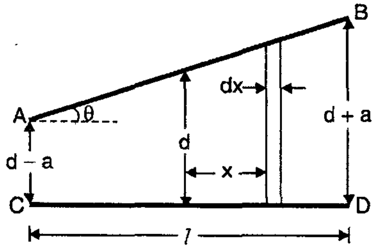


Fig. 5.23. Example 7.

Example 8. *A circular disc of radius a is placed with its plane inclined at a small angle α to another plane. The centre of the disc is at a perpendicular distance d from this plane. The disc and the plane are maintained at a potential difference V. Show that the mechanical force acting on the disc produce a force* $\pi\varepsilon_0 V^2a^2/2d^2$ *perpendicular to its plane and a couple* $\pi\varepsilon_0 V^2\alpha a^4/4d^3$.

Let us divide the disc in large no. of strips. Consider a strip of thickness dx at a distance x from the centre of disc, as shown in Fig. 5.24. If this strip is at a distance t from the plane, then the intensity E between it and the plane can be considered as constant. The potential difference

$$V = Et = \sigma t/\varepsilon_0 \text{ or } \sigma = \varepsilon_0 V/t.$$

Hence the mechanical force experienced by the elementary strip per unit area $= \sigma^2/2\varepsilon_0 = \varepsilon_0 V^2/2t^2$.

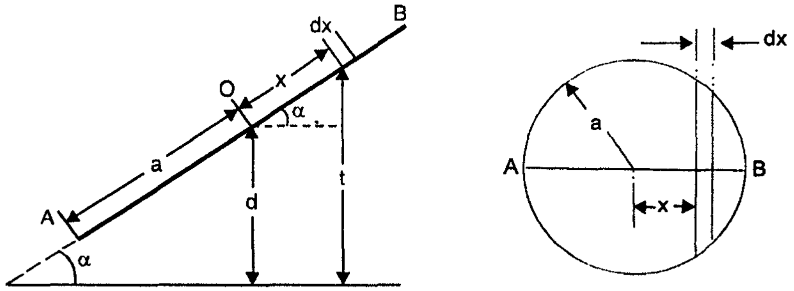


Fig. 5.24. Example 8.

$\therefore$ Force experienced by this strip $= \varepsilon_0(V^2/2t^2)\ 2(a^2 - x^2)^{1/2}dx$.

$\therefore$ Total force experienced by the circular plate $= \int_{-a}^{a} \frac{\varepsilon_0 V^2(a^2 - x^2)^{1/2}}{(d + x\alpha)^2}dx$ (As $t = d + x\alpha$)

$$= \frac{\varepsilon_0 V^2}{d^2}\int_{-a}^{a}(a^2 - x^2)^{1/2}\left(1 + \frac{x\alpha}{d}\right)^{-2}dx = \frac{\varepsilon_0 V^2}{d^2}\cdot\frac{a^2\pi}{2}$$

$$= \pi\varepsilon_0 V^2 a^2/2d^2.$$

Magnitude of the couple $= \int_{-a}^{a}\frac{\varepsilon_0 V^2(a^2 - x^2)^{1/2}dx}{(d + x\alpha)^2}.x = \frac{\pi\varepsilon_0 V^2\alpha a^4}{4d^3}$.

Example 9. *Show that if the equation of the rim of the movable plates of a rotary variable capacitor is* $r^2\theta^3 = const.$, *the change in* $1/\sqrt{C}$ *is proportional to the change in angle of rotation.*

We know that the capacitance of the rotary variable capacitor varies directly as the overlapping area. If the movable plates are semi-circles and the axis of rotation passes through their centres. $A = \frac{1}{2}\theta r^2$, where r is the constant radius of the plates and θ is the angle through which the movable plates have entered the fixed ones. Hence the change in capacitance dC = constant $r.r\ d\theta$.

$$\therefore \quad dC/d\theta = \text{const. } r^2.$$

If $1/\sqrt{C}$ varies linearly with θ, then $\theta^2 C$ will be constant and hence

$$2\theta\, d\theta C + \theta^2\, dC = 0 \quad \text{or } dC/d\theta = -2C/\theta = \text{const.}/\theta^3.$$

Equating these values of $dC/d\theta$, we get

$$\text{const. } r^2 = \text{const.}/\theta^3 \text{ or } r^2\theta^3 = \text{const.}$$

This is the required equation of the rim of the movable plates of rotary variable capacitor. This capacitor is known as a *straight-line frequency capacitor*.

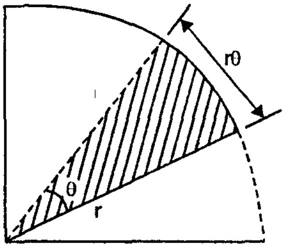


Fig. 5.25. Example 9.

Example 10. *If half the space between two concentric conducting spheres be filled with dielectric of dielectric constant k and the rest is filled with air. Show that the capacitance of the capacitor thus formed will be same as if the whole part is filled with the dielectric of dielectric constant* $\frac{1}{2}(1 + k)$.

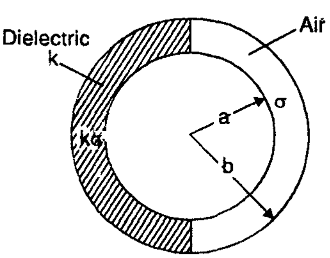


Fig. 5.26. Example 10.

If the space between two concentric conducting spheres is filled with air and σ is the surface density of charge on the inner sphere, then the potential difference between the spheres of radii a and b

$$\int_b^a dV = -\int_b^a \mathbf{E}\cdot d\mathbf{r} = -\int_b^a E dr = -\int_b^a \frac{q}{4\pi\varepsilon_0}\frac{1}{r^2}dr$$

$$\text{or} \qquad V_a - V_b = \frac{4\pi a^2\sigma}{4\pi\varepsilon_0}\left[\frac{1}{a} - \frac{1}{b}\right] = \frac{\sigma a^2}{\varepsilon_0}\left[\frac{b-a}{ab}\right].$$

$$\therefore \quad \text{Capacity } C = \frac{\text{Charge}}{V_a - V_b} = \frac{4\pi a^2\,\sigma\,\varepsilon_0}{\sigma\,a^2}\,\frac{ab}{b-a} = 4\pi\varepsilon_0\,\frac{ab}{b-a}.$$

If the space is filled with a dielectric of dielectric constant k', the capacity $C' = k'C = 4\pi k'\varepsilon_0\,ab/(b-a)$.

Now if half of the space is filled with dielectric of dielectric constant k and half with air, then in order to maintain the same potential difference between the spheres, the surface charge density over one half of the inner sphere with dielectric will be $k\sigma$ and over the other half filled with air will be σ. Thus the capacity

$$C'' = \frac{\text{Total Charge}}{\text{pot. diff.}} = \frac{2\pi a^2\,\sigma k + 2\pi a^2\sigma}{\sigma a^2(b-a)/\varepsilon_0 ab} = 4\pi\varepsilon_0\,\frac{1+k}{2}\,\frac{ab}{b-a}$$

C'' must be equal to C', therefore

$$4\pi\varepsilon_0 k'\,ab/(b-a) = 4\,\pi\varepsilon_0(1+k)ab/2(b-a)$$

$$\therefore \quad k' = \tfrac{1}{2}(1+k).$$

Example 11. *A slab of thickness t and of dielectric constant k is inserted between the plates of a parallel plate capacitor with a separation d between the plates. Calculate the percentage increase in capacitance and discuss the result.*

The capacitance of the parallel plate capacitor before the slab is inserted

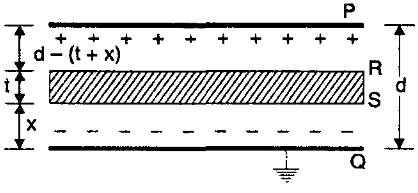


Fig. 5.27. Example 11.

$$C_0 = \varepsilon_0 A/d,$$

where A is the area of each plate. If σ be the charge per unit area on the plates. When the dielectric slab is inserted between these plates P and Q [Fig. 5.27], potential difference decreases and is given as

$$V = \int_Q^P \mathbf{E}\cdot d\mathbf{r} = \int_Q^P E dr = \int_Q^S E dr + \int_S^R E dr + \int_R^P E dr$$

$$= \sigma\ x/\varepsilon_0 + \sigma t/\varepsilon_0 k + \sigma\,[d - (t + x)]/\varepsilon_0$$

$$= \sigma\ [d - t\,(1 - 1/k)]/\varepsilon_0.$$

$\therefore$ Capacitance of the capacitor after inserting the dielectric

$$C = Q/V = \sigma A/V = A\varepsilon_0/[d - t(1 - 1/k)]$$

Thus the capacitance is increased in the ratio $C : C_0$

$$= d : [d - t\,(1 - 1/k)] = 1 : [1 - (1 - 1/k)t/d]$$

$$\%\ \text{increase in capacitance} = \frac{C - C_0}{C_0} \times 100 = \frac{t\,(1 - 1/k)}{d - t(1 - 1/k)} \times 100$$

If this slab is of metal, for which $k = \infty$, then we find the ratio $C/C_0 = d/(d - t)$.

This approaches to infinity, if $t = d$ (the space is completely filled with metal).

Example 12. *Determine the spacing d between the plates and their area A for a flat capacitor employing waxed paper as the insulator ($k = 4.3$, $E_{bd} = 15\ MVm^{-1}$) if its capacitance is 3800 pF and the operating voltage is 600V and the safety factor is 2.5. How will the capacitance and the safety factor of a capacitance change if the waxed paper fits loosely against the plates and forms a air gap of width 0.025 mm ?*

We know that the electric field with in the region between the parallel plates of a capacitor is given as $E = V_{AB}/d$, therefore the distance between the plates

$$d = \frac{\text{P. D. between the plates}}{\text{Electric field between the plates}} = \frac{\text{Operating voltage} \times \text{Safety factor}}{\text{Breakdown electric field}}$$

or
$$d = \frac{600 \times 2.5}{15 \times 10^6} = 10^{-4} \text{ m} = 0.1 \text{ mm}.$$

$$\text{Area of the each plate } A = \frac{Cd}{k\varepsilon_0} = \frac{3800 \times 10^{-12} \times 10^{-4}}{4.3 \times 8.854 \times 10^{-12}} = 0.01\ m^2.$$

When the waxed paper fits loosely with a air gap of 0.025 mm, its thickness will be 0.1 – 0.025 = 0.075 mm and the capacitor will act as a double layer capacitor or a system of two capacitors one with paper layer and other the air layer in series. The capacitance of a paper layer is

$$C_P = \frac{k\varepsilon_0 A}{d_P} = \frac{4.3 \times 8.854 \times 10^{-12} \times 0.01}{0.075 \times 10^{-3}} = 5.076 \times 10^{-9} F$$

and the capacitance of the air layer is

$$C_a = \frac{\varepsilon_0 A}{d_a} = \frac{8.854 \times 10^{-12} \times 0.01}{0.025 \times 10^{-3}} = 3.542 \times 10^{-9} F.$$

$$\text{Total capacitance } C = \frac{C_p C_a}{C_p + C_a} = \frac{5.076 \times 3.542}{5.076 + 3.542} \times 10^{-9} = 2.086 \times 10^{-9} F.$$

If V_p and V_a be the voltages across the paper and air layers respectively, then we have,

$$V_a/V_p = C_p/C_a = 5.076/3.542 = 1.433.$$

Since for a series combination $V_p + V_a = 600$ volts,

$$\therefore \quad V_p = 246.6 \text{ volts and } V_a = 353.4 \text{ volts.}$$

The electric field strength of each layer is uniform. Thus we have

$$E_p = V_p/d_p = 246.6/0.075 \times 10^{-3} = 3.288 \times 10^6\ V/m.$$
$$E_a = V_a/d_a = 353.4/0.025 \times 10^{-3} = 14.136 \times 10^6\ V/m.$$

The field strength in the air gap is much greater than that in the paper. Since the breakdown electric field in air = 30×10^6 V/m. Hence the safety factor = $30 \times 10^6/14.136 \times 10^6 = 2.122$.

Thus we see that the air gap drastically decreases the safety factor of the capacitor.

Example 13. *Calculate the capacitance of a parallel plate capacitor filled with two dielectrics of same dimensions but of dielectric constants k_1 and k_2 respectively.*

Two dielectrics can be inserted in the following two ways :

The arrangement (*a*) can be assumed as two capacitors in parallel. If *d* is the separation and *A* is the area of each plate, then each capacitor will be of area *A*/2 and separation *d*. The total capacitance

$$C_p = C_1 + C_2 = \frac{\varepsilon_0 (A/2) k_1}{d} + \frac{\varepsilon_0 (A/2) k_2}{d}$$
$$= \varepsilon_0 A(k_1 + k_2)/2d.$$

The arrangement (*b*) can be assumed as two capacitors in series. Each capacitor will be of area *A* and separation *d*/2.

$$\therefore \frac{1}{C_s} = \frac{1}{C_1} + \frac{1}{C_2} = \frac{(d/2)}{\varepsilon_0 A k_1} + \frac{(d/2)}{\varepsilon_0 A k_2}$$

or
$$C_s = \frac{2\varepsilon_0 A}{d}\left(\frac{k_1 k_2}{k_1 + k_2}\right).$$

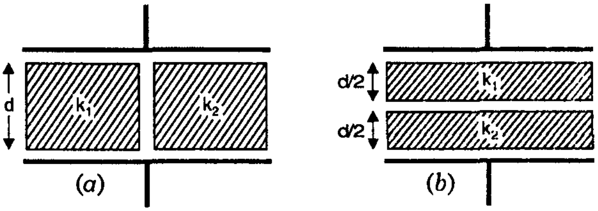


Fig. 5.28. Example 13.

Hence the ratio of capacitances in these arrangements

$$C_s/C_p = 4\ k_1 k_2/(k_1 + k_2)^2,$$

which is always smaller than one.

Example 14. *A parallel plate capacitor with a separation d = 1 cm has 29 kV applied when free space is with in the plates. Given that the air has a dielectric strength of 30 kV/cm. Show why the air breaks down when a thin sheet of glass (k = 6.5, thickness 0.20 cm) with a dielectric strength of 290 kV/m is inserted with in the plates of a capacitor.*

The problem becomes one of two capacitors in series. If A is the area of the each plate of the capacitor, then

$$C_1 = \frac{\varepsilon_0 A}{8\times10^{-3}} = 125\ \varepsilon_0 A \qquad \text{(Air as dielectric)}$$

$$C_2 = \frac{\varepsilon_0 . 6.5A}{2\times10^{-3}} = 3250\ \varepsilon_0 A. \qquad \text{(glass as dielectric)}$$

Since $C_1V_1 = C_2V_2 = Q$ and $V_1 + V_2 = V,$

$$\therefore \quad V_1 = \frac{C_2}{C_1+C_2}V = \frac{3250}{125+3250}\times 29\text{ kV} = 27.93\text{ kV}.$$

Therefore the electric field within the air region

$$E_1 = \frac{V_1}{d_1} = \frac{27.93\text{ kV}}{0.80\text{ cm}} = 34.97\text{ kV/cm},$$

which exceeds the dielectric strength of air. Therefore the air breaks down due to insertion of glass plate. Thus we see that the introduction of glass into the space between the plates worsens the safety factor of the capacitor despite the fact that the glass is electrically stronger then air.

Example 15. *A parallel plate capacitor has the space between the plates filled with a medium whose dielectric constant increases uniformly with distance. If d be the distance between the* plates and k_1 *and* k_2 *be the dielectric constants of the medium at the two plates respectively, show that the capacity of the capacitor per unit area is* $C = \varepsilon_0 (k_2 - k_1)/d \log_e (k_2/k_1)$.

Electric intensity at a point between the plates of a parallel plate capacitor is given by $E = \sigma/\varepsilon_0 k$.

Here k is not constant but varies from k_1 to k_2. As d is the separation of two plates, hence the increase in dielectric constant per unit length $= (k_2 - k_1)/d = \alpha$ (say). Hence the dielectric constant at point at a distance x from the first plate is $k = k_1 + \alpha x$ and the electric intensity at this point $= \sigma/\varepsilon_0(k_1 + \alpha x)$.

$\therefore$ Potential difference between the plates $V = -\int_d^0 \mathbf{E}\cdot d\mathbf{x} = \int_0^d E dx = \int_0^d \frac{\sigma dx}{\varepsilon_0(k_1+\alpha x)}$

$$= \frac{\sigma}{\varepsilon_0\alpha}\log_e\frac{k_1+\alpha d}{k_1} = \frac{\sigma d}{\varepsilon_0(k_2-k_1)}\log_e\frac{k_2}{k_1}.$$

Hence the capacitance $C = \frac{\sigma A}{V} = \frac{\varepsilon_0 A(k_2-k_1)}{d\log_e(k_2/k_1)}$

or the capacitance per unit area $= \varepsilon_0(k_2 - k_1)/d \log_e(k_2/k_1)$.

Example 16. *A capacitor is formed by two square metal plates of side a, separated by a distance d. Dielectrics of dielectric constants* K_1 *and* K_2 *are filled in the gap as shown in Fig. 5.29. Find the capacitance of the capacitor.*

Since the plates are square in size of side a. Therefore the area of each plate is a^2. Divide the space between the plates into large number of strips and consider a strip of area adx at a distance x from the edge O. Out of total height d, y is in dielectric K_2 and d-y in dielectric K_1. If E_1 and E_2 be the electric field in the medium of dielectric constants K_1 and K_2 repectively. The potential difference between the plates is therefore given as

$$V = E_1(d-y) + E_2(y) \qquad \ldots(i)$$

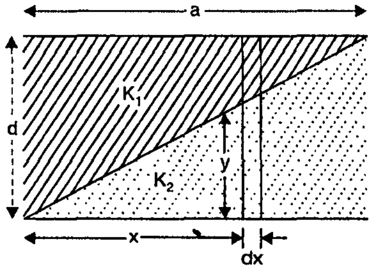


Fig. 5.29. Example 16.

The electric displacement is uniform in the strip, therefore

$$D = \varepsilon_0 K_1 E_1 = \varepsilon_0 K_2 E_2 \qquad \text{...}(ii)$$

If σ be the surface charge density on the positive plate, then $D = \sigma$. Thus we have

$$E_1 = \sigma/\varepsilon_0 K_1, \text{ and } E_2 = \sigma/\varepsilon_0 K_2$$

$$\therefore \quad V = \frac{\sigma}{\varepsilon_0}\left(\frac{d-y}{K_1}+\frac{y}{K_2}\right)$$

From the geometry of the Fig. 5.29, we have $y/x = d/a$ or $y = xd/a$.

$$\therefore \quad V = \frac{\sigma}{\varepsilon_0}\left[\frac{1}{K_1}\left(d-\frac{xd}{a}\right)+\frac{xd}{aK_2}\right] = \frac{\sigma d}{\varepsilon_0 K_1 K_2 a}\left[K_2(a-x)+K_1 x\right]$$

The charge on the strip capacitor is $\sigma a dx$, therefore its capacitance

$$dC = \frac{\sigma a dx}{V} = \frac{\varepsilon_0 K_1 K_2 a^2\, dx}{d\,[K_2(a-x)+K_1 x]} = \frac{\varepsilon_0 K_1 K_2 a^2\, dx}{d\,[aK_2 + x\,(K_1-K_2)]}$$

$$\therefore \quad C = \int dC = \frac{\varepsilon_0 K_1 K_2 a^2}{d}\int_0^a \frac{dx}{aK_2 + x(K_1-K_2)}$$

$$\text{or} \quad C = \frac{\varepsilon_0 K_1 K_2 a^2}{(K_1-K_2)d}\int_{aK_2}^{aK_1}\frac{dp}{p}, \quad \text{where } p = aK_2 + x\,(K_1-K_2)$$

$$= \frac{\varepsilon_0 K_1 K_2 a^2}{(K_1-K_2)d}\log_e\frac{K_1}{K_2}$$

Example 17. *If central plate, bearing total charge Q, can move as isolated between two parallel fixed earthed plates but makes a gastight seal where it slides on the walls. The air on both sides of the movable plate is initially at the same pressure p_0. Find the values of the displacement x where the plate can be in stable equilibrium.*

The three plates (side fixed two plates and central moving plate) form two parallel plate capacitors with capacitences C_1 and C_2. When the central plate is moved a distance x from the central position, the total capacitance of the system

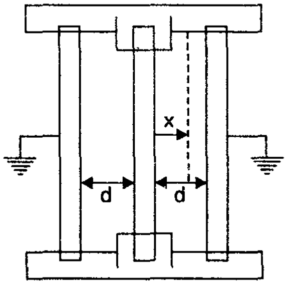


Fig. 5.30. Example 17.

$$C = C_1 + C_2 = \frac{A}{\varepsilon_0(d+x)} + \frac{A}{\varepsilon_0(d-x)} = \frac{2Ad}{\varepsilon_0(d^2-x^2)}$$

Hence the electrostatic energy of the system is

$$W_e = \frac{1}{2}\frac{Q^2}{C} = \frac{\varepsilon_0 Q^2(d^2-x^2)}{4\,Ad}$$

As the charge Q is distributed over the central plate, when this plate moves work is done against the electrostatic force, which is given by

$$F_e = -\partial W_e/\partial x = Q^2\varepsilon_0 x/2Ad.$$

As $F > 0$, the force is in the direction of increasing x. As an electronic conductor is also a good thermal conductor, the motion of the central plate can be considered isothermal. Let the pressures exerted by air on the left and right sides on the central plate be p_1 and p_2 respectively. Therefore by Boyl's law

$$p_1 = p_0 d/(d+x) \quad \text{and} \quad p_2 = p_0 d/(d-x).$$

The central plate will be in equilibrium position, if the electrostatic force is balanced by the force exerted by the pressure difference, *i.e.*,

$$F_e = (p_2 - p_1)A \quad \text{or} \quad Q^2\varepsilon_0 x/2\,Ad = 2\,Ap_0\, xd/(d^2-x^2)$$

$$\text{or} \quad x = \pm\, d\,(1 - 4\,p_0 A^2/\varepsilon_0 Q^2)^{1/2}.$$

Example 18. *Three flat vertical metallic plates form a double parallel plate capacitor the middle plate, of mass M is free to move vertically between the other two plates. Let the charge on this capacitor be q, which is discharging through an external load resistance R (Fig. 5.31). Neglecting the fringing field effects and the small internal resistance. The discharge is assumed to be slow enough so that the system is in static equilibrium at all times.*

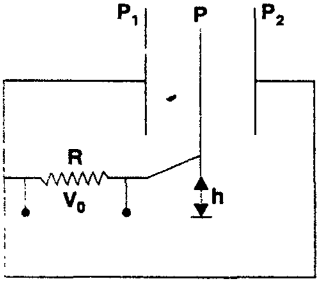


Fig. 5.31. Example 18.

(*a*) *How does the gravitational energy of the system depend on height h of the middle plate P?*

(*b*) *How does the electrostatic energy of the system depend on h and charge q?*

(*c*) *How does h varies with q?*

(*d*) *How does the output voltae varies due to this discharge?*

(*a*) The gravitational energy of the system $W_g = mgh$.

(*b*) Suppose that the distance between P and P_1 and that between P and P_2 are both d, each of the three plates has width a and length l. Also suppose that $h = h_0$ when the top of P coincides with those of plates P_1 and P_2. The system may be considered as two capacitors in parallel each with charge $q/2$ and capacitance when the height P is h is

$$C = \varepsilon_0 a(l + h - h_0)/d.$$

The electrostatic energy stores in the system $W_e = 2.\frac{1}{2}\frac{(q/2)^2}{C}$

$$= q^2 d/4\varepsilon_0 a(l + h - h_0)$$

(*c*) Therefore the total energy of the system $W = W_g + W_e = Mgh + q^2 d/4\varepsilon_0 a(l + h - h_0)$.

Since the discharge process is slow, P for each q will adjust to an equilibrium position h where the energy of the system is minimum. Therefore

$$(\partial W/\partial h)_q = 0$$

$$\text{or} \quad Mg - \frac{q^2 d}{4\varepsilon_0 a(l + h - h_0)^2} = 0 \quad \text{or} \quad h = \frac{q}{2}\left[\frac{d}{\varepsilon_0 a Mg}\right]^{1/2} + h_0 - l.$$

Thus we see that h varies linearly with q.

(*d*) As the system is discharging through R, q decreases and h decreases. Hence the output voltage

$$V_0 = \frac{\text{charge}}{\text{capacitance}} = \frac{q/2}{\varepsilon_0 a(l + h - h_0)/d} = \left[\frac{Mgd}{\varepsilon_0 a}\right]^{1/2}$$

which remains constant as the capacitor discharges.

Example 19. *In the capacitor shown in Fig. 5.32, the region between the plates is filled with a dielectric having $\varepsilon_r = 3.6$. Calculate the capacitance.*

Neglecting the fringing effect, the electric field between the plates in cylindrical coordinates will be along $\mathbf{a}_\phi$. The voltage of the plate $ABCD$ with respect to the plate $A'B'C'D'$

$$V_0 = -\int \mathbf{E}\cdot d\mathbf{l} = -\int_0^{\alpha} E_\phi \mathbf{a}_\phi \cdot r d\phi \mathbf{a}_\phi$$

$$= -E_\phi r \int_0^{\alpha} d\phi = -E_\phi r\alpha.$$

$\therefore \quad E_\phi = - V_0/r\alpha.$

$\therefore$ Charge density on the plate $\phi = \alpha$ is

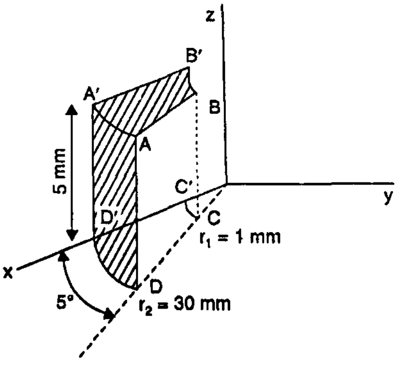


Fig. 5.32. Example 19.

$$\rho_s = D_n = -\varepsilon_r\varepsilon_0 E_\phi = \varepsilon_0\varepsilon_r V_0/r\alpha.$$

$\therefore$ Total charge on the plate

$$Q = \int \rho_s dS = \int_0^h \int_{r_1}^{r_2} \frac{\varepsilon_0\varepsilon_r V_0}{r\alpha}\, dr\, dz$$

$$= \frac{\varepsilon_0\varepsilon_r V_0 h}{\alpha} \log \frac{r_2}{r_1}.$$

Therefore
$$C = \frac{Q}{V_0} = \frac{\varepsilon_0\varepsilon_r h}{\alpha} \log \frac{r_2}{r_1} = \frac{8.854 \times 10^{-12} \times 3.6 \times 5 \times 10^{-3}}{(5 \times 3.14/180)} \log_e 30$$
$$= 6\ .12.$$

Example 20. *A cylindrical capacitor has radii a = 1 cm and b = 2.5 cm. If the space between the plates is filled with an inhomogeneous dielectric with* $\varepsilon_r = (10 + \rho)/\rho$, *where* ρ *is in centimeters. Find the capacitance per meter of the capacitor.*

For co-axial capacitor

$$V = -\int \mathbf{E} \cdot d\mathbf{l} = -\int_b^a \frac{q}{2\pi\varepsilon_0\varepsilon_r \rho L} \mathbf{a}_\rho \cdot d\rho\, \mathbf{a}_\rho$$

$$= -\frac{q}{2\pi\varepsilon_0 L} \int_b^a \frac{d\rho}{\rho(10+\rho)/\rho} = \frac{-q}{2\pi\varepsilon_0 L} \int_b^a \frac{d\rho}{10+\rho}$$

$$= \frac{q}{2\pi\varepsilon_0 L} \log_e \frac{10+b}{10+a}.$$

$\therefore$ Capacitance per meter $C = \dfrac{q}{V} = \dfrac{2\pi\varepsilon_0}{\log_e(12.5/11)} = 434.6\ pF/m.$

Example 21. *A potential of 50 volts is applied between two parallel plates of a capacitor which are 4 cm. apart. Obtain the force acting on charge of* 4.3×10^{-7} *coulomb placed between the plates.*

The electric field in between the parallel plate capacitor due to a potential V applied across the plates separated by a distance d is given by

$$E = V/d = 50/0.04 = 1250 \text{ volts/meter.}$$

The force acting on a charge q placed between the plates is the force due to this field E, *i.e.*, $F = qE = 4.3 \times 10^{-7} \times 1250 = 5.375 \times 10^{-4}$ newton.

Example 22. *A cylindrical capacitor has radii a and b. Show that half the stored electric energy lies with in a cylinder of radii* $\sqrt{ab}$.

The capacitance of a cylindrical capacitor of length l with inner and outer radii as a and b respectively, is given as

$$C = 2\pi\varepsilon_0 l/\log_e(b/a).$$

The electrostatic energy of this capacitor

$$U = \frac{1}{2}\frac{q^2}{C} = \frac{q^2 \log_e(b/a)}{4\pi\varepsilon_0 l}.$$

The capacitance of the capacitor with outer radius r (< b) and inner radius a is given by

$$C' = 2\pi\varepsilon_0 l/\log_e(r/a).$$

$$\therefore \qquad U' = \frac{1}{2}\frac{q^2}{C'} = \frac{q^2 \log_e(r/a)}{4\pi\varepsilon_0 l}$$

Given that $U' = \frac{1}{2} U$, therefore $\log_e(r/a) = \frac{1}{2} \log_e(b/a)$

or $\qquad r/a = (b/a)^{1/2}$ or $r = \sqrt{ab}$.

Example 23. *A capacitor consists of two metallic discs, each of 1 metre in diameter, placed parallel to each other at a distance 4 mm. apart. The potential difference between the plates is 10,000 volts. Calculate the energy stored by the capacitor.*

The capacitance of a parallel plate capacitor with plate area A, separated by a distance d is given by $C = \varepsilon_0 A/d$.

The energy stored by the capacitor $U = \frac{1}{2} CV^2 = \frac{1}{2} (\varepsilon_0 A/d)\, V^2$.

Substituting numerical values as : $\varepsilon_0 = 8.854 \times 10^{-12}$ cont/newton m^2, $A = \pi r^2 = 3.14 \times (0.5)^2$ m^2, $d = 0.004$ m and $V = 10{,}000$ volts,

we get
$$U = \frac{1}{2} \times 8.854 \times 10^{-12} \times 3.14 \times (0.5)^2 \times 10{,}000^2/0.004$$
$$= 8.688 \times 10^{-2} \text{ joules.}$$

Example 24. *Prove that the energy loss is always positive on sharing of charges in two capacitors.*

Let us consider two capacitors A and B of capacitances C_1 and C_2 and having charges q_1 and q_2 respectively. If these are connected by a wire, the charge will flow from the higher potential to lower potential. After sharing, let V be the common potential and q_1' and q_2' be the charges on A and B capacitors respectively.

$$\therefore \quad \text{Total charge } q_1 + q_2 = C_1V_1 + C_2V_2 = (C_1 + C_2)V,$$

or
$$V = \frac{C_1V_1 + C_2V_2}{C_1 + C_2} = \frac{q'_1}{C_1} = \frac{q'_2}{C_2}$$

$$\therefore \quad q_1' = C_1 \frac{C_1V_1 + C_2V_2}{C_1 + C_2} \text{ and } q_2' = C_2 \frac{C_1V_1 + C_2V_2}{C_1 + C_2}.$$

Hence total energy before contact $= \frac{1}{2} C_1 V_1^2 + \frac{1}{2} C_2 V_2^2$

and energy after contact $= \frac{1}{2} [C_1 + C_2] V^2 = \frac{1}{2} (C_1V_1 + C_2V_2)^2/(C_1 + C_2)$.

$\therefore$ Loss of energy due to sharing $\frac{1}{2}\ C_1C_2\, (V_1 - V_2)^2/(C_1 + C_2)$.

It is always positive, *i.e.*, there is always a loss of energy on sharing of charges. As total charge remains same, hence *this loss appears in the form of a spark and heat.*

Example 25. *Two capacitors AB and CD have capacities C_1 and C_2 respectively. They are uncharged initially and A is permanently connected to earth. The two capacitors are connected in series and D is charged to a potential V. CD is then disconnected and joined in parallel with AB. Show that the loss in energy is $\frac{1}{2} V^2 C_1 C_2 (C_1 - C_2)^2/(C_1 + C_2)^3$.*

When the capacitors are in series, the potential of the plates B and C are each equal to V' (say) and that of the plate D is V (given). If $+q$ be the charge on plate D, then the induced charges over plates C, B and A will be $-q$, $+q$ and $-q$ respectively. From the definition of capacity

$$V - V' = q/C_2 \text{ and } V' - 0 = q/C_1.$$

$$\therefore \quad q = C_1C_2V/(C_1 + C_2) = CV.$$

The electrostatic energy of the combination

$$W = \frac{1}{2} CV^2 = \frac{1}{2} C_1C_2V^2/(C_1 + C_2).$$

When the capacitor CD is disconnected and connected with AB in parallel, the capacity of the combination $C' = C_1 + C_2$. The charges are $+2q$ and $-2q$. Therefore the electrostatic energy

$$W' = \frac{1}{2} C'V^2 = \frac{1}{2} \frac{q'^2}{C'} = \frac{1}{2} \frac{(2q)^2}{C_1 + C_2}$$
$$= 2C_1^2C_2^2V^2/(C_1 + C_2)^3.$$

$$\therefore \quad \text{Loss of energy} = W - W' = \frac{1}{2} C_1C_2(C_1 - C_2)^2V^2/(C_1 + C_2)^3.$$

Example 26. *A parallel plate capacitor is charged to a potential of V volts and the charging battery is then disconnected. The capacitor is now placed vertically on the surface of a liquid of density ρ kg/m³. If the liquid is raised to a height of h meter inside the capacitor plates, find the expression for V.*

When the charged capacitor plates are vertically placed on the liquid surface, the liquid will rise in the space between the plates due to its polarization. The charge will remain constant while the capacitance will increase. Hence the stored energy will decrease. This decrease in electrostatic potential energy is compensated by the increase in gravitational potential energy of the liquid column held between the plates. If C_1 and C_2 be the capacitances before and after the rise in liquid. If $A = l \times b$ is the area of each plate and d the separation between the plates, then

$$C_1 = \frac{\varepsilon_0 A}{d} = \frac{\varepsilon_0 lb}{d}, C_2 = \frac{k\varepsilon_0 A'}{d} + \frac{\varepsilon_0 (A - A')}{d},$$

where A' = overlapping area covered by the dielectric (liquid) = $h \times b$.

$$\therefore \quad C_2 = \frac{k\varepsilon_0 hb}{d} + \frac{\varepsilon_0(lb - hb)}{d} = \frac{\varepsilon_0 b}{d}[h\,(k-1) + l],$$

Mass of liquid raised $m = hb\,d\,\rho$.

By the energy conservation principle

$$Q^2/2C_1 = Q^2/2C_2 + mg(h/2).$$

(The centre of gravity of the liquid column lies at the middle)

The charge stored in the capacitor $Q = C_1 V$.

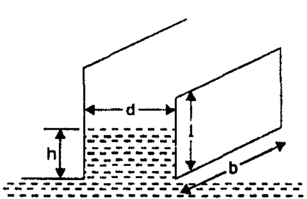


Fig. 5.33. Example 26.

$$\therefore \quad \frac{(C_1V)^2}{2C_1} = \frac{(C_1V)^2}{2C_2} + \frac{hbd\rho gh}{2} \text{ or } V^2\left[C_1 - \frac{C_1^{\,2}}{C_2}\right] = h^2bd\rho g$$

$$\text{or} \quad V^2 = \frac{h^2bd\rho gC_2}{C_1(C_2 - C_1)} = \frac{h^2bd\rho g\varepsilon_0 b\,[h\,(k-1) + l]/d}{(\varepsilon_0 lb/d)\,\varepsilon_0 bh(k-1)/d}$$

$$\therefore \quad V = hd\left[\frac{\rho g}{\varepsilon_0 l}\left(1 + \frac{l}{h(k-1)}\right)\right]^{1/2}.$$

Example 27. *Outside a spherical charged conductor A, there is a concentric insulated but uncharged conducting spherical shell, which consists of two segments. Prove that the two segments will not separate if the distance of the separating plane from the centre is less than* $bc/\sqrt{(b^2 + c^2)}$, *where b and c are the inner and outer radii of the shell respectively.*

If q is the charge on the conductor then the induced charges on the inner and outer surfaces of the shell will be $-q$ and $+q$ respectively. The surface charge densities σ_1 and σ_2 are thus given as $\sigma_1 = -q/4\pi b^2$ and $\sigma_2 = q/4\pi c^2$.

If this shell is divided into segments by the plane RS, then by symmetry the force acting on either segment will be ⊥ to the plane RS. Let us divide the segment RBS into annular discs by planes || to RS and now consider two closed discs at distance x and $x + dx$ from the centre O. If P and P' be the points of intersection of the plane at x with the inner and outer surfaces of the shell and $\hat{n}_1$ and $\hat{n}_2$ are the unit vectors along PO and OP' respectively as shown in Fig. 5.34. We know that the mechanical force per unit area is given by $\sigma^2/2\varepsilon_0$, hence the forces on an element of the inner surface dS_1 at P and on an element of the outer surface dS_2 at P' are $\sigma_1^{\,2}\, dS_1\, \hat{n}_1/2\varepsilon_0$ and $\sigma_2^{\,2}\, dS_2\, \hat{n}_2/2\varepsilon_0$ respectively. Since the resultant force on the segment RBS is along OB. The components along OB are $-\sigma_1^{\,2}\, dS_1 \cos\theta_1/2\varepsilon_0$ and $\sigma_2^{\,2}\, dS_2 \cos\theta_2/2\varepsilon_0$. Summing over all such elements on the two rings through P and P', we have

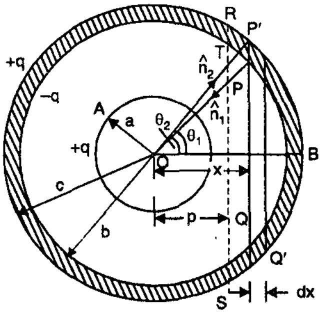


Fig. 5.34. Example 27.

$$dS_1 = 2\pi b \sin\theta_1\, b d\theta_1 \text{ and } dS_2 = 2\pi c \sin\theta_2\, c\, d\theta_2.$$

From the geometry of the Fig. 5.34, we have $b\cos\theta_1 = c\cos\theta_2 = x$

and hence $b \sin\theta_1\, d\theta_1 = c \sin\theta_2\, d\theta_2 = - dx$.

If p be the distance of RS from the centre O, then the total force on the segment RBS along OB is

$$= \frac{1}{2\varepsilon_0}\left[\int_p^c \sigma_2^{\,2} 2\pi c^2 \sin\theta_2 \cos\theta_2 d\theta_2 - \int_p^b \sigma_1^{\,2} 2\pi b^2 \sin\theta_1 \cos\theta_1\, d\theta_1\right]$$

$$= \frac{1}{2\varepsilon_0}\left[\int_p^c \sigma_2^{\,2} 2\pi x\, dx - \int_p^b \sigma_1^{\,2} 2\pi x\, dx\right] = \frac{q^2}{32\pi\varepsilon_0}\left[\frac{1}{c^4}(c^2 - p^2) - \frac{1}{b^4}(b^2 - p^2)\right]$$

The segment will not separate if this expression is negative, *i.e.*,

$$\sigma_2^{\,2} \int_p^c x\, dx < \sigma_1^{\,2} \int_p^b x\, dx \text{ or } \frac{1}{c^4}(c^2 - p^2) < \frac{1}{b^4}(b^2 - p^2).$$

$$\therefore \qquad p < bc/(b^2 + c^2)^{1/2}.$$

Example 28. *An insulated spherical conductor, formed of two hemispherical shells in contact, whose inner and outer radii are b and b′, has within it a concentric spherical conductor of radius a and outside of it another concentric spherical conductor of radius c. These two conductors are earthed connected and the middle one receives a charge. Show that the two shells will not separate if 2ac > bc + b′a.*

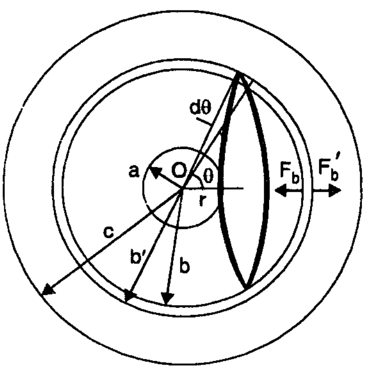


Fig. 5.35. Example 28.

If C_1 be the capacity between the surfaces $r = a$ and $r = b$, and C_2 that between $r = b'$ and $r = c$.

$\therefore \qquad C_1 = 4\ \pi\varepsilon_0\, ab/(b - a)$ and $C_2 = 4\pi\varepsilon_0 b'c/(c - b')$.

As both the surfaces of the middle conductor are at the same potential (say V). The charges shared by both surfaces are

$$q_b = C_1V = 4\pi\varepsilon_0\, abV/(b - a)$$

and $\quad q_b' = C_2V = 4\pi\varepsilon_0 b'\, cV/(c - b').$

$\therefore \quad \sigma_b = q_b/4\pi b^2 = \varepsilon_0 aV/b(b - a)$

and $\quad \sigma_b' = q_b/4\pi b'^2 = \varepsilon_0 cV/b'\,(c - b')$.

By symmetry, the force acting on either hemisphere is perpendicular to the vertical diametiral plane. The force on the inner or outer hemispherical shell can be determined by breaking its surface into a large number of concentric rings by planes parallel to the plane and then summing up the mechanical force on each ring. Thus

$$F_b = \int_0^{\pi/2} \frac{\sigma_b^{\,2}}{2\varepsilon_0}(2\pi b \sin\theta)\,(b d\theta)\cos\theta = \frac{\pi b \sigma_b^{\,2}}{2\varepsilon_0}.$$

Similarly the force on the outer surface $F_b' = \pi b'^2\, \sigma_b'^2/2\varepsilon_0$.

The two shells will not separate if $F_b > F_b'$, *i.e.*,

$$\frac{\pi b^2 \sigma_b^{\,2}}{2\varepsilon_0} > \frac{\pi b'^2 \sigma_b'^2}{2\varepsilon_0} \text{ or } \frac{b^2 a^2}{b^2(b - a)^2} > \frac{b'^2 c^2}{b'^2(c - b')^2}.$$

$$a^2/(b - a)^2 > c^2/(c - b')^2 \text{ or } 2ac > bc + ab'.$$

Example 29. *An attracted disc electrometer has a moving plate of area 100 cm², separated by a distance of 1 mm from the fixed plate. Calculate the force between the plates when the potential difference across them is 100 V. Calculate the sensitivity at this voltage in newton's per volt.*

The force between the plates of an attracted disc electrometer is given by

$$F = \frac{\varepsilon_0 V^2 A}{2d^2} = \frac{8.854 \times 10^{-12} \times 100^2 \times 100 \times 10^{-4}}{2 \times (1 \times 10^{-3})^2}$$

$$= 4.42 \times 10^{-4} \text{ newton.}$$

The sensitivity of this electrometer rises with the voltage V and is given by

$$\frac{dF}{dV} = \frac{\varepsilon_0 VA}{d^2} = \frac{8.854\times10^{-12}\times100\times100\times10^{-4}}{(1\times10^{-3})^2}$$
$$= 8.85\times10^{-6} \text{ newton/volt.}$$

Example 30. *The needle of a quadrant electrometer deflects through 20 scale divisions when it is connected idiostatically to a potential of 50 volts. It is then used heterostatically, needle is connected to 500 volts and the quadrants are maintained at smaller voltage. Find the voltage on the quadrants, if the deflection in the latter case reduces to 10 scale divisions.*

In the idiostatic use of electrometer, the needle is connected to one pair of quadrants, say, $V_0 = V_1$ and

$$\theta = \tfrac{1}{2}K(V_1 - V_2)^2$$
$$\therefore \quad K = 2\theta/(V_1 - V_2)^2 = 2\times20/50^2 = 1.6\times10^{-2}.$$

In the heterostatic use $V_0 >> V_1$ (or V_2) and $\theta = KV_0(V_1 - V_2)$, or the potential difference between two pairs of quadrants

$$V_1 - V_2 = \frac{\theta}{KV_0} = \frac{10}{1.6\times10^{-2}\times500} = 1.25 \text{ volts.}$$

ORAL QUESTIONS

1. A capacitor is connected across a potential source. (*a*) Why does each plate receive a charge of same magnitude ? (*b*) Is it true even if the plates are of different sizes ?
2. When one grasps the two wires leading from the two plates of a charged capacitor, he feels a shock of much more strength for a capacitor of greater capacitance but charged to the same potential difference. Explain the cause of sensation of shock given to the body.
3. How do you explain the effect of (*a*) a dielectric slab, (*b*) a conducting slab, when inserted between the plates of a parallel plate capacitor.
4. For a given potential difference does a capacitor store more or less charge with a dielectric than it does without a dielectric ?
5. What do you expect about the change of mass of an isolated conducting sphere if given a positive charge ?
6. A potential source is connected with a capacitor and is then disconnected. What happens to the charge, the capacitance and the potential difference when a dielectric slab is inserted in between the plates of the capacitor ?
7. What property of conductors accounts for their capacitance ?
8. Can a single conductor run above ground level be treated as a capacitor ?
9. How long will the electric field between the plates of a charged capacitor exist in an ideal dielectric ?
10. The device that stores an electric charge is known as (*a*) conductor, (*b*) capacitor, (*c*) electroscope.
11. The capacitors are used in (*a*) electric circuits, (*b*) heat, (*c*) optical phenomena.
12. The parallel plate capacitor connected to a battery stores twice as much charge with a given dielectric as it does with air as dielectric. The susceptibility of the dielectric is (*a*) 0, (*b*) 1, (*c*) 2.
13. If the earth is regarded a spherical capacitor, what is its capacitance ? Given R of earth is 6370 km.
14. Two capacitors C_1 and C_2 are charged to 120V and 200V respectively. When they are connected in parallel, it is found that the potential on each one of them is zero. Therefore (*a*) $5C_1 = 3C_2$, (*b*) $3C_1 = 5C_2$, (*c*) $3C_1 + 5C_2 = 0$.
15. Five thin rectangular conducting plates, each of same breadth 1m, arranged parallel to each other, separated by equal distance 1mm and height in decreasing order (8m, 4m, 2m, 1m, 0.5m).

The equivalent capacitance of the system is (*a*) $C/8$, (*b*) $C/12$, (*c*) $C/15$, where C is capacitance of the biggest capacitor.

16. How are capacitors connected in parallel ? The resultant of three capacitors, each of capacitance C, in parallel will be (*a*) $C/3$, (*b*) $3C$; (*c*) C^3.
17. How does the field energy of an air capacitor change if the plate spacing is varied : (*a*) after disconnecting the charged capacitor from the source of supply, (*b*) with the capacitor connected to the source ?
18. Keeping the capacitor in contact with a battery, a dielectric slab is slipped between the plates. Describe what happens to the charge, the capacitance, the electric field strength and the stored energy.
19. A parallel plate capacitor is charged to a charge Q. It is then disconnected from the battery, after which the plates are slowly pulled apart until the plate separation is doubled. Does Q change ? Does the energy stored in the capacitor change ?
20. How can you increase the sensitivity of the quadrant electrometer ?

Ans. 7. dimensions; **8.** yes; **9.** Infinite time; **10.** (*b*); **11.** (*a*); **12.** (*b*); **13.** 0.7078 mF; **14.** (*b*); **15.** (*c*); **16.** (*b*);

PROBLEMS

1. A 100μ μ*F* capacitor is charged to 100 volts. After the charging, battery is disconnected, the capacitor is connected in parallel to another capacitor. The final voltage is 20 volts. Calculate the capacity of second capacitor. (233.3μ μ*F*)
2. A 1μ*F* capacitor and a 2μ*F* capacitor are connected in parallel across a 1200*V* supply line. (*a*) Find the charge on each capacitor and the voltage across each, (*b*) The charged capacitors are disconnected from the line and from each other, and reconnected with terminals of like sign together. Find the final charge on each and the voltage across each.

   (*a*) 800μ*C*, 800*V*; 800μ*C*, 400*V*; (*b*) 533 μ*C*, 533*V*; 1066 μ*C*, 533*V*)
3. A charge q is distributed over two concentric hollow spheres of radii r and R, such that the surface charge densities are equal. Find the potential at the common centre. $q(r + R)/4\pi\varepsilon_0(r^2 + R^2)$
4. n' identical spherical drops of liquid are charged to the same potential V. If a large drop is formed by combining these. Find the potential of the large drop. ($n^{2/3}V$)
5. A parallel plate condenser is made up of 20 metal strips each 2cm × 3cm. separated by sheet of mica of dielectric constant 6 and uniform thickness 0.2mm. Calculate the capacity of the condenser. (8.39*pF*)
6. A capacitor is formed of two concentric spherical conducting shells of radii a and b. The inner sphere of radius a receives a thin coating of thickness t and of material of dielectric constant k. Show that the capacity is changed approximately by [$4\pi\varepsilon_0 b^2 (k - 1)t/k(b - a)^2$]
7. Find the capacitance of a cylindrical air capacitor having a plate diameter of 2.4 and 3.6 cm and length of 4cm. Calculate the maximum and minimum field strength if the capacitor is at a potential difference of 2000 volts. (5.5 *pF*, 4.12 *kV*/cm, 2.74 *kV*/cm)
8. Show that for a given potential difference the electric field at the surface of the inner cylinder of a cylindrical capacitor is least if the radius of this cylinder is $1/e$ of the internal radius of the outer cylinder.
9. A condenser is formed of a conducting sphere of radius r_0 and a conducting spherical shell of internal radius r, insulated from it. If n conducting concentric spherical shells are placed between these two spherical conductors, find the capacitance of the whole arrangement.

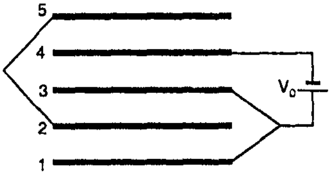
   ($4\pi\varepsilon_0/[1/r_0 - \Sigma\,(1/a_i) - 1/r + \Sigma\,(1/b_i)]$, where a_i and b_i be the inner and outer radii of i^{th} shell).

10. A metal sphere of radius a is surrounded by a spherical thick metal shell of inner and outer radii b and c. This shell is isolated with no net charge and is surrounded by a grounded spherical shell of radius d. Find the potential of the inner sphere, when a charge q is placed on it. What is the capacitance of the sphere ? [$C = 4\pi\varepsilon_0(1/a + 1/c - 1/b - 1/d)$]

11. A capacitor is formed of two concentric spherical conducting shells of radii a and b. If the medium between the spheres has a dielectric constant k_1 from a to r and k_2 from r to b, show that the capacity of the condenser is given by

$$C = 4\pi\varepsilon_0\left[\frac{1}{k_1 a} + \frac{1}{k_2 b} + \frac{1}{r}\left(\frac{1}{k_2} - \frac{1}{k_1}\right)\right]^{-1}.$$

12. A certain diode vacuum tube consists of a 0.2 mm diameter wire (the filament) on the axis of a 1.0 cm inner diameter hollow cylindrical plate. Both the plate and filament are 5.0 cm long. Find the capacitance of the vacuum tube. ($0.71 \times 10^{-12} F$)



13. Five identical conducting plates each of area A are placed parallel and equidistant at a distance d from each other and are connected as shown in Fig. 5.33. Find (*i*) the effective capacity of the system between the terminals of the battery and (*ii*) the charges on plates 3 and 5. ($5\varepsilon_0 A/3d$, $4\varepsilon_0 AV_0/3d$, $2\varepsilon_0 AV_0/3d$)

Fig. 5.33. Problem 13.

14. Show that if the equation of the rim of the movable plates of a rotary variable condenser is $0r^{-2}$ = const., the change in $\sqrt{C}$ is proportional to the change in angle of rotation.

15. In a variable air capacitor which is a pile of n plates of alternate polarity each having an area A and separated from adjacent plates by a distance d, alternate plates are connected together, one group being fixed in position and the other group being capable of rotation. Show that the maximum capacitance is $(n - 1)\varepsilon_0 A/d$.

16. A capacitor has square plates each of side a making an angle of θ with each other. Show that the capacitance is given by $C = \frac{\varepsilon_0 a^2}{d}\left(1 - \frac{a\theta}{2d}\right)$, where d is the minimum distance between the plates at one end.

17. At what distance should the circular plates of a parallel plate condenser, 40 mm in diameter, be placed in order to have the same capacity as a sphere of 100 cm. (0.1 mm)

18. Calculate the capacitance of a spherical capacitor using the energy relation. $4\pi\varepsilon_0 R_1 R_2/(R_2 - R_1)$

19. A copper slab of thickness b is inserted into a parallel plate capacitor exactly half way between the plates. If the separation of the plate is d and A is the area of each plate. Find the change in capacitance. $\varepsilon_0 Ab/(d - b)\, d$.

20. A parallel plate capacitor consists of two plates, each with area 200 cm^2, separated by a distance 5 mm. Calculate (*a*) its capacitance, (*b*) charge on the plate if the capacitor is connected to a 500V supply, (*c*) energy stored, (*d*) E between the plates. If a liquid with dielectric constant 2.5 is poured between the plates to fill the air gap, how much additional charge will flow to the capacitor from the supply. (35.4 pF, 17.7 nC, 4.425 μJ, 10^5 V/m, 26.55 nC)

21. Two capacitors (2μF and 4μF) are connected in parallel across a 300 volts potential difference. Calculate the total stored energy in the system and the total charge on it. (0.27 joule; 18×10^{-4} coul)

22. Calculate the force between the plates of parallel plate capacitor when the area of the plate is 300 cm^2 each, the separation is 0.5 cm and they are charged to p.d. 1000 volts. (5.31×10^{-3} newton)

23. A parallel plate capacitor, with area A and separation d, is connected with a battery of V_0 volt. The battery is then disconnected and a slab of dielectric constant k and of thickness d is introduced. Calculate the energy stored before and after the slab is introduced.

$$\left(\tfrac{1}{2}C_0V_0^2, \tfrac{1}{2}C_0V_0^2/k\right)$$

24. Two capacitors A and B have capacities C_1 and C_2. Capacitor A is charged by a battery and the battery is removed, it is discharged by allowing a spark to pass between the plates. Capacitor A is then charged as before, the battery is removed and the capacitor B is connected in parallel with A, a spark is appeared as the connection is made. These capacitors are then separated and each is discharged by a spark. Show that the energies of the four sparks are in the ratio.

$$(C_1 + C_2)^2 : C_2(C_1 + C_2) : C_1^2 : C_1C_2.$$

25. A circular gold leaf of radius b is laid on the surface of a charged conducting sphere of radius a ($a >> b$). Prove the loss of electric energy in removing the leaf from the conductor, assuming that it carries away its whole charge, is approximately $\pi\varepsilon_0 q^2b^2/a^3$, where q is the charge of the conductor and the capacity of the leaf is comparable to b.

26. Two capacitors of capacity C_1 and C_2 possessing charge q_1 and q_2 respectively are connected in parallel. Show that there is a loss of electrostatic energy amounting to $(C_2q_1 - C_1q_2)^2/2C_1C_2(C_1 + C_2)$.

27. A line charge of linear density λ is placed in a medium of dielectric constant k_1 parallel to and at a distance a from the plane boundary with another medium of dielectric constant k_2. Find the potentials in the two media and show that the force per unit length on the line charge is given by $F = \dfrac{\lambda^2}{4\pi\varepsilon_0 a}\left(\dfrac{k_1 - k_2}{k_1 + k_2}\right)\dfrac{1}{k_1}$.

28. A conducting sphere of radius r is electrified to a potential V. If the sphere consists of two separate hemispheres, show that the force between them is $\pi\varepsilon_0 V^2/2$ and if the whole be surrounded by an uninsulated concentric spherical conductor of internal radius R and the potential of the solid sphere is still V, prove that the force between the hemispheres is $\pi\varepsilon_0 V^2R^2/2(R - r)^2$.

29. A capacitor is formed by two coaxial cylinders of radii a and b. The axes of the cylinders are vertical and the inner cylinder is suspended from a balance so that it hangs only partly within the outer cylinder. Find an expression for the mass which must be added to the other pan of the balance to maintain equilibrium when a voltage V is connected between the two cylinders.

$$mg = \pi\varepsilon_0 V^2/\log_e(b/a)$$

30. The upper disc of an attracted disc electrometer is suspended by a spring. In equilibrium the separation between the two discs is x when a voltage V is applied and 'a' when $V = 0$. Show that the equilibrium in the former case is stable provided that $x > 2a/3$.

6

DIELECTRICS

6.1 DIELECTRICS

Upto this chapter we have considered the problems of electrostatics in the absence of matter. We must now consider the phenomena in a medium other than empty space (vacuum) such as solid or liquid *insulator, alternatively called dielectric.* The theory of dielectric was begun by Faraday and subsequently developed by Maxwell. From Faraday's experiments (*article* 5.3), we see that with a dielectric the capacitance of the capacitor is increased by a factor k which is greater than one. This factor is known as *dielectric constant or specific inductive capacitance* and is independent of the shape and size of the capacitor, but varies widely for different materials (medium). The following table gives the value of k for different materials.

Table 6.1. Dielectric Constant k

Material	k	*Material*	k
Vacuum	1.000	Mica (25°C)	2.5 – 7
Air (1 atm., 20°C)	1.006	Glass (25°C)	4 – 10
Air (100 atm., 20°C)	1.0548	Porcelains	5 – 10
Teflon	2.1	Bakelite (27°C)	5.5
Paraffin (20°C)	2.20	Bakelite (88°C)	18.2
Petroleum Oil (20°C)	2.2	Neoprene	6.9
Benzene (20°C)	2.28	Germanium (20°C)	16
Polyethylene	2.3	Ethanol	24.3
Polystyrene (20°C)	2.56	Liq. Ammonia (– 78°)	25
Ice (– 5°C)	2.9	Aceton	27
Rubber (27°C)	2.94	Methanol	33.6
Transformer oil	2 – 5	Glycerin (25°C)	42.5
Lucite	3.0	Water (25°C)	78.54
Paper (25°C)	3.5	Titanium dioxide	100
Nylon	3.6	Metal	∞

The comparatively high value of *k* for water suggests that water is a poor insulator, water is in fact a *semiconductor.*

The function of a solid dielectric between the plates of a capacitor is threefold. (*a*) It helps in maintaining *two large metal plates at a very small separation.* (*b*) *It increases the maximum potential difference which the capacitor can withstand without breakdown.* (c) *It increases the capacitance of a capacitor of given dimensions.*

6.2 DIELECTRIC AN ATOMIC VIEW

In any molecule there will be a distribution of protons and a distribution of electrons. The protons can be considered to act as the equivalent positive electric charge at some specific point in the molecule, the *centre of gravity* of the protons. Similarly there is a point at which a negative charge acts which is equivalent to the distributed electrons, the *centre of gravity* of electrons. If these two points coincide then the molecule is called *non-polar* and if they are separated by a short distance then the molecule is called *polar* molecule. Symmetrical molecules, H_2, N_2, O_2, CO_2, CCl_4, C_2H_6, etc. are *non-polar*. On the other hand molecules HCl, CO, NH_3, $CHCl_3$, CH_3OH, H_2O, N_2O, etc., are *polar*, as both hydrogen atoms or both nitrogen atoms lie on the same side of the oxygen atom in H_2O and N_2O molecules respectively. Thus the polar molecules work as dipoles and have dipole moments. If the dielectric (with non-polar molecules) is placed in an electric field, the charge centres of a non-polar molecule become displaced (Fig. 6.1). The molecules are then said to become *polarized* by the field and are called *induced dipoles*. The displacement is however limited by strong restoring force produced by the charge configuration in the molecule.

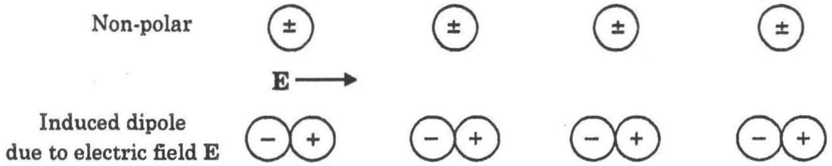
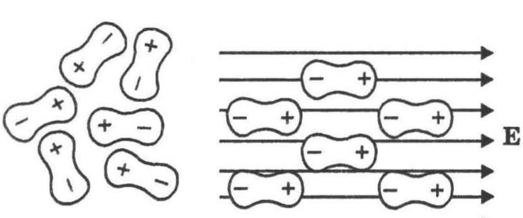


Fig. 6.1. Non-polar molecules in an E-field.

The polar molecules (permanent dipoles), in a dielectric are oriented at random in the absence of an external electric field. When an electric field is applied, the forces on a dipole give rise to a couple, whose effect is to orient the dipole *along the direction of electric field* (Fig. 6.2). The stronger the field, the greater is the aligning effect. Thus non-polar molecules become induced dipoles, where polar molecules are reoriented by the field and therefore have their dipole moments increased. The orientation of the induced dipoles or of the permanent dipoles in an external electrostatic field is such as to set the axis of the dipole along the field. This *phenomenon* is known as *electric polarization.*

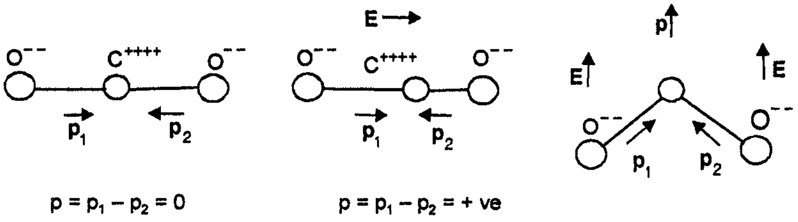


(*a*) Without electric field. (*b*) With electric field.

Fig. 6.2. Polar molecules in an E-field.

The main difference in above two mechanisms is the *temperature dependence.* The polarization of non-polar molecules is independent of temperature. As polar molecules are undergoing thermal motion, hence are randomly oriented. Thus the polar molecules can be aligned perfectly with the smallest external electric field at about absolute zero temperature.

Few Examples of Non-Polar Molecules: In a molecule of CO_2 the oxygen ions are symmetrically placed with respect to the carbon ion, hence the net dipole moment is zero (Fig. 6.3 *a*). If this molecule is placed in an electric field **E** along the line joining the ions, the oxygen ions get displaced with respect to the carbon ions and the net dipole moment (induced) is along the direction of E (Fig. 6.3 *b*). If the electric field E is applied perpendicular to the line joining the ions the direction of induced dipole moment is again along the field E (Fig. 6.3 *c*).



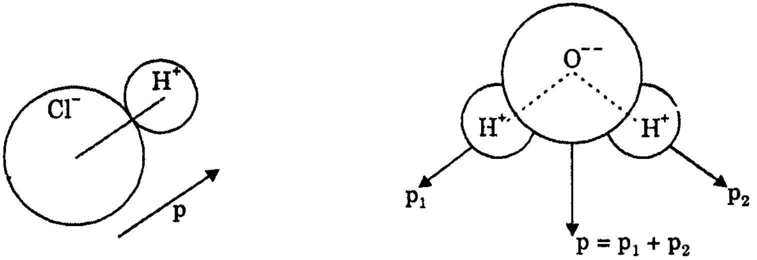
(*a*) (*b*) (*c*)

Fig. 6.3. (*a*) Non-polar molecule of CO_2.

Carbon tetrachloride (CCl_4) is another example of non-polar molecules. The external electric field changes the orientation of the *C — Cl* bond and thus produces induced dipole moment. Similarly H_2, O_2 *and* N_2 are non-polar molecules.

Few Examples of Polar Molecules : In a molecule of *HCl* there is an excess of positive charge on the *H*-ion and an equal negative charge on the *Cl*-ion. This molecule, therefore, has dipole moment at every instant of time and is a polar molecule.

Another interesting example of polar molecules is H_2O. In the water molecule two *O—H* bonds are not opposite to each other, unlike the case of CO_2 molecule, but are inclined at an angle of about 105°. As shown in Fig. 6.4, oxygen ion forms a dipole with each of the hydrogen ion, and there is a net dipole moment **($\mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2$)**.



(*a*) HCl (*b*) H_2O

Fig. 6.4. Polar molecules.

If a thin slab of dielectric, molecules of which are polar or non-polar, is placed in an electric field the slab as a whole becomes polarized. Due to the polarization, molecules are oriented so that the negative charges are in the left and positive charges are in the right, thus leaving the net positive charge at the right surface and the net negative charge at the left surface. Within the remainder of the dielectric net charge is zero. As the slab as a whole remains electrically neutral, the positive induced surface charge must be equal in magnitude to the negative induced surface charge. The charges so obtained on the surfaces, perpendicular to the direction of electric field E, are known as *polarization charges*. These charges are not free, but each is bound to a molecule lying in or near surface, that is why these are also named as *bound charges or fictitious charges*. Fig. 6.5 shows that the induced charges always appear in such a way that the electric field set up by these charges is in opposite direction to the external electric field E. Thus *when a dielectric is placed in an electric field, the induced surface charges weaken the original field within the dielectric.*

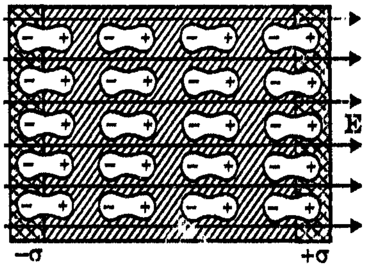


Fig. 6.5. Polarization Charges in a dielectric.

6.3 FORCES AND TORQUES ON DIPOLES

Before proceeding to polarized molecules in bulk, let us calculate expressions for the force **F** and the torque τ on a dipole in an external field. Let a dipole of length a with charges $+q$ and $-q$ is placed in a uniform external electric field **E**. If this dipole makes an angle θ with the direction of field and field of the dipole is negligible then the force acting on the poles of the dipole

$$\mathbf{F} = q\,\mathbf{E}$$

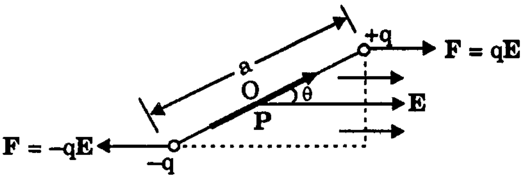


Fig. 6.6. Forces on the dipole.

and the torque acting on the dipole

$$\tau = \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F}_{+} + (-\mathbf{r}) \times \mathbf{F}_{-}.$$

Here the origin is taken at the centre of the dipole. τ is a vector perpendicular to the figure in the downward direction and of magnitude

$$\tau = \tfrac{1}{2}\, a \sin\theta qE + \tfrac{1}{2}\, a \sin\theta qE = qaE\sin\theta = pE\sin\theta,$$

or

$$\tau = p \times E, \qquad \ldots(1)$$

where $p\ (= qa)$ is the *dipole moment* of the electric dipole.

One can readily show that the torque τ is the same if any other point is chosen as the centre. If the field is not uniform in the neighbourhood of the dipole, so that its value at $+q$ is different from that at $-q$ an average value taken at the centre will be a good approximation. In true dipole (zero length) the torque is independent of any variation of **E**. The work done in rotating this dipole through an angle $d\theta$ in this field $dW = \tau\, d\theta$.

$\therefore$ Total work done in rotating it through an angle θ

$$W = \int_0^{\theta} \tau d\theta = \int_0^{\theta} pE \sin\theta\, d\theta = pE\,(1 - \cos\theta). \qquad \ldots(2)$$

The net force on the dipole is zero if the field is uniform, as in this case we have two equal and opposite forces. Thus the force must depend on the variation of the field, *i.e.*, on the gradients of the various components of the field. More specially, the net force in the x—direction $F_x = [qE_x]_{\text{at } q+} - [qE_x]_{\text{at } q-}$. The difference can be calculated by the differential formula involving the gradient. Thus

$$F_x = q\,\mathbf{a}\cdot\nabla E_x = \mathbf{p}\cdot\nabla E_x.$$

Similarly $F_y = \mathbf{p}\cdot\nabla E_y$ and $F_z = \mathbf{p}\cdot\nabla E_z$.

$$\therefore \qquad \mathbf{F} = \mathbf{p}\cdot\nabla E_x\mathbf{i} + \mathbf{p}.\nabla E_y\mathbf{j} + \mathbf{p}\cdot\nabla E_z\mathbf{k} = (\mathbf{p}\cdot\nabla)\,\mathbf{E}. \qquad \ldots(3)$$

Using vector relations this equation can also be written as

$$\mathbf{F} = \nabla\,(\mathbf{p}\cdot\mathbf{E}). \qquad \ldots(4)$$

We know that the conservative force can be defined as the negative gradient of the potential energy. Thus the *potential energy of the dipole*

$$U = -\,\mathbf{p}\cdot\mathbf{E}. \qquad \ldots(5)$$

This relation can also be obtained from relation (2) as the potential energy is equal to the work required in bringing the dipole from infinity to the point under consideration in such a way that +q and −q are at all times on the same equipotential surface, *i.e.*, $\mathbf{p} \perp \mathbf{E}$ always.

System of Two Dipoles : Let us now consider a dipole of moment $\mathbf{p}_1$ located at the origin of coordinates and a dipole of moment $\mathbf{p}_2$ located at a point **P** (r distance) from the origin. The *potential and field at the origin due to dipole* $\mathbf{p}_2$ are given by

$$\varphi = -\frac{1}{4\pi\varepsilon_0}\,\mathbf{p}_2\cdot\nabla\left(\frac{1}{r}\right) \text{ and } E = -\nabla\varphi = \frac{1}{4\pi\varepsilon_0}\nabla\left[\mathbf{p}_2\cdot\nabla\left(\frac{1}{r}\right)\right] \qquad \ldots(6)$$

Hence the *force on dipole* $\mathbf{p}_1$ *due to dipole* $\mathbf{p}_2$.

$$\mathbf{F} = (\mathbf{p}_1\cdot\nabla)\,\mathbf{E} = \frac{1}{4\pi\varepsilon_0}(\mathbf{p}_1\cdot\nabla)\left\{\nabla\left[\mathbf{p}_2\cdot\nabla\left(\frac{1}{r}\right)\right]\right\} \qquad \ldots(7)$$

The *interaction energy between the two dipoles, i.e.,* U_{12}, the energy of dipole $\mathbf{p}_1$ in the field of dipole $\mathbf{p}_2$ is given with the Eqn. (5) as

$$U_{12} = -\ \mathbf{p}_1 \cdot \mathbf{E} = -\frac{(\mathbf{p}_1 \cdot \nabla)}{4\pi\varepsilon_0}\left[\mathbf{p}_2 \cdot \nabla\left(\frac{1}{r}\right)\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\mathbf{p}_1 \cdot \nabla\left[\mathbf{p}_2 \cdot \left(-\frac{r}{r^3}\right)\right] = -\frac{1}{4\pi\varepsilon_0}\mathbf{p}_1 \cdot \nabla[-r^{-3}(\mathbf{p}_2 \cdot \mathbf{r})]$$

$$= \frac{1}{4\pi\varepsilon_0}\mathbf{p}_1 \cdot [\nabla(r^{-3})(\mathbf{p}_2 \cdot \mathbf{r}) + r^{-3}\nabla(\mathbf{p}_2 \cdot \mathbf{r})] = \frac{1}{4\pi\varepsilon_0}\mathbf{p}_1 \cdot [(-3r^{-5}\,\mathbf{r}(\mathbf{p}_2 \cdot r) + r^{-3}\mathbf{p}_2]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{r^3} - \frac{3}{r^5}(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})\right]. \qquad \ldots(8)$$

6.4 POLARIZATION AND CHARGE DENSITY

When a dielectric is placed in an electric field its molecules become polarized by this field, or oriented in the direction of field. This effect is known as *electric polarization* and is defined as the product of **p**, the component of the electric dipole moment of each molecule in the field direction and *n* the number of molecules per unit volume.

$$\mathbf{P} = n\ \mathbf{p}. \qquad \ldots(9)$$

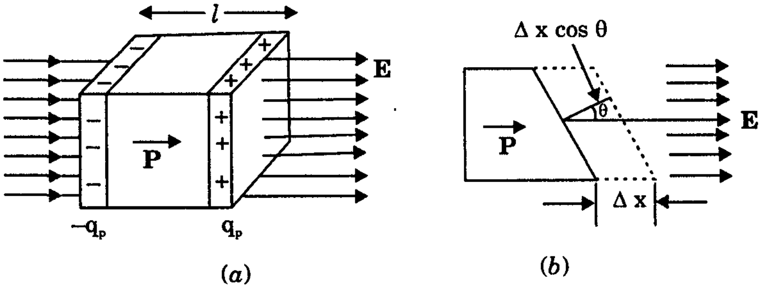
Dipole moment of a single molecule may be defined as

$$\mathbf{p} = \int_{\text{molecule}} \mathbf{r}\, dq. \qquad \ldots(10)$$

In a homogeneous and isotropic dielectric material the polarization vector **P** will be equal to the geometric sum of the electric dipole moments **p** per unit volume. If Δ **p** is the dipole moment associated with volume Δv then

$$\mathbf{P} = \underset{\Delta v \to 0}{\text{Limit}} \frac{\Delta p}{\Delta v} = \underset{\Delta v \to 0}{\text{Limit}} \frac{1}{\Delta v}\ \Sigma\, \mathbf{p}, \qquad \ldots(11)$$

where summation is taken over all individual dipoles in the volume Δv. It has the same direction as that of **p** and hence of electric field **E**. In SI units, unit of **P** is coulomb meter/meter^3 or *coul/meter*^{2}. Actually the term polarization has two meanings : *a qualitative one,* giving an idea about relative displacement of positive and negative charges and *a quantitative one,* giving resulting dipole moment of a unit volume of a polarized matter. Since the dipole moment per unit volume refers to the arrangement on an atomic scale, hence is not measurable directly by a macroscopic method. The simple method of measuring **P** is based upon its relationship with the *polarization charge* (q_p) on the surface of a dielectric slab placed in the field.



(*a*) (*b*)

Fig. 6.7. Polarized block of dielectric.

Fig. 6.7 (*a*) shows a polarized block of dielectric placed in a uniform electric field E. The net charges on the surfaces are also shown. If σ_p is the density of polarization charge on the surfaces, **P** the polarization vector, *A* the area of cross-section and *l* the length of the slab, then

Dipole moment of this block = *P Al.*

The second way is to consider this block as a single large dipole with surface charges q_p ($= \sigma_p A$) and $-q_p$ separated by a distance l, *i.e.*,

dipole moment = charge × length = $\sigma_p Al$.

Comparison of the two results gives

$$P = |\sigma_p|. \qquad \ldots(12)$$

The sign of the polarization surface charge depends on the direction of polarization and on which end of the block we consider. The magnitude of **P** depends on the charge density and on the relative displacement of the +ve and –ve charge arrays. Its direction is along the direction of the relative displacement of charge arrays.

If the dielectric surface is not perpendicular to **P**, but the normal to the surface makes an angle θ with the direction of **P** (or **E**). In this case the actual displacement of the array of positive charges within the dielectric is Δx. Its effective displacement is $\Delta x \cos \theta$. Thus for surfaces parallel to **P**, no surface polarization charge is induced and in general

$$P_n = \textit{component of } \mathbf{P} \textit{ normal to the surface} = |\sigma_p|. \qquad \ldots(13)$$

The polarization **P** is a vector field quantity having mathematical properties similar to **E**. For uniformly polarized matter we can apply Gauss's law to the lines originating from and ending on surface polarization charges. Thus the surface integral

$$\oint \mathbf{P} \cdot d\mathbf{S} = -q_p. \qquad \ldots(14)$$

Here –ve sign is included because the direction of lines of **P** is from negative to positive charges, lines of **P** emerge from a volume containing negative polarization charge and end in a volume containing positive polarization charge.

If the polarization of the medium is not uniform, the surface charge is accompanied by a volume distribution of charge in the interior of the dielectric and is given by

$$\oint \mathbf{P} \cdot d\mathbf{S} = \int_v \rho_p \, dv \text{ or } \int_v \operatorname{div} \mathbf{P} \, dv = -\int_v \rho_p \, dv.$$

$$\therefore \qquad \rho_p = -\operatorname{div} \mathbf{P} = -(\partial P_x/\partial x + \partial P_y/\partial y + \partial P_z/\partial z). \qquad \ldots(15)$$

Thus for the non-uniform polarization, the divergence of the polarization *vector* gives the net charge density produced in the dielectric. For the uniform polarization the charge per unit volume ρ_p in its interior is zero. To find the mean electric field in the vicinity of a dielectric (inside or outside the medium) we must add electric intensities due to outside charges, charge on the surface of the dielectric and the charge in the interior of the dielectric.

6.5 POTENTIAL AND FIELD OUTSIDE OF A DIELECTRIC

Let us consider a finite piece of a dielectric material which is polarized. The polarization gives rise to an electric field. To find it at point **r**, outside the dielectric, let us consider a volume element $\Delta v'$ at any point **r'** with in the dielectric of volume v, Fig. 6.8. If the polarization at the element is $\mathbf{P}(\mathbf{r}')$, the dipole moment associated with this volume element will be $\Delta \mathbf{p} = \mathbf{P}(\mathbf{r}')\,\Delta v'$. If the point A is far apart from the dielectric, *i.e.*, $\mathbf{r} - \mathbf{r}'$ is large compared with the dimensions of $\Delta v'$, the potential at point A due to the volume element $\Delta v'$ is

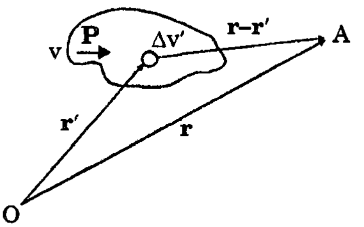


Fig. 6.8. Small element $\Delta v'$ of the polarized dielectric.

$$\Delta\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\Delta \mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{4\pi\varepsilon_0} \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')\Delta v'}{|\mathbf{r} - \mathbf{r}'|^3}.$$

The total potential due to the whole volume v of the dielectric is obtained by summing the contributions from all elements of the dielectric.

$$\therefore \qquad \varphi(\mathbf{r}) = \Sigma\Delta\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\mathbf{P}(\mathbf{r}') \cdot (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} dv'. \qquad \ldots(16)$$

We know the vector relation $\nabla' |\mathbf{r}-\mathbf{r}'|^{-1} = |\mathbf{r}-\mathbf{r}'|^{-3}(\mathbf{r}-\mathbf{r}')$,

where ∇' operator involves derivatives with respect to the primed co-ordinates. The Eq. (16) may thus be written as

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_v \mathbf{P}(\mathbf{r}')\cdot\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right)dv'. \qquad ...(17)$$

Using the vector identity $\nabla'.(f\mathbf{A}) = f\nabla'\cdot\mathbf{A} + \mathbf{A}\cdot\nabla' f$, and assuming $f = |\mathbf{r}-\mathbf{r}'|^{-1}$ and $\mathbf{A} = \mathbf{P}(\mathbf{r}')$, we get

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\int_v\left[\nabla'\cdot\left\{\frac{\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\right\} - \frac{1}{|\mathbf{r}-\mathbf{r}'|}\nabla'\cdot\mathbf{P}(\mathbf{r}')\right]dv'$$

$$= \frac{1}{4\pi\varepsilon_0}\int_S \frac{\mathbf{P}(\mathbf{r}')\cdot d\mathbf{S}'}{|\mathbf{r}-\mathbf{r}'|} + \frac{1}{4\pi\varepsilon_0}\int_v \frac{-\nabla'\cdot\mathbf{P}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}dv'. \qquad ...(18)$$

Here we have used Gauss's divergence theorem to the first term. If we write $d\mathbf{S}' = \hat{n}\, dS'$, where $\hat{n}$ is the outward normal to the surface element dS' and S is the surface enclosing volume v. Thus $\mathbf{P}(\mathbf{r}').\ d\mathbf{S}' = \mathbf{P}(\mathbf{r}').\hat{n}\, dS'$. The quantities $\mathbf{P}(\mathbf{r}').\ \hat{n}$ and $-\nabla'.\mathbf{P}(\mathbf{r}')$ are two scalar functions obtained from the polarization $\mathbf{P}$. They have the dimensions charge per unit area and charge per unit volume and are respectively represented by the symbols σ_p and ρ_p.

Thus $$\sigma_p = \mathbf{P}(\mathbf{r}')\cdot\hat{n} \text{ and } \rho_p \equiv -\nabla'\cdot\mathbf{P}(\mathbf{r}'). \qquad ...(19)$$

The quantities σ_p and ρ_p are respectively called *polarization surface charge density* and *polarization volume charge density*. The former is the component of polarization $\mathbf{P}$ normal to the surface and the latter is a measure of the non-uniformity of the polarization inside the dielectric material. Thus the expression for the electrostatic potential outside the dielectric material is given by

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_p dS'}{|\mathbf{r}-\mathbf{r}'|} + \int_v \frac{\rho_p dv'}{|\mathbf{r}-\mathbf{r}'|}\right] = \frac{1}{4\pi\varepsilon_0}\int \frac{dq'_p}{|\mathbf{r}-\mathbf{r}'|} \qquad ...(20)$$

The net polarization charge on the dielectric

$$q_p' = \int_S \sigma_p dS' + \int_v \rho_p dv' = \int_S \mathbf{P}(\mathbf{r}')\cdot\hat{n}\, dS' + \int_v[-\nabla'\cdot\mathbf{P}(\mathbf{r}')]dv'$$

$$= \int_v \nabla'\cdot\mathbf{P}(\mathbf{r}')\, dv' + \int_v[-\nabla'\cdot\mathbf{P}(\mathbf{r}')]\, dv' = 0, \qquad ...(21)$$

Thus the net polarization charge on the dielectric is always zero. The electrostatic field $\mathbf{E}(\mathbf{r})$ at point $\mathbf{r}$ outside the dielectric may be obtained as the negative gradient of a potential function $\varphi(\mathbf{r})$. Since φ is a function of $\mathbf{r}$, the appropriate gradient is $-\nabla$, we know that

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = -\nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) \text{and } \nabla'\left(\frac{1}{|\mathbf{r}-\mathbf{r}'|}\right) = \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}$$

$$\therefore \quad \mathbf{E}(\mathbf{r}) = -\nabla[\varphi(\mathbf{r})] = \frac{1}{4\pi\varepsilon_0}\left[\int_S \frac{\sigma_p(\mathbf{r}-\mathbf{r}')\, dS'}{|\mathbf{r}-\mathbf{r}'|^3} + \int_v \frac{\rho_p(\mathbf{r}-\mathbf{r}')\, dv'}{|\mathbf{r}-\mathbf{r}'|^3}\right] \qquad ...(22)$$

6.6 INDUCED CHARGE ON THE DIELECTRIC IN AN ELECTRIC FIELD

Let us apply Gauss's law to a parallel plate capacitor filled with a dielectric. Let $+q$ and $-q$ be the charges on the plates of this capacitor in both the cases, with and without the dielectric, each of area A.

When no dielectric is present (Fig. 6.9), the electric field $\mathbf{E}_0$ at any point on the gaussian surface is given by

$$\oint \mathbf{E}_0\cdot d\mathbf{S} = q/\varepsilon_0. \text{ or } \int E_0 dS = E_0\int dS = q/\varepsilon_0.$$

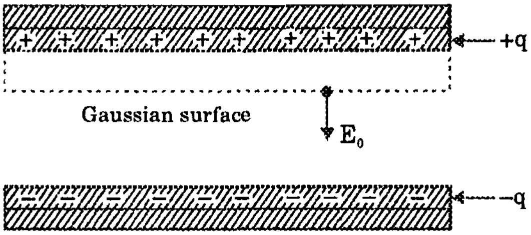


Fig. 6.9. Parallel plate capacitor without dielectric.

[Since $\mathbf{E}_0 \parallel d\mathbf{S}$ and constant over the gaussian surface.]

$$\therefore \quad E_0 A = q/\varepsilon_0 \text{ or } E_0 = q/\varepsilon_0 A. \qquad \text{...(23)}$$

In the second case, when dielectric is present (Fig. 6.10), the net charge within the gaussian surface is $q - q'$, where q' is the induced surface charge. The electric field E is given by

$$\oint \mathbf{E}\cdot d\mathbf{S} = (q - q')/\varepsilon_0, \text{ or } \int EdS = E\int dS = (q - q')/\varepsilon_0 \qquad \text{...(24)}$$

$$\therefore \quad EA = (q - q')/\varepsilon_0 \text{ or } E = q/\varepsilon_0 A - q'/\varepsilon_0 A. \qquad \text{...(25)}$$

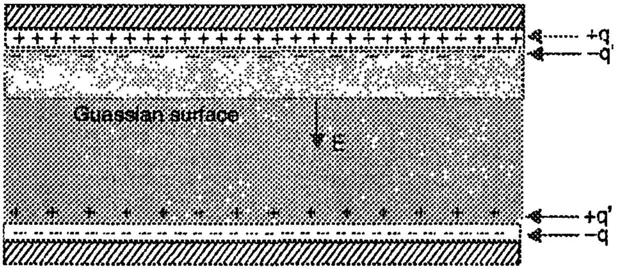


Fig. 6.10. Parallel plate capacitor filled with a dielectric.

Thus, the *induced charge q' tends to weaken the original field* (*i.e.*, $E < E_0$), which reveals itself as a reduction in potential difference between the plates. If V_d and V_0 are the potential differences with and without the dielectric respectively, then

$$E_0/E = V_0/V_d = k = \text{Dielectric constant.}$$

$$\therefore \quad E = E_0/k = q/k\varepsilon_0 A. \qquad \text{...(26)}$$

Combining equations (25) and (26), we get

$$q/k\varepsilon_0 A = q/\varepsilon_0 A - q'/\varepsilon_0 A$$

$$\text{or} \quad q' = q\,(1 - 1/k). \qquad \text{...(27)}$$

This shows that the induced surface charge q' is always less than the free charge q and is zero when $k = 1$ or the dielectric is not present.

5.7 GAUSS'S LAW IN A DIELECTRIC

Gauss's law establishes a relationship between the electric flux and the electrostatic charge. To extend it to a region containing charges embedded in a dielectric, we must consider polarization charges as well as the charges embedded.

Consider a gaussian surface S enclosing a volume v and bounded by a number of surface S_1, S_2 and S_3 inside a dielectric medium (Fig. 6.11). Let q_f be the charge

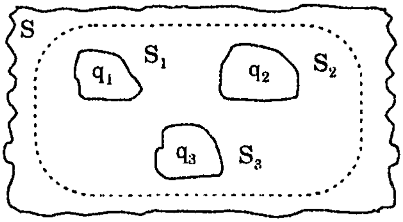


Fig. 6.11. Gaussian surface S in a dielectric medium.

embedded in this volume v, such that its three parts q_1, q_2 and q_3 exist on the surfaces S_1, S_2 and S_3 respectively. By Gauss's law, we have

$$\oint_S \mathbf{E}\cdot d\mathbf{S} = \oint_S \mathbf{E}\cdot\hat{n}dS = \frac{1}{\varepsilon_0}(q_f + q_p), \qquad \text{...(28)}$$

where q_f = free charge inside $S = q_1 + q_2 + q_3$ and q_p = Polarization charge, which may be due to surface and volume charge density distributions inside S.

$$q_p = \int_{S'} \sigma_p dS + \int_{v'} \rho_v dv,$$

where S' signifies all surfaces inside S where **P** is discontinuous and v′ is a subvolume of v exclusive of points where $\nabla.\mathbf{P}$ is infinite. We therefore have

$$q_p = \int_{S_1+S_2+S_3} \mathbf{P}\cdot\hat{n}dS + \int_v (-\nabla\cdot\mathbf{P})dv. \qquad \text{...(29)}$$

Here the surface integral does not contain a contribution from S, as there is no boundary of dielectric at S. The second term may be written in terms of a surface integral by using the Gauss's divergence theorem, as

$$\int_v (-\nabla\cdot\mathbf{P})\,dv = -\int_{S_1+S_2+S_3+S} \mathbf{P}\cdot\hat{n}\,dS, \qquad \text{...(30)}$$

where all surfaces bounding volume v are taken into consideration. The combination of Eqs. (29) and (30) gives

$$q_p = -\int_S \mathbf{P}\cdot\hat{n}\,dS. \qquad \text{...(31)}$$

Thus for any closed surface S, we have Eq. (28) as

$$\oint_S \mathbf{E}\cdot\hat{n}dS = \frac{1}{\varepsilon_0}\left(q_f - \oint_S \mathbf{P}\cdot\hat{n}dS\right)$$

or

$$\oint_S (\varepsilon_0\mathbf{E} + \mathbf{P})\cdot\hat{n}dS = q_f \qquad \text{...(32)}$$

The quantity $\varepsilon_0\mathbf{E} + \mathbf{P}$ has been given a special name, *the electric displacement* (by Maxwell) and is represented by a new macroscopic field vector **D** (as **E** and **P** both are macroscopic fields). Thus we have

$$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}. \qquad \text{...(33)}$$

Thus for an arbitrary closed surface S, the Gauss's law in dielectric becomes

$$\oint_S \mathbf{D}.\hat{n}dS = q_f \qquad \text{...(34)}$$

i.e. the surface integral of the normal component of **D** *over any closed surface (the flux of* **D***) is equal to the total free charge (only) within the surface.*

Here we see that **D** does not involve polarization charges, it depends only on free charges in contrast to **E** which depends on both free and polarization charges.

For an arbitrary distribution of charges, Eq. (34) may be written as

$$\oint_S \mathbf{D}\cdot\hat{n}dS = \int \rho dv. \qquad \text{...(35)}$$

Using the Gauss's divergence theorem, we get

$$\int div\,\mathbf{D}\,dv = \int \rho dv. \qquad \text{...(36)}$$

This holds for any volume. If the volume considered is reduced to an elementary volume, above relation becomes the point relation, as

$$div\,\mathbf{D} = \rho, \qquad \text{...(37)}$$

i.e., at every point in a medium the divergence of electric displacement is equal to the free charge density at that point. This is known as the differential form of Gauss's law in dielectric.

6.8 THREE ELECTRIC VECTORS

E, P and **D** are three electric vectors related with each other as shown by Eq. (33). These may vary in magnitude and direction from point to point in complicated problems of electrostatics.

Electric displacement vector **D** is connected with the free charge only. Like the electric field, the displacement field may be represented by lines known as *lines of displacement*. These lines begin and end on the free charges. The flux of **D** equals the free charge. **D** can not be regarded as the fundamental field of the status of **E**. It has not direct connection with the forces exerted on charges, therefore its conceptualization is difficult.

Polarization vector **P** is connected with the induced surface charge. It may also be represented by lines, known as lines of **P**. These lines begin and end on the polarization charges, *i.e.*, induced charges due to polarization. The flux of **P** equals the negative of the bound (*induced*) charge.

The electric field intensity **E** is connected with the charges actually present (*free and bound charges*). The lines of **E** depend upon the presence of both kinds of charges. The flux of **E** equals the total enclosed charge divided by ε_0. The unit of **E** is newton/coulomb while that of **P** and **D** was coulomb/meter^2.

The lines of three electric vectors **D**, **E** and **P** are shown in Fig. 6.12.

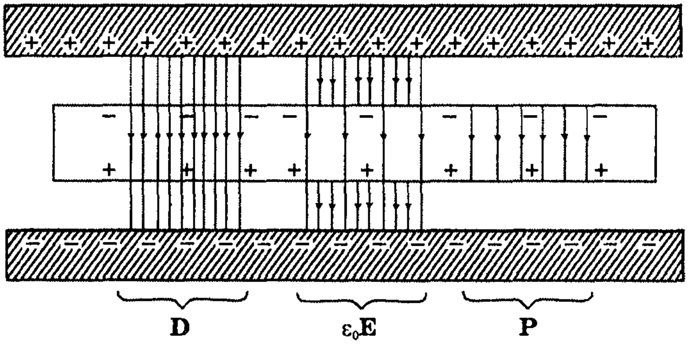


Fig. 6.12. The lines of three electric vectors.

In free space, where there is no dielectric and thus **P** = 0, we have

$$\mathbf{D} = \varepsilon_0 \mathbf{E}. \qquad ...(38)$$

The Electric Constitutive Relations : The relations between **P** (or) **D** and **E** are called electric constitutive relations. It has been seen earlier that the dielectric medium is polarized when placed in an electric field **E**. The degree of polarization **P** depends not only on **E**, but also on the properties of the dielectric. For most materials, **P** vanishes when **E** vanishes. For the isotropic material, **P** is proportional to and in the same direction as the electric field **E**, that is causing it.

$$\mathbf{P} \propto \mathbf{E} \text{ or } \mathbf{P} = \varepsilon_0 \chi \mathbf{E}, \qquad ...(39)$$

where χ is a scalar quantity, known as the *electric susceptibility* of the material.

The constant ε_0 is included only for the purpose of simplifying the form of latter relationship. χ is a dimensionless quantity and is a constant characteristic of the material.

The combination of Eqs. (39) and (33) gives

$$\mathbf{D} = \varepsilon_0 \mathbf{E} + \varepsilon_0 \chi \mathbf{E} = \varepsilon_0 (1 + \chi) \mathbf{E} = \varepsilon \mathbf{E}, \qquad ...(40)$$

where

$$\varepsilon = \varepsilon_0 (1 + \chi) \qquad ...(41)$$

is the *absolute permittivity* of the dielectric medium.

The electric behaviour of a material is completely specified by either the susceptibility χ or the permittivity ε. The ratio $\varepsilon/\varepsilon_0$ is a dimensionless quantity known as *relative permittivity* of the medium. It is denoted by a symbol k and is called the dielectric coefficient or simply the dielectric constant. Thus we have

$$k = \varepsilon/\varepsilon_0 = 1 + \chi. \qquad ...(42)$$

Combining Eqs. (42) and (39), we get

$$\mathbf{P} = \varepsilon_0(k - 1)\mathbf{E}. \qquad ...(43)$$

Thus we see that the three quantities ε, k and χ are simply the different ways of describing the same property of the dielectric. If χ (or k) does not depend upon the location of the piece of material, the material is called *homogeneous.*

Linear, Isotropic and Homogeneous Dielectrics : All above derived equations are for dielectric materials which are linear and isotropic. A material is said to be *linear* if **D** varies linearly with **E** and nonlinear otherwise. Materials are said to be *homogeneous* for which ε or σ does not vary in the region or same at all points. They are said to be inhomogeneous or non-homogeneous when ε is dependent of the space coordinates. Materials are said to be *isotropic* for which **D** and **E** are in the same direction. Thus the isotropic dielectrics have the same properties in all directions. For anisotropic or nonisotropic materials **D**, **E** and **P** are not all parallel. The susceptibility χ or permittivity ε is not a scalar but is a tensor, having nine components in three dimensional coordinate system. Crystalline materials and magnetized plasma are anisotropic. Many substances do not satisfy these conditions. For such cases, we have

$$\mathbf{P}_i = \varepsilon_0 \sum_j \chi_{ij} \mathbf{E}_j, \qquad ...(44)$$

where χ_{ij} are called *susceptibility coefficients.* The set of coefficients (9 in three dimensional coordinates) is known as *susceptibility tensor.*

Thus the dielectric material is linear if ε does not change with the applied **E** *field, isotropic if ε does not change with direction and homogeneous if ε does not change from point to point.*

6.9 BOUNDARY CONDITIONS AT THE DIELECTRIC SURFACE

Before we can solve more complicated problems, we must know how the field vectors **D** and **E** change in passing through a surface separating two media, may be two dielectrics or a dielectric and a free space. The rules governing the behaviour of **E** and **D** at the boundary between two dielectrics are embodied in *two boundary conditions.*

Let *AB* represents a small portion of the boundary between two media of absolute permittivities ε_1 and ε_2. The media are assumed to be homogeneous and isotropic. Let us consider an area dS on the boundary, which is so small that its curvature may be neglected. Let $\mathbf{D}_1$ and $\mathbf{D}_2$ be the electric induction vectors in the media on either side of dS, making angles θ_1 and θ_2 with the normal to dS, as shown in Fig. 6.13.

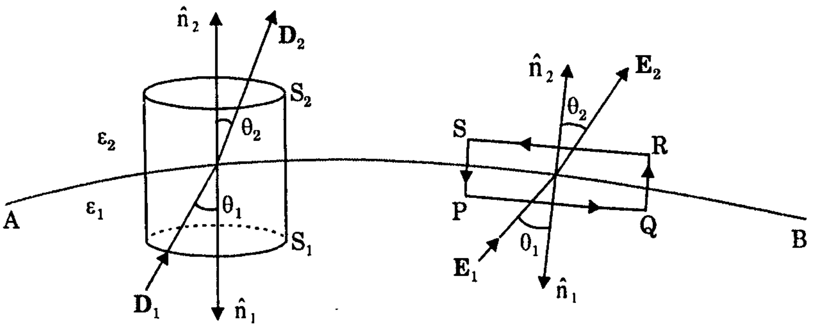


Fig. 6.13. Surface separating two media.

To find the boundary condition for **D**, let us apply Gauss's law to small cylinder which intersects the area dS on the boundary and of height very small compared to the diameter of the base. Assuming the volume of the cylinder very small, so that the free charge enclosed by the cylinder can be taken as the free charge on dS area of the boundary. Thus we have

$$\oint \mathbf{D}\cdot d\mathbf{S} = \int_C \mathbf{D}\cdot d\mathbf{S} + \int_{S_1} \mathbf{D}_1\cdot d\mathbf{S} + \int_{S_2} \mathbf{D}_2\cdot d\mathbf{S} = q$$

$$0 + (\mathbf{D}_1\cdot\hat{n}_1 + \mathbf{D}_2\cdot\hat{n}_2)\,dS = \sigma dS \quad \text{or} -D_1\cos\theta_1 + D_2\cos\theta_2 = \sigma. \qquad ...(45)$$

Here we have neglected the flux through the curved surface of the cylinder as its area is vanishingly small. In the majority of cases there is no free charge on the boundary, hence

$$D_2\cos\theta_2 - D_1\cos\theta_1 = 0 \text{ or } D_{2n} = D_{1n}. \qquad ...(46)$$

Therefore the normal component of the electric displacement is the same on both sides of the charge free boundary of two media of different dielectrics or the normal displacement is continuous across the boundary having no free charges. Thus we see that the discontinuity in the normal component of D arises due to the free charges on the boundary of two media.

The boundary condition for **E** is found by considering the work done in taking unit charge round a small rectangle of length dl and of negligible height, with its longest sides parallel to the surface of separation, one in each medium. The work done in taking a unit charge around the rectangle $PQRS$ must vanish, otherwise an infinite amount of work could be obtained by repeating the process indefinitely, hence

$$\oint \mathbf{E}\cdot d\mathbf{l} = \int_P^Q \mathbf{E}_1\cdot d\mathbf{l} + \int_Q^R \mathbf{E}\cdot d\mathbf{l} + \int_R^S \mathbf{E}_2\cdot d\mathbf{l} + \int_S^P \mathbf{E}\cdot d\mathbf{l} = 0,$$

$$E_1\sin\theta_1\,dl - E_2\sin\theta_2\,dl = 0 \text{ or } E_{1t} = E_{2t}. \qquad ...(47)$$

Here we have neglected the contributions of short sides QR and SP, as they are vanishingly small in comparison with that of other sides PQ and RS. Therefore the *tangential components of the electric intensities are the same on the both sides of the boundary surface or the tangential component of the electric field is continuous at the boundary between two dielectrics.*

Boundary Between Dielectric and a Conductor : We assume the medium 1 as a conducting medium. Since there is no electric field inside the conductor, hence $\mathbf{E}_1 = 0$ and therefore $\mathbf{D}_1 = 0$. Eq. (45) thus gives

$$D_2\cos\theta_2 = \sigma \text{ or } D_{2n} = \sigma, \qquad ...(48)$$

i.e., the normal component of **D** in a dielectric just outside the conductor is equal to the free surface charge density at the boundary. Since $\mathbf{E}_1 = 0$ in the conductor, therefore Eq. (47) gives

$$E_{2t} = 0, \qquad ...(49)$$

i.e., the tangential component of **E** just outside the conductor must also be zero or **E** is normal to the surface of a conductor.

Refraction of the lines of force or lines of displacement at an uncharged surface of discontinuity. We may use boundary conditions given above to show the effect of a dielectric (isotropic) boundary on the direction of an electric field or the electric displacement that crosses the boundary. Let the lines of force make angles θ_1 and θ_2 with the normal to the surface in regions 1 and 2 respectively (Fig. 6.14). Since in an isotropic dielectric medium $\mathbf{D} = k\varepsilon_0\mathbf{E}$, hence the displacement vector will coincide the electric field vector. Combining the conditions (46) and (47), we get

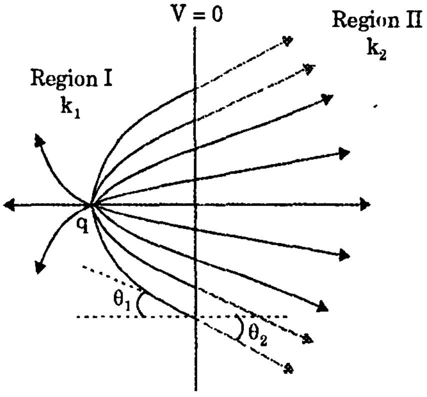


Fig. 6.14. Refraction of electric lines of force.

$$D_1\cos\theta_1/E_1\sin\theta_1 = D_2\cos\theta_2/E_2\sin\theta_2$$

$$\text{or} \quad (D_1/E_1)\cot\theta_1 = (D_2/E_2)\cot\theta_2.$$

$$\text{As} \quad D_1 = k_1\varepsilon_0 E_1 \text{ and } D_2 = k_2\varepsilon_0 E_2.$$

$$\therefore \quad k_1\cot\theta_1 = k_2\cot\theta_2 \qquad ...(50)$$

From this relation it is clear that in passing from empty space into a dielectric, the lines of force or displacement are bent away from the normal to the surface, *alike to the light rays entering the medium. The Eq. (50) may be called the law of refraction for electrostatic lines of force.*

6.10 EFFECTS OF DIELECTRICS

(*a*) **Dielectric relaxation** : Dielectric takes a finite time for the polarization to reach its maximum value. This is due to the forces between adjacent molecules which tends to prevent the alignment along the external field. The average time for *relaxation* varies from substance to substance, values from 10^{-10} sec to several thousand years.

Due to this relaxation, capacitor connected to a potential difference V acquires charge at a finite rate. Discharging process is also slow. If a capacitor is charged and discharged quickly and then left with terminals open, it will be found to have acquired a potential difference between its plates. Phenomenon of hysteresis with alternating field is also the consequence of the relaxation time.

(*b*) **Dielectric absorption** : On account of the finite conductivity of the electric materials noticeable energy loss is observed in the form of heat. This loss and the hysteresis loss are referred jointly as *dielectric losses.* Similar to the relaxation phenomenon absorption effect is observed in capacitors. If an initially charged capacitor is discharged by short circuiting it, it is found that the capacitor still retains a charge when the short circuit is removed. This is due to *electric absorption.* It is when the dielectric is not perfectly homogeneous. For a perfectly homogeneous dielectric, this residual charge should not occur. Thus a gas or a perfectly homogeneous crystal like calcite shows no residual charge.

If the capacitor is repeatedly charged and discharged, dielectric absorption causes heat to be produced. If this process is rapid, as with high frequency *ac*, energy losses in the dielectric are low, *i.e.* the dielectric absorption is small as the molecules are unable to rotate to any extent in this very short time (*i.e.*, the time of the cycle).

(*c*) **Dielectric breakdown** : It is the most important property of real dielectrics. For any material there is a maximum field intensity beyond which damage occurs that results in conduction, sparking and other breakdown phenomena. At relatively high fields, the electrons in the dielectric gain enough energy to knock other charged particles and make them available for conduction. This multiplying process occurs along certain paths in the material. Along these paths, local heating also occurs and a permanent damage results. Thus for any dielectric material there is a maximum electric field that the material can withstand without breaking down and losing its insulating quality. This field is called the *dielectric strength* of the material. Table 6.2 shows the dielectric strengths for some common materials.

Table 6.2. Dielectric Strengths of Materials (10^6V/m)

Material	*Dielectric Strength*	*Material*	*Dielectric Strength*
Air	3	Polyethylene	18
Titanium dioxide	6	Nylon	19
Glass (ordinary)	9	Lucite	18–20
Paraffin	10	Porcelain	20
Backlite	12	Rubber	40
Neoprene	12	Polystyrene	50
Transformer oil	12	Teflon	60
Paper	14	Mica	10–150

Electrostatic apparatus at high potentials requires a good insulation with respect to breakdown. Air is the most common insulating material. For a dry air at *N. T. P.* the breakdown occurs at fields of about 3×10^6 V/m. Fields somewhat less than this may produce some steady discharging current known as *corona*. When the field is not very large over most of the region it may come to or above the breakdown value near the point. The lines of force near the point are like those for a sphere, hence at the surface of the point $E = V/a$. Thus breakdown can occur even at 300 V at the point of 0.1 mm radius. The operation of lightning rods and brush discharge points in electrostatic generator is based on this principle.

6.11 POINT CHARGE IN A DIELECTRIC MEDIUM

Consider a point charge q in a homogeneous, isotropic and linear dielectric medium of dielectric constant k. If the point charge was situated in vacuum. The electric field would be pure radial. Since **E**, **P** and **D** are parallel to one another at each point, the radial nature of the field is not changed by the presence of the dielectric medium.

The molecules of the dielectric medium is polarized due to the electric field of point charge q as shown in Fig. 6.15. Applying Gauss's law to the spherical surface of radius r, assuming charge q at its centre, we get

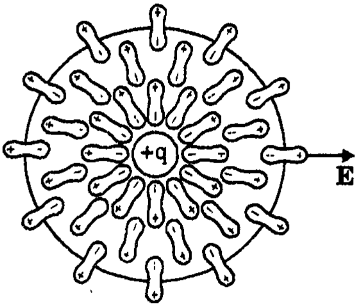


Fig. 6.15. Polarized molecules in a dielectric.

$$\oint \mathbf{D}\cdot d\mathbf{S} = \oint D\,dS = D4\pi r^2 = q$$

or $$D = q/4\pi r^2. \quad \text{...(51)}$$

Since $\mathbf{D} = \varepsilon\mathbf{E} = k\varepsilon_0\mathbf{E}$, therefore $\mathbf{E} = (q/4\pi\varepsilon_0 k r^3)\mathbf{r}$ ...(52)

On combining with Eq. (43), we get

$$\mathbf{P} = [(k-1)\,q/4\pi k r^3]\mathbf{r}. \quad \text{...(53)}$$

Eq. (52) shows that **E** is reduced by a factor of k due to the presence of dielectric medium. It is due to the presence of *polarization charge*, which is made up from two contributions, a volume density $\rho_p = -\nabla.\mathbf{P}$ and a surface density $\sigma_P = \mathbf{P}\cdot\hat{n}$. The Eq. (53) will give $\nabla.\mathbf{P} = 0$, for $r \neq 0$, so the volume density of polarization charge will be zero in this case.

Polarization Charge : The point charge q is a point in the macroscopic sense, although it will be large on a molecular scale. Let us assign it a radius a. The total surface polarization charge is then given by

$$q_P = \lim_{a \to 0} 4\pi a^2 (\mathbf{P}\cdot\hat{n})_{r=a} = -(k-1)\,q/k. \quad \text{...(54)}$$

$$\therefore \quad \text{Total charge } q_T = q + q_P = q/k. \quad \text{...(55)}$$

Thus we see that the effective charge is $(1/k)$ times the free charge in the dielectric medium.

Force Between Two Charges : We know that the polarization charge induced in the dielectric surface around the second charge q' will not affect the force on it, as the field inside a spherical distribution of charge due to that charge is zero. Hence the force between charges q and q' for the large value of their separation r will be given in the usual way as

$$F = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{k}\right)\frac{q'}{r^2} = \frac{qq'}{4\pi\varepsilon_0 k r^2} = \frac{qq'}{4\pi\varepsilon r^2}. \quad \text{...(56)}$$

This is same as obtained for point charges in vacuum except that ε_0 is being replaced by $\varepsilon\ (= \varepsilon_0 k)$.

6.12 ELECTRIC FIELDS IN CAVITIES OF A DIELECTRIC

The electric field inside a dielectric (polar or nonpolar) can be obtained if we consider a test charge placed in free space within the dielectric. According to Lord Kelvin an empty cavity is assumed within the dielectric where the test charge is placed. The result obtained depends on the shape of the cavity. Three cases are considered here, (*a*) a small *needle shaped cavity,* (*b*) *a disc-shaped cavity* and (*c*) a *spherical cavity.*

(*a*) Needle shaped cavity : Consider a dielectric placed in between the parallel charged plates producing the electric field **E**. A small portion of the dielectric is drawn so that we have needle shaped cavity *AB* [length greater than its area of cross section [Fig. 6.16 (*a*)]. Due to the presence of electric field the polarized bound charge will appear on the surfaces at *A* and *B*. These charges will be vanishingly small, as the area of cross section of the needle is taken very small. *Therefore there will be no contribution of bound charges to the electric field at point C inside such a cavity and the force on the charge at C is only due to the applied electric field* E. The same result is obtained when we apply boundary condition that the tangential components of electric intensity in the two media are continuous.

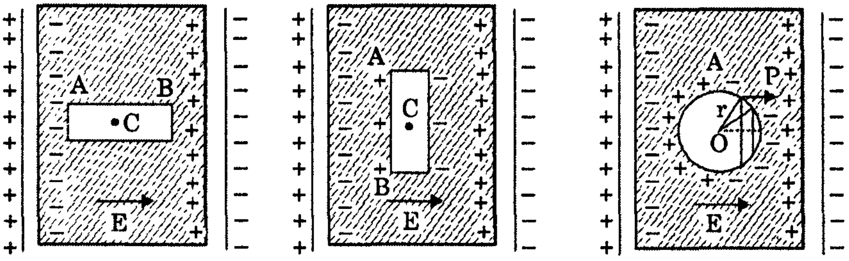
(*b*) Disc shaped cavity : Consider a disc shaped cavity *AB* [Fig. 6.16 (*b*)] within the dielectric, where the diameter of the disc is greater than its thickness in such a way that its thickness is in the direction of electric field. Due to this field polarized bound charges appear on the side surfaces, ⊥ to this electric field. Dissimilar to the previous case the bound charges also contribute to the electric force experienced by the test charge placed within this cavity. The electric field $\mathbf{E}_0$ inside the cavity is thus given by

$$\mathbf{E}_0 = \mathbf{E} + \mathbf{P}/\varepsilon_0,$$

or the electric displacement

$$\mathbf{D} = \varepsilon_0 \mathbf{E}_0 = \varepsilon_0 \mathbf{E} + \mathbf{P}.$$

This is the same result as for the displacement within the dielectric. This thing can also be obtained by applying boundary condition, the normal component of electric displacement **D** is continuous across the boundary.



(*a*) (*b*) (*c*)

Fig. 6.16. Cavities in the dielectric placed in an electric field.

In these two cases it has been assumed that the effect of the cavity in the dielectric does not make any difference to the displacement and field in immediately surrounding medium. This is only true for the small cavities.

(*c*) Spherical cavity-molecular field : The electric field which is responsible for polarizing a molecule of the dielectric (may be polar or non-polar) is called the *molecular field,* $\mathbf{E}_m$, a microscopic quantity. It differs from the external field and is therefore called *local field.* To find it, a small piece of dielectric is drawn, leaving a spherical cavity of radius *r*, surrounding the point under consideration Fig. 6.16. (*c*). Now we put the dielectric back into the cavity, molecule by molecule, except for the molecule where we wish to find $\mathbf{E}_m$. These molecules may be treated not as a continuum, but as individual dipoles. Let *O* be the centre of spherical cavity in the dielectric and its radius *r* be large compared with the intermolecular distance but small compared with the dimensions of the whole dielectric. If this dielectric is placed between two charged parallel plates, the electric field experienced by a molecule of the dielectric, if assumed to be placed at the centre of the cavity, is given by

$$\mathbf{E}_m = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \mathbf{E}_4. \qquad ...(57)$$

Here $\mathbf{E}_1$ is the field between two parallel charged plates with no dielectric, $\mathbf{E}_2$ is the field due to polarized charges on the outer surfaces of the dielectric (*i.e*, depolarizing field), $\mathbf{E}_3$ is the field due to polarized charges on the inner surface of the spherical cavity and $\mathbf{E}_4$ is the field due to all the dipoles inside this spherical cavity.

In this case $E_1 = \sigma/\varepsilon_0$, where σ is the surface charge density, $\mathbf{E}_2 = -\mathbf{P}/\varepsilon_0$, where $\mathbf{P}$ is the polarization vector due to the polarization of the dielectric when placed within the external field $\mathbf{E}_1$.

$$\therefore \qquad \mathbf{E}_m = \mathbf{E}_1 - \mathbf{P}/\varepsilon_0 + \mathbf{E}_3 + \mathbf{E}_4. \qquad ...(58)$$

If the dipoles in the cavity are located at the regular atomic positions of a cubic crystal, $\mathbf{E}_4 = 0$. Such as in gases or liquids, the dipoles may be oriented parallel but randomly distributed in position. The net electric field due to these dipoles in the cavity may be obtained as under :

We know that the electric field at a distance r from the dipole of dipole moment $\mathbf{p}$ is given as

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\left[\frac{\mathbf{p}}{r^3} - \frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5}\right]. \qquad ...(59)$$

Summing over all the dipoles within the sphere is equivalent to taking the spatial average of the x-component of the field

$$E_4 = \Sigma E_x = -\frac{1}{4\pi\varepsilon_0}\left[\frac{p_x}{r^3} - \frac{3(p_x x + p_y y + p_z z)x}{r^5}\right] \qquad ...(60)$$

For the *isotropic dielectric,* the x, y and z-directions are equivalent, *i.e.,*

$$\overline{x^2} = \overline{y^2} = \overline{z^2} = r^2/3 \text{ and } \overline{xy} = \overline{yz} = \overline{zx} = 0, \qquad ...(61)$$

hence the field $\mathbf{E}_4$ due to the dipoles within the sphere vanishes.

In case of anisotropic medium, $\mathbf{E}_4 \neq 0$. Due to this term we have anisotropic electric behaviour of calcite type of material.

Let $\mathbf{E}$ be the electric field in the dielectric when it is placed in an external electric field $\mathbf{E}_1$. From second boundary condition, we know that the normal component of electric displacement $\mathbf{D}$ is continuous, across the vacuum dielectric interface. As $\mathbf{D}$ is ⊥ to the surface, hence $\mathbf{D}$ in the vacuum just outside the dielectric is equal to that in the dielectric, *i.e.,*

$$\varepsilon_0\mathbf{E}_1 = \varepsilon_0\mathbf{E} + \mathbf{P}. \qquad ...(62)$$

Combining Eqs. (58) and (62) and using $\mathbf{E}_4 = 0$, we get

$$\mathbf{E}_m = \mathbf{E} + \mathbf{E}_3. \qquad ...(63)$$

The electric field $\mathbf{E}_3$ generated by the polarization is equivalent to the field produced by the charge distribution associated with the polarization. If the polarization through out the volume of dielectric sphere to be constant, then

$$\rho_p = -\nabla\cdot\mathbf{P} = 0 \text{ and}$$

$$\sigma_p = \mathbf{P}\cdot\hat{r} = P\cos\theta.$$

Thus the electric field at the centre *O due* to the charge over the surface area *S* of the surface at point *A* with polar coordinates (*r*, θ) is given by

$$d\mathbf{E}' = \frac{1}{4\pi\varepsilon_0}\frac{\sigma_p dS}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{P\cos\theta\, dS}{r^2}\hat{r}.$$

Let us assume dS as a ring shaped element formed by an arc at *A* between angles θ and θ + $d\theta$ [Fig. 6.17.]. Therefore $dS = 2\pi\,(r\sin\theta)\,(rd\theta)$. From symmetry it is clear that only the component of $d\mathbf{E}'$ along the direction of $\mathbf{P}$ will contribute to the total field $\mathbf{E}_3$.

$$\therefore \qquad E_3 = \Sigma dE' \cos\theta = \frac{1}{4\pi\varepsilon_0}\int\frac{P\cos\theta\cdot 2\pi r^2\sin\theta d\theta}{r^2}\cos\theta$$

or $$E_3 = \frac{P}{2\varepsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta d\theta = \frac{P}{3\varepsilon_0}, \quad ...(64)$$

or vectorially $\mathbf{E}_3 = \mathbf{P}/3\varepsilon_0$.

Thus Eq. (63) reduces to

$$\mathbf{E}_m = \mathbf{E} + \mathbf{P}/3\varepsilon_0. \quad ...(65)$$

Clausius-Mossotti relation : For ideal dielectrics $\mathbf{P} = (k-1)\,\varepsilon_0\mathbf{E}$, hence

$$\mathbf{E}_m = \mathbf{E} + \tfrac{1}{3}(k-1)\,\mathbf{E} = \tfrac{1}{3}\,(k+2)\,\mathbf{E}$$

or $$\mathbf{E} = [3/(k+2)]\,\mathbf{E}_m. \quad ...(66)$$

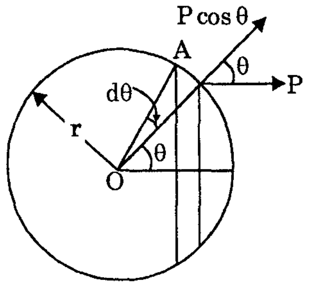


Fig. 6.17. Spherical cavity.

Substituting this value of E in the relation for the polarization P, we get

$$\mathbf{P} = (k-1)\,\varepsilon_0\,\frac{3}{k+2}\,\mathbf{E}_m = \frac{3\,(k-1)}{k+2}\varepsilon_0 E_m. \quad ...(67)$$

The dipole moment of a molecule per unit polarizing field is called the *polarizability* or the *molecular (or atomic, if we are dealing with atoms rather than molecules) polarizability, i.e.,*

$$\alpha = p_m/E_m. \quad ...(68)$$

If there are n molecules per unit volume, then polarization

$$\mathbf{P} = n\mathbf{p}_m = \alpha n\mathbf{E}_m. \quad ...(69)$$

Comparing Eqs. (67) and (69), we get

$$\alpha = \frac{3\varepsilon_0}{n}\,\frac{k-1}{k+2}. \text{ or } \frac{k-1}{k+2} = \frac{\chi}{\chi+3} = \frac{n\alpha}{3\varepsilon_0}. \quad ...(70)$$

$$\therefore \qquad \chi = \frac{n\alpha/\varepsilon_0}{1-(n\alpha/3\varepsilon_0)}. \quad ...(71)$$

Eq. (70) gives us the dielectric constant of a fluid in terms of α, the polarizability and is called the *Clausius Mossotti relation,* as Mossotti (in 1880) and Clausius (in 1879) independently established that for any given material $(k-1)/(k+2)$ should be proportional to the density of the material. The function $(k-1)/(k+2)$ is known as *volume polarizability.*

α can be determined by measuring k by experiment and using the relation $n = N_A\,\rho/M$, where M is the molecular weight of the dielectric material, ρ its density and N_A the Avogadro's number. The relation (70) may be written as

$$\alpha = \frac{3\varepsilon_0 M}{N_A\rho}\cdot\frac{k-1}{k+1} \text{ or } \frac{M}{\rho}\cdot\frac{k-1}{k+1} = \frac{N_A\alpha}{3\varepsilon_0}. \quad ...(72)$$

The quantity $(M/\rho)\ (k-1)/(k+1)$ is known as *molar polarizability* and $(k-1)/\rho\ (k+1)$ is called *specific polarizability.*

Since α is a property of the atoms (or molecules) of the dielectric medium, hence is a microscopic quantity. Thus we see that the *Clausius Mossotti relation relates a microscopic quantity* (α) *with a macroscopically measured physical quantity (k).*

At optical frequencies $k = \mu^2$, where μ is the refractive index of the medium. Thus Eq. (70) may be written as

$$(\mu^2-1)/(\mu^2+2) = n\alpha/3\varepsilon_0. \quad ...(73)$$

This is called the *Lorentz-Lorenz equation.*

The relation (70) shows that k will increase rapidly as n increases. For $n = 3\varepsilon_0/\alpha$, $k = \infty$. The dielectric constant is infinite corresponding to infinitely polarized material *i.e., conducting material.* This case will never occur in gases or dilute solutions. Thus the *Clausius Mossotti relation holds best for gases and dilute solutions. It is valid approximately for concentrate solutions and solids (k large). It does not hold in crystalline solids and polar molecules on accounts of the complicated interactions in their dipoles.*

6.13 INDUCED DIPOLES

A Simple Model : In the *non-polar molecule,* the centres of gravity of the positive and negative charge distributions coincide. When such a molecule is placed in an electric field, these opposite charges tend to be separated along the field direction, until a state of equilibrium is reached. At this stage the electrostatic attraction between these charges is equal to the force responsible for separation. The molecule thus becomes dipole, known as *induced dipole.*

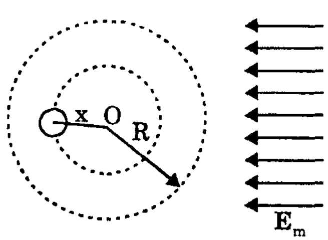


Fig. 6.18. Atom in an electric field.

For the simplicity, let us consider monoatomic molecules. Quantum mechanics tells us that the orbital electrons have finite probability of being situated in any part of the atom. Thus the electronic charge $-Ze$ is taken to be uniformly distributed in a spherical cloud of radius R with the positive nuclear charge Ze at its centre O. Under the influence of an external field E_m, the positive charge moves relative to the centre of electron cloud until the electrostatic attractive force between the cloud and the nucleus balances the force $Ze\,E_m$. At this equilibrium state, if the displacement of nucleus is x in the direction of E_m, then

$$Ze\,E_m = Ze\,q/4\pi\varepsilon_0 x^2. \qquad ...(74)$$

According to the Gauss's law, q the negative charge attracting the nucleus is that part of the cloud which is in the sphere of radius x. The density of electron cloud $\rho = Ze/\frac{4}{3}\pi R^3$ and $q = \frac{4}{3}\pi x^3\,\rho = Ze\,x^3/R^3$. Thus Eq. (74) gives

$$ZeE_m = \frac{1}{4\pi\varepsilon_0}\frac{(Ze)(Zex^3/R^3)}{x^2} \text{ or } Ze\,x = 4\pi\varepsilon_0 R^3 E_m.$$

The induced atomic dipole moment $\mathbf{p}_m = Ze\,\mathbf{x} = 4\pi\varepsilon_0 R^3 \mathbf{E}_m$. ...(75)

If α be the atomic deformation polarizability, then $\mathbf{p}_m = \alpha\,\mathbf{E}_m$, therefore

$$\alpha = 4\pi\varepsilon_0 R^3. \qquad ...(76)$$

For a diatomic molecule, the *molecular polarizability* may be taken as twice the polarizability (*i.e.*, 2α). It is also a constant quantity.

Effect of atomic interaction on molecular polarizability : Consider a diatomic molecule with two identical spherically symmetric atoms each of polarizability α, at a distance R from each other and placed in an electric field E_m. Each atom will experience an additional electric field E′ due to the induced dipole of the other atom. Thus for the each atom·

$$\mathbf{p} = \alpha(\mathbf{E}_m + \mathbf{E}'),$$

where $$\mathbf{E}' = \frac{2\mathbf{p}}{4\pi\varepsilon_0 R^3} = \frac{2\alpha}{4\pi\varepsilon_0 R^3}(\mathbf{E}_m + \mathbf{E}')$$

or $$\mathbf{E}' = \frac{2\alpha\,\mathbf{E}_m}{4\pi\varepsilon_0 R^3[1 - 2\alpha/4\pi\varepsilon_0 R^3]}$$

$$\therefore \quad \mathbf{p} = \alpha\mathbf{E}_m/[1 - (2\alpha/4\pi\varepsilon_0 R^3)].$$

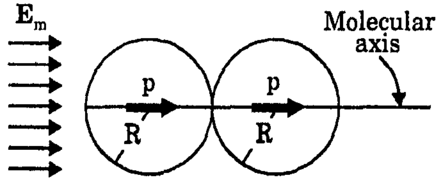


Fig. 6.19. Dipole-dipole interaction between atoms.

Total molecular polarization is twice the above result, therefore

$$\alpha' = 2\alpha/[1 - (2\alpha/4\pi\varepsilon_0 R^3)]. \qquad ...(77)$$

Thus we see that $\alpha' = 2\alpha$, if $2\alpha/4\pi\varepsilon_0 R^3 \ll 1$. In this case we have assumed the external electric field in the direction of molecular axis. If the field is perpendicular to the molecular axis, the induced atomic dipoles will be normal to the molecular axis. Thus α' will be different.

6.14 MOLECULAR POLARIZABILITY

When the molecules are placed in an electric field, the polarization can arise in two ways : (*a*) the field distorts the charge distribution and produces induced dipole moment in each molecule, (*b*) the field exerts torques on the randomly oriented permanent dipoles in case of polar molecules. Thus the *molecular polarizability* may have three types of contributions :

(*a*) Electronic polarizability : The electronic contribution may arise from the distortion of electronic cloud relative to the nucleus. The displacement x of the harmonically bound electron of mass m by the application of a field $\mathbf{E}_m$ is given by

$$-e\mathbf{E}_m = m\ \omega_0^2\mathbf{x} = \beta\mathbf{x},$$

where β is the force constant and ω_0 is the resonance frequency of the electron bound harmonically to an atom.

$$\therefore \quad \text{Electronic polarizability } \alpha_e = \frac{p}{E_m} = -\frac{ex}{E_m} = \frac{e^2}{m\omega_0^2}. \qquad ...(78)$$

The electronic polarizability depends on frequency. An electron having the resonance frequency ω_0 may be treated as a simple harmonic oscillator. The equation of motion in the field E_m sin ωt is given by

$$m\frac{d^2x}{dt^2} + m\omega_0^2 x = -eE_m \sin \omega t.$$

Therefore for $x = x_0 \sin \omega t$, we get $m(-\omega^2 + \omega_0^2)x_0 = -eE_m$

$$\therefore \quad \text{Dipole moment } p_0 = -ex_0 = \frac{e^2E_m}{m(\omega_0^2 - \omega^2)}$$

$$\text{and electronic polarizability} \quad \alpha_e = \frac{e^2/m}{\omega_0^2 - \omega^2}. \qquad ...(79)$$

In the visible region the frequency dependence is not usually very important in most transparent materials.

(*b*) Ionic polarizability : It is produced by opposite displacements of positive and negative ions in the material due to the applied field $\mathbf{E}_m$. Similar to the case of electronic polarizability the ionic polarizability is given by

$$\alpha_i = \frac{p}{E_m} = \frac{qx}{E_m} = \frac{q^2}{\omega_T^2}\left(\frac{1}{M_1} + \frac{1}{M_2}\right), \qquad ...(80)$$

where ω_T is the transverse optical phonon frequency, q the charge on an ion and M_1 and M_2 are the masses of ions *lie* on the odd numbered planes and on the even-numbered planes respectively.

(c) Orientational or dipolar polarizability : The orientational contribution arises when the substance is built up of molecules possessing a permanent dipole moment (polar dielectric).

Debye explained the dielectric constants of some substances consisting molecules possessing a permanent electric dipole moment. In the absence of an applied electric field these molecules point in random direction and the net polarization of the dielectric is zero.

$$\mathbf{P} = \frac{1}{\Delta v}\Sigma\mathbf{p}_m = 0, \qquad ...(81)$$

where the summation extends over all molecules in the volume element Δv. Here $\mathbf{p}_m$ is the permanent dipole moment of each molecule.

If an electric field is applied, the expression for the polarization can be obtained from *Langevin—Debye theory*, which is as under :

When the polar dielectric is subjected to an electric field the individual dipoles experience torques which tend to align them with the field. If the field is very strong, we may assume all dipoles to be aligned completely and the polarization may achieve the saturation value.

$$\mathbf{P}_s = n\ \mathbf{p}_m, \qquad ...(82)$$

where n is the number of molecules per unit volume.

At fields generally applied, the polarization of a polar dielectric is very small in comparison to its saturated value. It decreases with increase in temperature. This deviation from saturated value is due to the thermal energy of the molecules, which produces random dipole orientations. Statistical mechanics gives that the probability of finding a molecule with particular molecular energy U at temperature T is proportional to exp. $(-U/kT)$, where k is the Boltzmann constant, equal to 1.38×10^{-23} joule/degree.

If the dipole makes an angle θ with the direction of electric field $\mathbf{E}_m$, then the potential energy of a dipole is given as $U = -\mathbf{p}\cdot\mathbf{E}_m = pE_m \cos\theta$. Hence the number of dipoles per unit volume per unit solid angle making an angle θ with the direction of the electric field $\mathbf{E}_m$ is exp $(pE_m \cos\theta/kT)$. Thus the number of dipoles per unit volume with in the solid angle $d\Omega$ between directions 0 and $\theta + d\theta$ with $\mathbf{E}_m$ is given by

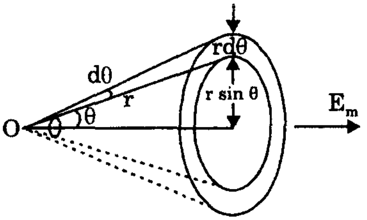


Fig. 6.20. Annular ring.

$$dn = C \exp(pE_m \cos\theta/kT)\, d\Omega$$

$$\because \quad d\Omega = 2\pi r \sin\theta\, r d\theta/r^2$$

$$\therefore \quad dn = C' \exp(pE_m \cos\theta/kT) \sin\theta\, d\theta$$

where $C' = 2\pi C$.

Hence the number of dipoles per unit volume

$$n = C' \int_0^{\pi} \exp(pE_m \cos\theta/kT) \sin\theta d\theta.$$

Each dipole contributes a dipole moment $p\cos\theta$ parallel to $\mathbf{E}_m$ and $p\sin\theta$ perpendicular to $\mathbf{E}_m$. It is clear from the spherical symmetry around $\mathbf{E}_m$ that the contribution of dipole moment in the direction ⊥ to $\mathbf{E}_m$ must be zero, hence the total dipole moment per unit volume is along the direction of $\mathbf{E}_m$ and is given by

$$P = \Sigma\, p\cos\theta\cdot dn = C'p \int_0^{\pi} \exp(pE_m \cos\theta/kT) \sin\theta\cos\theta d\theta$$

$$= np \frac{\int_0^{\pi} \exp(pE_m \cos\theta/kT) \sin\theta\cos\theta d\theta}{\int_0^{\pi} \exp(pE_m \cos\theta/kT) \sin\theta\, d\theta}$$

Substituting $pE_m/kT = u$, $u\cos\theta = x$ and hence $u \sin\theta\, d\theta = -dx$.

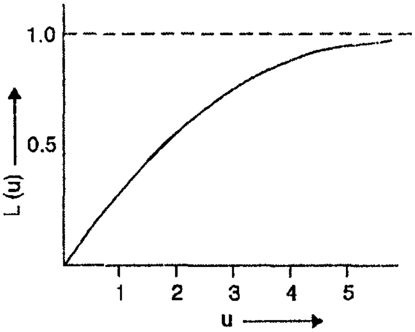


Fig. 6.21. Plot of Langevin function.

$$\therefore \quad P = \frac{np}{u} \frac{\int_{-u}^{u} e^x x dx}{\int_{-u}^{u} e^x dx} = \frac{np}{u} \frac{[xe^x - e^x]_{-u}^{u}}{[e^x]_{-u}^{u}}$$

$$= \frac{np}{u} \frac{[ue^u - e^u + ue^{-u} + e^{-u}}{e^u - e^{-u}} = np\left[\frac{e^u + e^{-u}}{e^u - e^{-u}} - \frac{1}{u}\right]$$

$$= np\left[\coth u - \frac{1}{u}\right] = np\left[\coth \frac{pE_m}{kT} - \frac{kT}{pE_m}\right], \qquad \text{...(83)}$$

which is known as *Langevin equation* and the bracket term as *Langevin function L(u)*. A plot of the function L(u) versus u shows that the polarization P tends to np, its saturation value, when $u \to \infty$. Thus high field E_m and low temperature T can give this saturation value. At smaller values of u, the curve is linear. Dipole moments are of the order of 3×10^{-30} coul m, so for $E_m = 10^7$ volts/meter we have $pE_m = 3 \times 10^{-23}$. At room temperature $kT = 4 \times 10^{-21}$ joule, so that $u \simeq 10^{-2}$. Thus we can write Langevin's function as

$$L(u) = \coth u - \frac{1}{u} = \frac{e^u + e^{-u}}{e^u - e^{-u}} - \frac{1}{u} \simeq \frac{u}{3}.$$

Hence

$$P = n\, p\, u/3 = np^2 E_m/3kT, \quad \text{or} \quad \mathrm{P} = (np^2/3kT)\mathrm{E}_m. \qquad \text{...(84)}$$

As polarizability is defined as the dipole moment of molecule per unit electric field, hence the dipolar or orientational polarizability per molecule

$$\alpha_d = \alpha_{0r} = P/nE_m = p^2/3kT, \qquad ...(85)$$

Thus we see that the orientational polarization depends inversely on the temperature.

The induced (electronic and ionic) polarization is also present in polar dielectrics. Thus the total polarization in polar dielectrics will be given by

$$\mathbf{P}_T = \mathbf{P}_i + \mathbf{P}_{or} = n\,\alpha_{in}\,\mathbf{E}_m + n\,(p^2/3kT)\mathbf{E}_m = (\alpha_{in} + p^2/3kT)n\,\mathbf{E}_m,$$

where α_{in} is the sum of the electronic and ionic polarizabilities. Hence the total polarizability becomes

$$\alpha = \alpha_{in} + p^2/3kT, \qquad ...(86)$$

This is the *Langevin-Debye equation* and is of great importance in interpreting molecular structures. The contribution from induced polarization is about one hundredth of that from permanent dipole moments and may be ignored in many cases.

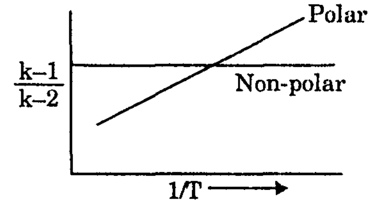


Fig. 6.22. Plot of Debye equation.

Debye modified the Clausius-Mossotti relation for polar molecules as

$$\frac{k-1}{k+1} = \frac{n\alpha}{3} = \frac{n}{3}\left(\alpha_{in} + \frac{p^2}{3kT}\right)$$

$$= \frac{p^2 n}{9k}\frac{1}{T} + \frac{\alpha_{in} n}{3}. \qquad ...(87)$$

This is called *Debye equation*. A graph between $(k-1)/(k+2)$ and $1/T$ will be a straight line, Fig. 6.22. The intercept of it on the ordinate will be $\alpha_{in}n/3$ and its slop will be $p^2n/9k$. The line will be parallel to abscissa for nonpolar molecules. Thus two types of polarization can be separated experimentally.

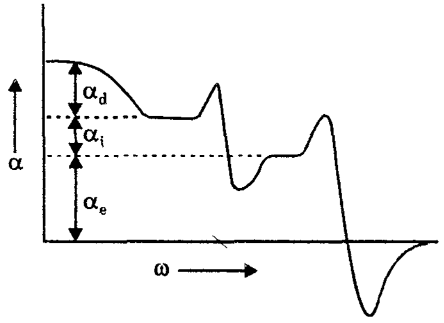


Fig. 6.23. Variation of polarizability with frequency.

The above theory is valid for gases and liquids. It is not perfectly true for all solids on account of the strong interactions between the molecules. Frequency dependence of the several contributions to the polarizability is shown in Fig. 6.23. It shows that at low frequencies, all the three types of polarizability contribute. The contribution of dipole polarization ceases when the frequencies are reached up in the microwave range (10^9 *Hz*). Ionic polarization continues to act up to infrared range (10^{13} Hz) and electronic polarization continues upto the ultraviolet range (10^{15} Hz).

6.15 PIEZOELECTRICITY

If a mechanical stress is applied to the crystal, the atoms are slightly displaced. The ionic displacements are symmetrical about the symmetry centres in the *centrosymmetric crystals* and the charge distribution inside the crystal is not changed appreciably by the applied stress. In the case of *acentric crystals, the* ionic displacements are asymmetric and the *electrical dipoles are produced in the crystal. This effect is known as piezoelectric effect and these crystals are called piezoelectric crystals.* Conversely when an electric field is applied to a piezoelectric crystal, dipoles are induced and the atomic displacements produce a mechanical strain. This is called the *inverse piezoelectric effect.* The displacement varies periodically when the alternating field is applied to a piezoelectric crystal. At a certain frequency called *resonance frequency,* these displacements are in phase with the applied field and hence are maximum. The frequency range at which resonance occurs is mostly very sharp hence such crystals are very much used for frequency controls in radio transmitters and in electronic clocks.

To obtain a relation between the mechanical stress and the electric polarization produced in a piezoelectric crystal, consider a crystal of width d placed between two metal plates A and B Fig. 6.24. If the crystal is compressed by an applied mechanical stress p, a mechanical strain s is produced in the crystal.

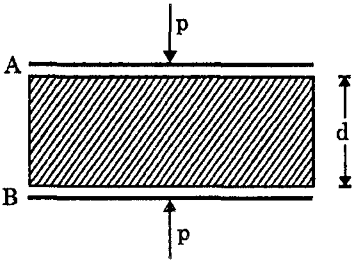


Fig. 6.24. Piezoelectric crystal.

$$s = \frac{\text{Mechanical stress } p}{\text{Young's modulus } Y} = \frac{\Delta d}{d} \qquad \text{...(88)}$$

The polarization density P in the crystal is proportional to the applied mechanical stress, therefore we have

$$P = \eta\, p, \qquad \text{...(89)}$$

where η is the *piezoelectric constant.*

On the other hand if an electric field **E** is applied to the crystal then the induced strain is proportional to **E**, *i.e.*

$$s_{in} = \eta' E, \qquad \text{...(90)}$$

When both an external field **E** and a stress p are applied to the crystal, the electric displacement is given by

$$D = \varepsilon_0 E + P = \varepsilon_0 E + \eta p \qquad \text{...(91)}$$

and the internal strain

$$s = \eta' E + p/Y. \qquad \text{...(92)}$$

Piezoelectric properties depend on temperature. These properties vanish at a certain temperature at which the crystal lattice is rearranged so that a centre of symmetry is formed.

Piezoelectric effect has numerous practical applications. Piezoelectric transducers are used for measuring rapidly varying pressures. Piezoelectric microphones are well known. Quartz ultrasonic vibrators are the well known examples of the inverse piezoelectric effect.

6.16 PYROELECTRICITY

Certain types of dielectric materials are asymmetric in such a way that they are spontaneously polarized in the absence of external fields. This self polarization results from the displacement of ions due to local electric fields as the forces on the ions produced by local fields are greater than the elastic restoring forces with in the crystal. The polarization due to permanent dipoles present in the dielectric is temperature dependent. A change in temperature of the dielectric changes its polarization. This effect is called *pyroelectric effect* and the crystals are called pyroelectrics. Every pyroelectric *is a piezoelectric, but the inverse is not true.* This is due to the fact that a pyroelectric has a preferred direction along which spontaneous polarization takes place, while a piezoelectric generally does not have such a direction. *The inverse pyroelectric effect* also exists (a variation of the electric field in an adiabatically isolated pyroelectric is accompanied by a change in its temperature).

6.17 FERROELECTRICITY

There are certain pyroelectric crystals in which one can reverse the direction of polarization by applying a sufficiently intense external field. Such crystals are called *ferro-electric* and this effect is called the *ferro-electric effect.* All the ferroelectric crystals are pyroelectric crystals but the reverse is not true.

The term *ferroelectricity* is applied to the ferroelectric phenomenon. A ferroelectric crystal generally consists of regions called *domains* within each of which all the molecular dipoles are parallel. In adjacent domains the polarization is in different directions. In the crystal as a whole the dipole moments of these domains are randomly oriented resulting in zero dipole moment. When an external electric field is applied these domains tend to align themselves in the field direction. The polarization induced does not vary linearly with the applied electric field E. The ferroelectric materials exhibit hysteresis phenomenon in a manner similar to ferromagnetic materials. They often exhibit large susceptibility. As the electric field is applied the total polarization increases rapidly until the saturation value is reached. It is the position where all the domains are parallel to the applied field. When the field decreases to zero, the polarization does not return to zero. The polarization left is called *residual or remanant polarization.* The *coercive field* required to reverse the polarization direction is not so large as to cause electric breakdown of the crystal. The larger the value of coercive field the better the residual polarization is attained by the ferroelectric.

As the temperature of a ferroelectric exceeds a certain value, its ferroelectric properties vanish and it becomes an ordinary polar dielectric. This temperature T_C is called *Curie temperature.* It is typical for each material. The point marking the transition between ferroelectric phase and polar electric phase is called the *Curie point.* In certain cases, such as Rochelle salt, there are two Curie points (*i.e.*, ferroelectric properties vanish with decreasing temperature also). The dielectric constant k in the vicinity of the Curie point is given by the relation

$$k = \frac{C}{T - T_0} + k_0, \qquad ...(93)$$

where C is the constant called the *Curie constant,* T_0 the *Curie-Weiss Temperature,* which is close to the Curie temperature T_C and k_0 is the contribution due to the electronic polarization. This relation is known as *Curie-Weiss law.* It shows that just above the critical temperature k is very large.

One of the best known ferroelectric materials is barium titanate ($BaTiO_3$), which is often used in capacitors where small size and large capacitance are required. Because of their low spontaneous polarizabilities ferroelectric crystals are frequently used in computers as memory cells. There are several groups of ferroelectric crystals. The examples are :

Rochelle salt $NaK\,(C_4\,H_4\,O_6)\,4H_2O$; *Potassium dihydrogen phosphate* KH_3PO_4; *Barium titanate* $BaTiO_3$; *Guanidine compounds such as* $C(NH_2)_3\,Al(SO_4)_2 \cdot 6H_2O$.

Para electric materials have a small positive susceptibility due to polarization in electric field. In fact all dielectric materials (except vacuum) are paraelectrics.

There is no dielectric which has negative susceptibility. Thus there is no existence of di-electric. It is because the polarization in the direction opposite to the polarizing field is not observed in any case.

6.18 ELECTRETS

When an external electric field is applied to a polar dielectric, the dipole moments get partly lined up. In general this induced polarization disappears when the external electric field is switched off. There are certain dielectrics (*organic waxes, plastic materials etc.*) in which the polarization remains when the field is removed. Such materials are called *electrets.* The duration for which these induced polarization persist depends on the nature of the material, its polarization condition and its surroundings. When an external electric field is applied to a dielectric in the *molten* state the molecular dipoles are partly aligned in the field direction. If this dielectric is allowed to solidify, the dipoles are frozen in their new orientations and do not return to their original orientations when the external field is removed. Electrets prepared by this method (applying field at the time of molten state and removing field when it is solidify) are called *thermoelectrets.* Electrets are also prepared by the simultaneous application of light and an electric field. Such electrets are called *photoelectrets.*

The electrets are very used in instruments such as microphones, high voltage generators, computers, radio and television receivers and other electronic devices. The photoelectrics are of very much use in technique for the reproduction of printed matter (xerography).

6.19 ELECTROSTATIC ENERGY WITHIN A LINEAR DIELECTRIC

Let us consider an arbitrary distribution of charge characterized by the volume charge density ρ and surface charge density σ. For convenience, we assume that the charge system is bounded, so that one can construct a closed surface of finite dimensions which encloses all of the charge. For simplicity, we further assume that all surface densities of charge σ reside on conductor surfaces. The densities ρ and σ are related to the electric displacement vector **D** as

$$\rho(\mathbf{r}) = \nabla\cdot\mathbf{D} \text{ throughout the dielectric regions}$$

and $$\sigma(\mathbf{r}) = \mathbf{D}\cdot\hat{n} \text{ on the conductor surfaces.}$$

We know that the electrostatic energy of an arbitrary charge distribution with charge densities ρ and σ is given by

$$U = \frac{1}{2}\int_{\text{volume}} \rho(\mathbf{r})V(\mathbf{r})\,dv + \frac{1}{2}\int_{\text{surface}} \sigma(\mathbf{r})\,V(\mathbf{r})\,dS. \qquad ...(94)$$

Substituting the values of $\rho(\mathbf{r})$ and $\sigma(\mathbf{r})$, we get

$$U = \frac{1}{2}\int_{\text{volume}} V(\mathbf{r})\nabla\cdot\mathbf{D}\,dv + \frac{1}{2}\int_{\text{surface}} V(\mathbf{r})\,\mathbf{D}.\hat{n}dS. \qquad ...(95)$$

The integrand in the first integral may be transformed by means of the vector identity div $(\varphi\mathbf{A}) = \varphi$ div **A** + **A**. grad φ, as

$$V(\mathbf{r})\,\nabla\cdot\mathbf{D} = \nabla\cdot[V(\mathbf{r})\,\mathbf{D}] - \mathbf{D}\cdot\nabla V(\mathbf{r}) = \nabla\cdot[V(\mathbf{r})\mathbf{D}] + \mathbf{D}\cdot\mathbf{E}$$

$$\therefore \qquad U = \frac{1}{2}\int_{\text{volume}} \nabla\cdot[V(\mathbf{r})\mathbf{D}]dv + \frac{1}{2}\int_{\text{volume}} \mathbf{D}\cdot\mathbf{E}dv + \frac{1}{2}\int_{\text{surface}} V(\mathbf{r})\mathbf{D}\cdot\hat{n}dS.$$

Using the Gauss's divergence theorem, the first integral may be converted into surface integral. Thus we have

$$U = \frac{1}{2}\int_{S+S'} V(r)\,\mathbf{D}\cdot\hat{n}dS + \frac{1}{2}\int_{\text{volume}} \mathbf{D}\cdot\mathbf{E}\,dv + \frac{1}{2}\int_{S} V(\mathbf{r})\,\mathbf{D}\cdot\hat{n}dS.$$

Here $S + S'$ is the entire surface bounding the volume. It consists of S, the surfaces of all conductors in the system and S' a surface which bounds our system from the outside. In both cases, the normal vector $\hat{n}$ is directed out of the volume. In the last integral the normal vector $\hat{n}$ is directed out of the conductor, hence into the volume. Thus the contribution of first integral due to surface S cancels with the last integral. Thus we have

$$U = \frac{1}{2}\int_{S'} V(\mathbf{r})\mathbf{D}\cdot\hat{n}dS + \frac{1}{2}\int_{\text{volume}} \mathbf{D}\cdot\mathbf{E}\,dv. \qquad ...(96)$$

If our charge distribution bears a net charge, then at large distances from the charge system the potential falls off inversely as the distance, the **D** falls off as r^{-2}. The area of a closed surface passing through a point at a distance r varies as r^2. Hence the integral over S' will vary as r^{-1}, which vanishes when S' is moved out to infinity. Thus the first term vanishes and the electrostatic energy is given by

$$U = \frac{1}{2}\int_{volume} \mathbf{D}\cdot\mathbf{E}\,dv. \qquad ...(97)$$

Here the integration is over the volume of the system external to the conductors. Since **E** equals zero inside a conductor, the integration may be extended to include all space. The energy may be assumed to be distributed with a density $\frac{1}{2}$ **D**·**E** per unit volume, *i.e.*,

$$u = \frac{1}{2}\mathbf{D}\cdot\mathbf{E}. \qquad ...(98)$$

The linear dielectric is characterized by a constant permittivity ε. Thus we have

$$u = \frac{1}{2}\varepsilon\,\mathbf{E}\cdot\mathbf{E} = \frac{1}{2}\varepsilon\,E^2 = D^2/2\varepsilon. \qquad ...(99)$$

The condition $\nabla V_1 = \nabla V_2$ gives $V = V_2$ + const., throughout the volume v. The presence of this arbitrary constant indicates that the potential is not unique. If the potential is specified at any point on S, then $V_1 = V_2$ at that point and the constant must be zero. Thus we see that Dirichlet boundary condition specified on any portion of surface S guarantees the uniqueness of the potential.

EXERCISES

Example 1. *Calculate the torque and force experienced by an electric dipole placed in the field of a point charge. The dipole axis makes an angle* α *with the radius vector from the point charge.*

Let charge q be placed at the origin O of the co-ordinate system and the dipole is assumed to be placed in $x-y$ plane. If r be the distance of the dipole from O and θ be the angle which radius vector makes with the x-axis. The torque on the dipole

$$\tau = \mathbf{p} \times \mathbf{E} = \mathbf{p} \times \left(\frac{q}{4\pi\varepsilon_0} \cdot \frac{1}{r^3}\mathbf{r}\right) = \frac{q}{4\pi\varepsilon_0 r^3}\mathbf{p} \times \mathbf{r}$$

or $$\tau = \frac{pq}{4\pi\varepsilon_0 r^2}\sin\alpha.$$

Its direction will be ⊥ to the plane consisting **p** and **r**.

The total force on the dipole due to non-uniform field **E** is given by

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\,\mathbf{E} = (\mathbf{p}\cdot\nabla)\,\frac{q}{4\pi\varepsilon_0 r^3}\mathbf{r}.$$

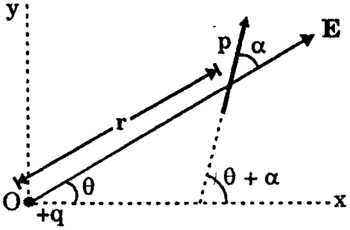


Fig. 6.25. Example 1.

Component of **F** along x-axis $F_x = (p_x\,\partial/\partial x + p_y\,\partial/\partial y)\,E_x$

Component of **F** along y-axis $F_y = (p_x\,\partial/\partial x + p_y\,\partial/\partial y)\,E_y$

Components of **E** along x and y-directions are

$$E_x = (q/4\pi\varepsilon_0 r^2)\cos\theta,\ E_y = (q/4\pi\varepsilon_0 r^2)\sin\theta$$

and that of **p** are $p_x = p\cos(\alpha+\theta)$ and $p_y = p\sin(\alpha+\theta)$, therefore

$$F_x = p\cos(\alpha+\theta)\,\frac{\partial}{\partial x}\left[\frac{qx}{4\pi\varepsilon_0(x^2+y^2)^{3/2}}\right] + p\sin(\alpha+\theta)\,\frac{\partial}{\partial y}\left[\frac{qx}{4\pi\varepsilon_0(x^2+y^2)^{3/2}}\right]$$

$$= \frac{pq}{4\pi\varepsilon_0}\left[\frac{y^2-2x^2}{(x^2+y^2)^{5/2}}\cos(\alpha+\theta) - \frac{3xy}{(x^2+y^2)^{5/2}}\sin(\alpha+\theta)\right]$$

Similarly we have

$$F_y = \frac{pq}{4\pi\varepsilon_0}\left[\frac{-3xy}{(x^2+y^2)^{5/2}}\cos(\alpha+\theta) + \frac{x^2-2y^2}{(x^2+y^2)^{5/2}}\sin(\alpha+\theta)\right]$$

In a particular case, if the dipole is on the axis of x at a distance a, then

$$F_x = -(2pq/4\pi\varepsilon_0 a^3)\cos\alpha \text{ and } F_y = (pq/4\pi\varepsilon_0 a^3)\sin\alpha.$$

–ve sign shows that the dipole experiences an attractive radial field. As a special case, if $\alpha = 0$, *i.e.* the dipole is along x-axis, F_y the transverse component of force becomes zero. For $\alpha = \pi/2$, $F_x = 0$.

Example 2. *A short electric dipole of dipole moment* $\mathbf{p}_1$ *is placed in an electric field of another dipole of dipole moment* $\mathbf{p}_2$ *at a distance d from it. Calculate the force and torque that* $\mathbf{p}_1$ *experiences, when (a)* $\mathbf{p}_1 \parallel \mathbf{p}_2$ *and (b)* $\mathbf{p}_1 \perp \mathbf{p}_2$.

The electric field along x-axis at a distance d from the dipole of dipole moment p_2 placed at the origin O is given by

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{2p_2}{d^3} \text{ and } E_y = 0, \text{ or } \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{2p_2}{d^3}\,\mathbf{i} = \frac{2\mathbf{p}_2}{4\pi\varepsilon_0 d^3}.$$

Here we have assumed $\mathbf{p}_2$ along x-axis.

(*a*) The torque experienced by the dipole of dipole moment $\mathbf{p}_1$ placed ∥ to x-axis due to the electric field of dipole $\mathbf{p}_2$ placed at O.

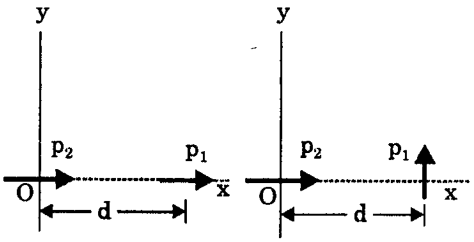


Fig. 6.26. Example 2.

$$\tau = \mathbf{p}_1 \times \mathbf{E}$$

or $\tau = p_1 E \sin 0 = 0.$

Force experienced by the dipole moment p_1 due to the electric field of the another dipole of dipole moment p_2 is given by

$$F_x = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y}\right) E_x \text{ and } F_y = \left(p_x \frac{\partial}{\partial x} + p_y \frac{\partial}{\partial y}\right) E_y$$

$$\text{or } F_x = p_1 \frac{1}{4\pi\varepsilon_0}\left(\frac{-6p_2}{d^4}\right) = \frac{-3p_1p_2}{2\pi\varepsilon_0 d^4} \text{ and } F_y = 0.$$

The –ve sign indicates that the force is attractive and along the line joining p_1 to p_2.

(*b*) If another dipole of dipole moment p_1 is placed in such a way that $\mathbf{p}_1 \perp \mathbf{p}_2$, the torque experienced by it is given as

$$\tau = \mathbf{p}_1 \times \mathbf{E}.$$

In this case $\mathbf{p}_1 = p_1 \mathbf{j}$ and $\mathbf{E} = E\mathbf{i}$. Therefore

$$\tau = p_1 E\mathbf{j} \times \mathbf{i} = -p_1 E\mathbf{k} = -(2p_1p_2/4\pi\varepsilon_0 d^3)\,\mathbf{k}.$$

Thus $\mathbf{p}_1$ experiences a clockwise torque of magnitude $2p_1 p_2/4\pi\varepsilon_0 d^3$.

Electric field at dipole of dipole moment p_2, due to dipole of dipole moment p_1 is given by

$$\mathbf{E} = -\frac{1}{4\pi\varepsilon_0}\frac{p_1}{d^3}\mathbf{j}, \qquad \text{(As } p_2 \text{ is in broad-on position)}$$

$\therefore$ Torque on $\mathbf{p}_2$, $\tau = \mathbf{p}_2 \times \mathbf{E} = p_2 E\mathbf{i} \times \mathbf{j} = -(p_1p_2/4\pi\varepsilon_0 d^3)\,\mathbf{k}$.

This torque is again in clockwise direction. If the system consisting of two dipoles is free, it will experience a net clockwise torque of magnitude $3p_1p_2/4\pi\varepsilon_0 d^3$.

Example 3. *An electric dipole of moment 2×10^{-8} coulomb-meter is placed in a uniform field of intensity 1.5×10^5 newton/coulomb.*

(*i*) *What maximum torque does the field exert on the dipole ?*

(*ii*) *How much work is done on turning dipole end to end ?*

We know that an electric dipole of moment p experiences a torque τ when placed in an electric field E. If the dipole makes an angle θ with the direction of E, then $\tau = p\,E \sin\theta$.

It is maximum for $\theta = 90°$. Therefore

$$\tau_{max} = p\,E = 2 \times 10^{-8} \times 1.5 \times 10^5 = 3 \times 10^{-3} \text{ newton meter.}$$

The total work done in rotating the dipole from θ_1 to θ_2 is

$$W = \int_{\theta_1}^{\theta_2} p\,E \sin\theta\, d\theta = -pE\,[\cos\theta_2 - \cos\theta_1]$$

In the present problem the dipole is rotated through 180°.

Thus by putting $\theta_1 = 0°$ and $\theta_2 = 180°$ in the above relation, we get

$$W = -p\,E\,[\cos 180° - \cos 0°] = 2p\,E$$
$$= 2 \times 2 \times 10^{-8} \times 1.5 \times 10^5 = 6 \times 10^{-3} \text{ joule.}$$

Example 4. *A right circular cone of semivertical angle α and height h has a uniform polarization P parallel to its axis of symmetry. Find the polarization charges every where.*

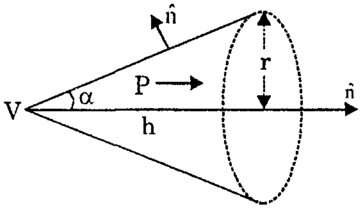


Fig. 6.27. Example 4.

Let us assume that the polarization vector **P** is directed from the vertex V to the base of the cone.

$$\sigma_P = \mathbf{P}\cdot\hat{n} = P$$

$\therefore$ Charge on the base $q_B = \sigma_P \pi r^2 = P\pi h^2 \tan^2\alpha$.

On the curved surface $\sigma_P = \mathbf{P}\cdot\hat{n}$

$$= P\cos\left(\frac{\pi}{2} + \alpha\right) = -P\sin\alpha.$$

$\therefore$ Charge on the curved surface $q_C = -P \sin\alpha \cdot \pi r l$

$= -P \sin\alpha \cdot \pi \cdot (h \tan\alpha)(h \sec\alpha) = -P\pi h^2 \tan^2\alpha.$

Thus the total charge on the cone $= P\pi h^2 \tan^2\alpha - P\pi h^2 \tan^2\alpha = 0.$

Example 5. *If a dielectric is introduced between the plates of a parallel plate capacitor. Show that the induced charge varies with the dielectric as $k = (1 - \sigma'/\sigma)^{-1}$. Also show that $k \to \infty$ for a metal dielectric.*

As discussed in article 6.6, the induced charge in the dielectric is given as

$$q' = q(1 - 1/k)$$

$$\therefore \quad \sigma' = \sigma(1 - 1/k) \text{ or } k = 1/(1 - \sigma'/\sigma).$$

In the metal dielectric, induced charge is same as that of the free charge but of opposite sign, *i.e.*, $\sigma' = \sigma$, hence

$$k = 1/0 = \infty.$$

Example 6. *Two parallel plates have equal and opposite charges and are separated by a dielectric 5 mm thick, of dielectric constant 3. If electric intensity in the dielectric is 10^6 V/m, calculate : (a) The displacement D in the dielectric, (b) The polarization P in the dielectric, (c) The bound charge per unit area on the surfaces of the dielectric, (d) The free charge per unit area on the conducting plate, and (e) The energy density in the dielectric.*

$$\text{Displacement vector } \mathbf{D} = k\varepsilon_0 E = 3 \times 8.854 \times 10^{-12} \times 10^6$$
$$= 26.55\ \mu\ C/m^2.$$

$$\text{Polarization vector } \mathbf{P} = \varepsilon_0(k-1)E$$
$$= 8.854 \times 10^{-12} \times (3-1)\, 10^6 = 17.71\ \mu\ C/m^2,$$

As polarization vector **P** is along **E** and thus ⊥ to the surface of the plate. Hence the bound charge per unit area on the surfaces of the dielectric

$$|\sigma'| = P_n = P = 17.71\ \mu\ C/m^2.$$

Similarly free charge per unit area $|\sigma| = D = 26.56\ \mu\ C/m^2$.

$$\text{Energy density in the dielectric} = \tfrac{1}{2} k\varepsilon_0 E^2 = \tfrac{1}{2} \times 3 \times 8.854 \times 10^{-12} \times 10^{12}$$
$$= 13.28 \text{ joules}/m^3.$$

Example 7. *A dielectric slab of thickness b and dielectric constant k is placed between the plates of a parallel plate capacitor of plate area A and separation d. A potential difference V_0 is applied with no dielectric present. The battery is then disconnected and the dielectric slab is inserted. Assume that $A = 100\ cm^2$, $d = 1.0$ cm, $b = 0.50$ cm, $k = 7.0$ and $V_0 = 100$ volts, then calculate : (a) The capacitance C_0 before the slab inserted, (b) The capacitance C with slab inserted, (c) The free charge q, (d) The electric field in the gap, (e) The electric field in the dielectric, (f) The potential difference between the plates, (g) The polarization P and electric displacement D in the dielectric and (h) The polarization P and electric displacement D in the air gap.*

(*a*) Capacitance of the parallel plate capacitor before the slab is inserted

$$C_0 = \frac{\varepsilon_0 A}{d} = \frac{8.854 \times 10^{-12} \times 100 \times 10^{-4}}{1 \times 10^{-2}}$$
$$= 8.854 \times 10^{-12} \text{ farad} = 8.854\ pF.$$

(*b*) Capacitance with *a* slab of thickness *b* and dielectric constant *k*

$$C_d = \frac{\varepsilon_0 A}{(d - b + b/k)} = \frac{8.854 \times 10^{-12} \times 100 \times 10^{-4}}{(1 \times 10^{-2} - 0.5 \times 10^{-2} + 0.5 \times 10^{-2}/7)}$$
$$= 15.49 \times 10^{-12} \text{ farad} = 15.49\ pF.$$

(*c*) The free charge

$$q = C_0 V_0 = 8.854 \times 10^{-12} \times 100 = 8.854 \times 10^{-10} \text{ coul.}$$

(*d*) The electric field in the gap

$$E_0 = \frac{q}{\varepsilon_0 A} = \frac{8.854 \times 10^{-10}}{8.854 \times 10^{-12} \times 100 \times 10^{-4}} = 10^4 \text{ volts/meter.}$$

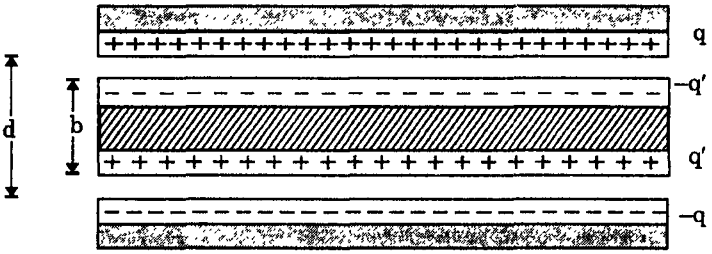


Fig. 6.28. Example 7.

(*e*) The electric field in the dielectric

$$E = \frac{E_0}{k} = \frac{10^4}{7} = 1428.6 \text{ volts/meter.}$$

(*f*) The potential difference between the plates.

$$V = \frac{q}{C_d} = \frac{8.854\times10^{-10}}{15.49\times10^{-12}} = 57.1 \text{ volts.}$$

(*g*) The electric displacement in the dielectric

$$D = k\varepsilon_0 E = 7 \times 8.854 \times 10^{-12} \times 1428.6$$
$$= 8.854 \times 10^{-8} \text{ coul/meter}^2.$$

and polarization

$$P = \varepsilon_0(k-1)E = 8.854 \times 10^{-12}\,(7-1) \times 1428.6$$
$$= 7.589 \times 10^{-8} \text{ coul/meter}^2.$$

(*h*) The polarization is zero in the air gap and the displacement

$$D_0 = k\varepsilon_0 E_0 = 8.854 \times 10^{-12} \times 10^4$$
$$= 8.854 \times 10^{-8} \text{ coul/meter}^2.$$

Example 8. *If a dielectric slab is partially withdrawn from between two charged plates of the parallel plate capacitor, calculate the force required to pull it.*

In order to calculate the force F, let us first see the effect on the stored energy of an elementary displacement dx of the slab. Since this placement modifies the capacitance of the capacitor, there will be a change dq in the charge on the capacitor. If it is held at constant potential a work $dW = V\,dq$ is done by the battery. Thus the total change in stored energy in the capacitor

$$dU = - Fdx + Vdq.$$

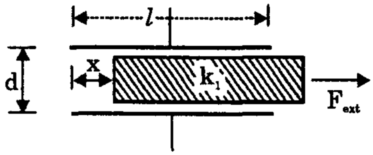


Fig. 6.29. Example 8.

We know that the total stored energy per unit volume in the capacitor is equal to $\frac{1}{2}\,\varepsilon E^2$ or $\frac{1}{2}\,\varepsilon_0 E^2$, if the capacitor is filled with the dielectric or the medium is vacuum respectively. If b is the width of the dielectric, l the length and d the plate separation, then

$$U = \tfrac{1}{2}\,\varepsilon_0 E^2 bd\,x + \tfrac{1}{2}\,\varepsilon E^2 bd\,(l-x) = \tfrac{1}{2}\,E^2\,bd\,[\varepsilon l - (\varepsilon - \varepsilon_0)x],$$

where x is the length of the slab outside the capacitor.

By differentiating this equation, we get the change in U produced by a small displacement dx of the slab.

$$\therefore \qquad dU = -\tfrac{1}{2}\,E^2 bd\,(\varepsilon - \varepsilon_0)dx.$$

We know that σ_1 $(= \varepsilon_0 E)$ and σ_2 $(= \varepsilon E)$ are the charge densities in the portion with no dielectric and with dielectric respectively. Hence the total charge on the capacitor plates

$$q = \sigma_1 A_1 + \sigma_2 A_2 = \varepsilon_0\,Ebx + \varepsilon Eb\,(l-x).$$

$\therefore$ Change in q with change in x is

$$dq = \varepsilon_0 Eb\,dx - \varepsilon\,Eb\,dx = -\,(\varepsilon - \varepsilon_0)\,Eb\,dx$$

Since

$$V = Ed, \text{ hence}$$

$$Fdx = -\,dU + Vdq = \tfrac{1}{2}\,E^2 bd\,(\varepsilon - \varepsilon_0)\,dx - Ed\,(\varepsilon - \varepsilon_0)\,Ebdx.$$

or

$$F = -\tfrac{1}{2}\,E^2 bd\,(\varepsilon - \varepsilon_0) = V^2 b\varepsilon_0(k-1)/2d,$$

where k is the dielectric constant of the slab material.

Here –ve sign indicates that the force is in the direction of decreasing x. Hence force is required to pull the slab and is independent of l and x.

Above relations also show that dU is half of the Vdq, hence half the energy contributed to the system by the battery is used in doing external work.

Example 9. *A capacitor is made of two plane parallel plates of width a and length l separated by a distance d (d << a, l). A dielectric slab of relative dielectric constant K is placed between the plates. (a) The capacitor is connected to a battery of emf V. The dielectric slab is partially pulled out of the plates such that a length x remains inside. Calculate the force on the dielectric slab which tends to pull it back into the plates. (b) With the dielectric slab fully inside the plates, the capacitor plates are again charged to potential* V_0 *and the battery is disconnected. Again the slab is pulled out of the plates such that a length x remains inside the plates. Calculate the force on the slab which tends to pull it back into the plates.*

Part (*a*) is same as example 8. It can also be solved by assuming two capacitors in parallel. The total capacitance

$$C = \frac{\varepsilon_0 (l-x)a}{d} + \frac{\varepsilon_0\, kxa}{d} = \frac{\varepsilon_0 a}{d}[(k-1)x + l] \qquad \ldots(i)$$

The conservation of energy principle gives

$$VdQ = d\left(\tfrac{1}{2}V^2C\right) + Fdx$$

As V = const, $Q = CV$ gives $VdQ = V^2dC$

$$\therefore \qquad F = \frac{1}{2}V^2\frac{dC}{dx} = \frac{\varepsilon_0(k-1)aV^2}{2d} \qquad \ldots(ii)$$

Since $k > 1$, therefore $F > 0$ *or the force will pull the slab back into the plates.*

(*b*) In the second case, the voltage is initially V_0 and then the battery is disconnected. Thus the plates become isolated or $dQ = 0$. As initially $C_0 = k\varepsilon_0 la/d$ and $Q = C_0V_0$. The energy principle gives

$$F = -\frac{d}{dx}\left(\frac{1}{2}V^2C\right) = -VC\frac{dV}{dx} - \frac{V^2}{2}\frac{dC}{dx}$$

$$\text{As} \qquad \frac{dV}{dx} = \frac{d}{dx}\left(\frac{Q}{C}\right) = -\frac{Q}{C^2}\frac{dC}{dx}$$

$$\therefore \qquad F = \frac{QV}{C}\frac{dC}{dx} - \frac{V^2}{2}\frac{dC}{dx} = \frac{Q^2}{2C^2}\frac{dC}{dx}. \qquad \ldots(iii)$$

From relation (*i*), we get $\dfrac{dC}{dx} = (k-1)\varepsilon_0 a/d$.

Substituting the values of $Q = C_0V_0 = k\varepsilon_0 laV_0/d$, C and dC/dx, we get

$$F = \frac{\varepsilon_0 k^2 (k-1) a l^2 V_0^2}{[(k-1)\,x + l]^2\, 2d}.$$

As F is again greater than 0, the force F will tend to pull back the slab into the plates.

Example 10. *A volume between the two concentric conducting spherical surfaces of radii a and b (a < b) is filled within inhomogeneous dielectric constant* $\varepsilon = \varepsilon_0/(1 + kr)$, *where* ε_0 *and k are constants and r is the radial distance. A charge Q is placed on the inner surface, while the outer surface is grounded. Find (a) the displacement in the region a < r < b, (b) the capacitance of the device, (c) the polarization charge density in a < r < b and (iv) the polarization surface charge density at r = a and r = b.*

The Gauss's law for the spherical symmetric system gives

$$\mathbf{D} = (Q/4\pi r^2)\,\hat{r} \qquad (a < r < b)$$

(*b*) The electric field intensity **E** is thus given by

$$\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}(1+kr)\,\hat{r} \qquad (a < r < b)$$

Hence the potential difference between the inner and outer spheres

$$V = \int_a^b \mathbf{E}\cdot d\mathbf{r} = \frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{1+kr}{r^2}\hat{r}.d\mathbf{r}$$

$$= \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}+k\log_e\frac{b}{a}\right)$$

The capacitance of the system

$$C = \frac{Q}{V} = \frac{4\pi\varepsilon_0\, ab}{(b-a)+abk\log_e(b/a)}.$$

(*c*) The polarization $\mathbf{P} = (\varepsilon-\varepsilon_0)\,\mathbf{E} = \left(\frac{\varepsilon_0}{1+kr}-\varepsilon_0\right)\frac{Q\,(1+kr)}{4\pi\varepsilon_0 r^2}\,\hat{r}$

$$= -\,(Qk/4\pi r)\,\hat{r}\,.$$

Therefore the volume polarization charge density at $a < r < b$ is given by

$$\rho_P = -\nabla\cdot\mathbf{P} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(\frac{Qkr}{4\pi}\right) = \frac{Qk}{4\pi r^2}.$$

(*d*) The surface polarization charge densities at $r = a$ and $r = b$ are

$$\sigma_P = \mathbf{P}\cdot\hat{n} \quad = Q\;k/4\pi a \qquad \text{at } r = a$$
$$= Qk\;/4\pi b \qquad \text{at } r = b.$$

Example 11. *Given a long cylindrical coaxial capacitor with an inner and outer conductors of radii a and b and a dielectric with a dielectric constant k(r), which varies with the cylindrical radius r. The capacitor is charged to voltage V. Calculate the radial dependence of k(r) such that the energy density in the capacitor is constant. Also calculate E(r).*

Let λ be the charge per unit length carried by the inner conductor. Gauss's law gives

$$\mathbf{D}(r) = (\lambda/2\pi r)\,\hat{r}$$

The energy density at r is

$$U(r) = \frac{1}{2}\mathbf{E}\cdot\mathbf{D} = \frac{D^2}{2\varepsilon_0 k(r)} = \frac{\lambda^2}{8\pi^2\varepsilon_0 r^2 k(r)}$$

This is given to be constant or independent of r, therefore $r^2k(r) = \text{const}\ (K)$ or $k(r) \propto 1/r^2$.

The voltage across the two conductors is

$$V = -\int_a^b \mathbf{E}\cdot d\mathbf{r} = -\frac{\lambda}{2\pi\varepsilon_0 K}\int_a^b r\,dr = \frac{\lambda}{4\pi\varepsilon_0 K}\left(b^2-a^2\right)$$

Therefore $\quad \lambda = -\,4\pi\varepsilon_0\, KV/(b^2-a^2)$

This gives $\quad E(r) = 2rV/(b^2-a^2)$

Example 12. *Calculate the voltage drop across each dielectric, in Fig. 6.30, where $k_1 = 2$ and $k_2 = 5$. The inner conductor is at $r_1 = 2$ cm and the outer at $r_2 = 2.5$ cm, with the dielectric interface half way between.*

As clear from the figure, that the segment with angle α will have a capacitance $\alpha/2\pi$ times that of the complete coaxial capacitor, *i.e.*,

$$C_1 = \frac{\alpha}{2\pi}\frac{2\pi\varepsilon_0 k_1 l}{\log_e(2.25/2.0)} = 1.5\times10^{-10}\,\alpha l$$

$$\text{and} \quad C_2 = \frac{\alpha}{2\pi}\frac{2\pi\varepsilon_0 k_2 l}{\log_e(2.5/2.25)} = 4.2 \times 10^{-10}\,\alpha l.$$

Since $Q = C_1V_1 = C_2V_2$ and $V_1 + V_2 = V$, therefore

$$V_1 = \frac{C_2}{C_1+C_2}V = \frac{4.2}{1.5+4.2}\times 100 = 73.7\ V$$

$$V_2 = \frac{C_1}{C_1+C_2}V = \frac{1.5}{1.5+4.2}\times 100 = 26.3\ V.$$

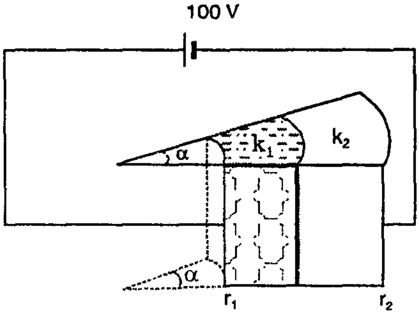


Fig. 6.30. Example 12.

Example 13. *Find the electric field and polarization due to a charge in a dielectric fluid.*

Let us consider a point charge q in the homogeneous isotropic medium of infinite extent and of dielectric constant k. If the point charge q was situated in a vacuum, the electric field would be a pure radial field. Since **E, D** and **P** are all parallel to one another at each point, hence the radial nature of the field is not changed by the presence of the medium.

Let us apply Gauss's law to a gaussian spherical surface of radius r, at the centre of which charge q is placed. Thus

$$k\varepsilon_0 \int E \cdot d\mathbf{S} = q.$$

$$\therefore \quad k\varepsilon_0 \int E dS = k\varepsilon_0 4\pi r^2 E = q \text{ or } E = q/4\pi\varepsilon_0 k r^2$$

or Vectorially $\mathbf{E} = (q/4\pi\varepsilon_0 k r^3)r$,

and hence $\mathbf{P} = \varepsilon_0(k-1)\,\mathbf{E} = \dfrac{(k-1)q}{4\pi k r^3}r.$

Thus the field is smaller by the factor k than would be the case if the medium was absent.

Example 14. *The surface separating two dielectrics k_1 and k_2 has a surface charge density σ. The electric intensities on the two sides of the boundary are E_1 and E_2 making angles θ_1 and θ_2 with the common normal. Prove that*

$$k_2 \cot\theta_2 = k_1 \cot\theta_1\,(1 - \sigma/\varepsilon_0 k_1 E_1 \cos\theta_1).$$

We know that the tangential components of electric intensity remains the same while crossing the common surface of two media. Hence

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \text{ or } E_2 = E_1 \sin\theta_1/\sin\theta_2. \qquad ...(i)$$

We also know that the sum of normal components of the displacement vector is equal to σ, hence

$$D_1\cos\theta_1 - D_2\cos\theta_2 = \sigma \text{ or } k_1\varepsilon_0 E_1\cos\theta_1 - k_2\varepsilon_0 E_2 \cos\theta_2 = \sigma.$$

$$\therefore \quad k_2E_2\cos\theta_2 = k_1E_1\cos\theta_1 - \sigma/\varepsilon_0. \qquad ...(ii)$$

Substituting value of E_2 in this Eqn and solving, we get

$$k_2\cot\theta_2 = k_1\cot\theta_1\,(1 - \sigma/\varepsilon_0 k_1 E_1\cos\theta_1).$$

It is the required relation.

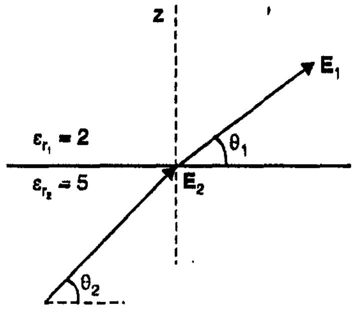


Fig. 6.31. Example 15.

Example 15. *If $\mathbf{E}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z$ V/m is the electric field intensity at the charge free electric interface of Fig. 6.31. Find $\mathbf{D}_2$ and the angles θ_1 and θ_2.*

The interface is a z = constant plane. Thus x and y-components are tangential and z-components are normal. The continuity condition requires $E_{1t} = E_{2t}$ and $D_{1n} = D_{2n}$.

$$\text{Since} \quad \mathbf{E}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z$$

$$\text{Therefore} \quad \mathbf{E}_2 = 2\mathbf{a}_x - 3\mathbf{a}_y + E_{z2}\mathbf{a}_z$$

$$\text{Since} \quad \mathbf{D}_1 = \varepsilon_0\varepsilon_{r1}\mathbf{E}_1 = 2\varepsilon_0(2\mathbf{a}_x - 3\mathbf{a}_y + 5\mathbf{a}_z)$$

$$= 4\varepsilon_0\mathbf{a}_x - 6\varepsilon_0\mathbf{a}_y + 10\varepsilon_0\mathbf{a}_z$$

$$\text{Therefore} \quad \mathbf{D}_2 = D_{x2}\mathbf{a}_x + D_{y2}\mathbf{a}_y + 10\varepsilon_0\mathbf{a}_z.$$

The unknown components may be found using the relation

$$\mathbf{D}_2 = \varepsilon_0 \varepsilon_{r2} \mathbf{E}_2$$

or $D_{x2}\mathbf{a}_x + D_{y2}\mathbf{a}_y + 10\,\varepsilon_0 \mathbf{a}_z = \varepsilon_0 5\,(2\mathbf{a}_x - 3\mathbf{a}_y + E_{z2}\,\mathbf{a}_z)$

$\therefore \quad D_{x2} = 10\varepsilon_0,\ D_{y2} = -15\varepsilon_0$ and $E_{z2} = 10/5 = 2.$

Hence $\quad \mathbf{D}_2 = 10\varepsilon_0\mathbf{a}_x - 15\varepsilon_0\mathbf{a}_y + 10\varepsilon_0\mathbf{a}_z.$

The angle made with plane of interface may be obtained as

$\mathbf{E}_1 \cdot \mathbf{a}_z = |\,\mathbf{E}_1\,| \cos(90° - \theta_1)$ or $5 = \sqrt{2^2 + 3^2 + 5^2}\,\sin\theta_1$

$\therefore \sin\theta_1 = 5/\sqrt{38}$ or $\theta_1 = 54.2°.$

$\mathbf{E}_2 \cdot \mathbf{a}_z = |\,\mathbf{E}_2\,| \cos(90° - \theta_2)$ or $2 = \sqrt{2^2 + 3^2 + 2^2}\,\sin\theta_2$

$\therefore \sin\theta_2 = 2/\sqrt{17}$ or $\theta_2 = 29°.$

Example 16. *The dielectric free space interface has the equation* $3x + 2y + z = 12$ *m,* $\mathbf{E}_1 = 2\mathbf{a}_x + 5\,\mathbf{a}_z$ *V/m and* $k_1 = \varepsilon_{r_1} = 3$*, find* $\mathbf{E}_2$.

The unit normal vector on the free space side is

$$\mathbf{a}_n = (3\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z)/\sqrt{3^2 + 2^2 + 1^2}$$

The component of $\mathbf{E}_1$ on $\mathbf{a}_n$ or the normal component of $\mathbf{E}$ at the interface

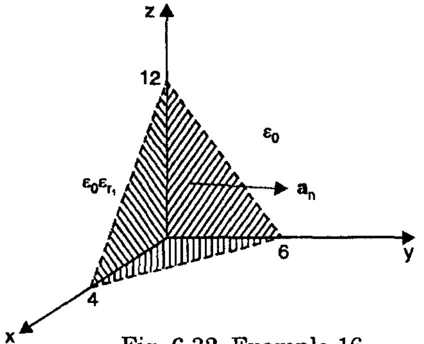


Fig. 6.32. Example 16.

$$\mathbf{E}_1 \cdot \mathbf{a}_n = (6 + 5)/\sqrt{14} = 11/\sqrt{14}.$$

or $\quad \mathbf{E}_{n_1} = (11/\sqrt{14})\ \mathbf{a}_n = 2.36\,\mathbf{a}_x + 1.57\,\mathbf{a}_y + 0.79\,\mathbf{a}_z.$

$$\mathbf{E}_{t_1} = \mathbf{E}_1 - \mathbf{E}_{n_1}$$

$$= -0.36\mathbf{a}_x - 1.57\mathbf{a}_y + 4.21\,\mathbf{a}_z = \mathbf{E}_{t_2}$$

$$\mathbf{D}_{n_1} = \varepsilon_0 \varepsilon_{r_1} \mathbf{E}_{n_1}$$

$$= 7.08\varepsilon_0\mathbf{a}_x + 4.71\varepsilon_0\mathbf{a}_y + 2.37\varepsilon_0\mathbf{a}_z = \mathbf{D}_{n_2}$$

$\therefore \quad \mathbf{E}_{n_2} = (1/\varepsilon_0)\,D_{n2} = 7.08\mathbf{a}_x + 4.71\mathbf{a}_y + 2.37\,\mathbf{a}_z$

Therefore

$$\mathbf{E}_2 = \mathbf{E}_{t_2} + \mathbf{E}_{n_2} = 6.72\mathbf{a}_x + 3.14\mathbf{a}_y + 6.58\,\mathbf{a}_z \text{ V/m}.$$

Example 17. *If the negative charge in hydrogen atom is uniformly distributed in the spherical volume of radius a* $(= 0.5 \times 10^{-8}$ *cm) around the proton. Calculate the separation between its +ve and –ve charges and polarizability* α*, when the atom is placed in an electric field of 30 kV/cm. Assuming that the sphere of negative charge remains undistorted.*

When the electric field is applied to the atom, the +ve charge (proton) will displace from its original position (the centre of atom). The centre of the sphere of –ve charge will be the centre of the atom. As the negative charge is uniformly distributed in the sphere of radius a, hence the electric field at a point P, at a distance r from the centre $(r < a)$ due to this charge distribution is given by

$$E = \frac{q'}{4\pi\varepsilon_0 r^2} = \frac{1}{4\pi\varepsilon_0 r^2}\left(e\,\frac{r^3}{a^3}\right) = \frac{er}{4\pi\varepsilon_0 a^3}.$$

If due to electric field the +ve charge is displaced to this point P, the field acting over this charge due to distribution of –ve charge will be same as that applied by an external source.

$\therefore$ Separation $\quad r = 4\pi\varepsilon_0 E a^3/e$

$$= \frac{4 \times 3.14 \times 8.854 \times 10^{-12} \times 30 \times 10^5 \times (0.5 \times 10^{-10})^3}{1.6023 \times 10^{-19}}$$

$$= 2.61 \times 10^{-16} \text{ metre}.$$

This result shows that a large electric field (30×10^5 V/m) can produce a very small separation of the order of 10^{-16} m, which is very small even in comparison of nuclear radius 10^{-15} m.

The dipole moment of this arrangement.

$$p = er = 1.6023 \times 10^{-19} \times 2.61 \times 10^{-16} = 4.18 \times 10^{-35} \text{ coul. meter.}$$

The polarizability $\quad \alpha = \dfrac{p}{E} = \dfrac{4.18 \times 10^{-35}}{30 \times 10^5} = 1.39 \times 10^{-41}$ coul. m^2/V.

Example 18. *Prove by the method of boundary conditions that the field inside a spherical cavity with in an isotropic dielectric is given as*

$$E_i = E + P/3\varepsilon_0.$$

Consider a spherical cavity of radius a existed in an isotropic dielectric of infinite extent. Let the field intensity $\mathbf{E}_i$ be constant everywhere and parallel to the original field **E**. Electric intensity at the points in the dielectric be assumed to be E plus that due to a dipole of moment **p** at the centre of the cavity. At any point on the cavity, the tangential component of the electric field intensity is continuous. Therefore

$$E_i \sin\theta = E \sin\theta - (p/4\pi\varepsilon_0 a^3) \sin\theta.$$

Another boundary condition that the normal component of electric displacement is continuous, gives

$$\varepsilon_0 E_i \cos\theta = \varepsilon_0 kE \cos\theta + \varepsilon_0 (2p/4\pi\varepsilon_0 a^3) \cos\theta.$$

Eliminating p/a^3 from these equations, we get

$$E_i = \frac{1}{3}(k+2)\,E.$$

At points far away from the cavity the polarization P is constant and is given by

$$D = k\varepsilon_0 E = \varepsilon_0 E + P.$$

$$\therefore \quad E_i = \frac{2}{3}E + \frac{1}{3}(E + P/\varepsilon_0) = E + P/3\varepsilon_0.$$

It is the required relation.

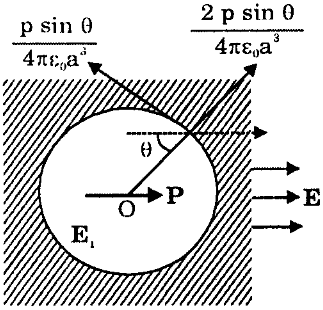

$\dfrac{p \sin\theta}{4\pi\varepsilon_0 a^3}$
$\dfrac{2p \sin\theta}{4\pi\varepsilon_0 a^3}$
θ
O
P
E
E_i

Fig. 6.33. Example 18.

Example 19. *Calculate the electric intensity at any point on the axis of a uniformly polarized bar electret.*

Let us consider a long cylindrical bar of dielectric that is uniformly and permanently polarized with the constant vector **P** parallel to the axis. The surface densities are $\sigma = P$ and $-\sigma = -P$ at the ends of the bar. The value of **E** on the axis of a uniformly charged disc of radius R at a distance x from its centre is given by

$$E = (\sigma/2\varepsilon_0)\,[1 - x/(x^2 + R^2)^{1/2}].$$

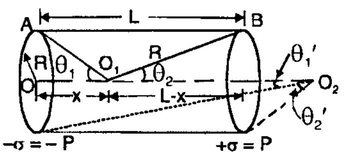


Fig. 6.34. Example 19.

Thus the field E due to charges on the discs A and B at a point O_1 inside the bar at a distance x from the –vely charged disc A is given by

$$E = \frac{-P}{2\varepsilon_0}\left(1 - \frac{x}{(x^2+R^2)^{1/2}}\right) - \frac{P}{2\varepsilon_0}\left(1 - \frac{L-x}{[(L-x)^2+R^2]^{1/2}}\right)$$

As $\quad \cos\theta_1 = x/(x^2 + R^2)^{1/2}$ and $\cos\theta_2 = (L-x)/[(L-x)^2 + R^2]^{1/2}$.

$$\therefore \quad E = -P/\varepsilon_0 + (P/2\varepsilon_0)(\cos\theta_1 + \cos\theta_2).$$

When point O_1 is outside the bar, *i.e.*, at O_2 at a distance x' from O, we get

$$E = \frac{-P}{2\varepsilon_0}\left(1 - \frac{x'}{(x'^2+R^2)^{1/2}}\right) + \frac{P}{2\varepsilon_0}\left(1 - \frac{x'-L}{[(x'-L)^2+R^2]^{1/2}}\right)$$

$$= (P/2\varepsilon_0)(\cos\theta_1 + \cos\theta_2).$$

Field at point O, the centre of disc A, is

$$E = -\frac{P}{2\varepsilon_0} - \frac{P}{2\varepsilon_0}\left(1 - \frac{L}{(L^2 + R^2)^{1/2}}\right) = -\frac{P}{2\varepsilon_0} - \frac{P}{2\varepsilon_0}\left[1 - \left(1 - \frac{R^2}{L^2}\right)^{-1/2}\right]$$

$$= -(P/2\varepsilon_0)(1 + R^2/2L^2).$$

Example 20. *Find expressions for electric potential and field intensity in all the regions due to a uniformly polarized spherical shell of inner and outer radii a and b respectively.*

We know that the density of polarization **P** is equal to the dipole moment per unit volume. Hence the dipole moment of a uniformly polarized sphere of radius a is $\frac{4}{3}\pi a^3\mathbf{P}$. Electric potential and field outside the polarized sphere are identical with those due to the short dipole of moment $\frac{4}{3}\pi a^3\mathbf{P}$ assumed to be placed at the centre of the sphere.

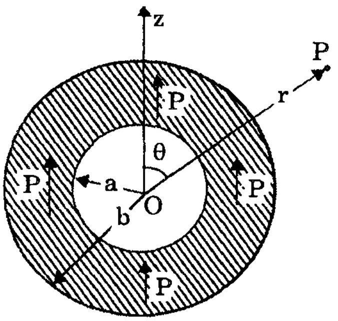


Fig. 6.35. Example 20.

$\therefore$ V_0 at a point $P(r, \theta) = \frac{4}{3}\pi a^3 P\cos\theta/4\pi\varepsilon_0 r^2$.

$= P\ a^3\cos\theta/3\varepsilon_0 r^2$.

Potential at the inner point $= \frac{4}{3}\pi Pr\cos\theta/4\pi\varepsilon_0$

$= Pr\ \cos\theta/3\varepsilon_0$.

In our problem **P** is directed along z-axis. The electric potential and field will be computed by subtracting the contribution due to inner sphere from the contribution due to whole sphere of radius b. Hence

(*i*) At the outer point P ($r > b$)

$$V = V_b - V_a = \frac{Pb^3\cos\theta}{3\varepsilon_0 r^2} - \frac{Pa^3\cos\theta}{3\varepsilon_0 r^2} = \frac{P\cos\theta}{3\varepsilon_0 r^2}(b^3 - a^3).$$

Hence $$E_r = -\frac{\partial V}{\partial r} = \frac{2P\cos\theta}{3\varepsilon_0 r^3}(b^3 - a^3)$$

and $$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{P\sin\theta}{3\varepsilon_0 r^3}(b^3 - a^3)$$

(*ii*) At the point inside the dielectric ($a < r < b$)

$$V = V_b - V_a = \frac{Pr\cos\theta}{3\varepsilon_0} - \frac{Pa^3\cos\theta}{3\varepsilon_0 r^2} = \frac{P\cos\theta}{3\varepsilon_0}\left[r - \frac{a^3}{r^2}\right].$$

Hence $$E_r = -\frac{\partial V}{\partial r} = \frac{P\cos\theta}{3\varepsilon_0}\left[1 + \frac{2a^3}{r^3}\right].$$

and $$E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = \frac{P\sin\theta}{3\varepsilon_0}\left[1 - \frac{a^3}{r^3}\right].$$

(*iii*) At the point inside the inner sphere ($0 < r < a$)

$$V = V_b - V_a = \frac{Pr\cos\theta}{3\varepsilon_0} - \frac{Pr\cos\theta}{3\varepsilon_0} = 0.$$

Hence $E_r = 0$ and $E_\theta = 0$.

Example 21. *If all the molecular dipoles in 0.1 cm radius drop of water are pointed in the same direction, calculate (a) the intensity of polarization and (b) maximum field strength at the point 20 cm away from the drop and at the surface. Given that the dipole moment for each molecule of water is 6×10^{-30} coulomb-meter.*

From the hypothesis of Avogadro's 18×10^{-3} kg of water must consists 6.023×10^{23} number of water molecules. Hence the number of molecules in a drop of radius 0.1 cm and thus of weight $\frac{4}{3}\pi\ (0.001)^3 \times 10^3$ is $6.023 \times 10^{23} \times \frac{4}{3}\pi\ (0.001)^3 \times 10^3/18 \times 10^{-3} = 1.4 \times 10^{20}$.

As all the dipoles are pointed in the same direction, hence the dipole moment of the drop

$$p = 1.4 \times 10^{20} \times 6 \times 10^{-30} = 8.4 \times 10^{-10} \text{ coul meter.}$$

(*a*) Intensity of polarization = dipole moment per unit volume

$$= 8.4 \times 10^{-10}/\tfrac{4}{3}\pi\,(0.001)^3 = 0.2 \text{ coul/m}^2.$$

(*b*) Maximum field will be at the point on the extended diameter along the direction of polarization, hence

$$E = \frac{2p}{4\pi\varepsilon_0 r^3} = \frac{2 \times 8.4 \times 10^{-10} \times 9 \times 10^9}{[20 \times 10^{-2}]^3} = 1.89 \times 10^3\ N/C$$

At the surface $E = \dfrac{2p}{4\pi\varepsilon_0 a^3} = \dfrac{2 \times 8.4 \times 10^{-10} \times 9 \times 10^9}{[10^{-3}]^3} = 1.51 \times 10^{10}\ N/C.$

Example 22. *Calculate the average radius of the atom of an air molecule if the polarizability of atoms in the air molecules is 9.7 × 10^{-41} coul. m^2/volt. Also compare this result with the relative displacement of the nitrogen nucleus and electron cloud at a field strength $E = 3 \times 10^6$ volt/m.*

We know an important relation between polarizability α and the radius R,

$$\alpha = 4\pi\varepsilon_0 R^3 \text{ or } R = [\alpha/4\pi\varepsilon_0]^{1/3}$$

$$\therefore \quad R = (9.7 \times 10^{-41} \times 9 \times 10^9)^{1/3} = 9.6 \times 10^{-11} \text{ meter.}$$

If x is the relative displacement of the nitrogen nucleus and electron cloud, then the force due to electric field is balanced by the electrostatic force and we have

$$Zex = p = \alpha E.$$

$$\therefore \quad x = \frac{\alpha E}{Ze} = \frac{9.7 \times 10^{-41} \times 3 \times 10^6}{7 \times 1.6 \times 10^{-19}} = 2.6 \times 10^{-16} \text{ meter.}$$

It shows that $x << R$ for ordinary field strengths.

Example 23. *The dipole moment of the water molecule is 6.2 × 10^{-30} Cm. (a) Calculate the polarizability α of the water molecule and (b) the dielectric constant for saturated water vapour at 20°C. Given the vapour density for saturated water vapour at 20°C as 17.3 gm/m^3.*

Since water molecule is a polar molecule, having permanent dipole moment. Hence the polarizability of the water molecule due to permanent dipole moment p_m at temperature T is given by

$$\alpha = \frac{p_m^2}{3kT} = \frac{(6.2 \times 10^{-30})^2}{3 \times 1.38 \times 10^{-23} \times 293} = 3.17 \times 10^{-39}\ Cm^2/V.$$

The number density of saturated water vapour at 20°C

$$N = \frac{17.3 \text{ gm/m}^3}{18.0 \text{ gm/mol}} \times 6.023 \times 10^{23} \text{ molecules/mol} = 5.79 \times 10^{23} \text{ molecules/m}^3.$$

Using the Clausius Mossotti relationship, we get

$$\frac{k-1}{k+2} = \frac{N\alpha}{3\varepsilon_0} = \frac{5.79 \times 10^{23} \times 3.17 \times 10^{-39}}{3 \times 8.85 \times 10^{-12}} = 6.91 \times 10^{-5}$$

$$\therefore \quad k = 1.000207.$$

Example 24. *The polarizability of NH_3 molecule in gaseous state, from the measurement of dielectric constant, is found 2.5 × 10^{-39} coul^2 m/N and 2.0 × 10^{-39} coul^2 m/N at temperatures 300°K and 400°K respectively. Calculate the contribution to the polarizability because of deformation of molecules and the contribution because of permanent dipole moment at each temperature.*

From Langevin–Debye's theory the total polarizability

$$\alpha = \alpha_1 \text{ (due to deformation)} + \alpha_2 \text{ (due to permanent dipole)}$$

$$\alpha = \alpha_1 + p^2/3\,kT = \alpha_1 + \beta/T.$$

$\therefore$ $2.5 \times 10^{-39} = \alpha_1 + \beta/300$ and $2.0 \times 10^{-39} = \alpha_1 + \beta/400.$

Hence $\beta = 6 \times 10^{-37}$ and $\alpha_1 = 0.5 \times 10^{-39}\ C^2m/N.$

Polarizability due to permanent dipole at temperature 300°K

$$\alpha_2 = \frac{\beta}{T} = \frac{6 \times 10^{-37}}{300} = 2 \times 10^{-39}\ C^2m/N.$$

and at 400°K $\alpha_2 = 6 \times 10^{-37}/400 = 1.5 \times 10^{-39}\ C^2.m/N.$

ORAL QUESTIONS

1. The capacitance of a capacitor when filled completely with a dielectric (*a*) increases, (*b*) decreases, (*c*) remains unchanged.
2. When a dielectric is placed in an external electric field, the field with in the dielectric is (*a*) stronger than the external field, (*b*) weaker than the external field, (*c*) same as the external field.
3. The force between two point charges in a dielectric medium is (*a*) greater, (*b*) smaller, (*c*) same as that in vacuum.
4. The dielectric constant of a polar dielectric is (*a*) higher (*b*) smaller, (*c*) same as that of a non-polar dielectric.
5. Which of the following are polar molecules ? H_2, O_2, H_2O, NO, N_2O, HCl, NH_3, CO_2.
6. Distinguish between the terms free charge, bound charge, polarization charge and fictitious charge.
7. A free atom, a dipole molecule and a small particle of dielectric, all move into the region of stronger field if placed in an inhomogeneous electric field. What will happen to their centres of mass if placed in a uniform field ?
8. Define (*a*) polarization and (*b*) molecular field.
9. Which of the following is wrong ? Electric field is stronger in (*a*) the dielectric of lower susceptibility, (*b*) of lower permittivity; (*c*) high polarization of the dielectric.
10. Explain the physical nature of the change occurring in the charge on the capacitor when the dielectric is replaced by one of different permittivity.
11. Which is not an example of convection current ? (*a*) moving charged belt, (*b*) current flowing in a Cu-wire; (*c*) electron beam in CR-tube.
12. If $\nabla.D = \varepsilon\nabla.E$ and $\nabla \cdot J = \sigma\ \nabla \cdot E$ in a given material, the material is same to be (*a*) linear, (*b*) homogeneous, (*c*) isotropic; (*d*) linear & homogeneous.
13. Which one of the following is incorrect ? (*a*) The conductivity of conductors and insulators vary with temperature; (*b*) Nonpolar molecules have no permanent dipoles; (*c*) **E** is always tangential to the conductor.
14. Claussius Mossotti relation is (*a*) $\alpha = \varepsilon_0(k - 1)/n(k + 2)$; (*b*) $\alpha = n(k + 1)/3\varepsilon_0\ (k - 1)$; (*c*) $\alpha = 3\varepsilon_0\ (k - 1)/n(k + 2)$.
15. Langevin-Debye equation gives (*a*) total polarizability; (*b*) polarization charge; (*c*) dielectric constant.
16. A dielectric block (Fig. 6.57) is uniformly polarized. If the polarization is **P**, find the polarization charge density on the faces 1, 2, and 3.
17. What are the values of **P** and χ for a metal as a polarizable body ?
18. At the boundary between two dielectric materials, show that although lines of **D** are continuous the magnitude of **D** in different media is generally different. Give the condition when magnitude of **D** is same in both the media.
19. Refraction of lines of force and displacement is alike to the light rays. Explain.
20. How can you check the correctness of the Classius–Mossotti relation ?
21. Should the dielectric constant of the mercury gas be close to the liquid metal ($\sim\infty$) or be close to the gas ($\sim$1) ?
22. Why the lines of flux do not strike the dielectric normally, similar to the metal surface ?
23. Correct the statement–conductor behaves like a dielectric.

Ans. 1. (*a*); **2.** (*b*); **3.** (*b*); **4.** (*a*); **9.** (*c*); **11.** (*b*); **12.** (*d*); **13.** (*c*), **14.** (*c*); **15.** (*a*);

PROBLEMS

1. A parallel plate capacitor with area 0.20 m^2 and separation 5.5 mm contains three dielectrics with interfaces normal to **E** and **D**, as $k_1 = 3$, $d_1 = 1$ mm; $k_2 = 4$, $d_2 = 2$ mm and $k_3 = 6$, $d_3 = 2.5$ mm. Calculate the capacitance. (1.41 *nF*)
2. A parallel plate capacitor is connected to a constant voltage source. What will happen to C, Q, W if a dielectric of $k = 2$ is inserted between the plates. ($2C_0$, $2Q_0$, $2W_0$)
3. A dielectric cube of side L and centre at the origin has a radial polarization given by $\mathbf{P} = a\mathbf{r}$, where a is a constant and $\mathbf{r} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z$. Find all bound charge densities. Show that the total bound charge vanishes. ($3aL^3$, $-3aL^3$)
4. A dielectric sphere ($\varepsilon_r = 5.7$) of radius 0.1 *m* has a point charge 2*pC* placed at its center. Calculate : (*a*) The surface density of polarization charge on the surface of the sphere, (*b*) The force exerted by the charge on a –4*pC* point charge placed on the sphere.

   (13.12 pC/m^2, $-1.263\ \mathbf{a}_r\ pN$)
5. A line of charge density λ per meter is placed along the *z*-axis and a short dipole of dipole moment *p* is kept at a distance *d* from it. If the axis of dipole lies in the plane containing the point of its location and the *z*-axis, calculate the couple and the translatory force on the dipole.

   $2\lambda p \sin\alpha/(4\pi\varepsilon_0 d)$, $2\lambda p \cos\alpha/(4\pi\varepsilon_0 d^2)$ *towards the line charge, when α is the angle which dipole makes with the direction of line charge.*
6. A dipole of dipole moment 3×10^{-30} coulomb meter is placed at a distance of 10 A.U. from a nucleus of charge 5 electronic units. Calculate the force and torque experienced by the dipole, (*a*) when it is placed radially to the nucleus with its negative pole towards the nucleus and (*b*) when *p* is perpendicular to **r**.

   (*a*) $F_r = 4.32 \times 10^{-11}$ *Nt.*, $F_\theta = 0$, $\tau = 0$

   (*b*) $F_r = 0$, $F_\theta = 2.16 \times 10^{-11}$ *Nt*, $\tau = 2.16 \times 10^{-20}$ *Nt* meter.
7. A dielectric slab of thickness 0.5 cm and dielectric constant 7.0 is placed between the plates of a parallel plate capacitor of plate area 100 cm^2 and separation 1.0 cm. A potential difference of 100 volts is applied with no dielectric slab. The battery is disconnected and the dielectric slab is inserted. Calculate *E*, *D* and *P*, (*a*) in the dielectric and (*b*) in the air gap.

   (*a*) 1.43×10^3 volts/*m*, 8.9×10^{-8} C/m^2, 7.5×10^{-8} *C*/m^2.

   (*b*) 1.00×10^4 volts/*m*, 8.9×10^{-8} C/m^2, 0.
8. A sphere of radius *a* is polarized along the radius vector so that $P = kr$. Find the volume and surface charge densities. Show that the total charge is zero. ($-3k$, ka)
9. A region between the plates of a parallel plate capacitor is filled with a dielectric slab of dielectric constant *k* and of thickness *d*. The capacitor is first connected with a pd *V* and then disconnected. If now the dielectric is partially withdrawn in the direction of length *l* of the plate through a distance $l - x$, calculate the pd across the capacitor. $klV/[l + (k - 1)x]$
10. A parallel plate capacitor is filled with three dielectrics having dielectric constant in the ratio 1:2:3. All the three dielectrics and plates of the capacitor have equal area. Thickness of the dielectrics is in the ratio 6:2:1. Calculate the capacitance of the capacitor thus formed, if the combined thickness of the dielectric is less than the distance between the plates of the capacitor.

    $\varepsilon_0 A/[(d - 6t) + 25t/6k]$.
11. A spherical drop of water carrying a charge of 3×10^{-11} coulomb has a potential 500 volts at its surface. What is the radius of drop ? If two such drops of the same charge and radius combine to form a single drop. What is the potential at the surface of the new drop so formed ?

    (0.54 *mm*, 840 *volts*)
12. A conducting sphere floats half submerged in a liquid dielectric of permittivity ε_1. Above this liquid is a gas of permittivity ε_2. If the total free charge on the sphere is *q*, calculate a radial electric field satisfying all boundary conditions and find the free, bound and total charge densities at all points on the surface of the sphere. $E = q/2\pi(\varepsilon_1 + \varepsilon_2)r^2$.
13. A point charge *q* is placed at a small distance *c* from the centre of a spherical cavity of radius *a* in an infinite dielectric of dielectric constant *k*. Show that charge experiences a force approximately equal to $2(k - 1)q^2c/4\pi\varepsilon_0\,(2k + 1)a^3$, away from centre.

14. If the space between concentric conducting spheres of radii a and b $(a < b)$ is filled with two dielectrics of constants k and k', their common surface being a concentric sphere of radius c, find the capacity of the capacitor. $4\pi\varepsilon_0 kk'\, abc/[bk'\,(c-a) + ak\,(b-c)]$

15. The space between two concentric conducting spheres is filled on one side of a diametral plane with dielectric of constant k and on the other side with dielectric of constant k'. The inner sphere is of radius a and has a charge q. Show that the force on it perpendicular to this diametral plane is $(k-k')q^2/8\pi\varepsilon_0(k+k')^2a^2$.

16. Find an expression for the force between two halves of a sphere charged to a potential V. $(\pi\varepsilon_0 V^2)$

17. A flat disc type cavity has been made in a uniformly polarized sphere with its bases perpendicular to the polarization density **P**. The thickness of the removed disc may be taken small in comparison with its face area. Calculate the electric field inside this cavity. $(2\mathbf{P}/3\varepsilon_0)$

18. Dielectric constant of helium at N.T.P. is 1.000074. What is the atomic polarizability ? How much is the dipole moment per helium atom when the gas is kept in an electric field of 1500 volt/cm? Take the Avogadro's number as 60×10^{22} per gm. mole. $2.2 \times 10^{-31}\ m^3$, 1.1×10^{-24} *stat-coul. cm.*

19. The atomic polarizability of helium is $0.23 \times 10^{-40}\ C.m^2/V$. Calculate (*a*) the electric susceptibility χ for helium gas at 20°C and 1 atm, (*b*) the radius of a helium atom and (*c*) the separation between the nucleus of a helium atom and the centre of its electronic cloud when $E = 10^5\ V/m$.
$(65.1 \times 10^{-6}, 5.61 \times 10^{-11} m, 7.2 \times 10^{-18}\ m)$

20. Calculate the polarizability of hydrogen gas. If gas is placed in an electric field with a strength of $10^6\ V/m$, What will be the effective displacement between the centres of the positive and negative charge in a hydrogen molecule ? Given that the density of hydrogen gas = 0.0899 kg/m^3 and the susceptibility of hydrogen gas = 254×10^{-6}. $(0.829 \times 10^{-40}\ Cm^2/V, 5.7 \times 10^{-16}\ m)$

21. An elementary dipole of strength **p** is placed at point P, outside and at a distance r from the centre O of an earthed conducting sphere of radius R. The axis of the dipole is in the direction OP. Prove that its image system consists of a point charge pR/r^2 and a dipole strength pR^3/r^3, both situated at the point P' which is inverse to P in the sphere.

22. If a dielectric sphere of dielectric constant k is placed in a uniform field $\mathbf{E}_0$ in vacuum. Prove that the field outside the sphere is equal to the field $\mathbf{E}_0$ plus a term due to a dipole at the centre of the sphere whose moment is equal to the total polarization of the sphere.

23. Show that the field inside a cylindrical cavity in a dielectric of constant k is $[2k/(k+1)]\ \mathbf{E}_0$, when the axis of the cylinder is at right angles to a uniform field $\mathbf{E}_0$.

24. A spherical conductor with charge q surrounded by a dielectric of constant k is subjected to a uniform external field E_0. Find the stress on the surface and the total force on the sphere.
$\frac{1}{2}\, k\varepsilon_0[3E_0 \cos\theta + q/4\pi\varepsilon_0 ka^2]^2$, qE_0.

7

ELECTROSTATIC BOUNDARY VALUE PROBLEMS

In the preceding chapters, the electric field **E** has been determined by using either Coulomb's law or Gauss's law, when the charge distribution or the potential V is known ($\mathbf{E} = -\nabla V$) throughout the region.

In most of the problems, neither the charge distribution nor the potential distribution is known. The charge and potential at some boundaries are known and it is desired to find **E** and V throughout the region. Such problems are solved using Poisson's or Laplace's equation or the method of images. They are usually referred to as boundary value problems.

7.1 POISSON AND LAPLACE EQUATIONS

Gauss's law in the differential form in which the space rate of variation of electric induction components are related to the distributed space charge, can be written as

$$\text{div}\,\mathbf{D} = \rho.$$

We know that $\mathbf{E} = -\text{grad}\,V$ and in an isotropic medium

$$\mathbf{D} = \varepsilon\mathbf{E},$$

$$\therefore \quad \text{div}\,\mathbf{D} = \text{div}\,\varepsilon\mathbf{E} = \varepsilon\,\text{div}\,(-\text{grad}\,V) = -\varepsilon\nabla^2 V = \rho$$

$$\text{or} \quad \nabla^2 V = -\rho/\varepsilon. \qquad \text{...(1)}$$

This is known as *Poisson's equation*. If there is no free charge present, $\rho = 0$ and we get

$$\nabla^2 V = 0. \qquad \text{...(2)}$$

This is known as *Laplace's equation*.

The operator ∇^2 is called the Laplacian operator. Poisson's and Laplace's equations are partial differential equations, which can be solved by appropriate boundary conditions. These relations are very useful for solving electrostatic problems.

In cartesian coordinates (x, y, z), Poisson's equation becomes

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}. \qquad \text{...(3)}$$

In cylindrical polar coordinates (ρ, θ, z), we have

$$\frac{1}{r}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right)+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}+\frac{\partial^2 V}{\partial z^2} = -\frac{\rho}{\varepsilon}. \qquad ...(4)$$

In spherical polar coordinates (r, θ φ), we have

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = -\frac{\rho}{\varepsilon}. \qquad ...(5)$$

Let us prove some important properties of the solution to Laplace's equation in terms of theorems :

Theorem 1. *If V_1, V_2 ... V_n are the solutions of Laplace's equation, then*

$$V = K_1V_1 + K_2V_2 + ... K_nV_n,$$

where the K's are arbitrary constants, is also a solution.

$$\nabla^2 V = \nabla^2 (K_1 V_1) + \nabla^2 (K_2V_2) + ... + \nabla^2 (K_nV_n)$$
$$= K_1 \nabla^2 V_1 + K_2 \nabla^2 V_2 + ... + K_n \nabla^2 V_n.$$

Since V_1, V_2, ... V_n are the solutions of Laplace's equation, therefore

$$\nabla^2 V_1 = \nabla^2 V_2 = ... = \nabla^2 V_n = 0$$

$$\therefore \qquad \nabla^2 V = 0. \qquad ...(6)$$

Thus we see that V is also a solution, *i.e., we can superimpose* two or more solutions of Laplace's equation in such a way that the resulting solution satisfies the given boundary conditions.

Theorem 2. *(Uniqueness Theorem). This theorem states that the Laplace's equation satisfying boundary conditions has one and only one solution, i.e., unique solution, i.e., two solutions of Laplace's equation that satisfy the same boundary conditions are identical or differ at most by an additive constant.*

Consider a problem in which a charge density ρ is contained in volume υ surrounded by a surface S. Let V_1 and V_2 are two potentials (solutions) satisfying Poisson's equation. Therefore

$$\nabla^2 V_1 = -\rho/\varepsilon \text{ and } \nabla^2 V_2 = -\rho/\varepsilon_0. \qquad ...(7)$$

Let V_0 be the difference potential $(V_2 - V_1)$. Equation (7) thus gives the Laplace equation as

$$\nabla^2 V_0 = \nabla^2(V_2 - V_1) = \nabla^2 V_2 - \nabla^2 V_1 = 0. \qquad ...(8)$$

It is convenient to introduce the identity

$$\nabla\cdot(V_0 \nabla V_0) = V_0 \nabla^2 V_0 + \nabla V_0 \cdot \nabla V_0 = 0 + |\nabla V_0|^2 = |\nabla V_0|^2. \qquad ...(9)$$

This equation may be integrated over volume υ as;

$$\int_\upsilon \nabla\cdot(V_0 \nabla V_0) d\upsilon = \int_\upsilon |\nabla V_0|^2 \, d\upsilon.$$

Applying the divergence theorem to the left hand term, we get

$$\int_S V_0 \nabla V_0 \cdot d\mathbf{S} = \int_\upsilon |\nabla V_0|^2 \, d\upsilon. \qquad ...(10)$$

The component of the gradient normal to S may be expressed as $\partial/\partial n$. We thus have

$$\int_S V_0(\partial V_0/\partial n) \, dS = \int_\upsilon |\nabla V_0|^2 \, d\upsilon. \qquad ...(11)$$

The uniqueness of the electrostatic field means that $\nabla V_1 = \nabla V_2$ or $\nabla V_0 = 0$ everywhere in the volume. Thus Eq. (11) gives either $V_0 = 0$ or $\partial V_0/\partial n = 0$ at all points on surface S. The condition $V_0 = 0$ gives $V_1 = V_2 = V_S$ on surface S. This is known as *Dirichlet boundary condition.* It corresponds to the *specification of potential function.* The condition $\partial V_0/\partial n = 0$ gives $\partial V_1/\partial n = \partial V_2/\partial n = -E_S$ on surface S. This is known as *Neumann boundary condition.* It corresponds to the *specification of normal electric field or charge density.*

The condition $\nabla V_1 = \nabla V_2$ gives $V = V_2$ + const., throughout the volume υ. The presence of this arbitrary constant indicates that the potential is not unique. If the potential is specified at any point on S, then $V_1 = V_2$ at that point and the constant must be zero. Thus we see that Dirichlet boundary condition specified on any portion of surface S guarantees the uniqueness of the potential.

7.2 EARNSHAW'S THEOREM

Earnshaw's Theorem states that a charged body connot exist in stable equilibrium under the influence of the electrostatic forces alone. This theorem can be proved by taking use of Poisson's equation as under :

For a positively charged body to be in stable equilibrium, it must be located at a point of minimum potential, *i.e.*, $\nabla\varphi = 0$ and $\nabla^2\varphi$ must be positive. But Poisson's equation shows that for a positive charged body $\nabla^2\varphi$ is negative. This is in contradiction of our assumption. Thus we see that the positively charged body can not be in stable equilibrium.

For a negatively charged body, the stability requires $\nabla\varphi = 0$ and $\nabla^2\varphi$ be negative. Since ρ is negative for negatively charged body, there the Poisson's equation shows that $\nabla^2\varphi$ is positive in this case. This is again in contradiction.

We therefore conclude that the charged body (positively or negatively charged) can not exist in stable equilibrium. This is what the Earnshaw's Theorem states.

7.3 MEAN VALUE AND MAXIMUM/MINIMUM VALUE THEOREMS

The following important properties of the potential in a charged free region can be obtained from Laplace's equation :

(1) The potential at the center of an included circle or sphere is equal to the average of the values it assumes on the circle or sphere.

(2) The potential can not have a maximum or minimum with in the region, *i.e.*, any maximum of potential must occur on the boundary of the region.

7.4 SOLUTIONS OF LAPLACE'S EQUATION

One Dimension

The Laplace equation $\nabla^2 V = 0$ reduces to an ordinary differential equation, if V is a function of a single variable.

In **cartesian coordinates**, if V is a function of x, then Laplace's equation reduces to

$$\partial^2 V/\partial x^2 = 0 \quad \text{or} \quad d^2V/dx^2 = 0 \qquad \text{...(12)}$$

On integration twice we get

$$V = Ax + B, \qquad \text{...(13)}$$

where A and B are constants to be determined from the boundary conditions.

In **cylindrical coordinates**, if V is a function of r, then Laplace's equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0. \qquad \text{...(14)}$$

Multiplying by r and integrating, we get

$$r\frac{dV}{dr} = A \quad \text{or} \quad \frac{dV}{dr} = \frac{A}{r},$$

where A is a constant. On further integration, we get

$$V = A \log_e r + B. \qquad \text{...(15)}$$

The constants A and B may be determined from the boundary conditions.

In **spherical coordinates**, if V is a function of r, then Laplace's equation reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) = 0. \qquad \text{...(16)}$$

Multiplying by r^2 and integrating, we get

$$r^2 \frac{dV}{dr} = A \quad \text{or} \quad \frac{dV}{dr} = \frac{A}{r^2},$$

where A is a constant. On further integration, we get

$$V = -\frac{A}{r} + B. \qquad \text{...(17)}$$

A and B are constants to be determined from the boundary conditions.

Two Dimensions

In **cartesian coordinates**, if V is a function of x and y, then Laplace's equation reduces to

$$\nabla^2 V = \partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 = 0. \qquad \text{...(18)}$$

This equation can be solved by the method of separation of variables by assuming

$$V(x, y) = X(x)\ Y(y), \qquad \text{...(19)}$$

where X is a function of x only and Y is a function of y only, substituting this value in equation (18), we get

$$X''Y + Y''X = 0.$$

Dividing through by XY and separating X from Y, gives

$$-\frac{X''}{X} = \frac{Y''}{Y} = \lambda \text{ (say).}$$

where λ is a constant, known as separation constant. We thus have

$$X'' + \lambda X = 0 \quad \text{and} \quad Y'' - \lambda Y = 0 \qquad \text{...(20)}$$

There are following three possible cases depending on λ.

(*a*) $\lambda = 0$: If $\lambda = 0$, then from equation (20)

$$X'' = 0 \quad \text{or} \quad d^2X/dx^2 = 0$$

which on integration twice yields

$$X = Ax + B \qquad \text{..}$$

Similarly $$Y = Cy + D \qquad \text{...}$$

(*b*) $\lambda < 0$: If λ is negative, say $-\alpha^2$, then equation (20) becomes

$$X'' - \alpha^2 X = 0 \quad \Rightarrow \quad d^2X/\partial x^2 - \alpha^2 X = 0, \text{ or } (D^2 - \alpha^2)X = 0.$$

$$\therefore \quad DX = \pm\, \alpha X \quad \text{or} \quad dX/dx = \pm\, \alpha X.$$

Therefore there are two possible solutions corresponding to the plus and minus sig. or the plus sign

$$\frac{dX}{dx} = \alpha X \quad \text{or} \quad \int \frac{dX}{X} = \alpha \int dx$$

$$\therefore \quad \log_e X = \alpha x + A = \alpha x + \log_e A_1$$

or $$X = A_1 e^{\alpha x}.$$

Similarly for the minus sign, the solution is given by

$$X = A_2 e^{-\alpha x}$$

$\therefore$ Total solution

$$X = A_1 e^{\alpha x} + A_2 e^{-\alpha x} \qquad \text{...(23)}$$

$$= B_1 \cosh \alpha x + B_2 \sinh \alpha x. \qquad \text{...(24)}$$

where $$B_1 = A_1 + A_2 \text{ and } B_2 = A_1 - A_2.$$

(*c*) $\lambda > 0$: Let $\lambda = \beta^2$, then equation (20) becomes

$$X'' + \beta^2 X = 0 \quad \text{or} \quad (D^2 + \beta^2)X = 0$$

$$\therefore \quad DX = \pm j\, \beta X \quad \text{or} \quad dX/dx = \pm j\beta X. \qquad \left(\because\ j = \sqrt{-1}\right)$$

As discussed earlier, there are two possible solutions. The total solution

$$X = C_1\, e^{j\beta x} + C_2\, e^{-j\beta x} \qquad ...(25)$$

$$= D_1 \cos \beta x + D_2 \sin \beta x. \qquad ...(26)$$

where $D_1 = C_1 + C_2$ and $D_2 = C_1 - jC_2$.

Therefore the general solution in two dimensional cartesian coordinates is

$$V = (B_1 \cosh \alpha x + B_2 \sinh \alpha x)\,(D_1 \cos \alpha y + D_2 \sin \alpha y) \qquad \text{for } \alpha < 0$$

$$= (B_1' \cos \alpha x + B_2' \sin \alpha x)\,(D_1' \cosh \alpha y + D_2' \sinh \alpha y) \qquad \text{for } \alpha > 0 \qquad ...(27)$$

In **cylindrical coordinates** (r, θ, z), when the potential V has a symmetry along the z-axis, the potential V is a function of r and θ only. Thus the Laplace equation reduces to

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0. \qquad ...(28)$$

This can be solved by the method of separation of variables. Now substituting $V(r, \theta) = R(r)\, S(\theta)$ in equation (28), we get

$$\frac{r}{R}\frac{d}{dr}\left(r\frac{dR}{dr}\right) = -\frac{1}{S}\frac{d^2S}{d\theta^2} = k^2 \text{ (say)}$$

$$\therefore \qquad \frac{d}{dr}\left(r\frac{dR}{dr}\right) - \frac{k^2}{r}R = 0 \quad \text{and} \quad \frac{d^2S}{d\theta^2} + k^2 S = 0. \qquad ...(29)$$

Since the function S is a linear combination of $\sin k\theta$ and $\cos k\theta$. If $k = n$, n being zero or positive integer.

The radial equation (33) becomes

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \frac{n^2}{r^2}R = 0. \qquad ...(30)$$

This is the Euler equation with the solutions r^n and r^{-n} (for $n > 0$) and is $\log_e r$ for $n = 0$. Therefore the general solution for V is

$$V = \sum_{n=1}^{\infty} [(A_n \cos n\theta + B_n \sin n\theta) r^n + (C_n \cos n\theta + D_n \sin n\theta) r^{-n}] + A_0 + C_0 \log_e r. \qquad ...(31)$$

Spherical Coordinates

In cases of azimuthal symmetry, the potential V depends on r and θ only. Laplace equation in two dimensional spherical coordinates is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) = 0. \qquad ...(32)$$

The solution is obtained by the method of separation of variables. We may set

$$V(r, \theta) = R(r)\, \Theta(\theta). \qquad ...(33)$$

Substituting this value and dividing by $R(r)\, \Theta(\theta)$, we get

$$\frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = -\frac{1}{\Theta \sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) = k \text{ (constant)} \qquad ...(34)$$

$$\therefore \qquad \frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = kR \quad \text{and} \quad \frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + k\Theta = 0. \qquad ...(35)$$

The second equation, which is the function of θ only, is called Legendre's equation. The solution of this equation is beyond the scope of the book. The solutions for all values of θ from 0 to π are acceptable only if $k = n(n + 1)$, where n is a positive integer. The solutions, known as Legendre's polynomials are written as $P_n(\cos\theta)$. The legendre's polynomials are orthogonal, *i.e.*,

$$\int_{-1}^{1} P_n(x)\, P_m(x)\, dx = \frac{2\delta_{nm}}{2n+1}, \qquad \text{(where } x = \cos\theta\text{)}. \qquad ...(36)$$

where δ_{nm} is the kronecker delta which is zero for $n \neq m$ and is unity for $n = m$. $P_0(\cos\theta) = 1$, $P_1(\cos\theta) = \cos\theta$, $P_2(\cos\theta) = \frac{1}{2}(3\cos^2\theta - 1)$ and $P_3(\cos\theta) = \frac{1}{2}(5\cos^3\theta - 3\cos\theta)$. The equation $(2n+1)x\,P_n(x) = (n+1)P_{n+1}(x) + n\,P_{n-1}(x)$ may be used to find the high order of polynomials.

The radial function may be obtained by solving

$$\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = n\,(n+1)\,R. \qquad \text{...(37)}$$

This equation has two independent solutions for each n. Such as

$$R = A_n r^n + B_n r^{-(n+1)} \qquad \text{...(38)}$$

Therefore the potential function $V_n(r, \theta) = [A_n r^n + B_n r^{-(n+1)}]\,P_n(\cos\theta)$. ...(39)

The general solution can be obtained by superposing these solutions for all possible values of n (*i.e.*, 0 to ∞) as

$$V(r, \theta) = \sum_{n=0}^{\infty}[A_n r^n + B_n r^{-(n+1)}]\,P_n(\cos\theta) \qquad \text{...(40)}$$

The coefficients A_n and B_n can be obtained by the boundary conditions.

The electrostatic problems are in general geometrically symmetricals therefore do not require three dimensional solutions of Laplace's equations.

7.4 APPLICATIONS OF LAPLACE'S EQUATION

Solution of Laplace's equation is very tedious. It requires the knowledge of Legendre's differential equation, which is beyond the scope of this text. The readers are advised to consult any standard book on differential equations. A few simple cases are discussed here.

(*a*) Potential between two parallel planes : Consider two parallel planes P and Q of infinite extent in the $x-y$ planes and separated by a distance d in the z-direction, as shown in Fig. 7.1. If V_1 and V_2 be the potentials of the planes P and Q respectively. The potential V at any point between these planes may be given by Laplace's equation $\nabla^2 V = 0$. In cartesian coordinates, we get

$$\nabla^2 V = \partial^2 V/\partial x^2 + \partial^2 V/\partial y^2 + \partial^2 V/\partial z^2 = 0.$$

From symmetry it is clear that V does not vary with x or y. Thus Laplace's equation in one dimensional cartesian coordinates reduces to

$$\nabla^2 V = \partial^2 V/\partial z^2 = 0. \qquad \text{...(41)}$$

On successive integration, we get

$$V = Az + B,$$

where A and B are arbitrary constants. Using the boundary conditions $V = V_1$ at $z = 0$ and $V = V_2$ at $z = d$, we get

$$V = \frac{V_2 - V_1}{d}z + V_1. \qquad \text{...(42)}$$

The electric field strength

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial z}\hat{z} = -\frac{V_2 - V_1}{d}\hat{z}. \qquad \text{...(43)}$$

Thus we see that E is constant in the region between the plates and is directed along the z-direction towards the plate of lower potential.

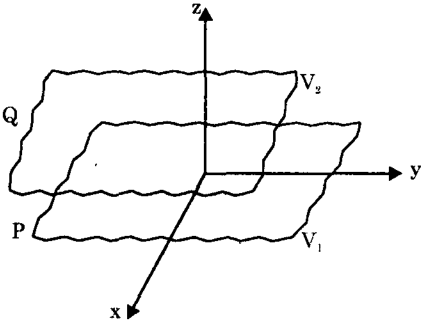


Fig. 7.1. Two parallel planes.

(*b*) Potential between two concentric cylinders : Consider two concentric very long cylinders with their common axis along z-direction. If the inner and outer cylinders are of radii

a and b and are at potentials V_a and V_b respectively. It is clear from the symmetry that the potential at points in between the cylinders will depend upon r only and will be independent of 0 and z. Thus the Laplace's equation in cylindrical coordinates becomes

$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) = 0 \text{ or } r\frac{\partial V}{\partial r} = \text{constant } A.$$

Integration of this relation ($dV = A\, dr/r$) gives $V = A \log_e r + B$. ...(44)

Using the boundary conditions $V = V_a$ at $r = a$ and $V = V_b$ at $r = b$, we get

$$V = \frac{V_a - V_b}{\log_e(a/b)}\log_e r + \frac{V_b \log_e a - V_a \log_e b}{\log_e(a/b)}. \quad ...(45)$$

The electric field in the region

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} = -\frac{A}{r}\hat{r} = -\frac{V_a - V_b}{r\log_e(a/b)}\hat{r}. \quad ...(46)$$

Thus the electric field in the region between the concentric cylinders varies inversely with the distance r. It is radial toward the cylinder of lower potential.

(*c*) Potential between two concentric spheres : Consider two concentric spheres with radii a and b which are at potentials V_a and V_b respectively. It is clear from symmetry that the potential at point in between these spheres will be independent of θ and ϕ. Thus Laplace's equation in spherical coordinates reduces to

$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{dV}{\partial r}\right) = 0 \text{ or } r^2\frac{\partial V}{\partial r} = \text{const } A. \quad ...(47)$$

Its integration gives us

$$V = -A/r + B. \quad ...(48)$$

Using the boundary condition $V = V_a$ at $r = a$ and $V = V_b$ at $r = b$, we get

$$V = \frac{(V_a - V_b)ab}{b - a}\cdot\frac{1}{r} + \frac{bV_b - aV_a}{b - a}. \quad ...(49)$$

The electric field in the region

$$\mathbf{E} = -\nabla V = -\frac{\partial V}{\partial r}\hat{r} = -\frac{A}{r^2}\hat{r} = \frac{(V_a - V_b)ab}{(b - a)r^2}\hat{r}. \quad ...(50)$$

Thus the electric field in the region between the concentric spheres varies inversely as the square of distance. It is radial toward the sphere of lower potential.

(*d*) Conducting sphere in a uniform field : Consider an uncharged conducting sphere of radius a placed in a uniform electric field $\mathbf{E}_0$, Fig. 7.2. *There is an equipotential plane cutting through the middle of the sphere and perpendicular to the direction of* $\mathbf{E}_0$. We define this plane to be at zero potential. Since net charge on the sphere is zero, hence $\nabla^2 V = 0$ and sphere is an equipotential surface with $V = 0$. The sphere distorts the field in its neighbourhood, but the field at a great distance remains unchanged. If the origin is taken at the centre of the sphere and the x-axis in the direction of original electric field $\mathbf{E}_0$, potential at large distances is that of the field $\mathbf{E}_0$ alone. Hence at point far from the sphere $V = -E_0 x = -E_0 r \cos\theta$. Thus we must look for a solution of Laplace's equation which satisfies the two boundary conditions.

$$V = -E_0 r\cos\theta \text{ for } r \to \infty \quad \text{and} \quad V = 0 \text{ for } r = a.$$

To satisfy these boundary conditions, let us choose the solution as

$$V = -E_0 r\cos\theta + (A/r^2)\cos\theta. \quad ...(51)$$

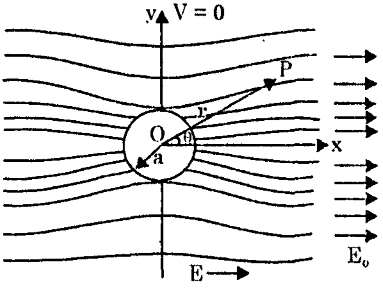


Fig. 7.2. Conducting sphere in a uniform field $\mathbf{E}_0$.

Both the terms are solutions of Laplace's equation $\nabla^2 V = 0$, the second term tends to zero for $r \to \infty$. From the boundary condition, $V = 0$ at $r = a$, we get $A = E_0 a^3$, hence the general solution of Laplace's equation for our particular case reduces to

$$V = -E_0 r\cos\theta + (E_0 a^3/r^2)\cos\theta = -E_0 r\cos\theta\,[1 - a^3/r^3]. \qquad \text{...(52)}$$

This expression is valid for all $r > a$. Inside the sphere the potential is everywhere zero.

The second term, $(E_0 a^3/r^2)\cos\theta$, amounts to the perturbation of the original uniform field by the conducting sphere. It corresponds to the effect of a dipole of dipole moment $p = E_0 a^3$ placed at the origin, with $\mathbf{p}$ along x-axis.

The components of electric intensity outside the sphere are :

$$\textit{along radius vector } E_r = -\frac{\partial V}{\partial r} = \left(1 + \frac{2a^3}{r^3}\right) E_0\cos\theta, \qquad \text{...(53)}$$

$$\perp \textit{ to radius vector } E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -\left(1 - \frac{a^3}{r^3}\right) E_0 \sin\theta. \qquad \text{...(54)}$$

At $r = a$, *i.e., the surface of conducting sphere,* $E_r = 3E_0\cos\theta$ and $E_\theta = 0$. Along the x-axis the field at the surface is thus $3E_0$ and at point on the surface along y-axis, both V and E are zero.

We know that the surface charge density on the surface of a conductor immersed in a medium of dielectric constant k is given by

$$\sigma = D = k\varepsilon_0 (E_r)_{r=a} = 3k\,\varepsilon_0\,E_0\cos\theta. \qquad \text{...(55)}$$

It is positive on the right hand hemisphere and negative on the left. If the sphere has a charge q, then we have

$$V = -\left(1 - \frac{a^3}{r^3}\right) E_0 r\cos\theta + \frac{q}{4\pi\varepsilon_0 r}. \qquad \text{...(56)}$$

(*e*) Dielectric sphere in a uniform field : Consider an uncharged dielectric sphere of radius a and dielectric constant k_1, surrounded by a medium of dielectric constant k_2 and placed in a uniform electric field $\mathbf{E}_0$, Fig. 7.3. Let V_1 and V_2 be the potential functions inside the outside the sphere respectively. The solution to this problem must satisfy the following requirements :

(*i*) $\nabla^2 V_1 = 0$ and $\nabla^2 V_2 = 0$, both inside and outside respectively, as the net charge on the sphere is zero.

(*ii*) V_1 must remain finite for all $r \le a$. V_2 must also remain finite at infinity.

(*iii*) $V_1 = V_2$ for $r = a$ at all angles θ, otherwise discontinuity would give an infinite electric field there.

(*iv*) Both boundary conditions must satisfy at the surface of the sphere.

$$D_{1n} = D_{2n} \text{ and } E_{1p} = E_{2p}. \qquad \text{...(57)}$$

As in the problem of the conducting sphere

$$V_2 = -E_0 r\cos\theta + (A/r^2)\cos\theta. \qquad \text{...(58)}$$

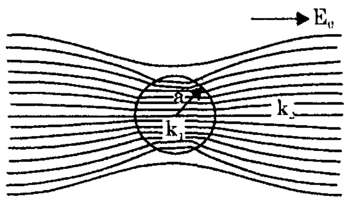


Fig. 7.3. Dielectric sphere in uniform electric field E_0.

As there is no volume distribution of free charge inside the sphere the potential function inside the sphere may be assumed of the same form as that used outside, hence

$$V_1 = B\,r\cos\theta + (C/r^2)\cos\theta. \qquad \text{...(59)}$$

Evidently, C must be zero, otherwise the potential would become infinite at the origin ($r = 0$), hence

$$V_1 = B\,r\cos\theta. \qquad \text{...(60)}$$

At the surface of the sphere ($r = a$), we have $V_1 = V_2$.

$$\therefore\ Ba\cos\theta = E_0 a\cos\theta + (A/a^2)\cos\theta \text{ or } B = -E_0 + A/a^3. \qquad \text{...(61)}$$

The boundary condition, that the normal components of the electric displacement at the surface must be continuous, gives

$$D_{1n} = D_{2n} \text{ or } k_1\varepsilon_0 E_{1n} = k_2\varepsilon_0 E_{2n}.$$

As $E_n = -\partial V/\partial r$, hence $-k_1(\partial V_1/\partial r)_{r=a} = -k_2(\partial V_2/\partial r)_{r=a}$

$$-k_1 B\cos\theta = -k_2[-E_0\cos\theta - (2A/a^3)\cos\theta]$$

or

$$k_1 B = -k_2(E_0 + 2A/a^3). \qquad \text{...(62)}$$

Solving Eqs. (61) and (62), we get

$$A = \frac{k_1 - k_2}{k_1 + 2k_2}E_0 a^3 \text{ and } B = \frac{-3k_2}{k_1 + 2k_2}E_0.$$

Thus the potential functions inside and outside the sphere are :

$$V_1 = -\frac{3k_2}{k_1 + 2k_2}E_0 r\cos\theta \qquad \text{...(63)}$$

$$V_2 = -\left(1 - \frac{k_1 - k_2}{k_1 + 2k_2}\cdot\frac{a^3}{r^3}\right)E_0 r\cos\theta. \qquad \text{...(64)}$$

As the potential inside the sphere can be written as $V_1 = Bx$, hence the field inside the sphere is uniform. It is parallel to x-axis, *i.e.*, along $\mathbf{E}_0$ and is of magnitude

$$E_1 = -\frac{\partial V_1}{\partial x} = \frac{3k_2}{k_1 + 2k_2}E_0. \qquad \text{...(65)}$$

For the dielectric sphere in a vacuum ($k_2 = 1$), the original uniform field E_0 has been reduced to a value $[3/(k_1 + 2)]E_0$. This reduction is due to the field of the polarization charges on the surface of the sphere acting in opposite direction. This reverse field is known as the *depolarizing field* E_{dep} and is given by

$$E_{dep} = E_0 - E_1 = \left(1 - \frac{3}{k_1 + 2}\right)E_0 = \left(\frac{k_1 - 1}{k_1 + 2}\right)E_0. \qquad \text{...(66)}$$

It may also be expressed in terms of a depolarizing factor L* as

$$\varepsilon_0 E_{dep} = LP = L\varepsilon_0(k_1 - 1)E_1. \qquad \text{...(67)}$$

Substitution of values gives $L = 1/3$. ...(68)

The potential distribution outside the sphere is that of a dipole of magnitude $4\pi k_2\varepsilon_0 a^3 E_0 (k_1 - k_2)/(k_1 + k_2)$, situated at the centre of the sphere, superimposed on that due to the uniform field.

If $k_1 \to \infty$, the solutions tend to those obtained for the conducting sphere.

7.5 ELECTRICAL IMAGES

The distribution of induced charge on a surface, hence the field intensity and potential distribution may be found in many cases by the method of electrical images, developed originally by Lord Kelvin in 1848. If two equal point charges of opposite signs are separated by a certain distance $2a$, the plane passing through the mid-point of the line joining them and perpendicular to this line is an equipotential surface (at zero potential). Now if we replace this equipotential surface by a conducting surface, and transfer one of the charges to this surface, the field between the other charge and the surface remains unaltered although that between the first charge and the surface is wiped out. Conversely, if a point charge is placed in front of an infinite conducting plane, the resultant electric field in front of the plane will be the same as that produced by the original charge plus an equal charge of opposite sign if placed at an equal distance from the plane on the opposite side. This charge is the *electric image* of the original charge in the plane, analogous to the theory of optical images. Thus an *electrical image is a set of one or more point charges on one side of a surface which produces on the otherside of the surface the same electric field as is produced by the actual electrification of the surface.*

---

* The depolarizing factor L depends on the shape of the dielectric. It is maximum ($L = 1$) for dielectric plate placed ⊥ to the external field $\mathbf{E}_0$. It is minimum ($L = 0$) for dielectric rod with axis parallel to $\mathbf{E}_0$.

The method of images thus consists in replacing a conductor by a point charge such that the conducting surface is still an equipotential surface ($V = 0$). Thus the Laplace's equation is still satisfied at all points outside the conductor and by the principle of uniqueness, the problem of a point charge and a conducting surface can be replaced by the problem of a point charge and its image. Let us apply this method for the following cases :

(1) Point charge and infinite conducting plane : To find the field due to a point charge q placed before a conducting plane AB at a distance d from it, we consider the field due to this charge q and its image, the charge $-q$ placed on the perpendicular dropped from q to the plane on the left at a distance d from the plane Fig. 7.4. The potential at an arbitrary point P (r, θ) is thus given by

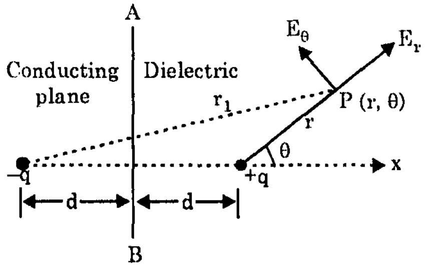


Fig. 7.4. Point charge before a conducting plane.

$$V = \frac{1}{4\pi\varepsilon}\frac{q}{r} - \frac{1}{4\pi\varepsilon}\frac{q}{r_1}$$

$$= \frac{q}{4\pi\varepsilon}\left[\frac{1}{r} - \frac{1}{\sqrt{(r^2 + 4d^2 + 4rd\cos\theta)}}\right], \quad \text{...(69)}$$

where ε is the permittivity of the medium outside the conductor.

The electric intensity at point P in the direction of $\mathbf{r} = E_r = -\partial V/\partial r$ and in the direction perpendicular to $\mathbf{r} = E_\theta = -(1/r)(\partial V/\partial\theta)$, therefore the component of electric intensity at point P normal to the plane

$$E_x = E_r\cos\theta - E_\theta\sin\theta = -\frac{\partial V}{\partial r}\cos\theta + \frac{1}{r}\frac{\partial V}{\partial\theta}\sin\theta$$

$$= \frac{q}{4\pi\varepsilon}\left[\frac{\cos\theta}{r^2} - \frac{r\cos\theta + 2d}{(r^2 + 4d^2 + 4rd\cos\theta)^{3/2}}\right]$$

If the point P is on the plane, then $r\cos\theta = -d$ and the field

$$E_x = -qd/2\pi\varepsilon r^3. \quad \text{...(70)}$$

The induced charge per unit area on the plane is then $\sigma = \varepsilon E_x = -qd/2\pi r^3$. Hence the surface density of the induced charge on the plane varies inversely as the cube of the distance of the position on the plane from the original charge $+q$ and is also opposite in sign to $+q$.

The force exerted on the point charge by the induced charge on the plane

= Force exerted on the charge by its image

$= -q^2/16\pi\varepsilon d^2$. ...(71)

Here the negative *sign indicates that the charge is attracted towards the plane.*

(2) Point charge and conducting sphere: *Case 1 : When the sphere is at zero potential (or grounded).* Let a charge q be placed at a point A at a distance f from the centre of an earthed spherical conductor of radius a (Fig. 7.5). To solve the problem we have to determine the magnitude and position of a point charge q' inside the sphere, which together with the charge q makes the sphere a equipotential surface (zero potential). As an approach to the solution, consider the charge q' at A' inside the conductor such

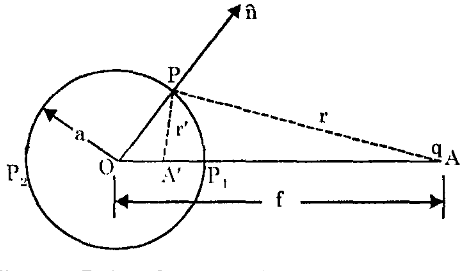


Fig. 7.5. Point charge and conducting sphere.

that $OA\ OA' = a^2$. Then $OA/a = a/OA'$, thus the triangles $A'OP$ and POA are similar where P is any point on the surface of the sphere.

$\therefore$ $$OA/a = a/OA' = PA/PA', \text{ or } PA/PA' = f/a.$$

The sphere will be an equipotential surface at zero potential, hence

$$\frac{q}{4\pi\varepsilon(PA)} + \frac{q'}{4\pi\varepsilon_0(PA')} = 0, \text{ or } q' = -q\,\frac{PA'}{PA} = -q\,\frac{a}{f}. \quad ...(72)$$

The charge induced on the surface is clearly the charge of image, *i.e.*, $-qa/f$. For investigating the field distribution this image may be used instead of the induced charge on the sphere. The surface density of induced charge σ may be found by using Coulomb's law, *i.e.*,

$$\frac{\sigma}{\varepsilon_0}\hat{n} = \text{Electric field at point } P = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r} - \frac{1}{4\pi\varepsilon_0}\frac{q\,a}{fr'^3}\mathbf{r}',$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the vectors **AP** and **A'P** respectively and $\hat{n}$ denotes a unit vector along the normal **OP** to the sphere at P. Now we have

$$\mathbf{r} = \mathbf{AO} + \mathbf{OP} = \mathbf{AO} + a\hat{n} \text{ and } \mathbf{r}' = \mathbf{A'O} + \mathbf{OP} = \mathbf{A'O} + a\hat{n}.$$

Remembering that **E** is radial (along $\hat{n}$) at P, the terms involving **AO** and **A'O** cancel and we have

$$\frac{\sigma}{\varepsilon_0}\hat{n} = \mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q\,a}{r^3}\hat{n} - \frac{1}{4\pi\varepsilon_0}\frac{q\,a}{f}.\frac{a}{r'^3}\hat{n}.$$

Using relation $PA/PA' = r/r' = f/a$, we get

$$\frac{\sigma}{\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}.\frac{qa}{r^3}\left(1 - \frac{f^2}{a^2}\right) \text{ or } \sigma = -\frac{1}{4\pi}\frac{q}{ar^3}(f^2 - a^2). \quad ...(73)$$

The force between charge q and the sphere is given by

$$F = \frac{qq'}{4\pi\varepsilon_0(AA')^2} = \frac{q(-qa/f)}{4\pi\varepsilon_0(f - a^2/f)^2} = -\frac{q^2af}{4\pi\varepsilon_0(f^2 - a^2)^2} \quad ...(74)$$

The negative sign indicates that the charge is attracted towards the sphere.

Case II : When the sphere is insulated and without charge. When a point charge q is placed near a insulated conducting sphere, the conditions are : (*i*) The gaussian conditions in the region outside the sphere remains the same, (*ii*) The total charge induced on the sphere is zero, as the sphere is insulated, (*iii*) The potential of the sphere is uniform all over the surface and is not zero.

The charge $+q$ at an external point A and its image charge $-qa/f$ at point A' satisfy the gaussian conditions. An additional image charge of qa/f must be distributed uniformly over the surface of the sphere, in order to satisfy condition (*ii*), which may be assumed to be placed at the centre O.

As regards the potential of the surface of the sphere, the two charges $+q$ at A and $-qa/f$ at A' together produce zero potential on the sphere and the net potential on the sphere is due to the charge qa/f at the centre O, which is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{\text{Charge}}{\text{Radius}} = \frac{1}{4\pi\varepsilon_0}\frac{qa/f}{a} = \frac{1}{4\pi\varepsilon_0}\frac{q}{f}, \quad ...(75)$$

which is uniform as required by the condition (*iii*).

Surface density of charge at point P = surface density due to charge $(q, -qa/f)$ + due to the charge (qa/f) at O, *i.e.*

$$\sigma = -\frac{1}{4\pi}\frac{q}{ar^3}(f^2 - a^2) + \frac{qa/f}{4\pi a^2} = -\frac{1}{4\pi}\frac{q}{a}\left[\frac{f^2 - a^2}{r^3} - \frac{1}{f}\right] \quad ...(76)$$

From this result we see that when point P is at P_1, for which $r = f - a$, we get

$$\sigma_{P_1} = -\frac{1}{4\pi}q\,\frac{3f - a}{f(f - a)^2}, \quad ...(77)$$

which is negative, and when point P is at P_2 for which $r = f + a$, we get.

$$\sigma_{P_2} = \frac{1}{4\pi} q \frac{3f + a}{f(f + a)^2}, \qquad \text{...(78)}$$

which is positive.

Thus there is a line between these positions P_1 and P_2 through point P' say, on which surface charge density $\sigma = 0$, such as

$$\frac{f^2 - a^2}{(P'A)^3} - \frac{1}{f} = 0, \quad \text{or } (P'A)^3 \doteq f(f^2 - a^2) = f^2\left[f - \frac{a^2}{f}\right]$$

$$\text{or} \qquad (P'A)^3 = O\ A^2 \cdot AA' \qquad \text{...(79)}$$

The force of attraction between the point charge $+q$ and the sphere is thus given as

$$F = \frac{1}{4\pi\varepsilon_0} \frac{q^2 a}{f(AA')^2} - \frac{1}{4\pi\varepsilon_0} \frac{q^2 a}{f(AO)^2} = \frac{1}{4\pi\varepsilon_0} \frac{q^2 a}{f}\left[\frac{f^2}{(f^2 - a^2)^2} - \frac{1}{f^2}\right]$$

$$= \frac{q^2 a^3 (2f^2 - a^2)}{4\pi\varepsilon_0 f^3 (f^2 - a^2)^2}. \qquad \text{...(80)}$$

(3) Point charge and a semi-infinite slab of dielectric : Let a point charge $+q$ is placed in a medium of permittivity ε_1 in front of a semi-infinite slab of dielectric of permittivity ε_2 at a distance d. *This problem will be same as that of two very long dielectric media having permittivities ε_1 and ε_2 with a plane interface. A point charge q is located in the first medium at a distance d from the interface.* This charge induces a charge on the dielectric slab. If this induced charge is replaced by an image charge (or charges) situated in the dielectric, then it must satisfy the following conditions : (*a*) Gaussian conditions hold good both for the dielectric and medium, (*b*) Boundary conditions are also valid, (*c*) The potential is zero at infinitely distant points.

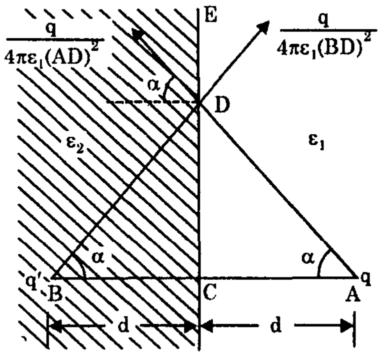

E
$\frac{q}{4\pi\varepsilon_1(AD)^2}$
$\frac{q}{4\pi\varepsilon_1(BD)^2}$
α
D
ε_2
ε_1
α
α
q′
B
C
A
q
d
d

Fig. 7.6. Point charge before a dielectric slab.

To solve this problem let us consider a charge q' placed at point B in the dielectric such that $AC = BC$. The value of q' is so chosen that the combination of q and q' gives the correct field distribution on the R.H.S. of the boundary and fulfils the condition (*c*). For the correct field distribution within the dielectric, *i.e.*, on the L.H.S. of the boundary only the charge q is replaced by the another charge q''. The act of placing charge q'' does not conflict with gaussian condition in dielectric and condition (*c*).

The charges q' and q'' are so chosen that boundary conditions are also satisfied. For this purpose we have to assume charges q and q' when we deal with R.H.S. of the boundary and charge q'' alone when with L.H.S. The electric field at point D on the boundary due to charge q is $q/4\pi\varepsilon_1 (AD)^2$ along AD and due to charge q' is $q'/4\pi\varepsilon_1 (BD)^2$ along BD. In the second case *we have taken ε_1 instead of ε_2* as q' *is an image assumed in medium not in dielectric.* Hence the total tangential component of electric intensity due to the presence of charges q and q'

$$= \frac{q}{4\pi\varepsilon_1(AD)^2}\sin\alpha + \frac{q'}{4\pi\varepsilon_1(BD)^2}\sin\alpha = \frac{\sin\alpha}{4\pi\varepsilon_1(AD)^2}(q + q').$$

The tangential component of the field due to the charge q'' supposed to be placed at point A, giving field distribution on L.H.S.

$$= \frac{q''}{4\pi\varepsilon_1(AD)^2}\sin\alpha.$$

Thus for the continuity of the tangential components of electric intensity, we have

$$\frac{(q+q')\sin\alpha}{4\pi\varepsilon_1(AD)^2} = \frac{q''}{4\pi\varepsilon_1(AD)^2}\sin\alpha,$$

or $$q + q' = q''. \qquad \text{...(81)}$$

The normal components of the displacements should also be continuous, hence

$$\frac{\varepsilon_1 q}{4\pi\varepsilon_1(AD)^2}\cos\alpha - \frac{\varepsilon_1 q'}{4\pi\varepsilon_1(BD)^2}\cos\alpha = \frac{\varepsilon_2 q''}{4\pi\varepsilon_1(AD)^2}\cos\alpha$$

or $$q - q' = (\varepsilon_2/\varepsilon_1)\,q'' = (k_2/k_1)\,q''. \qquad \text{...(82)}$$

Solving Eqs. (81) and (82), we get

$$q' = -q\,\frac{k_2 - k_1}{k_2 + k_1} \text{ and } q'' = \frac{2qk_2}{k_2 + k_1}. \qquad \text{...(83)}$$

Thus we see that for the positive value of q, q' is negative and q'' is positive. These two charges are the image charges each less than q and satisfying boundary conditions.

The force of attraction between the original charge q and the dielectric

$$= \frac{q\cdot q[(k_2-k_1)/(k_2+k_1)]}{4\pi\varepsilon_1(2d)^2} = \frac{q^2(k_2-k_1)}{4\pi\varepsilon_1(4d^2)\,(k_2+k_1)}$$

$$= q^2/4\pi\varepsilon_1\,(4d^2) \qquad (\text{for } k_2 = \infty) \qquad \text{...(84)}$$

This is same as obtained for the conductor and a point charge given by Eq. (71). Thus we see that a *conductor behaves like a dielectric with a permittivity of infinity.*

7.6 SYSTEM OF CONDUCTORS

In this article we shall discuss the general theory of an electrostatic field due to large number of conductors. Let us consider a system consisting of n independent isolated conductors in any medium. Any charge placed on the one conductor will not only increase the potential of this conductor but also of the other $(n-1)$ conductors. If the potentials of 1st, 2nd, 3rd ... nth conductors due to a unit charge on the 1st conductor are designated as $p_{11}, p_{12}, p_{13}, \ldots\ldots p_{1n}$ respectively. If a charge Q_1 is placed on the 1st conductor, then potentials become $p_{11}Q_1, p_{12}Q_1, p_{13}Q_1 \ldots\ldots p_{1n}Q_1$. If charge Q_2 is placed upon the second conductor, instead of first, then potentials will be $p_{21}Q_2$, $p_{22}Q_2$, $p_{23}Q_2$, ... $p_{2n}Q_2$. From this discussion it is clear that the first number of the suffix represents the conductor upon which the charge has been placed and the second number represents the conductor upon which the resultant potential has been produced.

Thus the contribution to potential due to the charge Q_n on the n^{th} conductor is $p_{n1}Q_n$, $p_{n2}Q_n$, $p_{n3}Q_n$, ... $p_{nn}Q_n$, The actual potential of any one conductor will be given by an addition of the terms corresponding to the charges on all conductors. Thus the potential of the 1st conductor is given by

$$V_1 = p_{11}Q_1 + p_{21}Q_2 + p_{31}Q_3 + \ldots\ldots P_{n1}Q_n$$

for 2nd conductor $$V_2 = p_{12}Q_1 + p_{22}Q_2 + p_{32}Q_3 + \ldots\ldots P_{n2}Q_n$$

$$\vdots \qquad \vdots \quad \vdots$$

for nth conductor $$V_n = p_{1n}Q_1 + p_{2n}Q_2 + p_{3n}Q_3 + \ldots\ldots P_{nn}Q_n,$$

These n equations can be written as

$$V_i = \sum_{j=1}^{n} p_{ji}Q_j. \qquad \text{...(85)}$$

These coefficients p_{ji} are called *potential coefficients.*

Above linear equations can be solved for the charges as linear functions of potentials. Suppose these are

$$Q_j = \sum_{i=1}^{n} q_{ij} V_i, \qquad ...(86)$$

where the coefficients q_{ij} are called the *induction coefficients.*

If all the conductors other than the r^{th} are earthed, then $Q_r = q_{rr} V_r$ and all other V terms are zero. Hence the term q_{rr} will be nothing but the capacitance. Thus the coefficients of the type q_{ii} are called *the coefficients of capacity.*

Since a positive charge creates in an isolated conductor a positive potential, all the capacitance coefficients (with similar indices such as q_{ii}) are positive.

The reciprocity relation for the coefficients of potential : The potential of one conductor due to a unit charge on another is equal to the potential of the second due to a unit charge on the first, *i.e.*, $p_{12} = p_{21}$, or in the generalised form

$$p_{ij} = p_{ji}, \qquad ...(87)$$

This is known as the *reciprocity relation for the coefficients of potential.*

To prove it, let us consider a simple case of three conductors, labelled as 1, 2, 3. Let charges Q_i give rise to potentials V_i and charges Q_i' give rise to potentials V_i' (i = 1, 2, 3). The potential V_1 on the first conductor depends upon the charge on these three conductors. In terms of charge densities on their surfaces, the potential on the conductor 1 is given by

$$V_1 = \frac{1}{4\pi\varepsilon_0}\left[\int_{S_1} \frac{\sigma_1 dS_1}{r_{11}} + \int_{S_2} \frac{\sigma_2 dS_2}{r_{12}} + \int_{S_3} \frac{\sigma_3 dS_3}{r_{13}}\right]$$

where σ_1, σ_2 and σ_3 are the surface charge densities, r_{11} the distance from the surface element dS_1 of the first conductor to a fixed point inside it, r_{12} the distance from the area dS_2 of the second conductor to the same point and r_{13} is the distance from the area element dS_3 of the third conductor to the same point.

Since $Q_1' = \int_{S_1} \sigma_1' \, dS_1$ and $Q_1 = \int_{S_1} \sigma_1 dS_1$, hence

$$\begin{aligned}
Q_1'V_1 + Q_2'V_2 + Q_3'V_3 &= \frac{1}{4\pi\varepsilon_0}\int_{S_1} \sigma_1' \, dS_1\left[\int_{S_1} \frac{\sigma_1 dS_1}{r_{11}} + \int_{S_2} \frac{\sigma_2 dS_2}{r_{12}} + \int_{S_3} \frac{\sigma_3 dS_3}{r_{13}}\right] \\
&+ \frac{1}{4\pi\varepsilon_0}\int_{S_2} \sigma_2' \, dS_2\left[\int_{S_1} \frac{\sigma_1 dS_1}{r_{21}} + \int_{S_2} \frac{\sigma_2 dS_2}{r_{22}} + \int_{S_3} \frac{\sigma_3 dS_3}{r_{23}}\right] \\
&+ \frac{1}{4\pi\varepsilon_0}\int_{S_3} \sigma_3' \, dS_3\left[\int_{S_1} \frac{\sigma_1 dS_1}{r_{31}} + \int_{S_2} \frac{\sigma_2 dS_2}{r_{32}} + \int_{S_3} \frac{\sigma_3 dS_3}{r_{33}}\right] \\
&= \frac{1}{4\pi\varepsilon_0}\int_{S_1} \sigma_1 \, dS_1\left[\int_{S_1} \frac{\sigma_1' dS_1}{r_{11}} + \int_{S_2} \frac{\sigma_2' dS_2}{r_{21}} + \int_{S_3} \frac{\sigma_3' dS_3}{r_{31}}\right] \\
&+ \frac{1}{4\pi\varepsilon_0}\int_{S_2} \sigma_2 \, dS_2\left[\int_{S_1} \frac{\sigma_1' dS_1}{r_{12}} + \int_{S_2} \frac{\sigma_2' dS_2}{r_{22}} + \int_{S_3} \frac{\sigma_3' dS_3}{r_{32}}\right] \\
&+ \frac{1}{4\pi\varepsilon_0}\int_{S_3} \sigma_3 \, dS_3\left[\int_{S_1} \frac{\sigma_1' dS_1}{r_{13}} + \int_{S_2} \frac{\sigma_2' dS_2}{r_{23}} + \int_{S_3} \frac{\sigma_3' dS_3}{r_{33}}\right] \\
&= Q_1V_1' + Q_2V_2' + Q_3V_3'. \qquad ...(88)
\end{aligned}$$

Here we have interchanged the order of integration and taken $r_{ij} = r_{ji}$. The relation (88) may be extended for more conductors. Thus we have

$$\Sigma Q_i' V_i = \Sigma Q_i V_i'. \qquad ...(89)$$

The relation (88) or (89) is known as *Green's Reciprocal theorem.*

As a particular case, let us consider various charges.

$$Q_1 = 1,\ Q_2 = 0,\ Q_3 = 0;\ Q_1' = 0,\ Q_2' = 1,\ Q_3' = 0, \qquad \text{...(90)}$$

$$\therefore \quad V_2 = V_1'.$$

Its comparison with equation (85) gives

$$V_2 = p_{12} \text{ and } V_1' = p_{21},$$

$$\therefore \quad p_{12} = p_{21}.$$

In the generalised form, it may be written as $p_{ij} = p_{ji}$. ...(91)

Hence is the required relation.

The Screening Theorem : Let us consider three conductors; conductor 1 is surrounded by an earthed conductor 2 and the conductor 3 is in the neighbourhood. As conductor 2 is earthed, $V_2 = 0$ and thus the charges on these conductors are given as

$$Q_1 = q_{11}V_1 + q_{31}V_3$$

$$Q_2 = q_{12}V_1 + q_{32}V_3$$

$$Q_3 = q_{13}V_1 + q_{33}V_3. \qquad \text{...(92)}$$

Let us assume $Q_1 = 0$. Since the conductor 1 is enclosed within an earthed conductor 2, potential V_1 of the conductor 1 will be same as that of conductor 2 (*i.e.*, zero). Therefore

$$Q_1 = q_{11} \times 0 + q_{31}V_3 = 0.$$

This must be true for all values of V_3. It will be possible for $q_{31} = 0$. Thus we see that the conductor 3 has no electric influence on conductor 1 and vice-versa, *i.e.*, the earthed conductor 2 around conductor 1 has electrically screened it from external charges (charge on the external conductor 3). Thus to protect an electrostatic apparatus from electrical disturbances, it should be enclosed in a grounded metal conductor. This is known as the *screening theorem*.

7.7 FIELD DUE TO A SET OF COLLINEAR POINT CHARGES

Let point charges $q_1, q_2, q_3, \ldots$ be situated at points A_1, A_2, A_3 ...... on a line LM, as shown in Fig. 7.7. Let P be any point having polar co-ordinates $(r_1, \theta_1), (r_2, \theta_2), (r_3, \theta_3)$ ..... with respect to point A_1, A_2, A_3, ... respectively.

The electric field intensities at point P due to these charges are respectively $q_1\mathbf{r}_1/4\pi\varepsilon_0 r_1^3$, $q_2\mathbf{r}_2/4\pi\varepsilon_0 r_2^3$, $q_3\mathbf{r}_3/4\pi\varepsilon_0 r_3^3$, ... The sum of their resolutes along normal to the line of force at P must vanish, or

$$\Sigma\ (q_i \sin \phi_i/4\pi\varepsilon_0 r_i^2) = 0,$$

where ϕ_i is the angle which the tangent to the line of force at P makes with the radius vector r_i. As $\sin \phi_i = r_i d\theta_i/ds$, hence

$$\Sigma \frac{q_i d\theta_i}{4\pi\varepsilon_0 r_i ds} = 0 \text{ or } \Sigma \frac{q_i d\theta_i}{r_i} = 0.$$

From the geometry of figure, we have

$r_1 \sin \theta_1 = r_2 \sin \theta_2 = r_3 \sin \theta_3 \ldots = \lambda$ (say).

Therefore

$$(1/\lambda)\ \Sigma q_i \sin \theta_i\, d\theta_i = 0 \text{ or } \Sigma q_i \sin \theta_i\, d\theta_i = 0,$$

which gives on integration $\Sigma q_i \cos \theta_i =$ constant

$$\text{or } q_1 \cos \theta_1 + q_2 \cos \theta_2 + q_3 \cos \theta_3 + \ldots\ldots = \text{constant}. \qquad \text{...(93)}$$

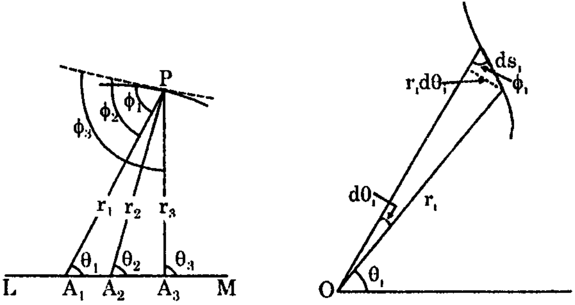


Fig. 7.7. Field due to collinear point charges.

7.8 FIELD DUE TO A NUMBER OF LONG CHARGED PARALLEL FINE WIRES

Let there be a number of long charged parallel fine wires carrying charges (per unit length) $\lambda_1, \lambda_2, \lambda_3$ ... and cut the plane of paper at right angles at points A_1, A_2, A_3 ...... The electric field

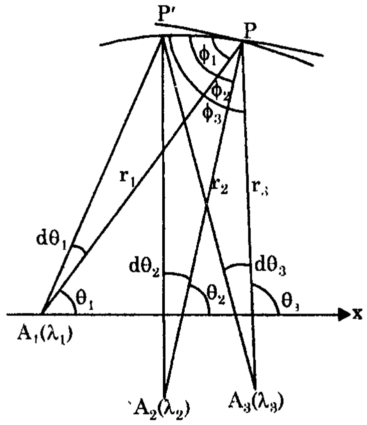


Fig. 7.8. Field due to long charged parallel wires.

at any point P due to these current carrying wires are $\lambda_1 \hat{r}_1 / 2\pi\varepsilon_0 r_1$, $\lambda_2 \hat{r}_2 / 2\pi\varepsilon_0 r_2$, $\lambda_3 \hat{r}_3 / 2\pi\varepsilon_0 r_3$ ...... respectively.

The sum of their resolutes along the normal to the line of force PP' must vanish.

or $\Sigma\ (\lambda_i/2\pi\varepsilon_0 r_i) \sin\theta_i = 0$, where ϕ_i, the angle which the tangent to the line of force PP' makes with the radius vector r_i, is given by

$$\sin\theta_i = r_i\, d\theta_i/PP'.$$

$$\therefore \Sigma\lambda_i\, d\theta_i/2\pi\varepsilon_0\, PP' = 0$$

or $$\Sigma\lambda_i\, d\theta_i = \lambda_1\, d\theta_1 + \lambda_2\, d\theta_2 + \lambda_3\, d\theta_3 + \ldots = 0$$

On integration, we get

$$\lambda_1\theta_1 + \lambda_2\theta_2 + \lambda_3\theta_3 + \ldots\ldots = \text{const.} \qquad \ldots(94)$$

It must be noted that if any charge lies on the opposite side of the line of force, the sign of the term due to this must be changed.

EXERCISES

Example 1. *Two parallel conducting planes in free space are at $y = 0$ and $y = 0.02m$, and the zero voltage reference is at $y = 0.01$ m. If $\mathbf{D} = 253\ \mathbf{a}_y\ nC/m^2$ between the conductors, determine the conductor voltages.*

Since V is not a function of x or z, Laplace's equation reduces to

$$d^2V/dy^2 = 0.$$

Its solution is $V = Ay + B$.

Therefore $\mathbf{E} = \mathbf{D}/\varepsilon_0 = -\nabla V = -A\mathbf{a}_y$

or $$\frac{253 \times 10^{-9}}{8.854 \times 10^{-12}}\mathbf{a}_y = -A\mathbf{a}_y \text{ or } A = -2.86 \times 10^4\ V/m.$$

$\therefore$ $V = -2.86 \times 10^4\, y + B$.

Since $V = 0$ at $y = 0.01m$, therefore

$B = 2.86 \times 10^2\ V$.

$\therefore$ $V = -2.86 \times 10^4 y + 2.86 \times 10^2$ volt

$V(\text{at } y = 0) = 286\ V$

$V(\text{at } y = 0.02) = -576 + 286 = -286\ V$.

Example 2. *The electrostatic potential in free space is given by $\varphi = a - b\ (x^2 + y^2)$ where a and b are constants. Find the charge density in the region.*

The charge density ρ can be obtained by Poisson's equations $\nabla^2\varphi = -\rho/\varepsilon_0$ or $\rho = -\varepsilon_0\nabla^2\varphi$

$$\varphi = a - b\ (x^2 + y^2),\ \therefore\ \frac{\partial\varphi}{\partial x} = -\ b(2x) \text{ and } \frac{\partial^2\varphi}{\partial x^2} = -2b$$

Similarly $\partial^2\varphi/\partial y^2 = -\ 2b$

$$\therefore \qquad \rho = -\varepsilon_0\left[\partial^2\varphi/\partial x^2 + \partial^2\varphi/\partial y^2 + \partial^2\varphi/\partial z^2\right] = -\varepsilon_0\ (-2b - 2b) = 4\varepsilon_0 b.$$

Example 3. *In cylindrical coordinates two ϕ = const semi-infinite conducting planes are insulated along the z-axis. Neglecting fringing, calculate V and E in the region between the planes $\phi = 0$ ($V = 0$) and $\phi = \pi/5$ ($V = 100$ V).*

In this problem potential is constant with ρ and z and depends only on ϕ. Thus the Laplace's equation becomes

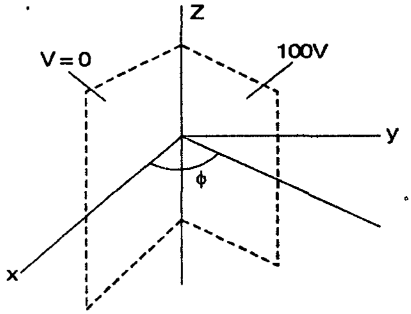


Fig. 7.9. Example 3.

$$\nabla^2 V = \frac{1}{\rho^2}\frac{d^2V}{d\phi^2} = 0 .$$

Since $\rho = 0$ is excluded due to insulating gap. We can therefore multiply by ρ^2 to obtain

$$d^2V/d\phi^2 = 0.$$

Integrating twice, we get $V = A\phi + B$.

Applying the boundary conditions

(*i*) $\phi = 0,\ V = 0 \quad \therefore \quad 0 = 0 + B \quad \text{or} \quad B = 0$

(*ii*) $\phi = \pi/5,\ V = 100 \quad \therefore \quad 100 = A\pi/5 \quad \text{or} \quad A = 500/\pi.$

$\therefore \qquad V = (500/\pi)\phi$

Therefore $$\mathbf{E} = -\nabla V = -\frac{1}{\rho}\frac{d}{d\phi}(V)\,\mathbf{a}_\phi = -\frac{1}{\rho}\frac{d}{d\phi}\left(\frac{500\,\phi}{\pi}\right)\mathbf{a}_\phi = -\frac{500}{\rho\pi}\,\mathbf{a}_\phi .$$

Example 4. *Find the potential and hence the electric intensity due to a cylindrical conductor placed in an uniform field with the Laplace equation.*

Consider an infinitely long uncharged conducting cylinder of circular cross section placed in a uniform electric field E_0 with its axis ⊥ to the field. If a be the radius of the cylinder and the field E_0 is along x-axis. It is clear from the symmetry that the potential is not a function of z and thus the problem is two-dimensional and can be written in the form of polar coordinates r and θ.

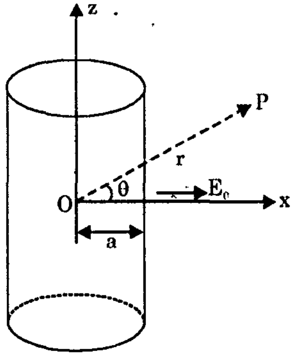


Fig. 7.10. Example 4.

At a very large distance from the origin, the potential must be

$$-E_0 x = -E_0 r\cos\theta.$$

Thus boundary conditions are

$$V = -E_0 r\cos\theta \qquad \text{for } r \to \infty$$
$$= 0 \qquad \text{for } r = a.$$

To find a potential function at a point P outside the cylinder, we must add second term consisting of first power of $\cos\theta$ and satisfying the Laplace's equation $\nabla^2 V = 0$, or

$$\frac{\partial}{\partial r}\left(r\,\frac{\partial V}{\partial r}\right) + \frac{1}{r}\frac{\partial^2 V}{\partial\theta^2} = 0. \qquad \text{(in two dimensional case)}$$

Thus the potential function must be of the form

$$V = -E_0 r\cos\theta + (A/r)\cos\theta.$$

The second term becomes zero at very large distances, and $V = 0$ at the surface of the cylinder, *i.e.*, at $r = a$, hence $A = E_0 a^2$. Thus we get the relation

$$V = -E_0 r\cos\theta + (E_0 a^2/r)\cos\theta = -(1 - a^2/r^2)\,E_0 r\cos\theta.$$

The first term is potential of the external field and the second term is as due to an extended dipole consisting of two parallel lines of positive and negative charges close to the z-axis.

The components of electric intensity in the direction of increasing r and θ are

$$E_r = -\frac{\partial V}{\partial r} = \left(1 + \frac{a^2}{r^2}\right)E_0\cos\theta \text{ and } E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -\left(1 - \frac{a^2}{r^2}\right)E_0\sin\theta.$$

The charge surface density (charge per unit area) of the conducting surface is

$$\sigma = k\varepsilon_0\,(E_r)_{r=a} = 2k\,\varepsilon_0\,E_0\cos\theta.$$

If the cylinder has a charge λ per unit length, then

$$V = -\left(1 - \frac{a^2}{r^2}\right)E_0 r\cos\theta - \frac{\lambda}{2\pi\varepsilon_0 k}\log_e\frac{r}{a}.$$

The second term increases E_r and σ. It is left as an exercise.

Example 5. *Two conducting co-axial cones ($\theta_1 = \pi/10$ and $\theta_2 = \pi/6$) of infinite extent are separated by an infinitesimal gap at $r = 0$. If $V = 0$ at $\theta = \pi/10$ and $50V$ at $\theta = \pi/6$, find the expression for V and* **E** *between the cones.*

As shown in Fig. 7.11, there are two concentric cones, where the gap acts as insulator between two conducting cones at the vertices. Thus potential is constant with r and ϕ. It depends on θ only. Thus the Laplace's equation in spherical coordinates becomes

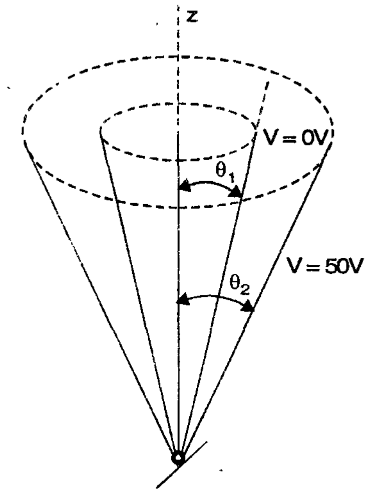


Fig. 7.11. Example 5.

$$\nabla^2 V = \frac{1}{r^2 \sin\theta}\frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0.$$

Since $r = 0$ and $\theta = 0, \pi$ are excluded, therefore we can multiply by $r^2 \sin\theta$ to obtain

$$\frac{d}{d\theta}\left(\sin\theta \frac{dV}{d\theta}\right) = 0.$$

On integration, we get

$$\sin\theta\,(dV/d\theta) = A \quad \text{(const.)}$$

or $$dV/d\theta = A/\sin\theta. \qquad ...(i)$$

On integration, we get

$$V = A\int \frac{1}{\sin\theta}d\theta = A\int \frac{d\theta}{2\sin(\theta/2)\cos(\theta/2)}$$

$$= A\int \frac{1}{2}\frac{\sec^2(\theta/2)}{\tan(\theta/2)}d\theta = A\log_e \tan(0/2) + B. \qquad ...(ii)$$

Using the boundary conditions, integration constants can be obtained as :

$$V = 0 \quad \text{at } \theta = \pi/10, \quad 0 = A\log_e \tan(\pi/20) + B \;\therefore\; B = -A\log_e \tan(\pi/20).$$

$$V = 50 \quad \text{at } \theta = \pi/6, \quad 50 = A\log_e \tan(\pi/12) - A\log_e \tan(\pi/20)$$

$$= A \log_e [\tan(\pi/12)/\tan(\pi/20)]$$

or $$A = 50/\log_e [\tan(\pi/12)/\tan(\pi/20)] = 95.1$$

$\therefore$ $$V = 95.1\left[\log_e \tan\frac{\theta}{2} - \log_e \tan\frac{\pi}{20}\right]$$

$$= 95.1 \log_e [\tan(\pi/2)/0.1584]V$$

Therefore $$\mathbf{E} = -\nabla V = -\frac{1}{r}\frac{dV}{d\theta}\mathbf{a}_\theta = -\frac{95.1}{r\sin\theta}\mathbf{a}_\theta \; V/m.$$

Example 6. *Find the potential and hence the electric field due to a spherically symmetric charge distribution of total charge q which has the radial dependence $\rho(r) = \rho_0 e^{-\alpha r}$.*

On account of the spherical symmetry, the potential will be a function of r only in the spherical coordinates. At an arbitrary position r, the potential V satisfies the Poisson's equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = -\frac{\rho(r)}{\varepsilon_0}.$$

$$r^2\frac{dV}{dr} = -\int\frac{\rho(r)}{\varepsilon_0}r^2dr = -\frac{\rho_0}{\varepsilon_0}\int e^{-\alpha r}r^2dr = \frac{\rho_0}{\varepsilon_0}\frac{e^{-\alpha r}}{\alpha^3}(2 + 2\alpha r + \alpha^2 r^2) + A, \qquad ...(i)$$

where A is a constant. The potential V can be found as under :

$$dV = \left[\frac{\rho_0}{\varepsilon_0}\frac{e^{-\alpha r}}{\alpha^3}\left(\frac{2}{r^2} + \frac{2\alpha}{r} + \alpha^2\right) + \frac{A}{r^2}\right]dr$$

or $$V = \frac{q}{4\pi\varepsilon_0 r}e^{-\alpha r} - \frac{\alpha q}{8\pi\varepsilon_0}e^{-\alpha r} - \frac{A}{r} + B. \qquad ...(ii)$$

where B' is another constant and $q = 8\pi\,\rho_0/\alpha^3$.

Let us apply the boundary conditions to evaluate A and B. Since the charge density $\rho \to 0$ as $r \to \infty$, the potential may be set as zero at $r \to \infty$. This will give $B = 0$. To evaluate A, let us apply Gauss's law on a closed shell of radius r and centre at the origin. The electric field $E = -(dV/dr)\,\hat{r}$. Thus equation (*i*) will give us **E**, therefore

$$\oint \mathbf{E}\cdot\hat{n}\, dS = \frac{1}{\varepsilon_0}\int \rho dv = \frac{4\pi\rho_0}{\varepsilon_0}\int_0^r e^{-\alpha r} r^2 dr.$$

Which gives $A = q/4\pi\varepsilon_0$. Thus the potential is given by

$$V(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}[1 - e^{-\alpha r}] - \frac{q\alpha}{8\pi\varepsilon_0} e^{-\alpha r}.$$

The electric field is thus given by

$$E(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r^2}\left[1 - (\alpha r + 1)e^{-\alpha r}\right]\hat{r} - \frac{q\alpha^2}{8\pi\varepsilon_0} e^{-\alpha r}\,\hat{r}.$$

Example 7. *Find the capacitance between the oppositely placed cones of Fig. 7.12. The space is free from dielectric. The potential is V_1 at θ_1 and zero at θ_2.*

If the fringing is neglected. As discussed in example 4, the potential V is constant with r and ϕ. It depends on q only and is obtain by solving θ dependent part of Laplace's equation in spherical coordinates. The solution is obtained as

$$V = A \ln \tan(\theta/2) + B,$$

where A and B are constants, can be obtained by boundary conditions.

In the present problem $\theta_2 - \theta_1 = 30°$ and $\theta_2 + \theta_1 = 180°$, therefore $\theta_1 = 75°$ and $\theta_2 = 105°$. The boundary conditions are :

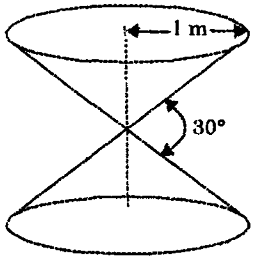


Fig. 7.12. Example 7.

$$V = V_1 \text{ at } \theta_1 = 75° \text{ and } V = 0 \text{ at } \theta_2 = 105°.$$

$$\therefore \quad V_1 = A \ln\tan(75/2) + B$$

and $$0 = A \ln\tan(105/2) + B$$

Solving these equations, we get

$$V_1 = A\,[\ln\tan(37.5°) - \ln\tan(52.5°)]$$

or $$A = V_1/[\ln(\tan 37.5°) - \ln(\tan 52.5°)]$$

and $$B = -V_1 \ln(\tan 52.5°)/[\ln(\tan 37.5° - \ln(\tan 52.5°)]$$

$$\therefore \quad V = \frac{V_1\,[\ln(\tan\theta/2) - \ln(\tan 52.5°)]}{\ln(\tan 37.5°) - \ln(\tan 52.5°)}$$

$$= -1.89\,V_1 \ln(\tan\theta/2) + 0.5\,V_1$$

The electric field $$\mathbf{E} = -\frac{1}{r}\frac{dV}{d\theta}\hat{\theta} = \frac{1.89\,V_1}{r\sin\theta}\hat{\theta}$$

$$\therefore \quad \mathbf{D} = \varepsilon\mathbf{E} = (1.89\,\varepsilon_0 V_1/r\sin\theta)\,\hat{\theta}.$$

The charge density on the upper plate $\sigma_s = D_n = 1.89\,\varepsilon_0 V_1/r \sin 75°$.

$$= 1.96\,\varepsilon_0 V_1/r.$$

Total charge on the upper plate $Q = \int \sigma_s dS = \int_0^{2\pi}\int_0^{75°} \frac{1.89\,\varepsilon_0 V_1}{r\sin 75°}\, r\sin 75°\, dr\, d\phi$

$$= 12.28\,\varepsilon_0 V_1.$$

$$\therefore \quad \text{Capacitance } C = Q/V_1 = 12.28\,\varepsilon_0.$$

Example 8. *A sphere of dielectric constant* ε *is placed in a uniform electric field* E_0. *Show that the induced surface charge density is*

$$\sigma(\theta) = [\varepsilon - \varepsilon_0]/(\varepsilon + 2\varepsilon_0)]\, 3\varepsilon_0 E_0 \cos\theta,$$

where θ *is the angle from the* E_0 – *direction. If the sphere is rotated at an angular velocity* ω *about the direction of* E_0, *will a magnetic field be produced?*

As discussed earlier, the electrostatic potential at a point inside and outside the sphere be φ_1 and φ_2. The general solution of Laplace equation using z-symmetry is given as

$$\varphi_1 = \sum_{n=0}^{\infty} [A_n r^n + B_n r^{-(n+1)}]\, P_n(\cos\theta) \qquad r < a$$

and

$$\varphi_2 = \sum_{n=0}^{\infty} [C_n r^n + D_n r^{-(n+1)}]\, P_n(\cos\theta) \qquad r \geq a$$

where A_n, B_n, C_n, and D_n are constants and P_n are legendre polynomials. The boundary conditions are :

(1) φ_1 is finite at $r = 0$,(2) φ_2 at $r \to \infty = -E_0 r \cos\theta = -E_0 r\, P_1(\cos\theta)$

(3) At $r = a$ $\varphi_1 = \varphi_2$ and $\varepsilon_1(\partial\varphi_1/\partial r) = \varepsilon_2(\partial\varphi_2/\partial r)$.

From conditions (1) and (2) we get $B_n = 0$, $C_1 = -E_0$, $C_n = 0$ $(n \neq 1)$.

From condition (3), we get

$$\sum_n A_n a^n P_n(\cos\theta) = -Ea\, P_1(\cos\theta) + \sum_n \frac{D_n}{a^{n+1}} P_n(\cos\theta)$$

$$\varepsilon \sum_n A_n n a^{n-1} P_n(\cos\theta) = -\varepsilon_0 \left[E_0 P_1(\cos\theta) + \sum_n (n+1) \frac{D_n}{a^{n+2}} P_n(\cos\theta) \right]$$

These equations must satisfy for each possible value of θ, *i.e.*, the coefficients of $P_n(\cos\theta)$ on the both sides of equation must be equal for every value of n. This gives

$$A_1 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0,\ D_1 = \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 a^3,\ A_n = D_n = 0\ (n \neq 1).$$

Hence the electric potentials inside and outside the sphere can be expressed as :

$$\varphi_1 = -\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} E_0 r \cos\theta$$

$$\varphi_2 = -\left[1 - \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \left(\frac{a}{r}\right)^3 \right] E_0 r \cos\theta.$$

Therefore the electric field intensites inside and outside the sphere

$$\mathbf{E}_1 = -\nabla\varphi_1 = \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} \mathbf{E}_0 \qquad (r < a)$$

$$\mathbf{E}_2 = -\nabla\varphi_2 = \mathbf{E}_0 + \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 \left[\frac{3(\mathbf{E}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{E}}{r^3} \right] r > a.$$

The electric field inside the sphere E_1 gives the polarization of the dielectric as

$$\mathbf{P} = (\varepsilon - \varepsilon_0)\, \mathbf{E} = \frac{3\varepsilon_0\, (\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} \mathbf{E}_0 .$$

The bound charge density on the surface of the dielectric sphere is

$$\sigma(\theta) = \mathbf{n} \cdot \mathbf{P} = \frac{3\varepsilon_0(\varepsilon - \varepsilon_0)}{\varepsilon + 2\,\varepsilon_0} E_0 \cos\theta$$

The total electric dipole moment is thus given by

$$\mathbf{p}_E = \tfrac{4}{3}\pi a^3\, \mathbf{P} = 4\pi\varepsilon_0\, (\varepsilon - \varepsilon_0)\, (\varepsilon + 2\varepsilon_0)^{-1}\, a^3 \mathbf{E}_0 .$$

τ has the same direction as $\mathbf{E}_0$. Thus when the sphere is rotated about the direction of $\mathbf{E}_0$, $\mathbf{p}_E$ will not change. This implies that the rotation will not give rise to a polarization current and therefore will not produce a magnetic field.

Example 9. *A surface charge density $\sigma(\theta) = \sigma_0 \cos\theta$ is glued to the surface of a spherical shell of radius R, where σ_0 is a constant and θ is the polar angle. There is a vacuum with no charges, both inside and outside of the shell. Calculate the electrostatic potential and the electric field both inside and outside of the spherical shell.*

Let φ_i and φ_o be the potentials inside and outside the shell respectively. Both these potentials satisfy the Laplace's equation. On account of the cylindrical symmetry they have the expressions

$$\varphi_i = \sum_{n=0} A_n r^n P_n(\cos\theta) \qquad r < R$$

$$\varphi_o = \sum_{n=0} B_n r^{-(n+1)} P_n(\cos\theta) \qquad r > R$$

The boundary conditions at $r = R$ for the potential and the displacement vector are

$$\varphi_i = \varphi_o$$

$$\sigma(\theta) = \sigma_0 P_1(\cos\theta) = \varepsilon_0 (\partial\varphi_i/\partial r - \partial\varphi_o/\partial r).$$

Substituting expressions for the potentials in the above equations and equating the coefficients of $P_n(\cos\theta)$ on the both sides of the equations, we get

$$A_n = B_n = 0 \qquad \text{for } n \neq 1$$

$$A_1 = \sigma_0/3\,\varepsilon_0,\ B_1 = \sigma_0 R^3/3\varepsilon_0 \qquad \text{for } n = 1$$

Therefore

$$\varphi_i = \sigma_0 r \cos\theta/3\,\varepsilon_0 \qquad r < R$$

$$\varphi_o = \sigma_0 R^3 \cos\theta/3\varepsilon_0\, r^2 \qquad r > R$$

Using $\mathbf{E} = -\nabla\varphi$, we get

$$E_i = -(\sigma_0/3\varepsilon_0)\,\hat{z} \qquad r < R$$

and

$$E_o = \frac{2\sigma_0 R^3}{3\varepsilon_0 r^3}\cos\theta\,\hat{r} + \frac{\sigma_0 R^3}{3\varepsilon_0 r^3}\sin\theta\,\hat{\theta} \qquad r > R.$$

Example 10. *If the potential of a spherical shell of zero thickness depends only on the polar angle θ and is given by $V(\theta) = V_0 \cos^2\theta$, inside and outside the sphere there being empty space. Find the expressions for the potentials inside and outside the sphere and an expression for the electric sources on the sphere.*

Since both inside and outside of the spherical shell are empty space, the potential will satisfy the Laplace's equation. Thus the potential functions inside and outside the sphere are given as

$$\varphi_{\text{inside}} = \sum_{n=0}^{\infty} A_n r^n P_n(\cos\theta) \qquad \text{...}(i)$$

$$\varphi_{\text{outside}} = \sum_{n=0}^{\infty} B_n r^{-(n+1)} P_n(\cos\theta) \qquad \text{...}(ii)$$

If a be the radius of the shell, then we have

$$V(\theta) = \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) \qquad \text{...}(iii)$$

Multiplying both side by $P_m(\cos\theta)\sin\theta\, d\theta$ and integrating from 0 to π, we get

$$\int_0^\pi V(\theta) P_m(\cos\theta)\sin\theta\, d\theta = \int_0^\pi \sum_{n=0}^{\infty} A_n a^n P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta. \qquad \text{...}(iv)$$

Using the orthogonal property of the Legendre polynomials $\int_0^\pi P_n(\cos\theta) P_m(\cos\theta)\sin\theta\, d\theta = \frac{2}{2n+1}$. The integral is zero for $n \neq m$. Therefore equation (iv) reduces to

$$A_n a^n \cdot \frac{2}{2n+1} = \int_0^\pi V(\theta)\, P_n\,(\cos\theta)\sin\theta\, d\theta.$$

$$\therefore \qquad \varphi_{\text{inside}} = \sum_{n=0}^{\infty}\left[\frac{2n+1}{2a^n}\int_0^\pi V(\theta)\, P_n\,(\cos\theta)\sin\theta\, d\theta\right] r^n\, P_n(\cos\theta)$$

and
$$\varphi_{\text{outside}} = \sum_{n=0}^{\infty}\left[\frac{(2n+1)a^{n+1}}{2}\int_0^\pi V(\theta)\, P_n\,(\cos\theta)\sin\theta\, d\theta\right] r^{-(n+1)}\, P_n(\cos\theta).$$

Given that
$$V(\theta) = V_0\cos^2\theta = \tfrac{1}{3}V_0\, P_0\,(\cos\theta) + \tfrac{2}{3}V_0\, P_2\,(\cos\theta)$$

$$\therefore \quad \varphi_{\text{inside}} = \sum_{n=0}^{\infty}\frac{2n+1}{2a^n}\left[\int_0^\pi \frac{1}{3}V_0\, P_0\,(\cos\theta)\, P_n(\cos\theta)\sin\theta\, d\theta + \int_0^\pi \frac{2}{3}V_0\, P_2\,(\cos\theta)\, P_n(\cos\theta)\sin\theta\, d\theta\right] r^n\, P_n(\cos\theta)$$

The first integral exists for $n = 0$ and the second one for $n = 2$, therefore

$$\begin{aligned}
\varphi_{\text{inside}} &= \frac{1}{2}\cdot\frac{1}{3}V_0\cdot 2 + \frac{5}{2}\cdot\frac{2}{3}\frac{V_0}{a^2}\cdot\frac{2}{5}\cdot r^2\, P_2\,(\cos\theta)\\
&= \frac{1}{3}V_0 + \frac{2}{3}V_0\frac{r^2}{a^2}P_2\,(\cos\theta) \qquad (r < R)
\end{aligned}$$

and
$$\varphi_{\text{outside}} = \frac{1}{3}\frac{V_0 a}{r} + \frac{2V_0}{3}P_2\,(\cos\theta)\frac{a^3}{r^3} \qquad (r > R)$$

The charge distribution on the spherical shell is given by the boundary condition for the displacement vector

$$\begin{aligned}
\sigma(\theta) &= \varepsilon_0\left(\frac{\partial\phi_{\text{inside}}}{\partial r}\right)_{r=a} - \varepsilon_0\left(\frac{\partial\phi_{\text{outside}}}{\partial r}\right)_{r=a}\\
&= \varepsilon_0\sum_{n=0}^{\infty}\left[\frac{(2n+1)^2}{2a}\int_0^\pi (V_0\cos^2\theta)\, P_n\,(\cos\theta)\sin\theta\, d\theta\right] P_n(\cos\theta)\\
&= \frac{\varepsilon_0 V_0}{3a}\left[1 + 5\, P_2(\cos\theta)\right].
\end{aligned}$$

Example 11. *A conducting sphere of radius a carrying a charge q is placed in a uniform electric field* E_0*. Find the potentials inside and outside of the sphere. What is the dipole moment of the induced charge on the sphere? Identify six energy terms produced by three electric fields.*

Let φ_{in} and φ_{out} be the total potentials inside and outside the sphere respectively. Then from Laplace's equation, we get $\nabla^2\varphi_{\text{in}} = \nabla^2\varphi_{\text{out}} = 0$ and $\varphi_1 = \varphi_0$ a constant.

The boundary conditions are

$$\begin{aligned}
\varphi_{\text{in}} &= \varphi_{\text{out}} && \text{for } r = a\\
\varphi_{\text{out}} &= -E_0\, r\, P_1\,(\cos\theta) && \text{for } r \to \infty.
\end{aligned}$$

Using the cylindrical symmetry of spherical coordinates, the general solution of Laplace's equation is given by

$$\varphi_{\text{out}} = \sum_n (A_n r^n + B_n r^{-(n+1)})\, P_n\,(\cos\theta)$$

Insertion of boundary conditions gives

$$A_1 = -E_0,\ B_0 = A_0\,\varphi_0,\ B_1 = E_0 a^3,$$

while all other coefficients are zero. Since the surface charge density

$$\sigma = -\varepsilon_0\,(\partial\varphi_{\text{out}}/\partial r)_{r=a}$$

$$\therefore \qquad q = \int_0^\pi (3\varepsilon_0 E_0\cos\theta + \varepsilon_0\varphi_0/a)\, 2\pi a^2\sin\theta\, d\theta = 4\pi a\,\varepsilon_0\varphi_0$$

or
$$\varphi_o = q/4\pi\varepsilon_0 a.$$

Thus the potentials inside and outside the sphere are

$$\varphi_{\text{in}} = q/4\pi\varepsilon_0 a \qquad r < a$$

$$\varphi_{\text{out}} = -E_0 r\cos\theta + q/4\pi\varepsilon_0 r + (E_0 a^3/r^2)\cos\theta \qquad r > a.$$

The field outside the sphere may be considered as the superposition of three fields namely, the uniform electric field $\mathbf{E}_0$, a field due to the charge q uniformly distributed over the sphere, and a dipole field due to charges induced on the surface of the sphere, *i.e.*, produced by dipole of moment $\mathbf{P} = 4\pi\varepsilon_0 a^3\,\mathbf{E}_0$ located at the spherical centre.

The energies of these three fields are the energies produced by each field alone and the interactions among these fields.

The energy density of the uniform external field $E_0 = \frac{1}{2}\,\varepsilon_0 E_0^{\,2}$. Its total energy over the entire space $= \frac{1}{2}\varepsilon_0 \int E_0^2\, dV$ is infinite, *i.e.*, $W_1 \to \infty$.

The total electrostatic energy of an isolated conducting sphere with charge q is

$$W_2 = \int_0^\infty \tfrac{1}{2}\,\varepsilon_0\,(q/4\pi\varepsilon_0 r^2)^2\, 4\pi r^2 dr = q^2/8\pi\varepsilon_0 a.$$

The electric intensity outside the sphere due to the dipole P is

$$\mathbf{E}_3 = -\nabla\left(\frac{E_0 a^3}{r^2}\cos\theta\right) = \frac{2a^3 E_0\cos\theta}{r^3}\,\hat{r} + \frac{a^3 E_0 \sin\theta}{r^3}\,\hat{\theta}.$$

∴ The corresponding energy density

$$W_3 = \frac{1}{2}\,\varepsilon_0 E_3^{\,2} = \frac{\varepsilon_0}{2}\,\frac{a^6 E_0^2}{r^6}\,(4\cos^2\theta + \sin^2\theta) = \frac{\varepsilon_0 a^6 E_0^2}{2r^6}(1 + 3\cos^2\theta)$$

As the dipole does not give rise to a field inside the sphere, the total energy of P is

$$W_3 = \int W_3 dV = \int_a^\infty \frac{\varepsilon_0 a^6 E_0^2}{2}\cdot\frac{dr}{r^4}\int_0^{2\pi} d\phi \int_{-1}^{1}(1 + 3\cos^2\theta)\, d\,(\cos\theta) = \tfrac{4}{3}\pi\varepsilon_0 a^3 E_0^2.$$

The interaction energy between the conducting sphere and the external field E_0 is given by

$$W_{12} = \int \sigma\,(-E_0\, a\cos\theta)\, 2\pi a^2 \sin\theta\, d\theta = \frac{qa\,E_0}{2}\int_{-1}^{1}\cos\theta\, d\,(\cos\theta) = 0.$$

(Here σ = surface charge density $= q/4\pi a^2$).

The interaction energy between the conducting sphere and the field of dipole $\mathbf{P}$ is

$$W_{23} = \int_0^\pi \sigma\left(\frac{E_0 a^3}{r^2}\cos\theta\right) 2\pi a^2 \sin\theta\, d\theta = 0\,.$$

The interaction energy between dipole $\mathbf{P}$ and the external field $\mathbf{E}_0$

$$W_{13} = -\ \tfrac{1}{2}\ \mathbf{P}\cdot\mathbf{E}_0 = -\,2\pi\varepsilon_0\, a^3\, E_0^{\,2}.$$

Thus we see that $W_1 \to \infty$, W_2 and W_3 are finite; $W_{12} = W_{23} = 0$ and W_{13} = finite.

Example 12. *The potential at a distance r from the axis of an infinite straight wire of radius a carrying a charge per unit length λ is given by λ ln (1/r) + const. If this wire is placed at a distance b >> a from an infinite metal plane, maintained at zero potential, calculate the capacitance per unit length of the wire of this system.*

In order to maintain the potential of the metal plane at zero potential, we consider an image of the given wire, which is an infinite straight wire with linear charge density $-\lambda$ symmetrically placed on the other side of the plane at the same distance b. Thus the capacitance of the system will be the capacitance of two parallel wires with equal and opposite charges and placed at a distance $2b$. The potential at a point between the two wires (wire and its image) at a distance r from the original wire is given by

$$\varphi(r) = \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{1}{r} - \frac{\lambda}{2\pi\varepsilon_0}\ln\frac{1}{2b - r}$$

∴ Potential difference between these two wires $V = \varphi(a) - \varphi\,(2b - a)$

$$= \frac{\lambda}{2\pi\varepsilon_0}\left[\ln\frac{1}{a} - \ln\frac{1}{2b-a} - \ln\frac{1}{2b-a} + \ln\frac{1}{a}\right]$$

$$= \frac{\lambda}{\pi\varepsilon_0}\left[\ln\frac{1}{a} - \ln\frac{1}{2b-a}\right] = \frac{\lambda}{\pi}\ln\frac{2b-a}{a} \approx \frac{\lambda}{\pi}\ln\frac{2b}{a}$$

Therefore the capacitance of the system per unit length of wire

$$C = \frac{q/l}{V} = \frac{\lambda}{V} = \frac{\pi\varepsilon_0}{\ln\,(2b/a)}.$$

Example 13. *Two large flat conducting plates separated by a distance d are connected by a wire. A point charge q is placed midway between the two plates. Find an expression for the surface charge density induced on the upper plate as a function of d, q and x, the distance from the charge q along the plate length.*

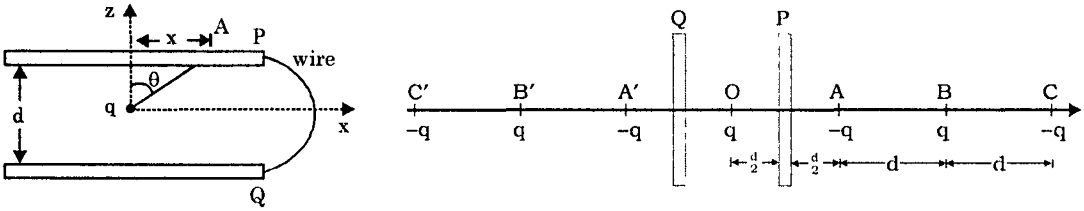


Fig. 7.13 (*a*) System, (*b*) Plates with image charges.

As the plates *A* and *B* are connected by a wire, therefore must have the same potential. The problem can be solved by the method of images. The image charges lying on the *z*-axis are shown in Fig. 7.13(*b*). If we consider an arbitrary point A_0 on the plate *P*. It can be seen that the electric field **E** at point A_0 has only the *z*-component and its magnitude as given by

$$E_z = \frac{q}{4\pi\,\varepsilon_0\,[(d/2)^2 + x^2]}\cdot\frac{d}{[(d/2)^2 + x^2]^{1/2}} - \frac{q}{4\pi\,\varepsilon_0[(3d/2)^2 + x^2]}\cdot\frac{3d}{[(3d/2)^2 + x^2]^{1/2}}$$

$$+ \frac{q}{4\pi\,\varepsilon_0[(5d/2)^2 + x^2]}\cdot\frac{5d}{[(5d/2)^2 + x^2]^{1/2}}\cdots$$

$$= \frac{qd}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{(-1)^n(2n+1)}{[(n+1/2)^2 d^2 + x^2]^{3/2}}$$

Therefore the surface charge density

$$\sigma(x) = -\,\varepsilon_0 E_z = -\frac{qd}{4\pi}\sum_{n=0}^{\infty}\frac{(-1)^n\,(2n+1)}{[(n+1/2)^2 d^2 + x^2]^{3/2}}$$

Example 14. *What is the least positive charge that must be given to a spherical conductor of radius a, insulated and influenced by an external point charge +q at a distance r > a from the center, in order that the surface charge density on the sphere may be everywhere positive? What will happen if the external point charge is −q?*

Consider the cartesian coordinates with the origin as the centre of the sphere and the *x*-axis along the line joining the centre of the sphere and the charge *q*. It is obvious that the greatest induced surface charge density, which is negative, on the sphere will occur at point *A* $(a, 0, 0)$. The action of the conducting spherical surface may be replaced by a point charge $-aq/r$ at $(a^2/r, 0, 0)$ and a point charge aq/r at the centre 0. Then the field at point *A* is given by

$$E_A = \frac{1}{4\pi\varepsilon_0}\left[\frac{aq/r}{a^2} - \frac{q}{(r-a)^2} - \frac{aq/r}{(a-a^2/r)^2}\right]\hat{x}$$

$$= \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{ar} - \frac{1}{(r-a)^2} - \frac{r}{a(r-a)^2}\right]\hat{x}.$$

Therefore the maximum negative induced surface charge density is

$$\sigma_A = \varepsilon_0 E = \frac{q}{4\pi}\left[\frac{1}{ar} - \frac{1}{(r-a)^2} - \frac{r}{a(r-a)^2}\right]$$

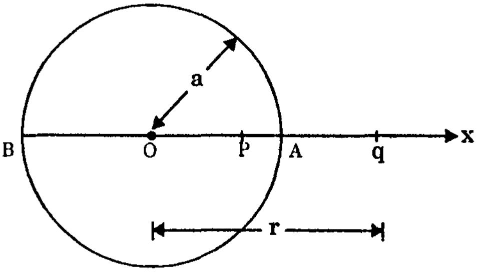


Fig. 7.14. Example 14.

If a positive charge Q is given to the sphere, it will distribute uniformly on the spherical surface with a surface charge density $Q/4\pi a^2$. Thus the net surface charge density is every where positive if

$$\frac{Q}{4\pi a^2} \geq -\sigma_A\left(= -\frac{q}{4\pi}\left[\frac{1}{ar} - \frac{1}{(r-a)^2} - \frac{r}{a(r-a)^2}\right]\right)$$

or $$Q \geq a^2\,(3r-a)\,q/r\,(r-a)^2.$$

On the other hand, the field at point $B\,(-a, 0, 0)$ is

$$E_B = -\frac{q}{4\pi\varepsilon_0}\left[\frac{1}{ar} + \frac{1}{(r+a)^2} - \frac{r}{a\,(r+a)^2}\right]\hat{x}$$

If q is replaced by $-q$, the maximum negative induced surface charge density will occur at $(-a, 0, 0)$. As discussed above the required positive charge

$$Q \geq -\sigma_B.\,4\pi\,a^2 = -\,\varepsilon_0\,4\pi\,a^2\left(\frac{-q}{4\pi\varepsilon_0}\right)\left[\frac{1}{ar} + \frac{1}{(r+a)^2} - \frac{r}{a(r+a)^2}\right]\hat{x}$$

$$= qa^2\,(3r+a)/r(r+a)^2.$$

Example 15. *A point charge q of mass m is released from rest at a distance a from an infinite grounded conducting plate. Show that the time taken by the charge to hit the plate is*

$$T = \frac{1}{q}\sqrt{2\pi^3\varepsilon_0 m a^3}.$$

At any instant of time t, let x be the distance of the point charge q from the infinite grounded conducting plate. Its image will be at a distance x behind the plate. Thus the distance between charge q and its image-q will be $2x$ and the force of attraction $F = q^2/4\pi\varepsilon_0(2x)^2$. Therefore the acceleration of q

$$f' = F/m = q^2/16\,\pi\varepsilon_0 m x^2.$$

The acceleration of q relative to its image $= 2f' = q^2/8\pi\varepsilon_0 m x^2$.

Since the acceleration $f = dv/dt = \frac{dv}{dx}\cdot\frac{dx}{dt} = -\,v\frac{dv}{dx}$

Here the negative sign appears because x decreases with increasing t.

$$\therefore \qquad -\,v\,dv = \frac{q^2}{8\pi\varepsilon_0 m x^2}\cdot dx.$$

On integrating, we get $\frac{v^2}{2} = \frac{q^2}{8\pi\varepsilon_0 m x} + C$

where C is the integration constant. At $t = 0$, $x = a$ and $v = 0$, therefore

$$C = -\,q^2/8\pi\varepsilon_0 m a.$$

$$\therefore \quad \frac{v^2}{2} = \frac{q^2}{8\pi\varepsilon_0 mx} - \frac{q^2}{8\pi\varepsilon_0 ma} = \frac{q^2}{8\pi\varepsilon_0 m}\left(\frac{1}{x}-\frac{1}{a}\right)$$

$$v = -\frac{dx}{dt} = \left[\frac{q^2}{4\pi\varepsilon_0 m}\left(\frac{1}{x}-\frac{1}{a}\right)\right]^{1/2} \quad \text{or } dt = -\left(\frac{4\pi\varepsilon_0 m}{q^2}\right)^{1/2} \frac{dx}{(1/x - 1/a)^{1/2}}$$

If the time taken by the charge to hit the plate is T, *i.e.*, at $t = T$, $x = 0$.

$$\therefore \quad \int_0^T dt = -\left(\frac{4\pi\varepsilon_0 m}{q^2}\right)^{1/2} \int_a^0 \frac{dx}{(1/x - 1/a)^{1/2}}.$$

Let $1/x = (1/a)\sec^2\theta$, *i.e.*, $x = a\cos^2\theta$, or $dx = -2a\cos\theta\sin\theta\, d\theta$,

$$\therefore \quad T = -\left(\frac{4\pi\varepsilon_0 m}{q^2}\right)^{1/2} \int_0^{\pi/2} \frac{-2a\cos\theta\sin\theta\, d\theta}{(1/a)^{1/2}\tan\theta} = \left(\frac{4\pi\varepsilon_0 m}{q^2}\right)^{1/2} a\sqrt{a} \int_0^{\pi/2} 2\cos^2\theta\, d0$$

$$= \left(\frac{4\pi\varepsilon_0 m}{q^2}\right)^{1/2} a^{3/2} \int_0^{\pi/2} (\cos 2\theta + 1)\, d\theta = \left(\frac{4\pi\varepsilon_0 m}{q^2}\right)^{1/2} a^{3/2} (\pi/2)$$

$$= \frac{1}{q}\sqrt{\pi^3 a^3 \varepsilon_0 m}$$

Example 16. *A point charge Q is located at point (a, 0, b) between two semi-infinite conducting planes intersecting at right angles. Determine the potential at point (x, y, z) and the force on Q.*

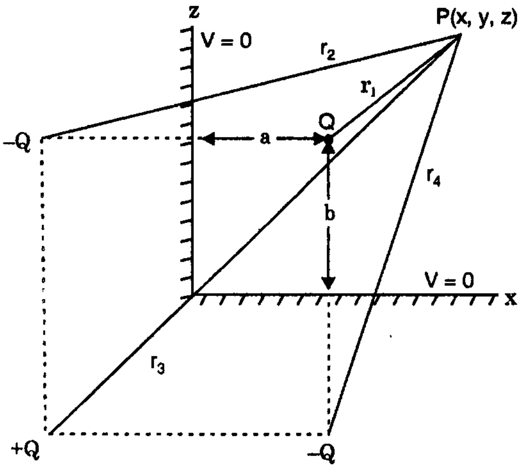


Fig. 7.15. Example 16.

Number of images due to two semi infinite conducting planes inclined at an angle 90° = 360/90–1 = 3. These images will be as shown in Fig. 7.15. The potential at any point $P\,(x, y, z)$ due to these four point charges (charge Q and its three images)

$$V = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2} + \frac{1}{r_3} - \frac{1}{r_4}\right),$$

where $r_1 = [(x-a)^2 + y^2 + (z-b)^2]^{1/2}$

$r_2 = [(x+a)^2 + y^2 + (z-b)^2]^{1/2}$

$r_3 = [(x+a)^2 + y^2 + (z+b)^2]^{1/2}$

$r_4 = [(x-a)^2 + y^2 + (z+b)^2]^{1/2}$

The net force on Q due to its three images

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

$$= \frac{Q^2}{4\pi\varepsilon_0(2b)^2}\mathbf{a}_z - \frac{Q^2}{4\pi\varepsilon_0(2a)^2}\mathbf{a}_x + \frac{Q^2(2a\,\mathbf{a}_x + 2b\,\mathbf{a}_z)}{4\pi\varepsilon_0[(2a)^2 + (2b)^2]^{3/2}}$$

$$= \frac{Q^2}{16\pi\varepsilon_0}\left[\left(\frac{a}{(a^2+b^2)^{3/2}} - \frac{1}{a^2}\right)\mathbf{a}_x + \left(\frac{b}{(a^2+b^2)^{3/2}} - \frac{1}{b^2}\right)\mathbf{a}_z\right].$$

Example 17. *Find the electric field due to a point charge q placed inside the sphere of zero potential.*

Let a be the radius of the earthed sphere and f be the distance from its centre to the point A at which charge q is placed. Let A' be the inverse point of A. Hence

$OA \cdot OA' = a^2$ or $OA/a = a/OA'$.

Thus ΔOAP and $\Delta OA'P$ are similar and hence

$$PA/PA' = a/(a^2/f) = f/a.$$

Let q' be the charge placed at point A'. As potential at P is zero,

$$\therefore \quad q/AP + q'/PA' = 0,$$

or

$$q' = -qPA'/PA = -qa/f.$$

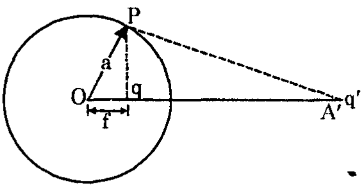


Fig. 7.16. Example 17.

Thus the electrical image required is a charge $-qa/f$ at point A' and the system is equivalent to a charge $+q$ at A and $-qa/f$ at A'.

Electric intensity at point P due to charge q at A

$$\mathbf{E}_1 = \frac{1}{4\pi\varepsilon_0}\frac{q}{AP^3}\cdot\mathbf{AP} = \frac{1}{4\pi\varepsilon_0}\frac{q}{AP^3}(\mathbf{AO}+\mathbf{OP})$$

and due to charge $-qa/f$ at A'

$$\mathbf{E}_2 = \frac{1}{4\pi\varepsilon_0}\frac{qa}{f}\frac{\mathbf{PA'}}{(PA')^3} = \frac{1}{4\pi\varepsilon_0}\frac{qa}{f(PA')^3}(\mathbf{PO}+\mathbf{OA'})$$

$\therefore$ Resultant intensity at point P

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{AO}}{AP^3}+\frac{\mathbf{OP}}{AP^3}+\frac{a\,\mathbf{PO}}{f(PA')^3}+\frac{a\,\mathbf{OA'}}{f(PA')^3}\right]\\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{OP}}{AP^3}\left\{1-\frac{a}{f}\left(\frac{AP}{PA'}\right)^3\right\}+\frac{\mathbf{AO}}{AP^3}\left\{1-\frac{a\,OA'}{f\,OA}\left(\frac{AP}{PA'}\right)^3\right\}\right]\\ &= \frac{q}{4\pi\varepsilon_0}\left[\frac{\mathbf{OP}}{AP^3}\left\{1-\frac{a}{f}\left(\frac{f}{a}\right)^3\right\}+\frac{\mathbf{AO}}{AP^3}\left\{1-\frac{a}{f}\left(\frac{a}{f}\right)^2\left(\frac{f}{a}\right)^3\right\}\right]\\ &= \frac{q}{4\pi\varepsilon_0}\left(1-\frac{f^2}{a^2}\right)\frac{\mathbf{OP}}{AP^3} = \frac{q}{4\pi\varepsilon_0}\frac{a^2-f^2}{aAP^3}\hat{n},\end{aligned}$$

where $\hat{n}$ is the unit vector at point P in outward direction.

Hence the surface density of induced charge on the sphere

$$\sigma = -\frac{q(a^2-f^2)}{4\pi a\,AP^3}.$$

Force between the charge q at A and the sphere

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{AA'^2} = \frac{q}{4\pi\varepsilon_0}\frac{(-qa/f)}{(a^2/f-f)^2} = -\frac{1}{4\pi\varepsilon_0}\frac{q^2af}{(a^2-f^2)^2}$$

Here –ve sign shows that the charge is attracted by the sphere.

Example 18. *With in a spherical hollow earthed conductor equal point charges q are placed at equal distances f from the centre on the same diameter. Calculate the force between them.*

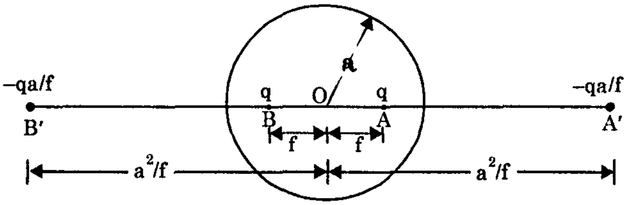


Fig. 7.17. Example 18.

Let us consider points A and B at a distance f from O on both the sides. To make the potential of the sphere zero, we have to put charges $-qa/f$ *at points* A' and B'.A' and B' are thus the electric images of charges at A and B respectively, Force on A due to all charges

$$\mathbf{F} = \frac{1}{4\pi\varepsilon_0}\frac{q(qa)}{f}\frac{1}{(AA')^2}\mathbf{i} + \frac{1}{4\pi\varepsilon_0}\frac{q^2}{(AB)^2}\mathbf{i} - \frac{1}{4\pi\varepsilon_0}\frac{q(qa)}{f}\frac{1}{(B'A)^2}\mathbf{i}$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{a}{f(AA')^2} + \frac{1}{(AB)^2} - \frac{a}{f}\frac{1}{(B'A)^2}\right]\mathbf{i}$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left[\frac{a}{f}\frac{1}{(a^2/f - f)^2} + \frac{1}{(2f)^2} - \frac{a}{f}\frac{1}{(a^2/f + f)^2}\right]\mathbf{i}$$

$$= \frac{q^2}{4\pi\varepsilon_0}\left\{\frac{4a^3f^3}{(a^4 - f^4)^2} + \frac{1}{4f^2}\right\}\mathbf{i}.$$

Example 19. *A point charge q is placed at a point A at a distance f from the centre of an insulated conducting sphere of radius a (< f). Calculate the charge required on the sphere in order that the line of no electrification may be inter section of the sphere by the polar plane of A and show that the numerical ratio of the greatest negative and positive surface densities of charge is* $[(f + a)/(f - a)]^{3/2}$.

We know that the polar plane of A is the plane ⊥ to OA and passing through its image A'. When this sphere carries a total charge Q, it can be considered to be concentrated at centre O. As the sphere is insulated, the total induced charge will be zero. This condition can be satisfied by a uniform distribution of the charge qa/f over the surface which can be assumed as concentrated at centre O. Hence field at any external point can be considered as due to three point charges $Q + qa/f$, $-qa/f$ and q at points O, A' and A respectively.

Similar to an example 17, one can prove that the surface charge density on the sphere due to charge q at A and $-qa/f$ at A' is $-q\,(f^2 - a^2)/4\pi a\,AP^3$. According to the problem, charge Q is of such a value that the circle through P and ⊥ to OA (dotted circle) becomes non-electrified. Thus

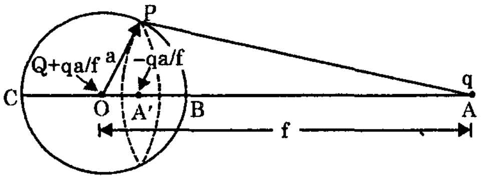


Fig. 7.18. Example 19.

$$\sigma_P = 0 = \frac{q(f^2 - a^2)}{4\pi a AP^3} + \frac{1}{4\pi a^2}\left[Q + \frac{qa}{f}\right]$$

or $\quad Q = a\, q\,[(f^2 - a^2)^{-1/2} - f^{-1}].$

Greatest negative and positive surface densities are at point B and C respectively. The surface density of charge at B

$$\sigma_B = -\frac{q\,(f^2 - a^2)}{4\pi a\,(AB)^3} + \frac{1}{4\pi a^2}\left[Q + \frac{qa}{f}\right].$$

Substituting value of Q and solving, we get

$$\sigma_B = -\frac{q}{4\pi a}\left[\frac{(f + a)^{3/2} - (f - a)^{3/2}}{(f - a)^2\,(f + a)^{1/2}}\right]$$

Similarly

$$\sigma_C = -\frac{q\,(f^2 - a^2)}{4\pi a(AC)^3} + \frac{1}{4\pi a^2}\left[Q + \frac{qa}{f}\right]$$

$$= \frac{q}{4\pi a}\left[\frac{-(f - a)^{3/2} + (f + a)^{3/2}}{(f + a)^2 (f - a)^{1/2}}\right]$$

Hence $\sigma_B/\sigma_C = -\,[(f + a)/(f - a)]^{3/2}$.

Example 20. *Calculate the capacity of a charged spherical conductor of radius a at a distance* $\frac{1}{2}$ *d from an earthed conducting plane of infinite extent.*

Let AB be the infinite plane in front of which a small sphere of radius a is placed. In this case we must make use of an infinite series of images. First suppose a charge q_1 placed at the centre O of the sphere. Due to this charge the surface of the sphere remains equipotential, but the surface of the plane is not equipotential. Add a charge $-q_1$, the image of charge q_1 to get the plane surface equipotential. As sphere is not equipotential now, add charge q_2, the image of $-q_1$, in the sphere. This restores the equipotential character of the spherical surface but disturbs that of the plane AB. Addition of the image $-q_2$ of q_2 in the plane makes the plane an equipotential surface again but disturbs the potential over the sphere. Continuation of the process gives us the desired state in which both the sphere and the plane are equipotential surfaces. Thus the total charge on the sphere is $Q = q_1 + q_2 + q_3 + \ldots$ and the charge on the plane is $-Q = -q_1 - q_2 - q_3 \ldots$

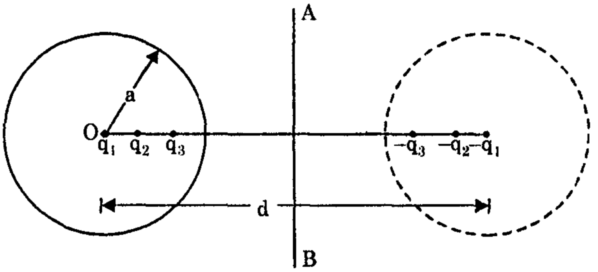


Fig. 7.19. Example 20.

Distance of $-q_1$ is d from the centre of the sphere O. Its image charge will be $q_2 = aq_1/d$ at a distance a^2/d from O. Thus its image will be a charge $-q_2$ at a distance $d(1 - a^2/d^2)$ from O.

$$\therefore \qquad q_3 = \frac{a/d}{1-a^2/d^2}q_2 = \frac{a^2/d^2}{1-a^2/d^2}q_1 \text{ at distance } \frac{a^2/d}{1-a^2/d^2} \text{ from } O.$$

Using $a/d = m$, one can write in general that charges q_1, mq_1, $m^2q_1/(1 - m^2)$, $m^3q_1/(1 - m^2)[1 - m^2(1 - m^2)^{-1}]$...... are placed at the distances 0, ma, $ma/(1 - m^2)$, $ma/[1 - m^2(1 - m^2)^{-1}]$... respectively from centre O.

$$\therefore \qquad \text{Total charge } Q = q_1\left[1 + m + \frac{m^2}{1-m^2} + \frac{m^3}{(1-m^2)[1-m^2/(1-m^2)]} + \ldots\right]$$
$$= q_1\,[1 + m + m^2 + m^3 + 2m^4 + 3m^5 + \ldots\ldots].$$

As q_1 is the image of $-q_1$. The potential of the surface of the sphere due to this pair of charges is zero. The same is true for the pairs q_3 and $-q_2$, q_4 and $-q_3$ ...... etc. Hence the potential of the sphere is due to the charge q_1 at the centre of the sphere alone and is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q_1}{a}.$$

Potential of the plane AB is zero, as equal and opposite charges are symmetrically placed on its both sides. Hence the capacity of the sphere relative to the earthed plane is

$$C = Q/V = 4\pi\varepsilon_0 a\,[1 + m + m^2 + m^3 + 2m^4 + \ldots\ldots].$$

Example 21. *Calculate the capacity of two infinitely long straight parallel wires of radius a, one has charge λ per unit length and the other an equal negative charge.*

Let us consider two parallel wires intersecting the plane of paper at O and O' and having charges λ and $-\lambda$ per unit length respectively. From symmetry it is clear that the potential at any point P depends only on the distance of P from the wires. Hence

$$V = -\frac{\lambda}{4\pi\varepsilon_0}\log_e r + \frac{\lambda}{2\pi\varepsilon_0}\log_e r' = \frac{\lambda}{2\pi\varepsilon_0}\log_e\left(\frac{r'}{r}\right).$$

This relation makes the potential zero midway between the two wires for each point of which $r = r'$.

Equipotential surfaces are corresponding to

$$\frac{r'^2}{r^2} = \frac{r^2 - 2rd\cos\theta + d^2}{r^2} = m^2 \text{ (constant)}$$

or $\qquad r^2 + 2r\cos\theta\, d/(m^2 - 1) = d^2/(m^2 - 1)$.

Transforming to rectangular coordinates with O as origin, $x = r\cos\theta$ and $y = r\sin\theta$, we get

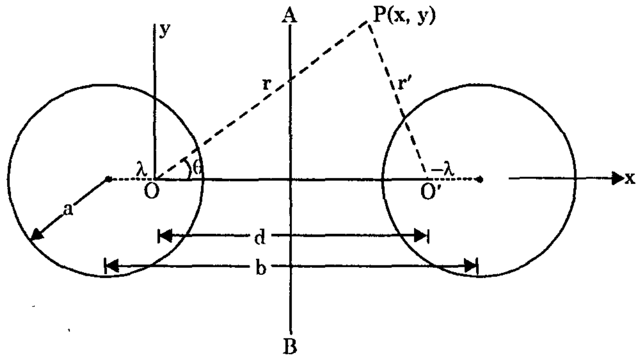


Fig. 7.20. Example 21.

$$\left(x+\frac{d}{m^2-1}\right)^2+y^2 = \frac{m^2d^2}{(m^2-1)^2}.$$

This is the equation of a circle of radius $md/(m^2-1)$ with centre $[-d/(m^2-1), 0]$. Thus equipotential surfaces are cylinders of circular cross-section. We may assume surface of the wires as equipotential surfaces. Hence radius of the wire $a = md/(m^2-1)$, and the distance between the axes of the two wires

$b = d + 2d/(m^2-1) = (m^2+1)d/(m^2-1)$.

Eliminating d between the above two relations, we get

$$\frac{b}{a} = \frac{(m^2+1)}{m},\ am^2 - bm + a = 0 \text{ or } m = \frac{b \pm \sqrt{(b^2-4a^2)}}{2a}.$$

Positive sign corresponds to +vely charged wire and –ve sign corresponds to the negatively charged wire. Hence the potential at the negatively charged wire

$$V_1 = \frac{\lambda}{2\pi\varepsilon_0}\log_e m_+ = \frac{\lambda}{2\pi\varepsilon_0}\log_e \frac{b+\sqrt{(b^2-4a^2)}}{2a}$$

and that of the –vely charged wire

$$V_2 = \frac{\lambda}{2\pi\varepsilon_0}\log_e m_- = \frac{\lambda}{2\pi\varepsilon_0}\log_e \frac{b-\sqrt{(b^2-4a^2)}}{2a}$$

$\therefore$ Potential difference $V_1 - V_2 = \dfrac{\lambda}{2\pi\varepsilon_0}\log_e \dfrac{b+\sqrt{(b^2-4a^2)}}{b-\sqrt{(b^2-4a^2)}}$

Capacity per unit length $\dfrac{\lambda}{V_1-V_2} = 2\pi\varepsilon_0/\log_e\left(\dfrac{b+\sqrt{(b^2-4a^2)}}{b-\sqrt{(b^2-4a^2)}}\right).$

Example 22. *If conductors A, B, C, D, ...,where C completely surrounds A and B only, are charged and then A and C are connected by a thin wire, show that the charge remaining on A bears constant ratio to the charge on B.*

Let the charges on the conductors A, B, C, D, ... be q_1, q_2, q_3, q_4, ..., respectively. As the conductor C encloses A and B, hence

$$p_{33} = p_{32} = p_{31}.$$

Now suppose that there is a unit charge on D and all others have zero charge, then

$$p_{43} = p_{42} = p_{41}.$$

Similarly $\quad p_{53} = p_{52} = p_{51}$, etc.

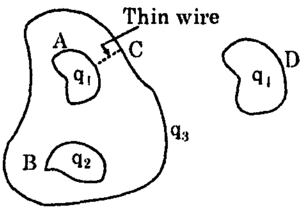


Fig. 7.21. Example 22.

When a conductor A is connected to C, the charges q_1 and q_3 are distributed, let new charges be q_1' and q_3' on these conductors and the charges on other conductors remain the same.

$$V_A = p_{11}q_1' + p_{21}q_2 + p_{31}q_3' + p_{41}q_4 + \ldots\ldots$$
$$V_C = p_{13}q_1' + p_{23}q_2 + p_{33}q_3' + p_{43}q_4 + \ldots\ldots$$

Let the new potential of A or C be V, hence

$$V_A = V_C = V \text{ or } V_A - V_C = 0.$$

$$\therefore \quad q_1' (p_{11} - p_{13}) + q_2 (p_{21} - p_{23}) + q_3' (p_{31} - p_{33}) + q_4 (p_{41} - p_{43}) + \ldots = 0$$

or
$$q_1' (p_{11} - p_{13}) + q_2 (p_{21} - p_{23}) = 0.$$

$$\therefore \quad \frac{\text{Charge remaining on } A}{\text{charge on } B} = \frac{p_{21} - p_{23}}{p_{13} - p_{11}} = \text{const.}$$

As we know that the coefficients of potential depend only on the shape and size and geometrical position of the conductors, hence the above ratio is constant.

Example 23. *If three equal spherical conductors are placed at the corners of an equilateral triangle and raised to potential φ_1, φ_2, φ_3. Show that the energy of the system is reduced by $\frac{1}{6}$ $[(\varphi_1 - \varphi_2)^2 + (\varphi_2 - \varphi_3)^2 + (\varphi_3 - \varphi_1)^2]$ W, where W is the energy of the system when the potentials are 1, –1 and 0 respectively.*

Since the three spherical conductors are equal, hence the coefficients of capacitance are equal.

$$q_{11} = q_{22} = q_{33} = a \text{ (say).}$$

As these are placed symmetrically, hence the coefficients of inductance are equal.

$$q_{12} = q_{13} = q_{23} = b \text{ (say).}$$

Thus the relations between the charges and potentials are

$$Q_1 = q_{11}\varphi_1 + q_{21}\varphi_2 + q_{31}\varphi_3 = a\,\varphi_1 + b\,\varphi_2 + b\varphi_3$$

$$Q_2 = q_{12}\varphi_1 + q_{22}\varphi_2 + q_{32}\varphi_3 = b\,\varphi_1 + a\,\varphi_2 + b\varphi_3$$

and
$$Q_3 = q_{13}\varphi_1 + q_{23}\varphi_2 + q_{33}\varphi_3 = b\,\varphi_1 + b\,\varphi_2 + a\varphi_3,$$

Hence total charge $Q_1 + Q_2 + Q_3 = (a + 2b)(\varphi_1 + \varphi_2 + \varphi_3)$.

∴ Initial energy of the system

$$W_1 = \tfrac{1}{2}(Q_1\varphi_1 + Q_2\varphi_2 + Q_3\varphi_3).$$

$$= \tfrac{1}{2}[a(\varphi_1^2 + \varphi_2^2 + \varphi_3^2) + 2b(\varphi_1\varphi_2 + \varphi_2\varphi_3 + \varphi_3\varphi_1)].$$

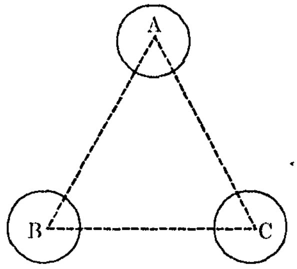


Fig. 7.22. Example 23.

Energy, when the potentials are (1, –1, 0), is given by

$$W = \tfrac{1}{2}[a(1 + 1 + 0) + 2b\{1 \times (-1) + 0 + 0\}]$$

$$= a - b.$$

When the conductors are interconnected, the charge distribution changes while the total charge remains constant. Let Q_1', Q_2' and Q_3' be the charges on the conductors A, B and C respectively and φ be the potential of each conductor. In this case

$$Q_1' = a\varphi + b\varphi + b\varphi = (a + 2b)\,\varphi = Q_2' = Q_3'.$$

$$\therefore \quad Q_1' + Q_2' + Q_3' = 3(a + 2b)\,\varphi = Q_1 + Q_2 + Q_3.$$

$$3(a + 2b)\,\varphi = (a + 2b)(\varphi_1 + \varphi_2 + \varphi_3) \text{ or } \varphi = \tfrac{1}{3}(\varphi_1 + \varphi_2 + \varphi_3).$$

Energy in this case
$$W_2 = \tfrac{1}{2}[Q_1'\varphi + Q_2'\varphi + Q_3'\varphi] = \tfrac{3}{2}(a + 2b)\,\phi^2$$

$$= \tfrac{1}{6}(a + 2b)(\varphi_1 + \varphi_2 + \varphi_3)^2.$$

∴ Loss of energy
$$W_1 - W_2 = \tfrac{1}{6}(a - b)[(\varphi_1 - \varphi_2)^2 + (\varphi_2 - \varphi_3)^2 + (\varphi_3 - \varphi_1)^2]$$

$$= \tfrac{1}{6} W[(\varphi_1 - \varphi_2)^2 + (\varphi_2 - \varphi_3)^2 + (\varphi_3 - \varphi_1)^2].$$

Example 24. *Positive and negative charges of same magnitude are situated at two points A and B. The line of force that leaves A at an angle α with AB meets the plane, which bisects AB at right angles at point P. Show that*

$$\sin \tfrac{1}{2}\alpha = \sqrt{2}\sin \tfrac{1}{2}(PAB).$$

Let Q be any point on the line of force, $\theta_1 = \angle QAB$ and $\theta_2 = \angle QBX$.

Then as $Q \to A$, $\theta_1 \to \alpha$, and $\theta_2 \to \pi$. The equation of a line of force is then

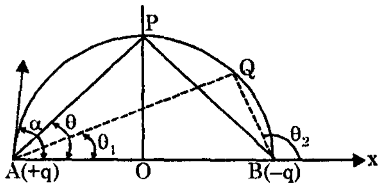


Fig. 7.23. Example 24.

$$q\cos\theta_1 - q\cos\theta_2 = \text{const.} = q\cos\alpha + q$$

or $$\cos\theta_1 - \cos\theta_2 = \cos\alpha + 1 = 2\cos^2\tfrac{1}{2}\alpha,$$

At point P on the bisecting plane

$$\theta_1 = \angle PAB = \theta \text{ and } \theta_2 = \angle PBX = \pi - \theta.$$

$$\therefore \cos\theta - \cos(\pi - \theta) = 2\cos^2\tfrac{1}{2}\alpha, \quad \text{or} \quad \cos\theta = \cos^2\tfrac{1}{2}\alpha$$

$$1 - 2\sin^2\tfrac{1}{2}\theta = 1 - \sin^2\tfrac{1}{2}\alpha \quad \text{or} \quad \sin\tfrac{1}{2}\alpha = \sqrt{2}\sin\tfrac{1}{2}\theta.$$

Example 25. *Positive and negative unit charges are situated at two points A and B. Find at what distance from AB the plane normal to and bisecting AB is cut by the lines of force which issue from A in a direction parallel to this plane.*

Proceeding as in the above example with the difference that $\alpha = \pi/2$ and $q = 1$, we have the equation of lines of force as

$$\cos\theta_1 - \cos\theta_2 = 1.$$

For point P, $\cos\theta - \cos(\pi - \theta) = 1$

or $$\cos\theta = \tfrac{1}{2} \quad \text{or} \quad \theta = 60°.$$

$$\therefore \qquad OP = OA\tan\theta = OA\sqrt{3} = \left(\sqrt{3}/2\right) AB.$$

Example 26. *If three infinitely long thin wires whose charges per unit length are 1, −2, 1 respectively, cut a plane at right angles in collinear points A, B, C, where AB = BC = a. Prove that the equations of the lines of force in this plane is $r^2 = a^2\cos(2\theta + \alpha)\sec\alpha$, where B is the origin with BC as axis and α a variable parameter.*

The equation of line of force is

$$\theta_1 - 2\theta + \theta_2 = \text{const.}$$

or $$(\theta_1 + \theta) - (\theta - \theta_2) = C\ .$$

If (x, y) be the cartesian coordinates of point P on the line of force with charge point B as origin, then

$$\tan\theta_1 = \frac{y}{a+x}, -\tan\theta_2 = \frac{y}{a-x} \text{ and } \tan\theta = \frac{y}{x}$$

or $$\theta_1 = \tan^{-1}\frac{y}{a+x}, -\theta_2 = \tan^{-1}\frac{y}{a-x}, \text{ and } \theta = \tan^{-1}\frac{y}{x}.$$

Thus equation of lines of force is given by

$$\left(\tan^{-1}\frac{y}{a+x} - \tan^{-1}\frac{y}{x}\right) - \left(\tan^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{a-x}\right) = C$$

or $$\tan^{-1}\left[\frac{y/(a+x) - y/x}{1 + y^2/x(a+x)}\right] - \tan^{-1}\left[\frac{y/x + y/(a-x)}{1 - y^2/x(a-x)}\right] = C$$

$$\tan^{-1} 2a^2xy/[a^2x^2 - (x^2+y^2)^2 - a^2y^2] = C$$

Using relations $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$\tan^{-1}[a^2\sin 2\theta/(a^2\cos 2\theta - r^2)] = C = \pi/2 - \alpha.$$

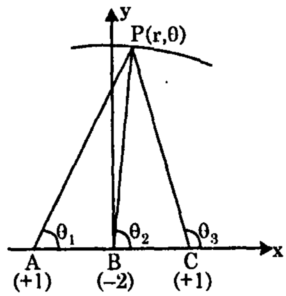


Fig. 7.24. Example 26.

Here constant has been replaced by $\pi/2 - \alpha$ in order to get the answer in the proper form. Above relation thus reduces to

$$a^2\sin 2\theta/(a^2\cos 2\theta - r^2) = \tan(\pi/2 - \alpha) = \cot\alpha \text{ or } r^2 = a^2\cos(2\theta + \alpha)\sec\alpha.$$

ORAL QUESTIONS

1. How is Laplace's equation related to Poisson's equation?
2. Define Earnshaw's Theorem.
3. What is Laplace's operator?
4. Define Uniqueness Theorem.
5. Explain the theory of electric images.
6. State the conditions, which must be satisfied in applying the electric images.
7. Write the Laplace's equation in two dimensional cylindrical coordinate system.
8. The number of images for a system consisting of a point charge between two semi infinite conducting planes inclined at an angle θ. ... ($360°/\theta - 1$)
9. Two potential functions V_1 and V_2 satisfy Laplace's equation within a closed region and assume the same values on its surface. Which one is correct?
   (*a*) $V_1 = V_2$, (*b*) $V_1 > V_2$, (*c*) $V_1 < V_2$.
10. In cylindrical coordinates equation $\partial^2\psi/\partial\rho^2 + (1/\rho)\,\partial\psi/\partial\rho + \partial^2\psi/\partial z^2 + a = 0$ is called ... equation. (Poisson's)
11. A charge placed in front of a metallic plane is (*a*) attracted to the plane by a mirror image of equal and opposite charge and (*b*) is repelled by the plane. (*a*)
12. Two similar charges are placed at a distance $2d$ apart. A grounded conducting sphere of radius a is placed midway between them. Find the electric images.
    $q' = -q\,(a/d)$ placed at two sides of the center of the sphere at distances $\pm\,(a^2/d)$.

PROBLEMS

1. In the electrohydrodynamic pump the region between the electrodes contains a charge with charge density $\rho = \rho_0\, x/a$, which is generated at the left electrode ($V = V_0$) and collected at the right electrode ($V = 0V$) at a distance a along the x-direction. Calculate the pressure of the pump. ($\rho^2 a^2/24\varepsilon_0 + \rho_0 V_0/2$)
2. A large conducting cone ($\theta = 45°$) is placed on a conducting plane with a tiny gap separating its vertex from the plane. If the cone is connected to a 50 V source, find V and $\mathbf{E}$ at $(-3, 4, 2)$ (22.13 V, $11.36\ \hat{\theta}$ *V/m*)
3. Show that the potentials $V = (z \cos\phi)/\rho$ and $V = a \cos\theta/r^2$ satisfy Laplace's equation.
4. Find V and $\mathbf{E}$ at (3, 0, 4) due to the two conducting cones of infinite extent, shown in Fig. 7.25. (11.7 V, $-17.86\ \hat{\theta}$ *V/m*)
5. A thin circular earthed conducting disc of radius a is placed in the field of a point charge q at a point P on the axis of the disc at a distance b from its centre. Show that the charge induced on the disc is $(2q/\pi)\tan^{-1}(a/b)$.
6. A long conducting cylinder of radius a, bearing a charge λ per unit length is oriented parallel to a grounded conducting plane of infinite extent. The axis of the cylinder is at a distance d from the plane. Find the position of the line image and find also the constant m in terms of a and d.
   $m = d/a + \sqrt{[(d/a)^2 - 1]}$, *image at* $d\,(m^2 - 1)/(m^2 + 1)$

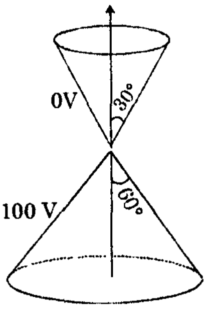


Fig. 7.25

7. A dipole **p** is oriented normal to and at distance d from an infinite conducting plane. The plane is grounded. Calculate the force exerted on the plane by the dipole. [$3p^2/32\pi\varepsilon_0 d^4$, attraction]
8. Two extensive homogeneous isotropic dielectrics meet on plane $z = 0$. $\varepsilon_{r1} = 4$ for $z \geq 0$ and is 3 for $z \leq 0$. A uniform field $\mathbf{E}_1 = 5\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z$ kV/m exists for $z \geq 0$. Find $\mathbf{E}_2$ for $z \leq 0$, and the energy densities in J/m^3 in both dielectrics ($5\mathbf{a}_x - 2\mathbf{a}_y + 4\mathbf{a}_z$ *kV/m*, 672 *μJ/m*^{3}, 597 *μJ/m*^{3})
9. Two equal spheres each of radius a are in contact. Show that the capacity of the conductor so formed is $8\pi\varepsilon_0 a \log_e 2$.

10. A conductor is formed by the outer surfaces of two equal spheres, the angle between the radii at a point of intersection being 120°. Show that the capacity of the conductor so formed is $(5\sqrt{3}-4)a/2\sqrt{3}$, where a is the radius of either sphere.

11. A line charge λ per unit length is placed parallel to and at a distance d from the axis of a long circular solid cylinder of uniform dielectric of constant k. The radius of the cylinder is $a(<d)$. Show that the line images can be found to determine the electric field and that there is a force of attraction between the line charge and the cylinder of amount $\frac{k-1}{k+1}\frac{8\pi\varepsilon_0\lambda^2 a^2}{d(d^2-a^2)}$ per unit length.

12. A coaxial conic capacitor is formed by two coaxial cones whose apexes are placed at the origin and whose axes are along the z-axis. The apex angles are θ_1 and θ_2. The cones are insulated from each other and the inner and outer cones are kept at V and zero potentials respectively. Determine the electric field inside the capacitor and the charge distribution on the cones. $-V\hat{\theta}/r \sin\theta \ln[(\tan\theta_1/2)/\tan\theta_2/2)]$, $-\varepsilon_0 V/r \sin\theta_1 \ln[(\tan\theta_1/2)/(\tan\theta_2/2)]$ at θ_1, $\varepsilon_0 V/r \sin\theta_2 \ln[(\tan\theta_1/2)/(\tan\theta_2/2)]$ at θ_2.

13. A point charge q is placed at a distance d from the centre of a grounded conducting sphere of radius a ($< d$). Calculate (*a*) The induced surface charge density on the sphere, (*b*) Show that the charge induced on the part of the sphere visible from q to that of the remaining part of the sphere is $[(d+a)/(d-a)]^{1/2}$, (*c*) Show that the ratio of maximum and minimum surface densities of the induced charge is $[(d+a)/(d-a)]^3$.

14. If n conductors in space be connected together by wires, prove that the capacity of the compound conductor thus formed is $\Sigma q_{rr} + 2\Sigma q_{rs}$. In the first summation r has values from 1 to n and in the second r and s are from 1 to n but $r \neq s$.

15. Three equal spheres are placed at the corners of an equilateral triangle. When their potentials are φ, 0, 0, their charges are Q, Q' and Q' respectively. Show that, when each sphere is at potential φ' each sphere has charge $(2Q'+Q)\varphi'/\varphi$.

16. Three insulated spherical conductors of the same size are placed at the corners of an equilateral triangle whose sides are large in comparison with the radii of the spheres. If the conductors are then touched in turn by another small charged spherical conductor, show that the charges they receive are in geometrical progression.

17. Find the capacitance per unit length between a cylindrical of radius a = 2.5 cm and a ground plane parallel to the conductor axis and a distance h = 6m from it. $[2\pi\varepsilon_0/\log_e(2h/a)]$

18. Point charges Q, $-Q'$ and $-Q'$ are placed at O, A *and* B respectively, which are collinear in such a way that $OA = OB$. If $Q > 2Q'$ show that the greatest angle which a line of force leaving O *and* entering A makes with OA is α, where $Q\sin^2(\alpha/2) = Q'$.

19. Two positive point charges q_1 and q_2 are placed at the points A and B respectively. Show that the tangent at infinity to the line of force which starts from q_1 making an angle α with BA, makes an angle $2\sin^{-1}\left[\sqrt{\{q_1/(q_1+q_2)\}}\sin\frac{1}{2}\alpha\right]$ with BA and passes through the point C in AB such that $AC/CB = q_2/q_1$.

20. Three infinite parallel straight lines, with charges q, $-q/2$, $-q/2$ per unit length, meet the plane of paper perpendicularly at A, B, C respectively. If $AB = AC$, prove that the lines of force reaching B, after having passed indefinitely closed to the equilibrium point make with BC an angle BAC.

21. In a Van de Graaff generator, the charge is transported on a belt of width 0.1m. If a surface charge density 0.5 μC/m^2 is transported at a velocity of 10 m/s and the leakage paths have resistance $10^{14}\Omega$, calculate the potential difference between the dome and the base. (50 mV)

22. A cone of semi vertex angle 30° is inserted in another core of semi vertex angle 45°. They are insulated at the vertex $r = 0$. The gap between these two is filled with a dielectric of relative permittivity $\varepsilon_r = 2.45$. At $r = 0.25\ m$ and $\theta = 30°$, $\mathbf{E} = -2.30 \times 10^3\ \mathbf{a}_\theta\ V/m$. Determine the potential difference V. Also determine the surface charge densities on the conducting cones at 30° and 45°. $[125.5V, -12.5/r\ nC/m^2, 8.84/r\ nC/m^2]$

8

STEADY CURRENT

8.1 CURRENT

In electrostatics we have been concerned with stationary electric charges. If a free charge is placed in an electric field it will be acted on by a force and will move in the direction of lines of force. In an isolated metallic conductor, the free electrons present in it are in random motion like the molecules of a gas and the net rate at which electrons pass through any hypothetical plane is zero. If the ends of the conductor are connected to a battery, an electric field $\boldsymbol{E}$ will be set up at every point with in the conductor. This field will act on the electrons and will give them a resultant motion in the direction $-\boldsymbol{E}$. This flow of electrons constitutes an *electric current and the strength of electric current also known as current is defined by the rate at which charge passes a specified point or through any specified surface area (i.e.,* $i = q/t$, where q is the net charge passes in time t). In **SI** units its unit is *ampere*, the coulomb per second. It is also the practical unit. If the rate of flow of charge with time is not constant, *i.e.*, the current varies with time, then at any instant t current is defines as

$$i = dq/dt. \qquad ..(1)$$

In metals the charge carriers are the free electrons. In electrolytes or in gaseous conductors the positive and negative ions travel in opposite directions. As the positive ions move slower and electrons, the negative ions, move faster, the net motion of charge is due to negatively charged electrons. This type of current is known as *conduction current*. It may be defined as *the steady and orderly movement of free charged particles occurring in a conductor under the influence of an electric field*. Liquids and gases may also undergo hydrodynamic motion, which will produce currents, known as *convection currents*. In this chapter we are not going to discuss convection currents, which are important in atmospheric electricity and in vacuum tubes.

8.2 CURRENT DENSITY, EQUATION OF CONTINUITY

Current is a characteristic of a particular conductor, like the mass, volume, etc. of an object and hence is a *macroscopic quantity*. A related microscopic quantity is the vector **j**, the current density, which is a characteristic of a point inside the conductor rather than of the conductor as a whole. *It is having direction of the flow of positive charge and a magnitude equal to the current per unit area through an infinitesimal area normal to the direction of current flow.*

Thus current density at a given point is the current through a unit normal area at that point. It is measured in amperes/meter^2 or coulomb/meter^2 × sec. If the current is distributed uniformly, the magnitude of **j** for all points on the cross sectional area S is given by $j = i/S$.

The current through any small area Δ **S** (Fig. 8.1) is given by $\Delta i = \mathbf{j}.\,\Delta\,\mathbf{S}$. It is same as through the area $\Delta S \cos\theta$ perpendicular to **j**. Thus for large areas, we can write

$$i = \int_S \mathbf{j}.d\mathbf{S}. \qquad ..(2)$$

The value of the integral is independent of the shape of surface taken. *The relation* (2) *shows that the current is the flux associated with* **j** *and is a scalar because the integrand* **j**. d**S** *is a scalar. The arrow often associated with the current in a conductor does not indicate that the current is a vector but shows only the sense of charge flow. Current is not a vector quantity as it does not obey the vector laws.* It remains unchanged if the wire is bent, tied into a knot or distorted. The current is known as *steady or stationary current if* **j** remains constant everywhere.

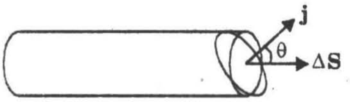


Fig. 8.1. Current through a small element.

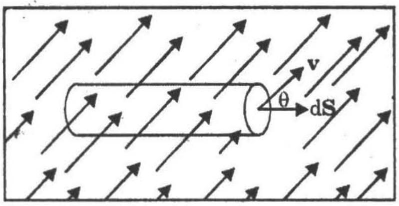


Fig. 8.2. Charge in a cylindrical volume.

To derive an expression for current density, consider a special case in which all the charge carriers have the same charge q and move with the same velocity **v**. In a small time interval Δt, each charge carrier will travel a distance $\mathbf{l} = \mathbf{v}\,\Delta t$. Let us consider a cylinder (or a prism) of cross sectional area d**S** and of length **l** in the conductor, Fig. 8.2. The volume of this cylinder will be $d\mathbf{S}\cdot\mathbf{l} = d\mathbf{S}\cdot\mathbf{v}\,\Delta t$. If n is the number of charge carriers per unit volume, then the charge in this cylindrical volume $= qn\,d\mathbf{S}\cdot\mathbf{v}\,\Delta t$. Hence the current or rate of flow of charge $= qnd\mathbf{S}\cdot\mathbf{v}$ and current density $\mathbf{j} = qn\mathbf{v}$.

If the charge carriers are having different charges and velocities, then the current through this cylinder will be the sum of the currents due to each of the charge carrier. Let per unit volume of the conductor consists of m groups, one having n_1 particles each of charge q_1 and velocity $\mathbf{v}_1$, second having n_2 particles each of charge q_2 and velocity $\mathbf{v}_2$..and the m^{th} group having n_m particles each of charge q_m and velocity $\mathbf{v}_m$.

∴ Total number of charge carriers $n = n_1 + n_2 + ..n_m$

and
$$\begin{aligned} I &= n_1q_1\,d\mathbf{S}\cdot\mathbf{v}_1 + n_2q_2\,d\mathbf{S}\cdot\mathbf{v}_2 + \ldots + n_mq_m\,d\mathbf{S}\cdot\mathbf{v}_m \\ &= d\mathbf{S}\cdot\sum_{i=1}^{m} n_iq_i\mathbf{v}_i. \end{aligned} \qquad ..(3)$$

Hence the current density $\mathbf{j} = \Sigma n_iq_i\mathbf{v}_i$. ..(4)

In solid conductors, the charge carriers are the free electrons, hence

$q_1 = q_2 = \ldots = q_m = -e$. Thus we have

$$\mathbf{j} = -e\Sigma n_i\mathbf{v}_i. \qquad ..(5)$$

We know that the average velocity

$$\mathbf{v}_{av} = \frac{n_1\,\mathbf{v}_1 + n_2\,\mathbf{v}_2 + \ldots n_m\,\mathbf{v}_m}{n_1 + n_2 + \ldots n_m} = \frac{\Sigma n_i\,\mathbf{v}_i}{n}$$

Hence
$$\mathbf{j} = -ne\mathbf{v}_{av}. \qquad ..(6)$$

Thus we see that the direction of **j** is opposite to $\mathbf{v}_{av}$ The quantity *ne* or *–ne* is the charge per unit volume ρ_v, the volume charge density. Thus the current density may be expressed as

$$\mathbf{j} = \rho_v \mathbf{v}_{av}. \quad ..(7)$$

Current sheet density : If the current is confined to the surface of a conductor, such as a current sheet. The rate of transport of charge per unit transverse length is called *current sheet density*, represented by **K** or $\mathbf{J}_s$. The total current *I* in the cylindrical sheet of radius r flowing in the positive z-direction (direction of length) is given by

$$\mathbf{K} = (I/2\pi r)\, a_z. \quad ..(8)$$

The vector **K** might vary from point to point. The current flowing through a contour **C** with in a current sheet is obtained by integrating the normal component of **K** along the contour as

$$I = \int_C K_n \, dl. \quad ..(9)$$

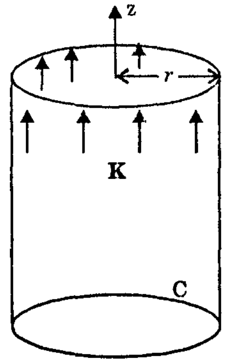


Fig. 8.3. Current sheet density.

8.3 CONDUCTIVITY AND RESISTIVITY

A conductor is characterized by a large amount of free electrons that provide conduction current due to applied electric field **E**. The force acting on an electron by this field is $\mathbf{F} = -e\mathbf{E}$. This electron is not in free space, thus will not be accelerated but suffer collision with atomic lattice and drifts from one atom to another. If the average drift velocity of electron of mass *m* is **u** and τ the average time interval between collisions, them from Newton's law we get

$$\mathbf{F} = -e\,\mathbf{E} = \Delta \mathbf{P} / \Delta t = m\mathbf{u}/\tau,$$

or

$$\mathbf{u} = -(e\,\tau/m)\,\mathbf{E}. \quad ..(10)$$

If there are *n* electrons per unit volume, the electronic charge density $\rho_v = -ne$.

∴ Conduction current density $\mathbf{j} = \rho_v \mathbf{u} = (ne^2 \tau/\mathrm{m})\,\mathbf{E}$

$$= \sigma\mathbf{E}, \quad ..(11)$$

where $\sigma = ne^2\tau/\mathrm{m}$ is the conductivity of the conductor. The conductivity of the material thus determines the ability to pass a conduction current under the influence of the electric field. The quantity $e\tau/\mathrm{m} = \mu$ is called the mobility has the units m^2/Vs.

In metallic conductors, the charge carriers are electrons, which drift in a direction opposite to that of the electric field, since both ρ_v and μ are negative for electrons, hence the conductivity σ is positive, as in the case of positive charge carriers. The unit of conductivity σ is ohm^{-1} meter^{-1} or mho/meter. It is also expressed as siemens per meter (*S*/m). It is a scalar quantity. The reciprocal of the conductivity σ is called the *resistivity ρ*. The dimensions of ρ are (volts/meter)/(ampere/meter^2) = volt-meter/ampere, or simply *ohm-meter*. These *constants σ and ρ are the characteristic of a material rather than of a particular specimen of a material.*

Consider first a particular piece of current carrying material like the wire. If at any instant, current is directed toward an insulated boundary of the conductor, it will produce steadily increasing added field due to the piling up of the charge at the boundary. Therefore, for the steady flow for which there is no source or sink of current the **j** *must be parallel to the conductor boundaries.* If we consider, two surfaces S_1 and S_2 everywhere perpendicular to **j**, each bounded by curves lying in the conductor boundaries the currents through them, $i_1 = \int_{S_1} \mathbf{j}\cdot d\mathbf{S}$ and $i_2 = \int_{S_2} \mathbf{j}\cdot d\mathbf{S}$ must be equal, otherwise charge will pile

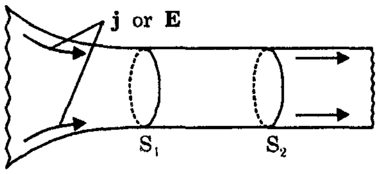


Fig. 8.4. E and j lines in a conductor.

up in between. Thus the current must be same through any surface that cuts across the wire. Current behaves like electric flux and we can draw lines of flow every where parallel to **j** that do not begin or end anywhere in the conductor. Since the vectors **E** and **j** are in the same direction, we can thus write

$$\mathbf{E} = \rho\mathbf{j}. \quad ..(12)$$

In many conducting materials the resistivity ρ is a constant, independent of **E**. The discovery that ρ is a constant for a metallic conductor at constant temperature was made by G.S. Ohm. Equation $\mathbf{j} = \sigma\mathbf{E}$ (or $\mathbf{E} = \rho\mathbf{j}$) is therefore known as the point form of *Ohm's law*. A material obeying Ohm's law is called an *ohmic conductor* or *a linear conductor.* Only metals and metal alloys are under this head. *Thus Ohm's law, like Boyle's law, describes a special property of certain materials and not a general property of all matter.* In non-isotropic conductors, the current may flow more easily in some directions than in others. In this case Eq. (11) can be written as

$$j_x = \sigma_x E_x, j_y = \sigma_y E_y, j_z = \sigma_z E_z, \quad ..(13)$$

where σ_x, σ_y and σ_z are the conductivities along the axes x, y, and z respectively. The resistivity of all metallic conductors increases with increase in temperature, as shown in Fig. 8.5 (*a*). It is practically independent of temperature for alloy *manganin*. The general expression for the resistivity of a pure metal at temperature t°C is

$$\rho_t = \rho_0(1 + \alpha t + \beta t^2 + ..), \quad ..(14)$$

where ρ_0 is the resistivity at 0°*C* and α, β, ...etc. are the constants in decending order of their values. When t is not very large, higher terms are negligible and the relation is thus given as $\rho_t = \rho_0(1 + \alpha t)$, where the constant α is termed as the temperature coefficient of the resistivity. We may therefore write

$$\alpha_t = \frac{\rho_t - \rho_0}{\rho_0 t} \text{ or } \alpha_T = \frac{\rho - \rho_0}{\rho_0(T - T_0)} \quad ..(15)$$

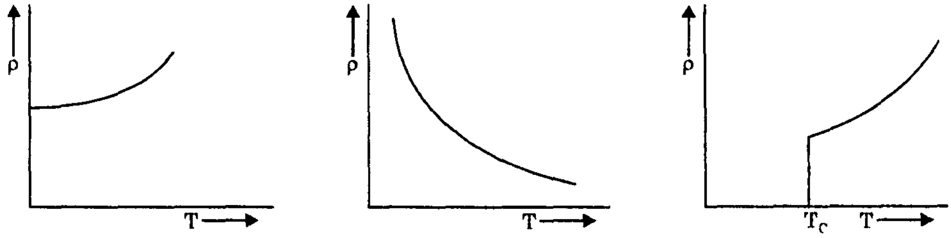
It may be rewritten as

$$\alpha_T = \frac{1\Delta\rho}{\rho_0\Delta T}. \quad ..(16)$$

Thus α_T may be defined as *the fractional increase in resistivity per degree increase in temperature.*

To maintain a finite current density **j** in a perfect conductor ($\sigma \to \infty$), ohm's law $\mathbf{j} = \sigma\mathbf{E}$ gives that the electrostatic field **E** inside the conductor must vanish. If some charges are introduced in the interior of such a conductor, the charges will move to the surface and redistribute quickly so that **E** inside the conductor vanishes. Gauss's law shows that volume charge density ρ_v must be zero for $\mathbf{E} = 0$. Thus inside a conductor

$$\mathbf{E} = 0, \rho_v = 0, V_{ab} = 0. \quad ..(17)$$



(a) Metal (b) Semiconductor (c) Superconductor

Fig. 8.5. Variation of resistivity with temperature.

The resistivity of semiconductors depends upon the impurities present and the temperature variation. A small amount of impurity in a semiconductor can decrease the resistivity by a very large factor. The resistivity of a semiconductor decreases very rapidly with increasing temperature as shown in Fig. 8.5 (*b*).

Table 8.1. The resistivities and the temperature coefficients of some substances at 20°C

Material	*Resistivity at 20°C* ($\Omega - m$)	α *at 20°C* (*per °C*)	*Material*	*Resistivity at 20°C* ($\Omega - m$)	α *at 20°C* (*per °C*)
Metals			***Insulators***		
Silver	1.6×10^{-8}	3.8×10^{-3}	Glass	$\sim 10^{12}$	
Copper	1.7×10^{-8}	3.9×10^{-3}	Mica	-10^{15}	
Aluminium	2.8×10^{-8}	3.9×10^{-3}	Wood	$10^{8} - 10^{14}$	
Gold	2.4×10^{-8}	3.4×10^{-3}	Amber	5×10^{14}	
Iron	11×10^{-8}	5×10^{-3}	Sulphur	10^{15}	
Tungsten	5.6×10^{-8}	4.5×10^{-3}	Quartz	$\sim 10^{17}$	
Alloys					
Brass	$\sim 7 \times 10^{-8}$	2×10^{-3}	Water (distilled)	$\sim 10^{4}$	
Constantan	4.9×10^{-7}	2×10^{-5}	Water (Sea)	$\sim .25$	
Manganin	4.4×10^{-7}	1×10^{-5} or 0	Earth (dry)	10^{5}	
Nichrome	1.1×10^{-6}	4×10^{-4}	Bakelite	10^{10}	
Semiconductors					
Carbon	3.5×10^{-5}	-5×10^{-4}	Paper	10^{11}	
Germanium	~ 0.5	-4.8×10^{-2}	Rubber (hard)	10^{15}	
Silicon	~ 2000	-7.5×10^{-2}	Porcelain	10^{12}	

The resistivity of a super conductor, such as alloy or compound decreases as the temperature is decreased, at first regularly like that of a metal and suddenly plunges to zero at a certain temperature, known as *critical temperature* ($\approx 0.1°K$ to $10°K$), T_c as shown in Fig. 8.5 (*c*).

The resistivities of various substances at 20°C are shown in Table 8.1. The large difference in the resistivities of conductors, semiconductors, and insulators is due to a large difference in the carrier concentrations rather than to a difference in the mobility of charge carriers.

8.4 CONTINUITY EQUATION AND RELAXATION TIME

Equation of continuity : The current density **j** and the charge density ρ are related at each point through a differential equation. The relation is based on the fact that electric charge can neither be created nor be destroyed and the rate of increase of the total charge inside any arbitrary volume must be equal to the net flow of charge into this volume.

$$\therefore \qquad i = \int_S \mathbf{j} . d\mathbf{S} = -\frac{\partial q}{\partial t} = -\frac{\partial}{\partial t}\int_V \rho dV. \qquad ..(18)$$

The minus sign comes here because the direction of $d\mathbf{S}$ is along outward normal and we wish current positive when the net charge is from the outside of V to within. Since we are dealing with a fixed volume V, hence we may put the *RHS* as $\int(\partial\rho/\partial t)\, dV$. Using divergence theorem the surface integral of Eq. (18) may be transformed into a volume integral. Thus we have

$$-\int_V div\, \mathbf{j}\, dV = \int_V \frac{\partial\rho}{\partial t}\, dV \text{ or } \int_V \left(div\ \mathbf{j} + \frac{\partial\rho}{\partial t}\right) dV = 0.$$

This integral must be zero for any arbitrary volume. It is only possible when integrand is zero. Thus,

$$div\, \mathbf{j} + \partial\rho/\partial t = 0. \qquad ..(19)$$

This differential equation is known as the *continuity equation*. If the region does not contain a source or sink of current, $\partial\rho/\partial t = 0$ and hence for *steady current*, we have $div\ \mathbf{j} = 0$.

or $$\oint_S \mathbf{j}.\, d\mathbf{S} = 0 \qquad \text{...(20)}$$

for all possible (closed) surfaces S. This means that the charge cannot accumulate anywhere so the *lines of current density are continuous.* Eq. (20) might be called *Gauss's law of currents.*

Charge relaxation time : If a charge is given to a conducting body, the charge will distribute itself in such a way that a zero electric field exists within and tangent to the conductor surface. The time required for this process is called charge *relaxation time.* For its qualitative *evaluation,* let us proceed as :

Consider a homogeneous conducting region with a conductivity σ. We know that the surface of the conductor bears charges whose surface density equals $\varepsilon_0 E_n$. If the conductor is placed in a dielectric medium, we must replace ε_0 by ε, the permittivity of the medium. In the process of conduction, valence electrons are free to move due to external electric field. During the motion of these electrons static conditions do not exist. These electrons should not be confused with net charge, which is zero as conduction electrons are balanced by protons in the nucleus.

If during a temporary imbalance, a region within a solid conductor has a net charge density ρ_0 at time $t = 0$, then from continuity equation.

$$\nabla\cdot\mathbf{j} = -\partial\rho/\partial t, \text{ and } \mathbf{j} = \sigma\mathbf{E}, \text{ we get}$$

$$\nabla\cdot(\sigma\,\mathbf{E}) = \sigma\,\nabla\cdot\mathbf{E} = -\,\partial\rho/\partial t.$$

From Gauss's law $\nabla\cdot\mathbf{E} = \rho/\varepsilon$, therefore

$$\sigma\rho/\varepsilon = -\,\partial\rho/\partial t.$$

On integration $$\frac{\sigma}{\varepsilon}\int_0^t dt = -\int_{\rho_0}^{\rho}\frac{d\rho}{\rho} \text{ or } \frac{\sigma}{\varepsilon}t = -\log_e\frac{\rho}{\rho_0}$$

$$\therefore \qquad \rho = \rho_0 e^{-\sigma t/\varepsilon}, \qquad \text{...(21)}$$

where ρ_0 is the initial charge density (ρ at $t = 0$).

Relation (21) shows that ρ decays exponentially, with a time constant $\tau = \varepsilon/\sigma$, known as the relaxation time or rearrangement time t_{re}. Thus the relaxation time is the *time required for the initial charge distribution to decay to 1/e or 36.8% of its initial value*. It is extremely short ($\sim 10^{-19}$ sec) for good conductors and is relatively large (few days) for insulators.

8.5 POTENTIAL AND RESISTANCE

The magnitude of the current between two surfaces S_1 and S_2 Fig. 8.4 of a conductor depends on the nature of the conductor and on the force which urge the free charges through it. The effect of all such forces is most easily calculated in terms of the work performed per unit positive charge. This quantity is called the *potential difference*. The charge flows through the conductor in the direction of decreasing potential. Hence the potential drop

$$V = V_1 - V_2 = -\int_2^1 \mathbf{E}\cdot d\mathbf{l} = \int_1^2 \mathbf{E}\cdot d\mathbf{l} = \int_1^2 E\,dl = \int_1^2 \rho j\,dl, \qquad \text{...(22)}$$

where the path of integration is along any line of flow from S_1 to S_2. If these points are brought together, the above relation reduces to zero. It is therefore clear that *no static distribution of charge can produce a potential difference around a circuit.* If we increase the field and therefore the potential everywhere in the same ratio, the flow lines will not change their pattern, but will increase everywhere in the same proportion as will i. Thus we can write above relation as

$$V = i\int_1^2 \rho(j/i)\,dl = R\,i, \qquad \text{...(23)}$$

where R is a quantity which remains constant but depends on the location of S_1 and S_2 on the flow pattern between these surfaces and on the value of ρ. This constant R is known as *resistance* of the piece of the conductor between surfaces S_1 and S_2 and is measured in ohms (volts/ampere). The relation (23) is also named as Ohm's law. Very large resistances are measured in megohms (1 megohm = 1 million ohms). The reciprocal of resistance is called the *conductance* of the conductor and is represented by G. Its unit is ohm^{-1} or mho.

Equation (11), (12) and (23) represent ohm's law in mathematical form as applied to a conductor where in the free electrons are influenced only by electric field. Ohm's law states that the *current in a conductor is directly proportional to the voltage or potential difference across the conductor and is inversely proportional to the resistance of the conductor.* The relationship between the current through a resistance and the voltage across the same resistance is called the volt ampere ($V - I$) characteristic. It will be a straight line passing through the origin for a linear resistance. The higher the conductance, greater the current and steeper the curve.

If the conductor is a straight and long wire of length l and area of cross section A, the lines of current flow and of electric field will be parallel, and thus **E** will be constant along any one line and hence, throughout the wire. The **j** is uniform in magnitude and direction and we can write $j = i/A$. Thus we have

$$R = \int_1^2 \rho\left(\frac{1}{A}\right)dl = \rho\left(\frac{1}{A}\right)\int_1^2 dl = \frac{\rho l}{A}$$

or

$$\rho = RA/l. \qquad \text{...(24)}$$

For $l = 1$ and $A = 1$, we get $\rho = R$.

The *resistivity of the material of a conductor may thus be defined as the resistance offered by a specimen of the material of unit length and of unit area of cross section.* It may be found by measuring R, l and A of a sample of the material in the form of wire.

V, i and R are macroscopic quantities as these are the properties of a body as a whole. The corresponding microscopic quantities are **E**, **j** and ρ. The former two may be found by integrating over the latter quantities, as indicated by the relations (20) and (2). The resistance of a conductor between two equipotential surfaces S_1 *and* S_2 may also be expressed in microscopic terms as

$$R = \frac{V}{i} = \frac{\int_1^2 \mathbf{E}.d\mathbf{l}}{\int_1^2 \mathbf{j}.d\mathbf{S}} \qquad \text{...(25)}$$

From Eqs. (10) and (12), we get

$$div\,\mathbf{j} = div\,(\sigma\mathbf{E}) = \sigma\, div\,\mathbf{E} = 0.$$

Since $\mathbf{E} = -$ grad V, hence $\nabla^2 V = 0$. ...(26)

Thus we see that Laplace's equation holds, as in electrostatics *and the lines of current flow in the medium are orthogonal to the lines of constant V and coincide with lines of* **E**.

Resistance for Radial Current Through a Circular Disc

Let us consider a circular slab of conducting material of thickness t and conductivity σ. Consider the radial flow of current from some radius a at a potential V_a to the outer radius b at a potential V_b. Total current through any cross section will be same as that through any other cross section. Current through a cross section ⊥ to the current flow at any radius r and thus of area $2\pi rt$ is given by

$$i = jA = \sigma AE = \sigma.\, 2\pi rt\,(-dV/dr),$$

or

$$dV = -\frac{i}{2\pi t\sigma}\frac{dr}{r}.$$

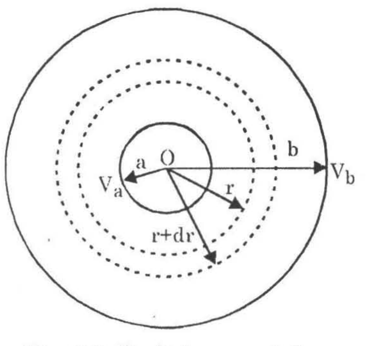


Fig. 8.6. Radial current flow.

$\therefore$ The potential difference $= \int_{V_a}^{V_b} dV = \dfrac{-i}{2\pi t\sigma}\int_a^b \dfrac{dr}{r}$

or
$$V_a - V_b = \frac{i}{2\pi t\sigma}\log_e \frac{b}{a}.$$

$$\therefore \quad \text{Resistance } R = \frac{V_a - V_b}{i} = \frac{1}{2\pi t\sigma}\log_e \frac{b}{a}. \qquad \text{...(27)}$$

If the current is not $\perp$ to the cross section, we must use $i = \mathbf{j}\cdot\mathbf{A}$ and the problem will be complicated.

Sphere partially immersed in a non uniform conducting medium : Consider a perfectly conducting sphere of radius a with half of the sphere in contact with the ground. Let σ_1 be the conductivity of the layer of earth of radius b that is in immediate contact with the sphere and σ_2 the conductivity of the rest of the ground. This is the simplified model of grounding an electric current using a spherical electrode. If the current i flows from the sphere to the ground, then the current density j in the medium 1 is given by

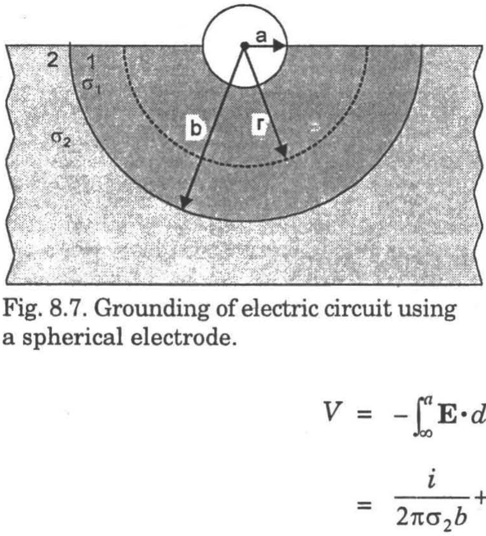


Fig. 8.7. Grounding of electric circuit using a spherical electrode.

$$\int_S \mathbf{j}\cdot d\mathbf{S} = \int j_1 dS = 2\pi r^2 j_1 = i$$

or
$$j_1 = i/2\pi r^2.$$

It is in the radial direction. Therefore the corresponding electric field in the region 1 is given by

$$\mathbf{E}_1 = \frac{\mathbf{j}_1}{\sigma_1} = \frac{i\,\hat{r}}{2\pi\sigma_1 r^2}$$

Similarly $\mathbf{j}_2$ and $\mathbf{E}_2$ for the region 2 are given as

$$\mathbf{j}_2 = \frac{i}{2\pi r^2}\hat{r} \text{ and } \mathbf{E}_2 = \frac{i}{2\pi\sigma_2 r^2}\hat{r}.$$

The potential of the sphere can be determined as

$$V = -\int_\infty^a \mathbf{E}\cdot d\mathbf{r} = -\int_\infty^b \frac{i}{2\pi\sigma_2 r^2}dr - \int_b^a \frac{i}{2\pi\sigma_1 r^2}dr.$$

$$= \frac{i}{2\pi\sigma_2 b} + \frac{i}{2\pi\sigma_1}\left(\frac{1}{a} - \frac{1}{b}\right).$$

$\therefore$ Resistance R between the sphere and the ground $= V/i$

$$= \frac{1}{2\pi\sigma_2 b} + \frac{1}{2\pi\sigma_1}\left(\frac{1}{a} - \frac{1}{b}\right). \qquad \text{...(28)}$$

8.6 STEADY CURRENTS IN MEDIA WITHOUT SOURCE OF CURRENT

Since charge can neither be piling up or disappearing at a boundary, the current towards the boundary on one side must be equal to that away from it on the other side. Thus

$$\mathbf{j}_1\cdot d\mathbf{S} = \mathbf{j}_2\cdot d\mathbf{S} \text{ or } j_{1n} = j_{2n}, \qquad \text{...(29)}$$

or *the normal component of* $\mathbf{j}$ *must be continuous across the interface.* This relation can also be written as

$$\sigma_1 E_{1n} = \sigma_2 E_{2n}. \qquad \text{...(30)}$$

This equation replaces the analogous equation for the continuity of D_n across the dielectric interfaces (*i.e.*, $D_{1n} = D_{2n}$).

So long as there are no sources of emf in either medium, $\oint \mathbf{E}\cdot d\mathbf{l} = 0$ for a closed path which links both media, therefore

$$E_{1t} = E_{2t}. \qquad \text{...(31)}$$

The equation is analogous to the equation for the continuity of E_t in the case of dielectric problems ($E_{1t} = E_{2t}$). It can be written as

$$j_{1t}/\sigma_1 = j_{2t}/\sigma_2. \qquad ...(32)$$

Combining Eqs. (29) and (32), we get

$$\sigma_1 \cot \theta_1 = \sigma_2 \cot \theta_2. \qquad ...(33)$$

It is the equation for the *law of refraction*. Thus we see that *at the boundary between two media of different conductivities the lines of flow are refracted in a manner analogous to that of the lines of force at the boundary between two dielectrics.*

If the two media have permittivities, ε_1 and ε_2, then the Gauss's law in dielectric requires that $D_{2n} - D_{1n} = \sigma_f$ or $\varepsilon_2 E_{2n} - \varepsilon_1 E_{1n} = \sigma_f$, where σ_f is the free surface charge density at the interface. These relations are very important dealing with boundary value problems that involve dielectrics with conductive properties.

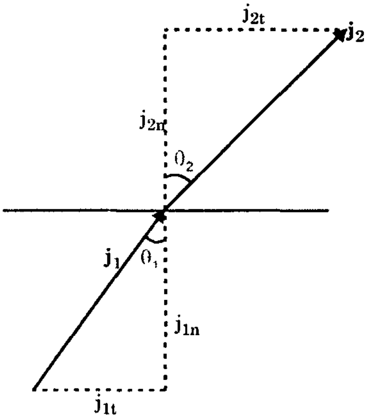


Fig. 8.8. Current densities at the boundary of two different media.

A second example of the similarity between conductors and the dielectrics is the *resistance capacitance analogy*. Consider two metallic conductors in a homogeneous ohmic medium of conductivity σ and permittivity ε. If a potential difference V is applied between these conductors, then the current is given by

$$i = V/R,$$

where R is the resistance of the medium. Current i can be written in the form of current density **j** as

$$i = \int_s \mathbf{j}\cdot d\mathbf{S} = \int_s \sigma \mathbf{E}\cdot d\mathbf{S} = \sigma \int_s \mathbf{E}\cdot d\mathbf{S}.$$

As by Gauss's law integral term is equal to q/ε, hence

$$V/R = \sigma q/\varepsilon \text{ or } q = \varepsilon V/\sigma R,$$

As $$q = CV,$$

hence $$C = \varepsilon/\sigma R \text{ or } RC = \varepsilon/\sigma. \qquad ...(34)$$

It is the relation between the resistance of the medium and the capacitance of the equivalent electrostatic problem. The resistance of the material filled within the coaxial line may easily be calculated by first calculating the capacitance of the line and then using the Eq. (33).

Duality Between j and D

Since **j** and **D** both are linearly related to the electric field **E** in many materials, hence dual relationship between **j** and **D** exist. In the regions, where non conservative fields are absent, we have

Conducting media	*Dielectric media*
$\mathbf{j} = \sigma\mathbf{E}$	$\mathbf{D} = \varepsilon\mathbf{E}$
$\nabla\cdot\mathbf{j} = 0$	$\nabla\cdot\mathbf{D} = 0$
$\nabla\times\mathbf{E} = 0$	$\nabla\times\mathbf{D} = 0$ ...(35)

In a homogeneous materials where ε and σ are constant, we have

$$\nabla\times\mathbf{j} = 0 \qquad \nabla\times\mathbf{D} = 0$$

and hence $$\mathbf{j} = -\nabla\varphi \qquad \mathbf{D} = -\nabla\varphi \qquad ...(36)$$

where φ, a potential function, satisfies Laplace's equation.

Above equations show that any solution of **j** can be transformed into a solution for **D** and vice versa, by interchanging as **j** ↔ **D** and σ ↔ ε.

This is also true as we have similar type of relations for the boundary of two media.

$$\begin{array}{cc} \textit{Conducting media} & \textit{Dielectric media} \\ j_{n_1} = j_{n_2} & D_{n_1} = D_{n_2} \\ j_{t_1}/\sigma_1 = j_{t_2}/\sigma_2 & D_{t_1}/\varepsilon_1 = D_{t_2}/\varepsilon_2 \end{array} \qquad ...(37)$$

Relation (37) shows that a measurement of resistance R will suffice to determine the capacitance C.

8.7 LORENTZ-DRUDE THEORY OF ELECTRICAL CONDUCTION AND OHM'S LAW

The theory of electrical conductivity in metals was developed by Paul Drude in 1900 and was modified by Lorentz. According to this theory the valence electrons are not attached to individual atoms but are free to move about within the lattices like the molecules of a perfect gas in a container and are called the *conduction electrons*. As the effective radius of an atom is of the order of 10^{-10} m, while that of an electron is of the order of 10^{-15} m, we may neglect the collisions between electrons in comparison with the collisions between electrons and ion cores. The electrons and ion cores collisions are elastic random collisions. As in the case of molecular collisions in the gas, we can describe these collisions by a *mean free path* λ, the average distance that an electron travel between two successive collisions and the *mean free time* τ, the average time between collisions. Because of the random nature of this motion, on the average no net charge is transferred in the absence of an external electric field.

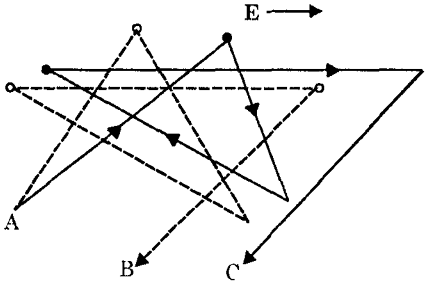


Fig. 8.9. Random motion of electron.

If now an electric field **E** is applied to the metal, the electrons are accelerated in the direction opposite to that of the field and acquire an average drift velocity **v**, anti-parallel to **E**. In the Fig. 8.9, the *dashed lines show the electron path in the presence of the electric field* **E**. The ratio v/E, the drift velocity in unit field is known as the *mobility*. The drift velocity is small compared to the thermal velocities so that we can assume these two as independent to each other. The latter is determined by the structure of the conductor and the temperature but not by **E**. As the acceleration experienced by an electron is $-e\mathbf{E}/m$, hence the increase in drift velocity between two collisions will be $-e\tau\mathbf{E}/m$. The effect of each collision is to reduce the drift velocity to zero and to restore the random thermal motion. The drift velocity increases between two collisions and reduces to zero after next collision. Thus the average drift velocity over the time between two consecutive collisions will be $-\frac{1}{2}e\tau\mathbf{E}/m$.

If n is the conduction electrons per unit volume, then the current density due to the drifting of electrons is given by

$$\mathbf{j} = -ne\mathbf{v} = \frac{1}{2}\, ne^2\tau\, \mathbf{E}/m.$$

If v_{av} is the average thermal velocity of the electrons, then the time between two consecutive collisions will be λ/v_{av}. Thus the above equation can be written as

$$\mathbf{j} = \frac{1}{2} ne^2\, \lambda\mathbf{E}/(mv_{av}), \qquad ...(38)$$

$$\text{The resistivity } \rho = E/j = 2mv_{av}/ne^2\lambda, \qquad ...(39)$$

$$\text{and the} \qquad \text{conductivity } \sigma = 1/\rho = ne^2\lambda/(2mv_{av}). \qquad ...(40)$$

This is in qualitative agreement with experiment. The quantities v_{av} and λ depend on the velocity distribution of the conduction electrons. Thus the right hand side is independent of **E** and is constant at constant temperature. It shows that metals obey Ohm's law.

Since v_d, the drift velocity ($\simeq 10^{-4}$ m/sec) is very small as compared to v ($\simeq 10^6$ m/sec), hence we have used $v + v_d \simeq v$ and

$$<|v+v_d|> \simeq <v> = v_{av}$$

in the above equations.

Ohm's law can directly be derived from the relation (38). At constant temperature λ and v_{av} are constants, hence

$$j \propto E \qquad \text{or} \qquad i/A \propto V/l$$

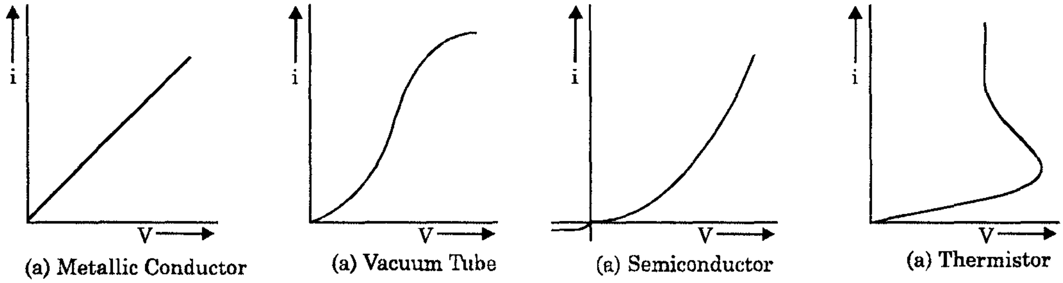
or

$$i \propto V, \qquad \ldots(41)$$

as for a given conductor l and A are fixed. This is the *Ohm's law.*

If the temperature is increased, v_{av} increases and thus the resistivity of a metal increases. In a semiconductor, the number of charge carriers per unit volume, n, increases rapidly with increasing temperature. This increase is greater than that in v_{av} and the resistivity thus decreases. At low temperature, n is very small and the resistivity becomes so large that the material can be considered an insulator.

Any resistor obeying ohm's law is called *Ohmic circuit element* or a *linear circuit element* (a plot of V versus i is a straight line). Many conductors do not obey Ohm's law and are called *non-ohmic circuit elements* or *non-linear circuit elements.* The curve between i and V for a Cu-conductor is a straight line, which shows that the resistance of a conductor is the same no matter what applied voltage is used to measure it. This holds good for metallic conductors, which are thus known as *linear circuit elements.* The non-ohmic circuit elements are the vacuum tubes, semiconductor devices and thermistors. The variations of the current i with the voltage V *applied* for metallic conductor, vacuum tube, semiconductor diode and thermistor are shown in Fig. 8.10. There are certain devices (*e.g.*, tunnel diode) which exhibit a negative slope in their $i-V$ characteristics at certain points.



(a) Metallic Conductor (a) Vacuum Tube (a) Semiconductor (a) Thermistor

Fig. 8.10. Examples of non-linear circuit elements.

Wiedemann Franz's Law : Wiedemann and Franz gave an empirical law which states that the ratio of the thermal and electric conductivities is proportional to the absolute temperature and is the same at constant temperature for all metals. Later on Lorentz showed that

$$\frac{\textit{Thermal conductivity } K}{\textit{Electrical conductivity } \sigma} = \frac{\frac{3}{2}k^2\, T\lambda n/mv_{av}}{ne^2\lambda/2mv_{av}} = 3\left(\frac{k}{e}\right)^2 T,$$

or

$$K/\sigma \propto T, \qquad \ldots(42)$$

where constant of proportionality, also known as *Lorentz number* $L = 3\,(k/e)^2$, where k is the Boltzmann constant.

For a number of pure metals this ratio was found experimentally and found to fall within the range 2.3×10^{-8} to 3.2×10^{-8} watt-ohm/deg^2. These results are in fair agreement with the value of $3\,(k/e)^2$, *i.e.*, 2.24×10^{-8} watt ohm/deg^2.

Inspite of the success of the free electron model to explain K and σ, it fails to explain the specific heat of metals. From this metal electron should have a thermal energy $\frac{3}{2}\,kT$ at temperature T and each electron should contribute to the specific heat of the metal.

At low temperatures, the Lorentz number tends to decrease. It is due to the difference in the collision averages involved in the thermal and electrical conductivities.

8.8 RESIDUAL RESISTIVITY

As discussed in the preceding article, the resistivity ρ of the conductor is due to the scattering of free electrons within the conductor, known as *thermal scattering*. It is true in an ideal lattices only where the ions are arranged with a perfect regularity. In practice, some impurity ions and/or defects are always present. The potential function undergoes a sudden change in the neighbourhood of these impurities and causes an electron to scatter. It gives rise to additional resistivity called residual resistivity ρ_R. *It is independent of temperature and depends only on the impurities and defects present in the metal hence is constant for a particular specimen.*

Total resisitivity ρ = Resisitivity due to thermal scattering $\rho_T + \rho_R$.

Hence at absolute zero or very low temperature $\rho = \rho_R$.

8.9 SUPER CONDUCTORS

Ordinary conductors of electricity become better conductors as the temperature is lowered. It was thought that the resistance of a metal would be zero at absolute zero. Using the liquefied helium to achieve lower temperatures. Kammerlingh Onnes in 1911 measured the resistivity of mercury in the neighbourhood of 4°*K*. He observed that mercury completely loses its resistance to electric current at 4.2 °*K*. This loss occurs abruptly within an interval of a few hundredths of a degree as shown in Fig. 8.5 (*c*). This phenomenon is known as *super conductivity* and the material under such conditions is said to be in the super conducting state. The temperature below which the material is in a super conducting state (or a *super conductor*) is called the *transition or critical temperature* T_c.

Kammerlingh Onnes succeeded later in observing that lead, tin and indium also become super conductors at 7.2, 3.7 and 3.4°*K* respectively. Now we have various super conductors. Very interesting fact is that certain alloys become superconductors at rather high temperatures. The resistivity of a super conductor is less than $10^{-25}\ \Omega$-*m*, which is negligible in comparison to the resistivity of the best normal conductor *Ag* (*i.e.*, $1.47 \times 10^{-8}\ \Omega\ m$). The supper conductivity depends on the crystal lattice structure. For example while tin is a super conductor while grey tin is not. Mercury is a super conductor only in the α – phase. The super conductivity can be explained in terms of the interaction among the electrons and the ionic lattice.

In 1914, Kammerlingh Onnes found that the super conducting state is destroyed by a magnetic field. The minimum magnetic field sufficient for this purpose is called *critical field.* Table 8.2 lists critical temperatures for some super conducting materials.

Table 8.2. Critical temperatures for various superconductors

Elements	T_c (°K)	*Compound*	T_c (°K)
Zinc	0.88	An Sb_2	0.58
Aluminium	1.19	Pd Sb_2	1.25
Tin	3.72	Ni Bi	4.25
Mercury	4.15	NbN	16.0
Lead	7.18	Nb_3 Sn	18.05
Technetium	11.2	Nb_3 Ge	23.2

The resistance of materials in the super-conducting state is zero and the current once established in closed super conducting circuits persist for weeks, even though there is no battery in the circuit. Such currents drop rapidly to zero if the temperature is raised slightly above the super conducting point. Since no heat is generated when a current flows through super conductors, hence are very much used, *e.g.*, in large electromagnets.

8.10 ENERGY DISSIPATION IN A RESISTANCE (*JOULE'S LAW*)

When current passes through a conductor of finite resistance, charge moves from higher potential to lower potential and the work is continuously done by the electric field. For each coulomb of charge that moves through one volt of potential drop, one joule of work is performed. If dq charge flows between two points differing in potential by V in a time dt, the power loss

$$P = Vdq/dt = Vi = VV/R = V^2/R = i^2 R, \qquad \text{...(43)}$$

where R is the resistance between the two points. The unit of power is volt-ampere known as the *watt* or *Joule/sec*. Eq. (43) represents three different forms of Joule's law.

In an extended medium of conductivity σ, the power dissipated in an element of cross section ΔS and length Δl is

$$\Delta P = \Delta V \cdot j\Delta S = E\Delta l \cdot j\Delta S.$$

or
$$\text{power dissipated/volume} = jE = \mathbf{j}\cdot\mathbf{E} = \sigma\mathbf{E}\cdot\mathbf{E} = \sigma E^2. \qquad \text{...(44)}$$

This is the differential form of Joule's law.

If the device is a motor, the dissipated energy appears as mechanical work done by the motor. If the device is a storage battery this energy appears as stored chemical energy. If the device is a resistor energy appears as heat. The flow of electrons through the resistor is like that of the stone through water, where the potential energy mgh of the stone appears as thermal energy in the stone and the surrounding water. On a microscopic scale the transfer of dissipated electric energy in the form of heat can be understood as; *collisions between the electrons and the lattice of ions increase the amplitude of the thermal vibrations of the lattice and this corresponds to a temperature increase. This effect is often called the Joule heating and Eq. (44) is known as Joule's law.*

Let us now derive the energy formula for resistance. Consider a conductor with end surfaces S_1, S_2 kept at constant potentials V_1 and V_2. The current density $\mathbf{j}$ flows normal to these end faces and parallel to sides of the conductor. From Eqs. (43) and (44), we have

$$P = i^2R = \int \mathbf{j}\cdot\mathbf{E}\,d\tau = \int \mathbf{j}\cdot\mathbf{j}(1/\sigma)\,d\tau = \int \sigma(\mathbf{E}\cdot\mathbf{E})\,d\tau.$$

$$\therefore \qquad R = \frac{1}{i^2}\int \mathbf{j}\cdot\mathbf{E}\,d\tau = \frac{1}{i^2\sigma}\int j^2\,d\tau = \frac{\sigma}{i^2}\int E^2\,d\tau, \qquad \text{...(45)}$$

where $d\tau$ is the elementary volume of the conductor.

Let
$$\mathbf{E} = -\nabla V$$

$$\mathbf{E}\cdot\mathbf{j} = (-\nabla V)\cdot\mathbf{j} = -\nabla\cdot(V\mathbf{j}) + V\,\nabla\cdot\mathbf{j}$$

Since $\nabla.\mathbf{j} = 0$, hence upon application of the divergence theorem above equation becomes

$$\int \mathbf{E}\cdot\mathbf{j}\,d\tau = -\int \nabla\cdot(V\mathbf{j})\,d\tau = -\int_S V\,\mathbf{j}\cdot\hat{n}\,dS, \qquad \text{...(46)}$$

where S is the total area of the conductor. Since $\mathbf{j}.\hat{n} = 0$ except on the constant potential surfaces, hence we have

$$i^2R = V_1\int_{S_1} \mathbf{j}\cdot(-\hat{n})\,dS - V_2\int_{S_2} \mathbf{j}\cdot\hat{n}\,dS = (V_1 - V_2)i. \qquad \text{...(47)}$$

It is same as Eq. (43).

8.11 ELECTROMOTIVE FORCE

We know that the closed line integral of electrostatic field $\oint \mathbf{E}\cdot d\mathbf{l} = 0$, hence electrostatic field can not cause current flow in the circuit. The source of energy in a circuit may be chemical, mechanical or thermal, or it may result from a changing magnetic field. In an electrolytic cell, the energy is chemical and in a Van de-Graaff generator the energy is mechanical. In a thermo-couple, a thermo-*emf* is produced when the junctions of a circuit consisting of dissimilar metals are at different temperatures and the source of energy is thermal. In a photovoltaic cell, the photo *emf* is produced when the junction $(p - n)$ is irradiated with light, infrared rays or other radiant energy.

In a device such as a drycell, storage battery, or electrostatic generator, in which nonelectrical energy is converted into electrical energy or *vice versa*, let us consider that the terminal a is maintained by the source at a higher potential than terminal b (Fig. 8.11). There is therefore an electrostatic field $\mathbf{E}_e$ at all points between and around the terminals. The source is itself a conductor and if only this electrostatic field exerts, the positive charges will move from a toward b and the negative charges from b toward a. The excess charges on the terminals would decrease and hence the *pd* between them would decease and eventually become zero. As it does not happen, we conclude that there must also exit at every point within the source a force $\mathbf{F}_n$ of non-electrostatic origin acting on every charged particle and which is equal and opposite to the electrostatic force $\mathbf{F}_e = q\mathbf{E}_e$. We can define the non-electrostatic field $\mathbf{E}_n$ as $\mathbf{F}_n = q\mathbf{E}_n$. When the source is an open circuit, the vector sum of these fields at every point must be zero, *i.e.*,

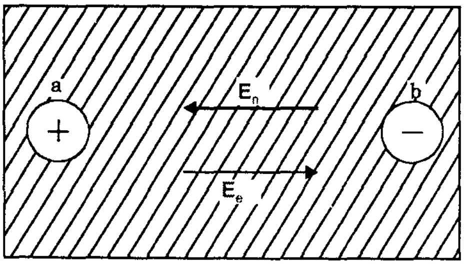


Fig. 8.11. Extraneous emf.

$$\mathbf{E} = \mathbf{E}_n + \mathbf{E}_e = 0 \text{ or } \mathbf{E}_e = -\mathbf{E}_n.$$

$$\therefore \int_a^b \mathbf{E}_e . d\mathbf{l} = -\int_a^b \mathbf{E}_n . d\mathbf{l} = \int_b^a \mathbf{E}_n . d\mathbf{l}.$$

$$\text{As } \int_a^b \mathbf{E}_e . d\mathbf{l} = V_{ab}, \text{ therefore } \int_b^a \mathbf{E}_n . d\mathbf{l} = V_{ab}. \qquad ...(48)$$

The line integral of the non-electrostatic field from b to a is called the *electromotive force* of the source. This name is totally misleading because the source refers to energy and not to force. Hence to avoid confusion the abbreviation *emf* is in common use. We will stick with it and represent the *emf* of a source by $\mathcal{E}$ (do not confuse it with E for electric field).

Thus *the emf $\mathcal{E}$ of the source equals the open circuit potential difference.* We should not confuse between terms *potential difference* and *emf*, as the former is the line integral of an electrostatic field and the latter is the integral of a non-electrostatic field. The electrostatic field, hence the potential difference depends on the current in the source. The non-electrostatic field, hence the *emf* of the source is a constant independent of the current and this represents the property of the source. *The non-electrostatic forces are called extraneous electromotive forces. Thus a direct current cannot be generated by the force which are of purely electrostatic origin. In order to create a direct current we must have extraneous electromotive forces.*

For a closed circuit, *i.e.*, when two terminals of a source are connected by a conductor, the resultant field $\mathbf{E}$ within the source is $\mathbf{E} = \mathbf{E}_e + \mathbf{E}_n$, hence

$$\int_a^b \mathbf{E} . d\mathbf{l} = \int_a^b \mathbf{E}_e . d\mathbf{l} + \int_a^b \mathbf{E}_n . d\mathbf{l}$$

or

$$Ir = -V_{ab} + \mathcal{E}, \qquad ...(49)$$

where r is the internal resistance of the source. Here the direction of integration is the direction of current within the source.

If R is resistance of the conductor connecting the terminals a and b, then Eq. (49) can be written as

$$Ir = -IR + \mathcal{E} \text{ or } i = \mathcal{E}/(R + r). \qquad ...(50)$$

If a circuit consists of any number of sources and conductors connected in series, then the relation can be written as

$$i = \Sigma\mathcal{E}/\Sigma R, \qquad ...(51)$$

where $\Sigma\mathcal{E}$ is the algebraic sum of emf's of the sources and ΣR is the sum of all resistances, external and internal.

8.12 RESISTORS

Resistors may be divided into two groups : resistance boxes and rheostates. The former is used when a definite and known resistance is required and the latter is used when we are concerned with regulation of currents and absorption of power.

In **resistance boxes** coils are connected in various combinations so that any required resistance upon the full capacity of the box may be obtained. The most common types are the *plug box* and the *dial box*. The plug box consists of a number of resistance elements of different magnitudes connected in series. Each junction between elements is connected to a metal block, so that if a metal plug is inserted between adjacent blocks the corresponding resistance is short circuited. Thus the total resistance of the box in circuit is the sum of the resistances whose plugs have been removed. Resistance material used for winding coils is manganin, which is an alloy 84% Cu + 12% Mn + 4% Ni. It has smaller temperature coefficients and higher resistivity. The dial box also consists of a number of resistance elements in series, the junctions being connected to a set of metal buttons over which a contact arm moves.

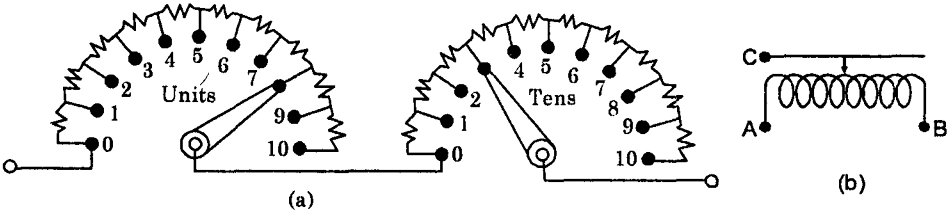


Fig. 8.12. (*a*) Dial type resistance box, (*b*) Sliding rheostat.

Usually in dial boxes the elements are arranged in groups of ten of equal magnitude, successive groups however differing in magnitude by a factor of ten as shown in Fig. 8.12.

A sliding rheostat usually consists of oxidised nichrome wire (an alloy of Ni, Cr and Cu) wound on an insulating cylinder. A sliding contact is arranged on a metal rod mounted above the winding, parallel to the cylinder. This alloy provides high resistivity. It is taken oxidised to provide insulation between turns. Besides the sliding contact terminal, two terminals are provided at the ends of the wire. These both terminals are used only when the rheostat is used as a potential divider. In this case the battery is connected across the fixed terminals and the potential difference across are fixed terminal and the variable terminal (sliding contact) is used as a potential source.

In electronic circuits resistors of **fixed resistances** are required. Most resistors are marked with the normal value of their resistances. The present practice is either to print the value at the body of the resistor in number or to put three coloured bands around the body, as shown in Fig. 8.13, which can be translated into a number by using the standard colour code* given below :

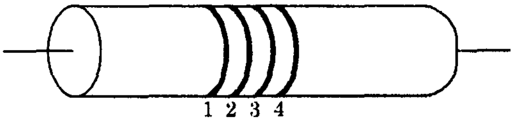


Fig. 8.13. Resistor colour code.

Black	0	Red	2	Yellow	4	Blue	6	Gray	8
Brown	1	Orange	3	Green	5	Violet	7	White	9

The first colour band represents the first figure of the resistance and the second band the second figure and the third band the number of zeros to add the first two figures to get the total resistance. For resistances between 1 and 10 ohms, gold is used for the third band. Thus green, orange and gold bands represent 5.3 ohms. The limit of accuracy is indicated by the colour of the fourth band, silver being 10% and gold being 5%. Ordinary carbon resistors should not be used when the signal level is 10^{-3} volt or less.

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* Bill Brown Realized Only Yesterday Good Boys Value Good Work where capital bold letters stand for colours.

Carbon Resistors : As wire wound resistors are expensive, not suitable to use at very high frequencies and can not be obtained in small size if are of high resistance values. The best alternatives are deposited carbon and carbon-composition resistors.

(a) Deposited-carbon resistors : The deposited-carbon resistor also known as carbon film resistor consists of a ceramic core on which a thin layer of crystalline metallic carbon is deposited from a hydrocarbon vapour at high temperature. These resistors are much cheaper, nearly stable, having temperature coefficient of resistivity of the order of 200 to 500 parts per million per °C, very small noise and have superior performance at radio frequencies.

(b) Carbon-composition resistor : These are of two types : *(i)* rod type and *(ii)* film type. *The carbon-composition-rod type resistor* consists of a solid conducting rod formed of a mixture of fine carbon particles and an insulating binding medium. Although these resistors are widely used in electronic equipments, but are not suitable for precision work because of their poor stability, high value of α, poor performance at high frequencies. *Carbon-composition film type resistors* consists of an insulating support such as a ceramic or glass rod or tube at the surface of which a conducting film of a mixture of carbon particles and insulating binding medium is provided. These are frequently called *metallized resistors*. As the value of α, stability with time and performance at high frequencies are concerned, these are superior to those of carbon rod type resistors. Although these are somewhat inferior to those of the deposited carbon resistors.

Noise : When *dc* is passed through carbon resistor, a noise voltage is developed across its terminals. It is almost proportional to the *dc* voltage applied to the resistor. It varies inversely as the frequency. Different types of resistors differ in the noise that they generate. No noise is observed in wire wound resistor, deposited carbon resistors have the lowest noise (<0.25 μV noise per *dc* volt), carbon rod types have the highest noise and the carbon film resistor have in between two and can thus be assumed as good resistors.

8.13 KIRCHHOFF'S LAWS

Most electrical circuits comprise a number of sources and resistors interconnected in a more or less complicated manner. These are named as *networks*. All networks can not be reduced to simple series-parallel combinations. There are number of techniques to solve such problems. We shall describe here only one of these, first developed by Gustav Robert Kirchhoff. Let us first define three terms : a *branch point* in a network is a point where three or more conductors are joined, *loop* is any close conducting path and *mesh* is a closed path (loop) which can not be subdivided into other loops. With the restriction that Ohm's law obeys for each part of the network, Kirchhoff gave following two laws :

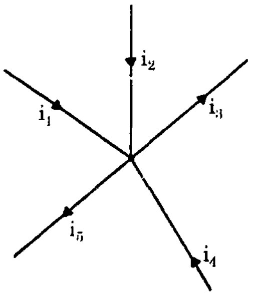


Fig. 8.14. Branch point in a network.

(1) The *algebraic sum of all the currents meeting at a branch point is zero.* It follows from the equation of continuity, that there can be no accumulation of charge at any point in the circuit. It can be written as

$$\Sigma i = 0. \qquad ...(53)$$

Hence the net current meeting at any branch point (Fig. 8.14) can be given as

$$i_1 + i_2 - i_3 + i_4 - i_5 = 0.$$

Here the convention is that the current flowing towards the branch point or the junction is considered positive, whereas out flowing current is considered negative.

(2) *In any loop in a network of conductors, the algebraic sum of emf's is equal to the algebraic sum of the products of the resistances of each link of the loop and the current flowing through them. It follows that the total fall of* potential round any simple circuit is equal to the total emf's in that circuit. It can be written as

$$\Sigma iR = \Sigma \mathcal{E}. \qquad \text{...(54)}$$

Before applying this law, the direction of current in the conductors of the loop is assumed and the product iR is given a positive sign if the current flowing through the resistance R is along the assumed current direction and the emf $\mathcal{E}$ is given a positive sign if the assumed current direction is from –ve to the +ve terminal inside the source.

8.14 WHEATSTONE'S BRIDGE

The Wheatstone's bridge is a special type of resistance network, commonly used for comparing resistances. It consists of four resistances R_1, R_2, R_3 and R_4 connected in series. The one pair of opposite junctions of the four resistances, such as A and C being connected by battery E of emf $\mathcal{E}$ and internal resistance r and the other pair of opposite junction B and D through a galvanometer of resistance R_G.

Let i, i_1, i_2, i_3, i_4 and i_g be the currents in the battery circuit, through resistances R_1, R_2, R_3, R_4 and the glavanometer respectively. Then by Kirchhoff's first law, we have for

the branch point A $\quad i - i_1 - i_3 = 0,$

the branch point B $\quad i_1 - i_2 - i_g = 0,$

the branch point D $\quad i_3 + i_g - i_4 = 0.$

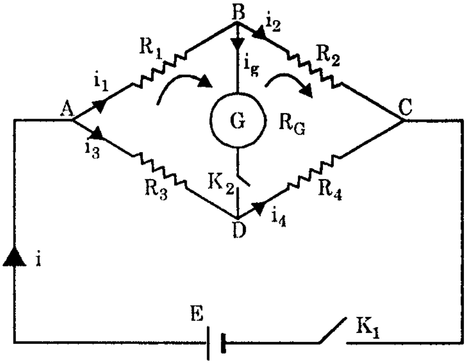


Fig. 8.15. Wheatstone's bridge.

Thus the currents in AB, BC, AD and DC will be i_1, $i_1 - i_g$, $i - i_1$ and $i - i_1 + i_g$ respectively. Assuming the clockwise direction of the current as positive in the meshes, and now applying Kirchhoff's second law to the meshes $ABCD$ and $BCDB$, we get

$$i_1R_1 + i_g i_G - (i - i_1)R_3 = 0$$

and

$$(i_1 - i_g)R_2 - (i - i_1 + i_g)R_4 - i_gR_G = 0.$$

If the bridge is balanced, *i.e.*, no current is flowing through the galvanometer ($i_g = 0$), then we get

$$i_1(R_1 + R_3) = iR_3 \text{ and } i_1(R_2 + R_4) = iR_4.$$

$$\therefore \qquad \frac{R_1 + R_3}{R_2 + R_4} = \frac{R_3}{R_4} \text{ or } \frac{R_1}{R_2} = \frac{R_3}{R_4}, \qquad \text{...(55)}$$

which is the well known condition for the bridge to be balanced.

Thus two resistances can be compared or an unknown resistance R_4 can be measured in terms of three known resistances R_1, R_2 and R_3. Although this relation is independent of the current in the battery circuit, but the currents must not be large enough to heat the resistances and so alter their values. It is a null method, thus does not require any calibration of glavanometer. It can be shown that when the positions of the galvanometer and battery are interchanged, the above condition for no deflection of galvanometer will still hold. The arms (known as 'battery arm' and the '*galvanometer arm*') are known as conjugate arms when the bridge is balanced. Key's in these arms are known as battery key and galvanometer key respectively.

Sensitivity of a Wheatstone's bridge : The accuracy in the measurement of resistances depends upon the sensitiveness of the bridge. For a better sensitivity of the bridge other resistances in the bridge arms should be so adjusted that the deflection in galvanometer is large even for small deviation from the balance position.

Sensitiveness of Wheatstone's bridge : We shall follow a treatment given by Callendar for the conditions of the bridge sensitiveness. The resistances in the four arms of the unbalanced bridge are written in the form $R_4 = r$ (unknown), $R_2 = mr_1$, $R_3 = nr_1$ and $R_1 = nmr_1$, where m *and* n are simple numbers. When $r = r_1$ the bridge is balanced and $i_g = 0$.

If i be the current through the resistance r to be measured when the bridge is not balanced and i_g be the current through the galvanometer of resistance R_G, then from Kirchhoff's first law, the current through resistance nr_1 will be $i + i_g$. If i_1 is the current through the resistance nmr_1, then the current through the resistance mr_1 will be $i_1 + i_g$, as shown in Fig. 8.16.

To solve the problem, let us assume the current in the clock-wise direction in the meshes *ABDA and BCDB* as positive. Now applying Kirchhoff's second law to these meshes, we have

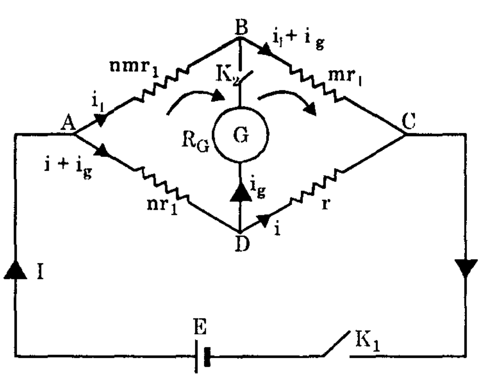


Fig. 8.16. Unbalanced Wheatstone's bridge.

$$nmr_1\, i_1 - R_G i_g - nr_1\,(i + i_g) = 0$$

and $$mr_1\,(i_1 + i_g) - ri + R_G i_g = 0.$$

Rearranging these relations and eliminating i_1 from these, we get

$$\frac{i_g}{i} = \frac{n(r - r_1)}{(n+1)R_G + n(1+m)r_1}$$

$$= \frac{r - r_1}{(1+m)r_1 + (1 + 1/n)R_G}. \qquad ...(56)$$

The ratio i_g/i is a measure of the sensitiveness of the bridge. It is independent of the emf and resistance in the battery circuit. For a given want of balance $r - r_1$, the sensitivity can be increased by (*i*) increasing current i, (*ii*) decreasing glavanometer resistance R_G, (*iii*) decreasing m and (*iv*) increasing n. The current i through the unknown resistance is limited by the heating effect of this current. Galvanometer resistance can not be decreased below a certain limit as it varies with the number of turns in the coil. Accuracy of the galvanometer is proportional to the number of turns in coil. Thus the decrease in R_G is possible at the cost of accuracy of the galvanometer.

Thus we see that the sensitivity of the bridge can only be increased by decreasing m and increasing n to any extent. The limiting values are $m = 0$ *and* $n = \infty$, in which case

$$\frac{i_g}{i} = \frac{r - r_1}{r_1 + R_G} \qquad ...(57)$$

This condition is impracticable because if $n = \infty$, the current in the bridge would be zero, as resistances in arms *AB* and *AD* are both infinite; and if $m = 0$, the branch *ADC* and the galvanometer arm are short circuited. If $m = n = 1$, we have

$$\frac{i_g}{i} = \frac{r - r_1}{2(r_1 + R_G)}. \qquad ...(58)$$

Thus we see that when $m = n = 1$, the sensitiveness is still half as great as in the ideal limit ($m = 0$ and $n = \infty$). The sensitiveness can be increased by increasing n and decreasing m. It will be greatly reduced if n is made small and m large. Prof. Callendar gave as a rule that *the battery and galvanometer should be connected in such a way that the resistance in series with the unknown resistance be greater than the resistance in parallel with it.*

As we know that the positions of the galvanometer and the battery can be interchanged but these two arrangements are not equally sensitive. Maxwell gave a rule regarding the best arrangements of these – *of the two resistances that of the battery and that of the galvanometer,* *the greater connects the junctions of the two greatest with that of the two least of the four arms of the bridge.*

8.15 COMPARISON OF TWO RESISTANCES

The Wheatstone's bridge method is the most accurate and frequently used method of comparing resistances. The great advantage in this method is the null detection.

A compact form of Wheatstone's bridge is a *Post Office Box*, originally designed to measure the resistances of electric cables and telegraph wires in Post Offices. It consists of three separate sets of coils of known resistances to form three arms *AB, BC* and *AD* of the wheatstone bridge. Two exactly alike and similarly placed sets *P and Q* are known as ratio arms of the bridge. In *P* and *Q*, usually each consists of three resistances of 10, 100, 1000 ohms respectively. The third set of resistance *R forms* the rheostat arm, consists of a series of coils of values from 1 to 5000 ohms. The unknown resistance *S* is connected in the fourth arm *CD*. Two spring keys K_1, K_2 are also provided, which when pressed make internal connections with terminals *A* and *B* respectively. Key K_1 is known as battery key and key K_2 as the galvanometer key.

To determine the unknown resistances, the connections are made as shown in Fig. 8.17. The ratio arms are adjusted to 10 ohms each and rheostat arm is set to 1 ohm. The battery key is first pressed then the galvanometer key is pressed and the direction of the deflection in the galvanometer is noted. Direction of the deflection is also noted for the resistance 2000 from the rheostat arm. If the deflection is in the direction opposite to the previous one, the unknown resistance must have value some between 1 and 2000. If two deflections are in same directions, there is either some faulty connections or the value of *S* exceeds 2000. By varying the resistance in the rheostat arm systematically, two resistances, differing by one ohm, with which the deflections are in opposite directions, are found by trial, Now $P = Q$, thus the unknown resistance must like between these two values, say n and $n + 1$.

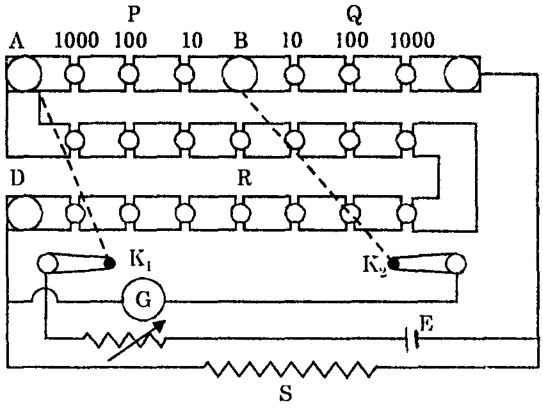


Fig. 8.17. Post-office box.

Now *P* is increased to 100 ohms keeping $Q = 10$ ohms, by varying resistance *R*, two values are similarly obtained, which give deflection in opposite directions. Using Eq. (58), the unknown resistance will be between $n_1/10$ and $(n_1 + 1)/10$ ohms. Similarly the accuracy can be more increased by taking $P = 1000$ ohms and keeping $Q = 10$ ohms.

For the measurements of very high resistance greater than the sum of all resistance in the rheostat arm, *Q* is increased to 100 and 1000 ohms by keeping $P = 10$ ohms.

8.16 KELVIN'S (THOMSON) METHOD OF DETERMINING GALVANOMETER RESISTANCE

This is the Wheatstone bridge method in which galvanometer is removed from galvanometer arm and the balance position is obtained by the galvanometer, whose resistance is to be measured and connected in the fourth unknown arm. The connections are made as shown in Fig. 8.18. Using Kirchhoff's first law, current distribution among the branches is obtained, which are shown in figure. Now applying Kirchhoff's second law to the meshes *ABDA* and *BCDB* and assuming clockwise direction of the current as positive in these meshes, we get

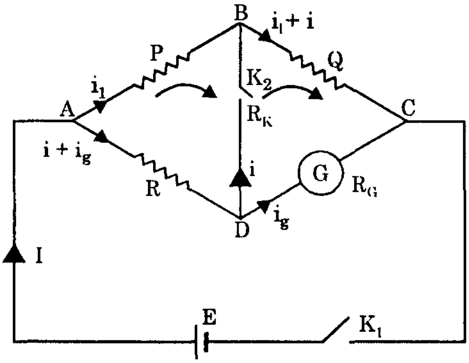


Fig. 8.18. Thomson method.

$$i_1 P - iR_K - (i + i_g) R = 0$$

and $$(i_1 + i) Q - i_g R_G + iR_K = 0.$$

If $i = 0$, *i.e.*, the current i_g through the galvanometer is independent of the resistance R_K or remains constant whether the key K_2 is closed or open. Then above relations reduce to

$$i_1 P = i_g R \text{ and } i_1 Q = i_g R_G,$$

$$\therefore \quad P/Q = R/R_G, \text{ or } R_G = RQ/P. \qquad ...(59)$$

Post Office box is used generally for the measurement of R_G. Galvanometer is connected across the unknown arm *CD* and the galvanometer arm *BD* is short circuited by connecting a thick *Cu* – wire. As soon as we press battery key K_1 the current will flow through the galvanometer and will give large deflection, particularly when the galvanometer is sensitive. Hence to check this large deflection a variable high resistance, say a resistance box adjustable up to 10,000 ohms or a *U*-tube containing water or dilute $CuSO_4$ solution should be inserted in series with the cell. As clear from the theory given above, relation (59) is valid when i_g the galvanometer current, is independent of *i*. Thus by taking any ratio from ratio arms the resistance should be adjusted from the rheostat arm for which deflection of galvanometer does not change when the galvanometer key K_2 is pressed. Knowing *P*, *Q* and *R* the resistance of galvanometer R_G can be calculated.

8.17 MANCE'S METHOD OF DETERMINING CELL RESISTANCE

In this method, the cell, resistance of which is to be measured, is connected in the fourth arm of the Wheatstone's bridge and the battery arm is short circuited by connecting a thick *Cu*-wire. The galvanometer is connected as usual. Applying first Kirchhoff's law the current distribution so obtained is shown in Fig. 8.19. Now applying Kirchhoff's second flow to the meshes *ABDA* and *BCDB* and assuming the clockwise direction of current in these meshes as positive, we get

$$(I - i_g - i) P - i_g R_G + (I - i_g) R = 0 \qquad ...(i)$$

and $$(I - i) Q + Ir + i_g R_G = \mathcal{E} \qquad ...(ii)$$

On rearranging, we get

$$I(P + R) - i_g (R_G + P + R) - iP = 0 \qquad ...(iii)$$

and $$I (Q + r) + i_g R_G - iQ = \mathcal{E} \qquad ...(iv)$$

Multiplying Eq. (*iii*) by $(Q + r)$ and Eq. (*iv*) by $(P + R)$ and then subtracting we get

$$i_g[R_G(P + R) + (R_G + P + R)(Q + r) - i[Q(P + R) - P(Q + r)] = \mathcal{E}(P + R).$$

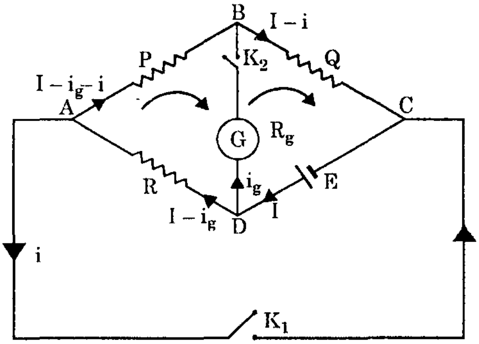


Fig. 8.19. Mance's method.

Since *i* is not zero, hence the galvanometer current i_g will be independent of the resistance in the battery arm *AC*, *or remains constant whether the battery key* K_1 is closed or open when the coefficient of *i* is zero, *i.e.*,

$$Q(P + R) - P(Q + r) = 0,$$

$$\therefore \quad QR = Pr \text{ or } r = RQ/P. \qquad ...(60)$$

To check high current through the galvanometer variable high resistance, say a resistance box of 10,000 Ω or a U-tube containing water, or very dilute solution of $CuSO_4$ should be placed in series with the galvanometer. In this experiment ratio P and Q is first taken as 10 : 10 and the resistances from rheostat arm R is so adjusted that the deflection obtained by pressing galvanometer key K_2 does not change while pressing battery key K_1. Now P, Q and R should be noted and r can be calculated.

If the cell resistance is less than 1 ohm as is often the case with a freshly prepared cell, keep the ratio $P/Q = 10$ and adjust R and take observations as given before.

8.18 CAREY FOSTER'S BRIDGE FOR MEASURING RESISTANCE

Most simple practical application of the Wheatstone's bridge is the *meter bridge* which consists of a straight and uniform wire of manganin or constantan stretched along a meter scale. The balance is obtained on the wire. If the balance point divides it into two parts of lengths l_1 and l_2, then the ratio of two resistances.

$$R_1/R_2 = l_1/l_2. \qquad \text{...(61)}$$

Carey Foster's bridge is a modified form of meter bridge in which effective length of the wire is considerably increased by including resistances X and Y in series with the bridge wire LM. Thus increases the accuracy of the result due to any error in reading the position of the balance, as well as the sensitiveness of the bridge. It is very useful for the comparison of two nearly equal resistances, whose difference is less than the resistance of the bridge wire LM. Two resistances X and Y to be compared are connected in the outer gaps of the bridge in series with the wire LM. Thus we have two arms of Wheatstone's bridge, one AD and other CD, where D is the position of null point on the wire. The remaining two arms are formed by two nearly equal resistances P and Q, connected in the inner gaps of the bridge. The cell and galvanometer are connected as indicated in Fig. 8.20. If the wire is of length L and the position of null point is at a distance l_1 from the left hand side, then at the time of balance we have

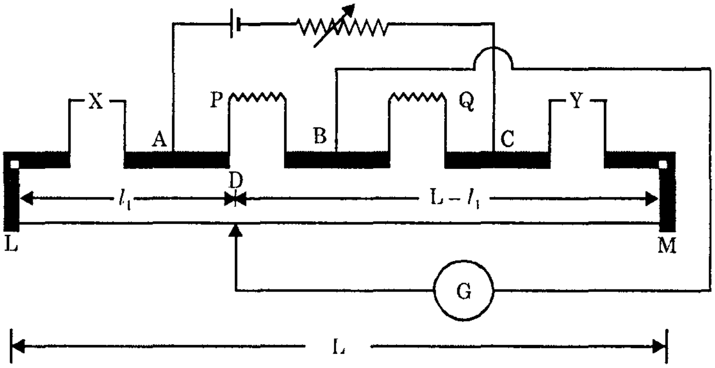


Fig. 8.20. Carey Foster's bridge.

$$\frac{P}{Q} = \frac{R_1 + \rho l_1}{R_2 + \rho(L - l_1)} = \frac{X + \alpha + \rho l_1}{Y + \beta + \rho(L - l_1)}, \qquad \text{...}(i)$$

where ρ is the resistance per unit length of the bridge wire and α and β are the end corrections at the left and right ends of the bridge wire respectively.

If X and Y are interchanged and the balance point is now obtained at a distance l_2 from the left end on the wire, we have

$$\frac{P}{Q} = \frac{Y + \alpha + \rho l_2}{X + \beta + \rho(L - l_2)}. \qquad \text{...}(ii)$$

Comparing these equations, we get

$$\frac{X + \alpha + \rho l_1}{Y + \beta + \rho(L - l_1)} = \frac{Y + \alpha + \rho l_2}{X + \beta + \rho(L - l_2)}.$$

Adding 1 to both sides, we see that the numerators are identical, hence their denominators must also be equal. Thus we can write

$$Y + \beta + \rho(L - l_1) = X + \beta + \rho(L - l_2).$$

or
$$Y - X = \rho\,[l_1 - l_2]. \qquad \text{...(62)}$$

This equation shows that the end corrections are eliminated without measuring and the difference of two resistances can be obtained by determining the resistance of the bridge wire between two positions of the null points.

ρ, the resistance per unit length can be measured by short circuiting one outer gap by putting a thick *Cu*-strip and including a standard fractional resistance in the other gap. Two positions of the null point l_1 and l_2 are thus obtained by putting resistance in left, *Cu*-strip in the right gap and by interchanging their positions. Thus from Eq. (62), we have

$$0 - R = \rho[l_1 - l_2] \text{ or } \rho = R/(l_2 - l_1). \qquad \text{...(63)}$$

Keeping X and Y constant, balance points may be obtained at different parts of the wire by small alterations of the ratio P/Q, thus testing of the uniformity of the wire can be done by seeing that $l_2 - l_1$ constant. This method is very useful for the measurement of low resistances. However, this method fails if the difference of two resistances Y–X exceeds the total resistance of the bridge wire *LM*. The unknown resistance X can be measured by connecting it in one gap and the known resistance Y in the other gap. By obtaining two positions of null point corresponding to the X in left and then in right, X can be calculated.

8.19 KELVIN'S DOUBLE BRIDGE FOR THE MEASUREMENT OF LOW RESISTANCES

Wheatstone's bridge is not suitable for the comparison of low resistances because it has low sensitivity in such circumstances and the resistances of the contact and connecting leads are not negligible in comparison to the resistance to be measured. In the Kelvin's double bridge these errors are eliminated. It is very useful for the measurement of resistances below 0.1 ohm and is also an excellent method for finding the specific resistance of a conducting material taken in the form of a rod of uniform cross-section. The principal diagram of the bridge is shown in Fig. 8.21.

R is the unknown low resistance and *S* a standard resistance of the same order of magnitude preferable a continuously variable resistance such as a slide wire. Both the resistances are joined together by a metal bar or a thick wire *BC* of low resistance *r* and are connected in series with a battery, rheostat and key. Sometimes *BC* is replaced by a cup filled with mercury. *M, N m* and *n* are four known non-inductive resistances. These are arranged together with a low resistance galvanometer *G to form a bridge. The ratio* M/N is kept the same as m/n, the resistance *S* is varied until zero deflection of the galvanometer is obtained.

Let the distribution of currents in the branches is as shown in Fig. 8.21. At the time of balance of the bridge the current I_1 is in the arms *M* and *N*, I_2 in *m* and *n* and I in *R* and *S*. Let us now apply Kirchhoff's second law for the meshes *AEFBA* and *CFEDC*, we get

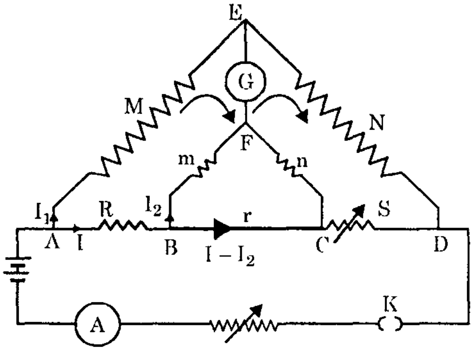


Fig. 8.21. Kelvin's double bridge.

$$MI_1 - mI_2 - RI = 0$$

and
$$NI_1 - nI_2 - SI = 0$$

As m and n are in parallel with the resistance r, the current I in R divides so that $[r/(r + m + n)]\,I$ passes through m and n. Substituting this value of I_2 in the above equations we have

$$MI_1 = Irm/(r + m + n) + RI$$

and
$$NI_1 = Irn/(r + m + n) + SI.$$

By solving, we get

$$R = \frac{M}{N}S + \frac{nr}{m+n+r}\left(\frac{M}{N} - \frac{m}{n}\right) \qquad \text{...(64)}$$

If $M/N = m/n$, then

$$R = MS/N. \qquad ...(65)$$

It gives the unknown resistance R in terms of the standard resistance S.

The *resistivity* ρ of the material can be calculated, by taking the material in the form of a wire. If A is the area of cross-section and l the length, then

$$\rho = AR/l. \qquad ...(66)$$

Different forms of this bridge are manufactured by different manufactures. The bridge, generally employed in laboratories, consists a compact form of potentiometer and double ratio box. The value of M/N is made equal to that of m/n by this box and resistance S is measured by measuring the length of the potentiometer wire in circuit knowing that 1 coil = 100 cm of wire = 0.1 ohm.

8.20 CALLENDER AND GRIFITH'S BRIDGE

The resistance variation with temperature is well defined in metals by the relation.

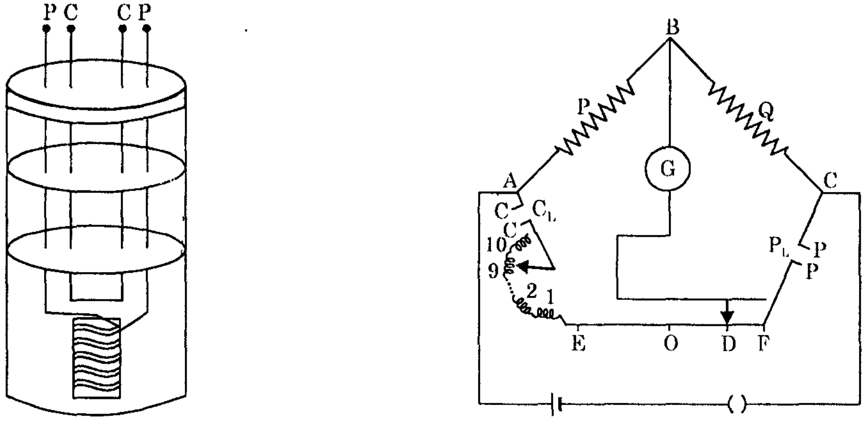
$$R_t = R_0(1 + \alpha t + \beta t^2), \qquad ...(67)$$

where R_t and R_0 are the resistances at $t°$ and $0°$ respectively and α and β are the constant characteristics of the metal. Platinum is the most suitable metal for measurement of temperature, because it does not combine with oxygen even at very high temperature and its resistance varies as given by Eq. (67).

For low temperature measurement the β may be neglected and one may have a relation $R_t = R_0(1 + \alpha t)$.

If R_{t_1} and R_{t_2} are the resistances of the specimen at temperatures t_1 and t_2 respectively, then the temperature coefficient of resistance α can be calculated by the relation

$$\alpha = \frac{R_{t_2} - R_{t_1}}{t_2 R_{t_1} - t_1 R_{t_2}} \qquad ...(68)$$



(a) (b)

Fig. 8.22 (*a*). Platinum resistance thermometer, (*b*) Callender and Grifith's bridge.

A platinum resistance thermometer consists of a fine platinum wire usually about 0.2 mm in diameter and of resistance about one ohm, wound non-inductively on a mica frame and placed in a glass tube for temperature upto 300° and in glazed porcelain or fused silica tube for higher temperatures. The ends of the wire are soldered to two copper leads which are connected to the two terminals PP at the other end on the top of the tube. The compensating leads or dummy leads, exactly similar in size and material to the leads connected with the platinum wire of the thermometer, are placed parallel to the tube, joined together at the lower end and connected to another pair of terminals *CC* at the top [Fig. 8.22 (*a*)].

Callender and Grifith's bridge is the modified Wheatstone's bridge for the accurate measurement of resistance of the thermometer. The ratio arms *P* and *Q* have equal resistance coils and C_L *and* P_L are the gaps for connecting the compensating and resistance leads of the thermometer. In series with a dial type adjustable resistance *r*, a stretched uniform wire *EF* of resistance 1 ohm is connected. A graduated scale with zero at the middle is associated with this wire, as shown in Fig. (8.22) (*b*).

The thermometer is placed in the bath whose temperature is to be measured, the *CC and PP* leads are connected across the gaps C_L *and* P_L respectively, a suitable resistance *r* is introduced in the dial box and the balance is thus observed at point *D* on the wire *EF*. At this stage, we have

$$P = Q \text{ and } C + r + \rho ED = R_t + C + \rho DF$$

or

$$R_t = r + \rho(ED - DF) = r + 2\rho OD, \quad \text{...(69)}$$

where R_t is the resistance of the platinum wire at *t*°C, *C* of the compensating leads, and *O* is the mid-point of the wire *EF*, having resistance per unit length, ρ. The distance *OD* is positive if the point *D* is right to the mid point *O*.

The resistance of the platinum wire is measured at 0°. 100° and the unknown temperature *t*°C. Neglecting β in comparison to α in Eq. (67), we thus have

$$t = \frac{R_t - R_0}{R_{100} - R_0} \times 100 \quad \text{...(70)}$$

This relation is not accurate for high temperature measurements. The coefficients β is also taken into account at these temperatures. For most accurate measurements of temperatures the temperature coefficients α and β are first obtained by measuring resistance of platinum wire at three known temperatures (ice, steam and boiling sulphur) and the use is made of Eq. (67).

8.21 POTENTIOMETER

The potentiometer is an instrument which can measure accurately the emf of source of supply without drawing any current from the source or the potential difference across a circuit element. An unknown potential difference is balanced against an adjustable, measurable potential difference.

The **principle** of the potentiometer is shown in Fig. 8.23. A uniform wire *AB* of length 100 cm, which in more accurate instruments is 1000 cm, is connected to the terminals of a source E_1 of emf $\mathcal{E}_1$. The current through the wire can be varied by means of an adjustable resistance R_1 connected in series with the wire. If terminal *A* is connected with the positive terminal of battery, there will be a continuous fall in potential from *A* to *B* along the wire. Consider a source E_2 of emf $\mathcal{E}_2$ in such a way that its +ve pole is connected to *A* and its negative pole through a galvanometer *G* to a contact slidding on the wire. If the emf of this source is less than the potential drop along the wire *AB*, it is always possible to find *X* on the wire at which galvanometer shows no deflection. At this position the potential drop along *AX* is equal to the emf $\mathcal{E}_2$ of the second source, *we call the potentiometer as balanced and the point X as the balance position.* If the wire is uniform and its properties are not affected by altering its electrical potential, the potential difference between any two points in the potentiometer wire is directly proportional to the length of wire between them.

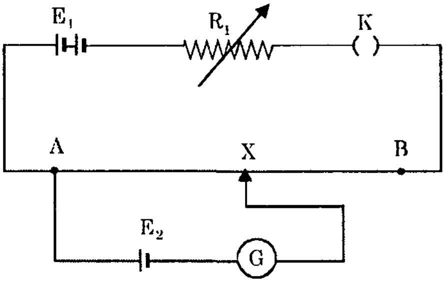


Fig. 8.23. Potentiometer.

To compare emf's $\mathcal{E}_1$ and $\mathcal{E}_2$ of two cells, these cells are connected in turn to the galvanometer by means of the two-way key and the position of balance point on the potentiometer wire is obtained for each cell. If the current through the potentiometer wire remains same in each case, then

$$\mathcal{E}_1/\mathcal{E}_2 = l_1/l_2. \qquad ...(71)$$

To measure the emf or a potential difference accurately a standard cadmium cell (emf = 1.01864 *volts as 20°C, roughly taken as 1.018 volts*) is used to standarise the potentiometer wire. The potential drop per unit length is thus measured with the help of a standard cell.

To measure the internal resistance r of a cell, its emf $\mathcal{E}$ is balanced against a length l_1 of the potentiometer wire. Now known resistance R from a resistance box is placed across the cell. The potential difference across the cell will thus fall to V. It is now balanced against a length l_2 of the potentiometer wire.

$$\mathcal{E} = \rho l_1 \text{ and } V = \rho l_2.$$

Since the current in the circuit i is given by

$$i = \frac{\mathcal{E}}{r+R} = \frac{V}{R}$$

$$\therefore \qquad r = \frac{\mathcal{E}-V}{V}R = \frac{l_1-l_2}{l_2}R. \qquad ...(72)$$

The l_1 is first observed for the open circuit, *i.e.*, across the cell terminals without resistance box in parallel. The resistance box (0–5Ω) is connected across the cell and l_2 is measured for different values of R from the resistance box. To check the continuous flow of current, a key is also connected in series with the resistance box.

The calibration of an ammeter is done by using it to measure the current through an accurately known resistance R. The pd V produced across this resistance due to the current I passing through it is measured by the potentiometer. The current V/R is measured against the reading in the ammeter I'.

For this purpose the connections are arranged as shown in Fig. 8.24. With the plug in the gap 1 of two way key K the emf $\mathcal{E}$ of the standard Cd cell is balanced against the pd across a length l_1 (1018 cm for easy calculations) of the potentiometer by adjusting rheostat Rh_1. Now the potential gradient, of the potentiometer wire will be $\rho = \mathcal{E}/l_1$. The key K is then switched to position 2, the potential V across the resistance R is balanced against the length l_2. Thus the current through the resistance R is

$$I = V/R = \rho l_2/R = \mathcal{E} l_2/l_1 R. \qquad ...(73)$$

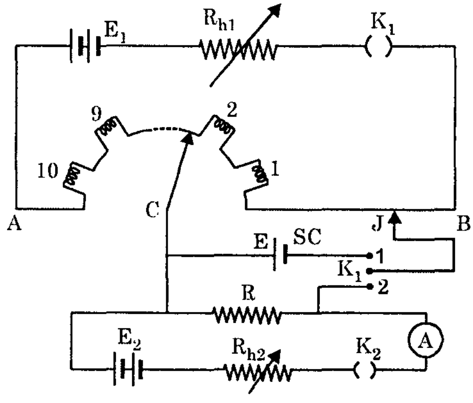


Fig. 8.24. Calibration of ammeter.

In practice R is generally taken as 1 ohm, l_1 = 1018 cm for cadmium cell hence current $I = l_2 \times 10^{-3}$ volt/ohm. By changing current through R by means of the rheostat Rh_2, several sets of observations for different values of *pd* across the resistance R are taken. The ammeter readings I' are also noted at the same time. In this way calibration of ammeter can be done very easily.

To determine the value of an unknown resistance, a steady current of known strength **I** is allowed to pass through the resistance R to be measured and the pd V across it is determined by means of a potentiometer, accurately calibrated by using a standard cadmium cell using the circuit diagram as shown in Fig. 8.24, R may be calculated with the help of the Eq. (73).

To compare two resistances, the standard source is not required as we do not want to measure the potential gradient of the potentiometer. Since the potential difference across a resistance is proportional to the product of its resistance and the current passing through it, the ratio of two resistances will be equal to the ratio of the potential differences across them when they are each carrying the same current, hence the ammeter is also not required. Connections are made as shown in Fig. 8.25. A steady current I is allowed to pass through the resistances R_1 and R_2, to be compared by connecting them in series. Now let the higher potential terminal of the resistance, one of say R_1 be connected to the variable terminal C of the potentiometer in dial type and to the terminal A in 10 wire or simple potentiometer and the lower potential terminal through a galvanometer to the jockey J of the potentiometer. Then if l_1 be the length of the potentiometer wire between C and J for no deflection in galvanometer, the potential difference

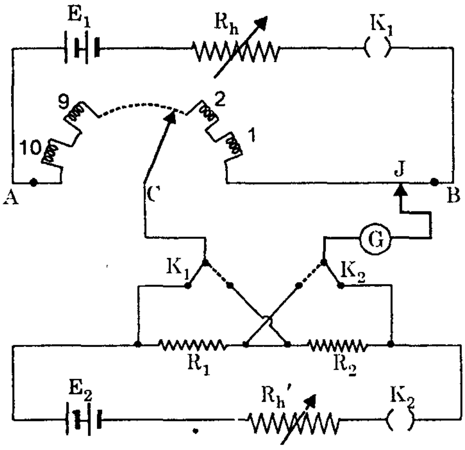


Fig. 8.25. Comparison of two resistances.

$$V_1 = \rho l_1 = IR_1. \qquad ...(i)$$

If similar observations are taken for the another resistance R_2, then

$$V_2 = \rho l_2 = IR_2. \qquad ...(ii)$$

$$\therefore \quad R_1/R_2 = l_1/l_2. \qquad ...(74)$$

At least six sets of observations are taken in the same manner by sliding rheostat Rh_2.

To determine a low resistance by means of a potentiometer, this unknown low resistance r is connected in series with a known resistance R and this combination is connected in series with a battery and a rheostat. The higher potential terminal of the resistance R is connected to the variable terminal C, the lower potential terminals of resistances R are r respectively through a two way key and a galvanometer to the jockey of the potentiometer. First of all the potential difference across the resistance R is balanced by the length l_1 of the potentiometer wire and then the pd across the combination $R + r$ is balanced by the length l_2. Thus we have

$$V_1 = IR = \rho l_1 \text{ and } V_2 = I(R + r) = \rho l_2,$$

$$\therefore \quad \frac{V_2}{V_1} = \frac{R+r}{R} = \frac{l_2}{l_1} \text{ or } r = \left(\frac{l_2}{l_1} - 1\right)R. \qquad ...(75)$$

8.22 TRANSDUCERS

A transducer is a device that converts one type of energy into another. A *microphone* is one kind of transducer which changes sound wave into an electrical signal. A *loudspeaker* is another kind of transducer which transforms electric energy into sound energy. Transducers are often used for measuring particular quantities. The *resistance thermometer* and *thermistor* convert temperature into an electrical property, electrical resistance. *Thermocouple* produces an electrical signal (emf) when subjected to differing temperature. One kind of *pressure transducer* makes use of the *piezoelectric effect* : When a mechanical force is applied on the two opposite faces of a quartz like crystal, the emf is produced across the other two opposite faces.

Strain Gauge : Another kind of pressure transducer is *strain gauge*, which makes use of the fact that the strain in the stretched wire is proportional to the stress (or force) applied to it. When stretched, its resistance must increase (as l is increased and A is decreased). A fine wire of a strain gauge is bounded to a flexible backing material, as shown in Fig. 8.26. When fixed tightly to a structure, the change in the resistance of the strain gauge is proportional to the change in stress on the structure. For the strain with in the elastic limit, the strain gauge may be calibrated using the Wheatstone bridge. The wire strain gauge is used by architects and engineers to determine the stress at critical points of the models of proposed structures. If a membrane or diaphragm is attached to the wire strain gauge, it can be used to measure pressure. This system is called *pressure transducer*.

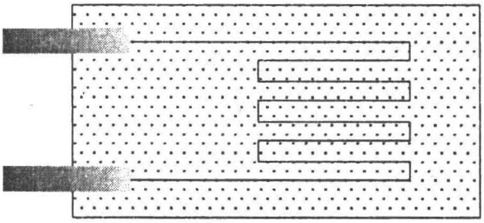


Fig. 8.26. Wire strain gauge.

8.23 ELECTROLYSIS

The phenomenon associated with the passage of electricity through liquid conductors was first studied by Faraday. The molecules of a salt in a solution are supposed to break up into two parts of opposite charges, called the ions. When an electric field is applied between two rods (or plates) immersed in the solution, these ions move towards the rods (electrodes) of opposite potentials. This phenomenon of transfer of charged ions by an electric current is called *electrolysis*. The solution is known as *electrolyte* and the vessel containing electrodes and electrolyte is called *voltameter*.

Faraday has Formulated Two Laws

(1) The mass of an element or ion deposited or liberated at an electrode during electrolysis is proportional to the total charge passing through the electrolyte, *i.e.*, $m \propto Q$ or $m = Z\ (It)$. Here the constant Z is known as *electrochemical equivalent* of the element.

(2) This mass is also proportional to the equivalent weight E of the element. This law also tells that the same quantity of electricity is needed to liberate gram-equivalent of any substance. This quantity of electricity is generally referred to as one Faraday.

$$\therefore \qquad 1 \text{ Faraday} = 96500 \text{ coulombs} = E/Z.$$

Two theories have been given to explain electrolysis phenomenon.

(*a*) Grotthus theory : Grotthus suggested that each molecule is consisted of two parts of opposite signs. When a potential difference is applied between the electrodes, the molecules arrange themselves along rows with all positive ends toward cathode and negative ends toward the anode. Thus a certain potential difference is required to bring the electrolysis phenomenon. But experimentally it is observed that even a very small pd can produce electrolysis.

(*b*) Dissociation theory : According to Arrhenious (1887), the molecules of the dissolved substance break up into atoms and also there is a constant interchange of atoms between the molecules. These atoms are electrically charged, even before the application of an electric field and are known as *ions*. The metallic ions and hydrogen (usually) being positive and the nonmetallic ones negative. When the pd is applied across electrodes dipped into the electrolyte, the positive ions move towards the cathode, hence called *cations* and the negative ions move towards the anode, hence called *anions*. The charges carried out by the ions are delivered up at the electrodes, leaving the free atoms.

The state of dynamic equilibrium in a potassium chloride solution can be expressed as

$$KCl \rightleftharpoons K^+ + Cl^- \qquad \text{...(76)}$$

The equilibrium will shift towards the right when the dilution increases. It is observed that the more dilute the solution the more complete are the dissociated. Thus practically all the molecules of the dissolved substance dissociate into free ions in a very dilute electrolyte. The degree of dissociation

$$\alpha = \frac{\text{Number of dissociated molecules in a solution}}{\text{Total number of molecules in a solution}}. \qquad \text{...(77)}$$

The above process can be explained with the help of electrostatic forces. Atoms in the molecules are held together by electrostatic attraction. When the molecules are immersed in a solvent, this force of attraction decreases, as the dielectric medium is changed from air to liquid. The solvents of high permittivities have thus more degrees of dissociation.

Electrical Conductivity of an Electrolyte : It is observed that in the cells in which the electrodes are composed of some other metal not chemically acted upon either by the salt or acid, back emf $\mathcal{E}_B$ is produced and the current is thus proportional to $(V - \mathcal{E}_B)$. This result is true if the current density is low and the applied potential is of very low frequency.

On the application of pd between the electrodes dipping into a solution, the ions will be accelerated and will reach a *terminal velocity*. While reaching to the electrodes, they collide with ions and molecules. This terminal velocity depends upon the size of the ion, the potential gradient and the viscosity of the solution.

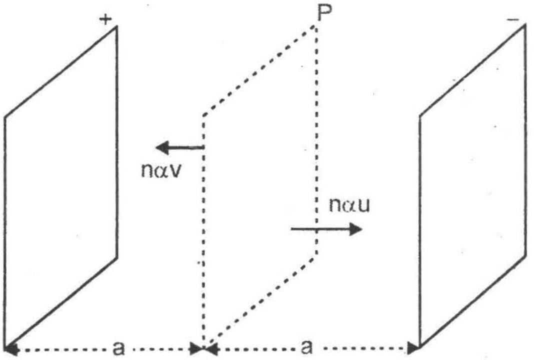


Fig. 8.27. Imaginary plane in between the electrodes.

Let n be the number of molecules per unit volume of solute in a solution and nx the number of molecules dissociated. Let u be the velocity of the cations (+ve ions) and v the velocity of the anions (–ve ions). If the electrodes are assumed as parallel planes, then on applying a pd between the electrodes, $n\,\alpha\,u$ cations and $n\,\alpha\,v$ anions will pass per unit time per unit area of an imaginary plane P, as shown in Fig. 8.27. If q be the charge on each ion, then the current passing per unit area will be $n\,\alpha\,(u + v)\,q$. Hence in unit time $n\,\alpha\,(u + v)$ cations must be deposited on the cathode and the same number of anions on the anode. As the number of cations reaching the cathode per unit time per unit area is $n\,\alpha\,u$ and that of deposited at the cathode is $n\,\alpha\,(u + v)$, hence the excess number $n\alpha v$ must come from the electrolyte in the immediate neighbourhood of the cathode. Therefore the electrolyte concentration must decrease. This number of cations must leave a similar number of anions. These anions move towards the anode and decrease electrolyte concentration in the neighbourhood of the anode in the similar way. Hence ratio of loss in concentration in the cathode region to that in anode region is v/u = *mobility of the negative ion* k_n / *mobility of the positive ion* k_p.

Instead of n, if the concentration C of molecules is used, then current density

$$j = C\alpha(u + v)\,q. \qquad \text{...(78)}$$

As ionic mobility is defined as the velocity of the ion acquired under unit potential gradient, hence for the potential gradient E, we have

$$u = Ek_p \text{ and } v = Ek_n.$$

$$\therefore \qquad j = C\,\alpha\,E\,(k_p + k_n)\,q,$$

or the specific conductivity $\sigma = j/E = C\alpha(k_p + k_n)\,q$. ...(79)

The ratio $\sigma/C = \alpha\,(k_p + k_n)\,q = \Lambda$ = *Equivalent conductivity.*

For dilute solution $\alpha = 1$ and $\Lambda = \Lambda_\infty$.

$$\therefore \qquad \Lambda / \Lambda_\infty = \alpha. \qquad \text{...(80)}$$

Hence the degree of dissociation α for a solution of any concentration is found from a graph of Λ against concentration C.

The specific conductivity can be determined by using *Kohlrausch's bridge*, will be discussed in chapter 16.

8.24 VOLTAIC CELL

Voltaic cells are simply electrolytic cells in which the electrodes react with the electrolyte in such a way as to produce an internal emf. For theoretical purposes the cells are classified as *reversible* and *irreversible*. The former is one which may be restored to it's original condition after discharge by passing a reverse current through it. It requires same energy for charging as obtained during the discharge. In irreversible cell, some chemical action takes place in the course of working which can not be reversed by sending the current in reverse direction. For practical purposes the cells are classified as : *primary* and *secondary*. The latter, often called storage cells or batteries, are necessarily reversible.

Kelvin suggested that the chemical reactions within the cells were the sole source of energy. By equating this with thermal energy the emf of a cell can be calculated. Let H_1 and H_2 are the heats of reaction at the two electrodes due to the unit charge passing through the cell at very slow rate so that joule heating can be avoided then,

$$\text{Energy liberated } J(H_1 \sim H_2)q = \text{Energy consumed } \mathcal{E}q. \qquad \text{...(81)}$$

From this equation emf of the cell $\mathcal{E}$ can be calculated.

Let us apply this equation to a Daniell cell, bearing in mind that the passage of 1 faraday, *i.e.*, 96,500 coulombs, corresponds to the formation of 1 gm. equivalent of the electrolyte. In this cell, zinc sulphate is formed at the zinc electrode. It can be determined that 1 gm of Zn develops 1670 calories in forming $ZnSO_4$. As atomic weight of Zn is 65.37 and it is divalent, hence,

$$H_1 = 1670 \times 65.37/2 = 54584 \text{ calories per gm equivalent.}$$

At the copper electrode, Cu, is deposited from $CuSO_4$ solution. It can be determined that 1 gm of Cu develops 909.5 calories in forming $CuSO_4$. As the atomic weight of Cu is 63.57 and it is divalent, hence,

$$H_2 = 909.5 \times 63.57/2 = 28908.5 \text{ calories per gm equivalent.}$$

$$\therefore \quad 4.18[54584 - 28908.5] = 96500\,\mathcal{E} \text{ or } \mathcal{E} = 1.112 \text{ volts} \qquad \text{...(82)}$$

Gibbs-Helmholtz Equation : In the above theory we have used the first law of thermodynamics. However, it has been experimentally determined that a cell has a temperature coefficient of emf. If the charge through the cell flows very slowly, second law of thermodynamics can be applied and the cell can be assumed as reversible engine.

Let a reversible cell of emf $\mathcal{E}$ be connected with external circuit and is maintained at a constant temperature T. The emf of the cell is balanced against an equal and opposite *pd*, derived from the external circuit. The balancing *pd* is reduced slightly so that the cell will derive a charge q at a slow rate round the circuit. This is represented by AB in an indicator diagram (Fig. 8.28). The cell will absorb an amount of heat h to keep its temperature constant.

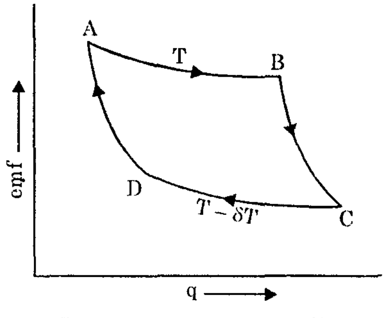


Fig. 8.28. Reversible cell.

If the cell is thermally insulated from the surroundings and allowed to send a small charge round the circuit. In this process the cell has to use its own energy and hence temperature of the cell falls and therefore emf will also fall from $\mathcal{E}$ to $\mathcal{E} - (d\mathcal{E}/dT)\,\delta T$. The change is represented by BC.

Now let the cell is maintained at this new temperature $T - \delta T$ in an enclosure. If q be the charge passed into the cell by means of the external circuit of infinitesimally greater emf so that the rate of flow of charge is very slow. It is represented by CD. If the cell is restored to its initial position A, then the work done in a complete cycle $ABCD$

$$= W_1 - W_2 = \text{Area } ABCD = q\,(d\mathcal{E}/dT)\,\delta T.$$

For a reversible cycle

$$\frac{W_1 - W_2}{W_1} = \frac{T_1 - T_2}{T_1} \text{ or } \frac{q(d\mathcal{E}/dT)\delta T}{h} = \frac{\delta T}{T}$$

or

$$h = qT\,(d\mathcal{E}/dT). \quad \text{...(83)}$$

Thus the Gibbs-Helmholtz theory modifies the Eq. (81) as

$$J(H_1 \sim H_2)\,q + h = J(H_1 \sim H_2)\,q + qT\,(d\mathcal{E}/dT) = \mathcal{E}q$$

or

$$\mathcal{E} = JH + T(d\mathcal{E}/dT). \quad \text{...(84)}$$

This is known as *Gibbs Helmholtz equation*. Here $H = H_1 \sim H_2$. It is clear from this equation that for $d\mathcal{E}/dT = 0$ it is same as Eq. (81).

If $d\mathcal{E}/dT > 0$, $\mathcal{E} > JH$ and the heat must therefore be taken in by the cell to keep the temperature constant. Thus the cell will cool while giving a current.

If $d\mathcal{E}/dT < 0$, $\mathcal{E} < JH$ and the heat must therefore be taken out by the cell to keep the temperature constant. Thus the temperature of the cell will increase while giving a current.

8.25 CONVERSION OF CHEMICAL TO ELECTRICAL ENERGY

Batteries or cells convert chemical energy into electric energy. The action is based on electrolytic dissociation. When a pure zinc plate or rod (electrode) is immersed in an aqueous solution of sulphuric acid (electrolyte), the zinc is dissolved and its atoms enter the solution as positive ions, leaving the electrons on the plate. As dissolving continues, more and more electrons are left on the electrode and positive ions go into the solution. As a result an electro-electric field set up by the ions and directed from the electrolyte to the electrode opposes the transfer of zinc ions into the solution. The equilibrium has been reached and the dissolving of zinc ions comes to an end. At this stage the electrostatic field E_e becomes equal and opposite to the field due to chemical forces responsible for electrolytic dissociation.

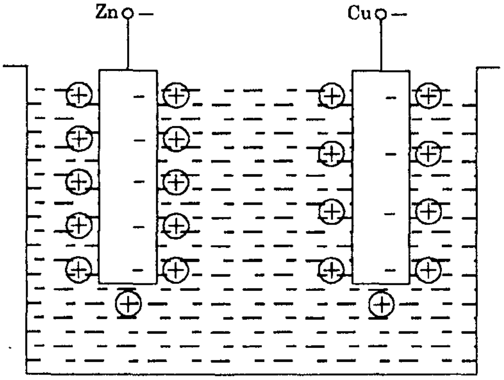


Fig. 8.29. Two dissimilar rods immersed in an acid solution.

$$\mathbf{E}_e + \mathbf{E}_{ch} = 0. \quad \text{...(85)}$$

Thus we see that the zinc rod being negative, is at a lower potential with respect to the solution. If another rod say of copper is also immersed in the solution, similar things happen to Cu-rod. When equilibrium is reached, the Cu-rod is also at a lower potential with respect to the solution. Since the Zn-rod dissolves faster in acid and therefore has more negative charge than the *Cu*-rod. Thus there exists a potential difference between the Cu- and Zn-rods, with *Cu*-rod being at a higher potential than Zn-rod. Since the field E'_{ch} due to chemical force acting on the *Zn*-rod is greater than that E''_{ch} acting on *Cu*-rod. These are in opposite directions. At equilibrium, we have

$$\mathbf{E}_e + \mathbf{E}'_{ch} + \mathbf{E}''_{ch} = 0. \quad \text{...(86)}$$

If φ_1 and φ_2 be the potential's of the *Zn* and *Cu*-electrodes at the time of equlibrium

$$\varphi_2 - \varphi_1 = \int_1^2 (\mathbf{E}'_{ch} + \mathbf{E}''_{ch}).d\mathbf{l}. \qquad ...(87)$$

If the electrodes are connected by a wire. The electrons will flow from the Zn-rod to *Cu*-rod and thus contribute to the electric current. As the electrons leave Zn-rod, the rod becomes less negative. Some of the positive ions leave this rod and go to the solution. The arrival of electrons to the *Cu*-rod through the wire helps in attraction of positive ions in the solution which get deposited on this rod. Thus the mass of Zn-rod decreases due to dissolving and the mass of *Cu*-rod increases due to depositing on it. *Cu*-rod thus requires cleaning. In primary cells the current is produced by irreversible electrochemical process and have a short useful life.

8.26 NUCLEAR BATTERY

In the *solar battery* the sun rays fall on a sensitive metal surface which consequently limits electrons via photoelectric effect. These electrons are collected at another nearby metal surface. This charge separation is responsible for producing emf.

Similarly in the *nuclear battery*, a radioactive source (α-source) is placed at one terminal and the charged α-particles are collected at another terminal. Here the source of energy is nuclear rather than electromagnetic (sunlight) in a solar battery.

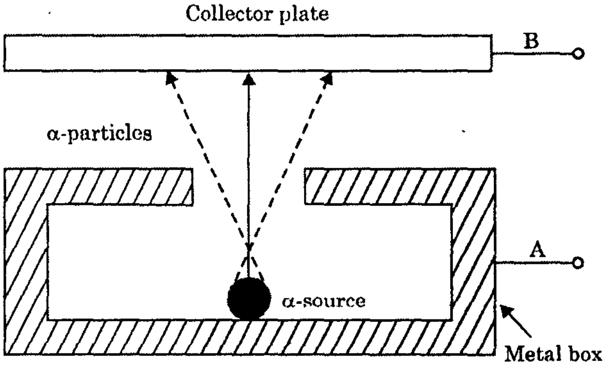


Fig. 8.30. Nuclear battery.

As shown in Fig. 8.30, the radioactive source is placed in a metallic box *A*. The α-particles are collimated into a beam and pass to a metal collector electrode *B*. The potential of *B* relative to the box will rise. For the open circuit, an equilibrium will reach when the emitted α-particles are just repelled back by the electrostatic field.

The non-conservative field $\mathbf{E}_n$ must be ascribed to a nuclear force which kicks the α-particles out of the nuclei of the radioactive source. Since the atom is neutral, hence two electrons are also released in the material with the emission of each α-particle. The work done by the nuclear force is equal to the *K.E* of the α-particle, which is about 5 MeV.

$$\tfrac{1}{2} m_\alpha v_\alpha^2 = 2e\int_A^B \mathbf{E}_n.d\mathbf{r} = 5\times 10^6 eV.$$

In the equilibrium situation

$$2e[\varphi(B) - \varphi(A)] = \tfrac{1}{2} m_\alpha v_\alpha^2 = 5 \times 10^6 eV.$$

$$\therefore \qquad \mathcal{E} = \int_A^B \mathbf{E}_n \cdot d\mathbf{r} = \varphi(B) - \varphi(A) = 2.5\times 10^6 V. \qquad ...(88)$$

If a resistance *R* is connected externally in between *A* and *B*, the current *I* will flow. Thus in an equilibrium situation

$$\varphi(B) - \varphi(A) = R\,I, \qquad ...(89)$$

where $I = 2\,e \times$ number of α-particles moving per sec from *A* to *B*

$$= 2 \times 1.6 \times 10^{-19} \times 10^6 = 3.2 \times 10^{-13} \text{ amp.}$$

For $\quad R = 10^7\ \Omega,\ RI = 10^7 \times 3.2 \times 10^{-13} = 3.2 \times 10^{-6}\ V.$

If *r* be the internal resistance of the emf-source, then

$$\mathcal{E} - Ir = IR \text{ or } r = (2.5 \times 10^6 - 3.2 \times 10^{-6})/3.2 \times 10^{-13} \simeq 10^{19}\ \Omega. \qquad ...(90)$$

Thus *r is* very large in comparison to *R*. This type of emf source is known as *current source*.

8.27 UNCONVENTIONAL ENERGY RESOURCES

Conventional fossil fuels, coal, oil and gas are wasting assets and are non-renewable by human effects. Thus the increase in industrialisation will require unconventional energy resources. Remarkable energy resources, including geothermal energy, solar energy, wind energy and ocean wave energy are seemingly inexhaustible. The better technology may provide energy from these sources for the industries tomorrow.

(*a*) **Geothermal Power :** Geothermal energy flows from the earth's interior. The sources of the internal heat of the earth are varied, including residual heat that has remained since the planet was born, elastic energy released by earthquakes, energy released by gravitational differentiation of magma and so on.

(*b*) **Solar Energy :** It is about 5000 times greater than the Geothermal energy. The earth's surface receives average solar energy at the rate of 200 W/m^2/hr. Some part is reflected by the surface and absorbed in between. the energy gained by the earth surface is 80-90 W/m^2/hr. Its use does not provide pollution in the environment. The harnessing of solar energy in the larger way has not become easy. Otherwise it seems to be unlimited source of energy.

(*c*) **Windmill Power :** The motion of the wind is conveyed to the dynamo which will generate electricity. For harnessing wind power, sites are to be identified where the wind speed is steady and powerful enough to operate the windmills.

(*d*) **Biogas Power :** Biogas from which energy is generated is formed by the fermentation of the dung of cattles, human excrete, garbage, crop waste, etc. The fermentation occurs at a temperature of above 28°C. The gases produced are mainly methane and carbon dioxide. Plants are in operation under Khadi and Village Industries Commission at some localities, but still not much popular.

EXERCISES

Example 1 : *If* $J = \frac{1}{r^3}(2\cos\theta\,\hat{r} + \sin\theta\hat{\theta})\ A/m^2$, *calulate the current passing through (a) a hemispherical shell of radius 50 cm and (b) a spherical shell of radius 25 cm.*

We know that $\mathbf{I} = \int \mathbf{J}.\, d\mathbf{S}$, *where* $d\mathbf{S} = r^2 \sin\theta\, d\theta\, d\phi\ \hat{r}$

(*a*) In the hemispherical shell $0 \le \theta \le \pi/2$, $0 \le \phi \le 2\pi$, $r = 0.5$ m.

$$I = \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\frac{1}{r^3}.r^2.2\cos\theta\sin\theta\, d\theta\, d\phi = \left(\frac{2}{r}\right)_{r=0.5}\int_0^{\pi/2}\sin\theta\, d(\sin\theta)\int_0^{2\pi} d\phi$$

$$= \frac{4\pi}{0.5}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 4\pi = 12.56\,A.$$

(*b*) In the spherical shell $0 \le \theta \le \pi$, $0 \le \phi \le 2\pi$, $r = 0.25$

$$I = \left(\frac{2}{r}\right)_{r=0.25}\int_0^{\pi}\sin\theta\, d(\sin\theta)\int_0^{2\pi} d\phi = \frac{4\pi}{0.25}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi} = 0.$$

Example 2 : *Find the current in the circular wire of radius 2 mm if the current density is* $j = 10(1 - e^{-1000r})\ \mathbf{a}_z\ (A/m^2)$.

Here $\mathbf{a}_z$ represents that the wire is placed along the z-axis. The current dI across the surface element $d\mathbf{S}$ is given by

$$dI = \mathbf{j} \,.\, d\mathbf{S} = 10(1 - e^{-1000r})\,\mathbf{a}_z.\ r dr d\phi\ \mathbf{a}_z$$
$$= 10\,(1 - e^{-1000r})\, r dr d\phi$$

$$\therefore \quad \text{Current } I = \int_0^{0.002} 10(1-e^{-1000r})r\,dr \int_0^{2\pi} d\phi$$
$$= 8.87 \times 10^{-5} A = 88.7\ \mu A.$$

Example 3 : *Calculate the drift velocity for the electrons in a silver wire which has a radius of 0.1 cm and carries a current of 2 amperes. Also find the current density in the wire.*

We know that $i = nq\mathbf{v}.\mathbf{A} = nqvA$.

Since silver is univalent, each atom can be assumed to contribute one electron. We know that one gm atom of silver (*i.e.*, 108 gm of silver) has 6.023×10^{23} atoms. As the density of Ag is 10.5 gm. per cm^3, hence the number of electrons per unit volume

$$n = \frac{6.023 \times 10^{23}}{108 \times 10^{-3}} \times 10.5 \times 10^{-3} \times 10^{6} = 5.856 \times 10^{28} \text{ per m}^2.$$

Cross-sectional area of the wire $A = \pi r^2 = 3.14 \times 10^{-6}$ m^2.

$$\therefore \text{ Drift velocity} \quad v = \frac{i}{rqA} = \frac{2}{5.856 \times 10^{28} \times 1.6 \times 10^{-19} \times 3.14 \times 10^{-6}}$$
$$= 6.80 \times 10^{-5} \text{ m/sec},$$

$$\text{and current density} \quad j = i/A = 2/3.14 \times 10^{-6}$$
$$= 6.37 \times 10^{5} \text{ amp/m}^2.$$

Example 4 : *Given two plane parallel electrodes, at potentials 0 and V_0 and separated by a distance d. Calculate the current density if an unlimited supply of electrons at rest is supplied to the lower potential electrode.*

Let us choose x-axis perpendicular to the plates. Therefore the charge and current density are functions of x only. In the steady state $\frac{\partial j(x)}{\partial x} = 0$.

Therefore $\mathbf{j} = -j_o\,\hat{x}$, where j_0 is a const.

If $v(x)$ be the velocity of the electrons at a distance x between the electrodes, given by $\frac{1}{2}mv^2(x) = eV(x)$

$\therefore$ Charge density $\rho(x) = -j_0/v(x)$

The potential $V(x)$ is given by the Poisson's equation as

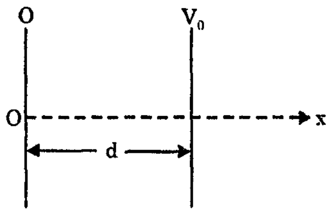


Fig. 8.31. Example 4.

$$\frac{d^2V(x)}{dx^2} = -\frac{\rho(x)}{\varepsilon_o} = \frac{j_o}{\varepsilon_o v(x)} = \frac{j_o}{\varepsilon_o}\sqrt{\frac{m}{2eV(x)}}$$

To solve this differential equation, let $dV/dx = u$, therefore

$$\frac{d^2V}{dx^2} = \frac{d}{dx}\left(\frac{dV}{dx}\right) = \frac{du}{dx} = \frac{du}{dV}.\frac{dV}{dx} = u\frac{du}{dV}$$

$$\therefore \quad u\,du = \frac{j_o}{\varepsilon_o}\sqrt{\frac{m}{2e}}V^{-1/2}dV$$

$$\text{or} \quad \frac{1}{2}u^2 = \frac{2j_o}{\varepsilon_o}\sqrt{\frac{m}{2e}}V^{1/2} \text{ or } u = 2\left[\frac{j_o}{\varepsilon_o}\sqrt{\frac{m}{2e}}\right]^{1/2} V^{1/4}$$

$$\int_0^{V_o} V^{-1/4}dV = 2\left[\frac{j_o}{\varepsilon_o}\sqrt{\frac{m}{2e}}\right]^{1/2}\int_0^d dx \quad \text{or} \quad \frac{4}{3}V_o^{3/4} = 2\left[\frac{j_o}{\varepsilon_o}\sqrt{\frac{m}{2e}}\right]^{1/2} d.$$

$$\therefore \quad j_o = \frac{4}{9}\frac{\varepsilon_0}{d^2}\left[\frac{2eV_o^3}{m}\right]^{1/2}$$

$$\text{Current density } \mathbf{j} = -j_o\,\hat{x} = -\frac{4}{9}\frac{\varepsilon_o}{d^2}\left[\frac{2eV_o^3}{m}\right]^{1/2}\hat{x}.$$

Example 5 : *In a copper wire of cross-sectional area 2 mm^2, a current of 30 amperes is flowing. Calculate the rms velocity of free electrons at 27°C. Also prove that the drift velocity is very small in comparison to it. Given that Boltzmann constant $k = 1.38 \times 10^{-23}$ joule/°K and density of Cu = 8.9 gm/cc.*

We know that the root mean square velocity of free electrons

$$\begin{aligned} v_{rms} &= \sqrt{\frac{3kT}{m}} = \sqrt{\frac{3 \times 1.38 \times 10^{-23} \times 300}{9.11 \times 10^{-31}}} \\ &= 1.17 \times 10^5 \text{ m/sec.} \end{aligned}$$

Drift velocity is given by the relation $v_d = -j/ne$.

Current density $j = i/A = 30/2 \times 10^{-6} = 1.5 \times 10^7$ amp/m^2, and the number of free electrons per unit volume.

$$n = 6.023 \times 10^{23} \times 8.9 \times 10^3 / 63.5 \times 10^{-3} = 8.44 \times 10^{28}.$$

$$\begin{aligned} \therefore \quad \text{Drift velocity } v &= \frac{i}{nqA} = \frac{1.5 \times 10^7}{8.44 \times 10^{28} \times 1.6 \times 10^{-19}} \\ &= 1.11 \times 10^{-3} \text{ m/sec.} \end{aligned}$$

Thus we see that the root mean square velocity is about 10^8 times greater than the drift velocity of free electrons in *Cu*.

Example 6 : *An aluminium wire whose diameter is 0.245 cm is welded end to end to a copper wire with diameter 0.1626 cm. The composite wire carries a steady current of 10 ampere. (a) What is the current density in each wire ? (b) Find the drift speed of the electrons in copper wire.*

Assume one free electron per atom in copper. Given Avogadro number = 6 × 10^{23} atoms per gm mole, Molecular weight of copper = 64, density of copper = 8.9 gm/cc.

(*a*) Since the same current is flowing through *Al* and *Cu*-wires, the current density will vary inversely with their area of cross sections.

$$\begin{aligned} (A_1)_{Al} &= \pi r_1^2 = 3.14 \times (0.245 \times 10^{-2}/2)^2 = 4.712 \times 10^{-6} \text{ m}^2. \\ (A_2)_{cu} &= \pi r_2^2 = 3.14 \times (0.1626 \times 10^{-2}/2)^2 = 2.075 \times 10^{-6} \text{ m}^2. \end{aligned}$$

Therefore the current densities in these wires

$$\begin{aligned} j_{Al} &= \frac{i}{A_1} = \frac{10}{4.712 \times 10^{-6}} = 2.122 \times 10^6 \text{ amp/m}^2. \\ j_{cu} &= \frac{i}{A_2} = \frac{10}{2.075 \times 10^{-6}} = 4.818 \times 10^6 \text{ amp/m}^2. \end{aligned}$$

(*b*) The number of electrons per unit volume *n* in copper wire is related with the Avogadro number *N*, *M* the molecular weight of *Cu* and *d* its density as

$$\begin{aligned} n &= \frac{Nd}{M} = \frac{6 \times 10^{23} \times 8.9}{64} = 8.344 \times 10^{22} \text{ electrons/cm}^3 \\ &= 8.344 \times 10^{28} \text{ electrons/m}^3. \end{aligned}$$

Since the current density *j* is related with the drift speed as

$$j = nqv_d$$

or

$$v_{cu} = \frac{j_{cu}}{nq} = \frac{4.819 \times 10^6}{8.344 \times 10^{28} \times 1.6 \times 10^{-19}} = 3.61 \times 10^{-4} \text{ m/sec.}$$

Example 7 : *In a copper there are 10^{22} free electrons per cc all of which contribute to a current of 1 amp in a wire of Cu of 0.01 cm^2 cross sectional area. Calculate (1) the average drift* *speed of the electrons in Cu and (2) the electric field in the wire. Given ρ of Cu as 1.7×10^{-8} ohm–m.*

Drift velocity $v_d = i/nqA$.

In our problem $i = 1$ amp., $n = 10^{22} \times 10^6$ electrons/m², $A = 0.01 \times 10^{-4}$ m² and charge on the electron $q = 1.6 \times 10^{-19}$ coul.

$$\therefore \quad v_d = 1/10^{28} \times 1.6 \times 10^{-19} \times 0.01 \times 10^{-4} = 6.25 \times 10^{-4} \text{ m/sec.}$$

We know that $j = \sigma E$ or $E = \rho j = \rho i/A$.

$$\therefore \quad E = 1.7 \times 10^{-8} \times 1/(0.01 \times 10^{-4}) = 1.7 \times 10^{-2} \text{ volt/m.}$$

Example 8 : *A rod of a certain metal is 100 cm long and 0.55 cm in diameter. The resistance between its ends at 20°C is 2.87×10^{-3} ohm. A round disc of thickness 1 mm and of diameter 2 cm is formed of this material. What is the resistance between the opposing round faces ? What is the material ?*

$$\text{Resistivity of the material } \rho = RA/l = 2.87 \times 10^{-3} \times 3.14 \times (0.275 \times 10^{-2})^2$$
$$= 6.82 \times 10^{-8} \text{ ohm-meter.}$$

It is the value for nickle, hence the rod is of the nickel.

Resistance of the disc $R = \rho l/A$.

Since the disc is of the same material. Therefore, we have

$$R = 6.82 \times 10^{-8} \times 10^{-3}/(3.14 \times 10^{-4}) = 2.17 \times 10^{-7} \text{ ohm.}$$

Example 9 : *Find the resistance between the curved surfaces of the silver block, shown in Fig. 8.32. Given the conductivity $\sigma = 6.17 \times 10^7$ S/m.*

Since the same current will pass through both the inner and outer curved surfaces, *i.e.*,

$$I = \mathbf{j}_1 \cdot d\mathbf{S}_1 = \mathbf{j}_2 \cdot d\mathbf{S}_2$$
$$= j_1\, dS_1 = j_2 dS_2$$

or $$j_1 = \frac{I}{dS_1} = \frac{I}{tA} = \frac{I}{t\phi r_a}$$

and $$j_2 = I/t\,\phi\, r_b.$$

Since $\mathbf{j}$ and $d\mathbf{S}$ are along $\mathbf{a}_r$, therefore we may write

$$\mathbf{j} = (k/r)\,\mathbf{a}_r$$

and $$\mathbf{E} = (k/\sigma r)\,\mathbf{a}_r$$

$\therefore$ Resistance between two curved surfaces

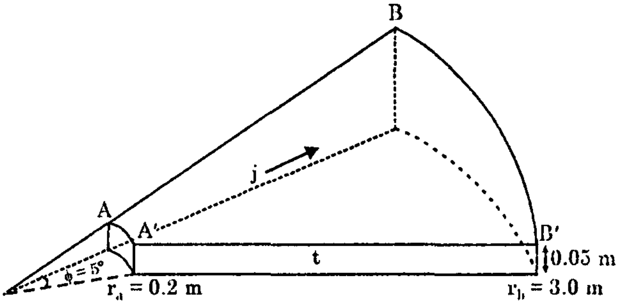


Fig. 8.32. Example 9.

$$R = \frac{\int \mathbf{E}.d\mathbf{l}}{\int \sigma \mathbf{E}.d\mathbf{S}} = \frac{\int (k/\sigma r)\mathbf{a}_r.dr\,\mathbf{a}_r}{\int\int (k/r)\mathbf{a}_r.rd\phi dz\,\mathbf{a}_r} = \frac{\int_{r_a}^{r_b}\left(\frac{k}{\sigma}\right)\left(\frac{dr}{r}\right)}{\int_0^{\phi}\int_0^{z} k\,dz\,d\phi} = \frac{\int_{0.2}^{3}\frac{k}{\sigma}\frac{dr}{r}}{\int_0^{\pi/36}\int_0^{0.05} k\,dz\,d\phi}$$

$$= \frac{1}{\sigma}\frac{(\log_e 3 - \log_e 0.2)}{0.05 \times \pi/36} = \frac{2.708}{6.17 \times 10^7 \times 0.05 \times 0.0873}$$

$$= 10.06\ \mu\Omega.$$

Example 10 : *A vacuum diode consists of plane parallel electrodes separated by a distance d and each having area A. When a potential V is applied to the anode with respect to the cathode, a current i flows through the diode. Assuming that the electrons are emitted with zero velocity and do not change the field between electrodes, calculate electron velocity v and charge density ρ at any point between the electrodes.*

We are neglecting the effect of space charge. Thus for steady current

$$\operatorname{div} \mathbf{j} = 0 \text{ or } \partial j_x/\partial x + \partial j_y/\partial y + \partial j_z/\partial z = 0.$$

Flow of electrons is from cathode to anode only in one direction. If it is taken in x-direction only, then $j_y = 0, j_z = 0$ and $\partial j_x/\partial x = 0$ or j_x = const. or independent of x. We know that $j_x = en_x v_x$, where v_x is the velocity of electrons at a distance x from the cathode and n_x is the number of electrons per unit volume at this distance.

Force acting on each electrons

$$F = eE = eV/d.$$

∴ Acceleration on the electrons

$$f = F/m. = eV/md.$$

Thus the velocity of electrons at a distance x from the cathode is given by

$$v_x^2 = 0 + 2fx = 2eVx/md$$

or $v_x = \sqrt{2eVx/md}$

Current $i = j_x A = en_x v_x A = en_x A\sqrt{2eVx/md}$

Therefore space charge density $\rho_x = en_x = i\sqrt{md/2eVxA^2}$.

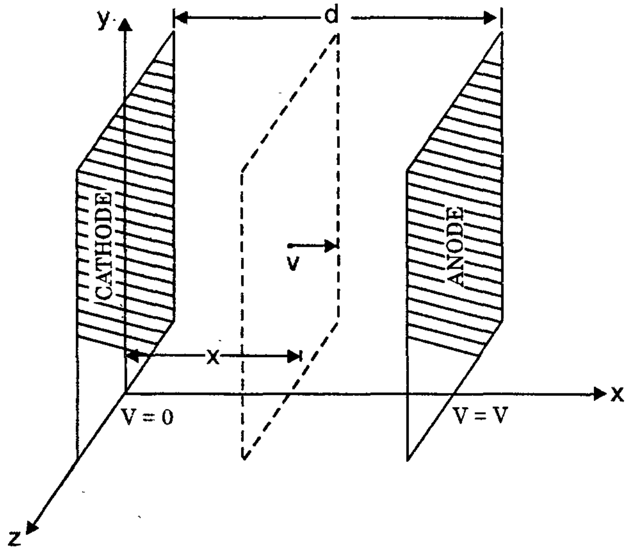


Fig. 8.33. Example 10.

Example 11 : *The region between two concentric spheres of radii a and b is filled with a conducting material. The inner sphere is maintained at a potential* V_a *and the outer at a lower potential* V_b*, so there is a current in the radially outward direction. Calculate the electric intensity at any point with in the material.*

Let us consider a cross-sectional area A at a distance r between the spheres. The resistance ΔR due to the material of thickness Δr is

$$\Delta R = \rho\Delta r/A = \rho\Delta r/4\pi r^2,$$

$$\therefore \quad R = \frac{\rho}{4\pi}\int_a^b \frac{dr}{r^2} = \frac{\rho}{4\pi}\left[\frac{1}{a} - \frac{1}{b}\right].$$

As the current between the spheres $i = (V_a - V_b)/R$, hence current density at any point on the sphere of radius r is

$$j = i/A = (V_a - V_b)/4\pi r^2 R.$$

Thus the corresponding electric field

$$E = \rho j = \frac{V_a - V_b}{r^2(1/a - 1/b)}.$$

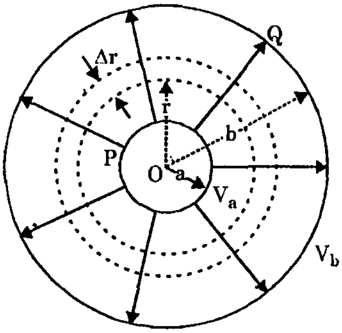


Fig. 8.34. Example 11.

It can be proved very easily that it is same as electrostatic field between two concentric spheres placed in vacuum.

Example 12 : *Two small, spherical, perfectly conducting electrodes of radii a and b are embedded in an infinite medium of conductivity* σ_e*. Their centres are separated by a distance l, such that l >> a, b. Find the resistance between them.*

To calculate the resistance between the given spheres, we assume that q and $-q$ be the charges on spheres A and B respectively at any instant. The electric field at any point P on the line connecting the centres of the electrodes is given by

$$\mathbf{E}_x = \mathbf{E}_{Ax} + \mathbf{E}_{Bx} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{x^2} + \frac{1}{(l-x)^2}\right]\hat{x}.$$

The potential difference between the electrodes

$$V_{AB} = -\int_{l-b}^{a} \mathbf{E}_x \cdot d\mathbf{x} = \int_{a}^{l-b} \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{x^2} + \frac{1}{(l-x)^2}\right]dx$$

$$= \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{x} + \frac{1}{l-x}\right]_a^{l-b} = \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{l-b} + \frac{1}{a} + \frac{1}{b} - \frac{1}{l-a}\right]$$

For $l >> a, b$, we have

$$V_{AB} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{a} + \frac{1}{b} - \frac{2}{l}\right]. \quad ...(i)$$

The current $i = \int \mathbf{j} \cdot d\mathbf{S} = \sigma \int \mathbf{E} \cdot d\mathbf{S}$

$$= \sigma q / \varepsilon_0, \quad ...(ii)$$

where i is the current in the medium and S is the closed surface surrounding one of the electrodes. Thus from Eqs. (*i*) and (*ii*), we get

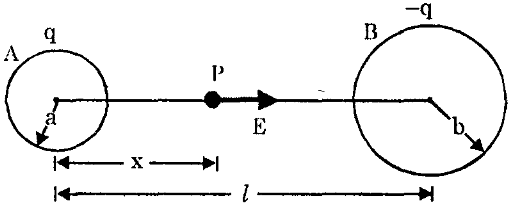


Fig. 8.35. Example 12.

$$R = \frac{V_{AB}}{i} = \frac{1}{4\pi\sigma}\left[\frac{1}{a} + \frac{1}{b} - \frac{2}{l}\right].$$

Example 13 : *Calculate the resistance between the center conductor of radius a and the coaxial conductor of radius b for the cylinder of length l >> b, which is filled with a dielectric of permittivity ε and conductivity σ. Also calculate the capacitance between the inner and outer conductors.*

One can calculate the electric field intensity **E** (r) between the two coaxial conductors with a potential difference of V as

$$\mathbf{E}(r) = [V/r \ln(b/a)]\hat{r}.$$

$$\therefore \quad \text{Current } I = 2\pi r l j = 2\pi r l (\sigma E)$$

$$= 2\pi r l \sigma V/r \ln(b/a) = 2\pi l \sigma V/\ln(b/a)$$

Hence the resistance between the conductors

$$R = \frac{V}{I} = \frac{\ln(b/a)}{2\pi l \sigma}.$$

Since the electric field is zero inside a conductor, the surface charge density ρ_s of the inner conductor can be obtained from the boundary condition

$$D_{2n} - D_{1n} = \rho_s, \text{ i.e., } \varepsilon E - 0 = \rho_s \text{ or } E = \rho_s/\varepsilon.$$

$$\therefore \quad \rho_s = \varepsilon E = \varepsilon V/a \ln(b/a)$$

and the total charge on the inner conductor $q = 2\pi a l \rho_s$

$$\therefore \quad \text{Capacitance} = \frac{q}{V} = \frac{2\pi a l \varepsilon}{a \ln(b/a)} = \frac{2\pi\varepsilon l}{\ln(b/a)}$$

Example 14 : *A parallel plate capacitor is filled with two media with constants (conductivities and permittivities) σ_1, ε_1, and σ_2, ε_2 respectively as shown in Fig. 8.36. Calculate the resistance of the system.*

In a parallel plate capacitor with the parallel slabs of two media, the fields $\mathbf{E}_1$ and $\mathbf{E}_2$ and currents j_1 and j_2 are expected to be normal to the plates and constant in each medium.

$$\therefore \quad V = -\int \mathbf{E} \cdot d\mathbf{r} = E_1 t_1 + E_2 t_2, \quad ...(i)$$

For conducting media, one can not use the relation $D_1 = D_2$, as free charge σ_f will collect at the interface between the media. Thus the boundary condition on j gives

$$j_{1n} = j_{2n} \text{ or } \sigma_1 E_1 = \sigma_2 E_2. \quad ...(ii)$$

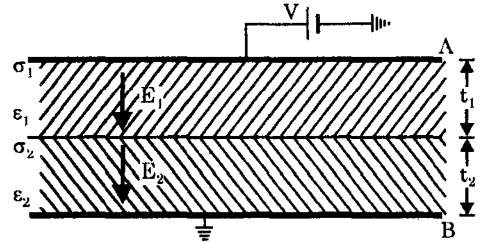


Fig. 8.36. Example 14.

Combination of Eqs. (*i*) and (*ii*) gives

$$E_1 = \frac{\sigma_2 V}{\sigma_2 t_1 + \sigma_1 t_2} \text{ and } E_2 = \frac{\sigma_1 V}{\sigma_2 t_1 + \sigma_1 t_2}.$$

The resistance R of the system of two media is given by

$$R = \frac{V}{jA} = \frac{V}{\sigma_1 E_1 A} = \frac{V}{\sigma_2 E_2 A} = \frac{\sigma_2 t_1 + \sigma_1 t_2}{\sigma_1 \sigma_2 A}.$$

One will recognize this configuration as equivalent to two resistors in series.

Example 15 : *Calculate the resistance of the strip ABCD (shown in Fig. 8.37) between electrodes CD and AB.*

In the medium where the conductivity is constant, the potential equation is $\nabla^2 V = 0$. In polar coordinators, $V = c\theta$ is a solution of $\nabla^2 V = 0$, if c is a constant. Equipotential surfaces are given by θ = const. The field is given by $\mathbf{E} = -\text{grad } V$ and hence $E = -c/r$ in the direction $\perp$ to the radius vector. The solution for the flow of current in a thin strip bounded by any two concentric circles and having the lines AB and CD as electrodes will be

$$\mathbf{j} = \sigma \mathbf{E}$$

$$\therefore \quad j = \sigma E = -\sigma c/r.$$

Hence the total current leaving AB is

$$i = \int \mathbf{j} \cdot d\mathbf{S} = -\int_a^b \frac{\sigma c}{r} t\, dr = -c\sigma t \log_e \frac{b}{a},$$

where t be the thickness of the strip.

As the *pd* between the electrodes AB and CD is $c\alpha$, where $\alpha = \angle AOC$, hence the resistance of the strip

$$R = \frac{-c\alpha}{-c\sigma t \log_e (b/a)} = \frac{\alpha}{\sigma t \log_e (b/a)}.$$

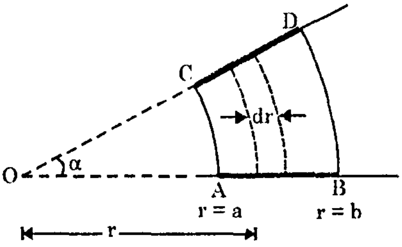


Fig. 8.37. Example 15.

Example 16 : *A 50 meter length of copper wire has an 80.8 mil diameter. If it carries a current of 20 amp, find electric field intensity, drift velocity, voltage drop and the resistance. Given $\sigma_{cu} = 5.8 \times 10^7$ S/m and $\mu = 0.0032$ m²/Vs.*

Since a mil = 10^{-3} inch = 2.54×10^{-5} m.

$\therefore$ Cross sectional area of *Cu* wire = $\pi r^2 = \pi (40.4 \times 2.54 \times 10^{-5})^2$

$= 3.31 \times 10^{-6}$ m².

Therefore the current density $j = \frac{i}{A} = \frac{20}{3.31 \times 10^{-6}} = 6.05 \times 10^6$ A/m².

For *Cu*, $\sigma = 5.8 \times 10^7$ *S*/m, thus we have

Electric field $E = \frac{j}{\sigma} = \frac{6.05 \times 10^6}{5.8 \times 10^7} = 0.104\ V/m.$

and voltage drop $= El = 0.104 \times 50 = 5.2\ V.$

$\therefore$ Resistance $R = \frac{V}{i} = \frac{5.2}{20} = 0.26\,\Omega.$

Since $\sigma = \rho\,\mu$, therefore the charge density

$$\rho = \frac{\sigma}{\mu} = \frac{5.8 \times 10^7}{0.0032} = 18.125 \times 10^9\ C/m^3.$$

The drift velocity v_d $= j/\rho = 6.05 \times 10^6 / 18.125 \times 10^9$

$= 3.34 \times 10^{-4}$ m/sec.

Example 17 : *Calculate the conductivity of n-type germanium (Ge) at 300 K, assuming one donor atom per 10^8 atoms of Ge. The density of Ge is 5.32×10^3 kg/m^3 and the atomic weight is 72.6 kg/kmol.*

Each donor atom produces one electron, called carrier. Therefore the carrier density

$$N_e = \frac{6.023 \times 10^{26}}{72.6} \times \frac{5.32 \times 10^3}{10^8}$$
$$= 4.41 \times 10^{20} \text{ electrons/m}^3.$$

We know that the intrinsic concentration n_i for *Ge* at 300 K is 2.5×10^{19} m^{-3}. The mass-action law, $N_e N_h = n_i^2$. Thus the density of holes

$$N_h = (2.5 \times 10^{19})^2/4.41 \times 10^{20} = 1.42 \times 10^{18} \text{ holes/m}^3.$$

Because $N_e >> N_h$, the conductivity will be mainly due to denoted electrons, whose mobility at 300 K is 0.38.

$$\therefore \qquad \sigma = N_e e\, \mu_e = (4.41 \times 10^{20})\,(1.6 \times 10^{-19})\,(0.38) = 26.8 \text{ S/m}.$$

Example 18 : *N lamps, each of resistance r, are fed by a machine of resistance R (including leads). If the light emitted by any lamp is proportional to the square of the heat produced, prove that the most efficient way of arranging the lamps is to place them in parallel arcs, each containing n lamps, where n is the integer nearest to* $(NR/r)^{1/2}$.

As each arc is containing n lamps, hence the resistance of each arc is nr and the number of arcs is N/n. Total resistance S is given by

$$\frac{1}{S} = \sum \frac{1}{nr} = \frac{N}{n}\left(\frac{1}{nr}\right) \text{ or } S = \frac{n^2 r}{N}.$$

$$\therefore \qquad \text{Total resistance } = R + S = R + n^2 r/N.$$

If $\mathcal{E}$ is the emf of the machine, the current entering the arcs is $\mathcal{E}/(R + S)$ and in the each arc is $n\mathcal{E}/(R + S)N$. Hence the current passing through each lamp

$$i = \frac{n\mathcal{E}}{N(R + n^2 r/N)} = \frac{\mathcal{E}}{N}\left[\frac{R}{n} + \frac{nr}{N}\right]^{-1}.$$

The heat produced per sec in the lamps is $H = Nri^2$ and the light emitted is proportional to H^2. Therefore the light produced is maximum when H^2 and hence H is maximum or factor $R/n + nr/N$ is minimum. We can write

$$\frac{R}{n} + \frac{nr}{N} = \left(\left[\frac{R}{n}\right]^{1/2} - \left[\frac{nr}{N}\right]^{1/2}\right)^2 + 2\left(\frac{Rr}{N}\right)^{1/2}.$$

It is minimum when $\sqrt{(R/n)} - \sqrt{(nr/N}$ $= 0$ or very small or n is closely equal to $(NR/r)^{1/2}$.

Example 19 : *Two concentric metal spheres of inner and outer radii a and b respectively are separated by a medium that has dielectric constant ε and conductivity σ. At time t = 0, an electric charge q is suddenly placed on the inner sphere. (a) Calculate the total current through the medium as a function of time. (b) Calculate the joule heat produced by this current and show that it is equal to the decrease in electrostatic energy that occurs as a consequence of the rearrangement of the charge.*

(*a*) At $t = 0$, when the inner sphere is given an electric charge q, the electric field intensity inside the medium is

$$E(0) = q/4\pi\varepsilon r^2.$$

At time t, when the inner sphere has electric charge $q(t)$, the electric field

$$E(t) = q(t)/4\pi\varepsilon r^2.$$

Ohm's law gives current density $j = \sigma \mathbf{E}$.

For the spherical surface of radius r (> a), we have from charge conservation

$$-\frac{dq(t)}{dt} = 4\pi r^2 j(t) = 4\pi r^2 \sigma E(t) = \frac{\sigma}{\varepsilon} q(t)$$

or
$$\frac{dq(t)}{q(t)} = -\frac{\sigma}{\varepsilon} dt.$$

This has solution $q(t) = q\, e^{-(\sigma/\varepsilon)t}$

$$\therefore \quad E(t) = (q/4\pi\varepsilon r^2)\, e^{-(\sigma/\varepsilon)t} \text{ and } j(t) = (q\sigma/4\pi\varepsilon r^2)\, e^{-(\sigma/\varepsilon)t}$$

The total current flowing through the medium at time t is

$$I(t) = 4\pi r^2 j(t) = (\sigma q/\varepsilon)\, e^{-(\sigma/\varepsilon)t}.$$

(b) The Joule heat loss per unit volume per unit time in the medium

$$\omega(t) = \mathbf{j}.\mathbf{E} = \sigma E^2(t) = [\sigma q^2/(4\pi\varepsilon r^2)^2]\, e^{-(2\sigma/\varepsilon)t}$$

$$\therefore \text{ Total joule heat } W = \int_0^\infty dt \int_a^b 4\pi r^2 \omega(t)\, dr = \frac{q^2}{8\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

The electrostate energy in the medium before discharging

$$W_o = \int_a^b \frac{1}{2}\varepsilon E_0^2 4\pi r^2 dr = \frac{q^2}{8\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

$$\therefore \quad W = W_o.$$

Example 20 : *A cable AB of length l of uniform wire develops a leak at certain point. To locate the fault two observations are made. The resistance between A and the earth through the cable when B is earthed is found to be that of a length a of the wire and that between B and the earth when A is earthed that of a length b. Show that the point at which the fault exists divides AB in the ratio* $[a(l-b)/b(l-a)]^{1/2}$ *and find the resistance of the leak.*

Let ρ be the resistance per unit length of the wire. If the fault exists at a distance x from the end A and r is the resistance of the leak.

(a) When B is earthed, the resistance between A and the earth through the cable

$$R = x\rho + \frac{1}{1/r + 1/(l-x)\rho} = a\rho \text{ (given)}$$

or
$$\frac{1}{\rho(a-x)} = \frac{1}{r} + \frac{1}{(l-x)\rho}. \qquad ...(i)$$

(b) When A is earthed, the resistance between B and the earth through the cable

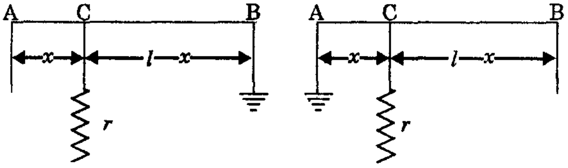


Fig. 8.38. Example 20.

$$R' = (l-x)\rho + \frac{1}{1/r + 1/x\rho} = b\rho \text{ (given)}$$

or
$$\frac{1}{\rho(b-l+x)} = \frac{1}{r} + \frac{1}{x\rho}. \qquad ...(ii)$$

Eliminating r from Eqs. (i) and (ii), we get

$$\frac{1}{\rho(a-x)} - \frac{1}{(l-x)\rho} = \frac{1}{\rho(b-l+x)} - \frac{1}{x\rho}$$

or
$$\frac{x}{l-x} = \left[\frac{a(l-b)}{b(l-a)}\right]^{1/2}$$

For the second part of the question, let us assume that the ratio $x/(l-x) = \mu$, or $x = \mu l/(1+\mu)$.

Substituting this value of x in Eq. (i), we get

$$\frac{1}{\rho[a-\mu l/(1+\mu)]} = \frac{1}{r}+\frac{1}{\rho[l-\mu l/(1+\mu)]}$$

$$\frac{1}{r} = \frac{l-\mu l/(1+\mu)-a+\mu l/(1+\mu)}{\rho[a-\mu l/(1+\mu)][l-\mu l/(1+\mu)]}$$

or $$r = \rho\, l[a(1+\mu)-\mu l]/(1+\mu)^2\,(l-a).$$

Example 21 : *Determine the relaxation time for silver, with conductivity $\sigma = 6.17 \times 10^7$ S/m. If a charge of volume charge density ρ_o is placed within a silver block, find ρ after five time constants.*

The relaxation time or time constant $$\tau = \frac{\varepsilon}{\sigma} = \frac{36\pi\times10^{-9}}{6.17\times10^{7}}$$
$$= 1.43 \times 10^{-19}\text{ s.}$$

The charge density at any time t is given by

$$\rho = \rho_o\, e^{-t/\tau}$$

For $t = 5\tau$, we get $\rho = \rho_o\, e^{-5} = 6.74 \times 10^{-3}\, \rho_o$.

Example 22 : *In a uniform submarine cable there is a leak resistance r per unit length and the resistance of the cable is R per unit length. At distance x along the cable the potential is V and the current is i. Show that $dV/dx = -Ri$ and $di/dx = -V/r$. Show also that if the cable is of length l and the two ends being at potential V_0 and 0, the potential V is given by the relation*

$$V = V_0 \cosh (R/r)^{1/2}\, l \sinh (R/r)^{1/2}\,(l-x),$$

Assuming the submarine cable as the line with continuous leak. As V is the potential at a distance x from end A along the cable and i the current in the cable at this point. The potential at a distance $x + \delta x$ from A will be $V + (dV/dx)\delta x$. The leakage current is thus $-(di/dx)\delta x$. Since the resistance of the insulation from x to $x + \delta x$ varies inversely as δx, hence at a distance x,

$$V = \left(-\frac{di}{dx}\delta x\right)\frac{r}{\delta x} = -r\frac{di}{dx},$$

or $$di/dx = -V/r. \qquad \ldots(i)$$

For the element of the cable from x to $x + \delta x$, we get

$$V-\left(V+\frac{dV}{dx}\delta x\right) = iR\delta x,$$

or $$dV/dx = -iR. \qquad \ldots(ii)$$

Eliminating i between Eqs. (i) and (ii), we get

$$V = -r\frac{d}{dx}\left[-\frac{1}{R}\frac{dV}{dx}\right] = \frac{r}{R}\frac{d^2V}{dx^2}.$$

or $$d^2V/dx^2 = (R/r)V. \qquad \ldots(iii)$$

The solution of this equation is

$$V = A \cosh (R/r)^{1/2}x + B \sinh (R/r)^{1/2}x.$$

If $V = V_0$, when $x = 0$ and $V = 0$ when $x = l$, we find

$$V = V_0 \cosh (R/r)^{1/2}l \sinh (R/r)^{1/2}\,(l-x).$$

Example 23 : *In Fig. 8.39, a second source of emf = 18 V and resistance 2 Ω has been connected to the first like terminals of the sources being connected together of emf 12 V and resistance 2Ω. Calculate the total power output. Find the effect of changing the polarity of the second source.*

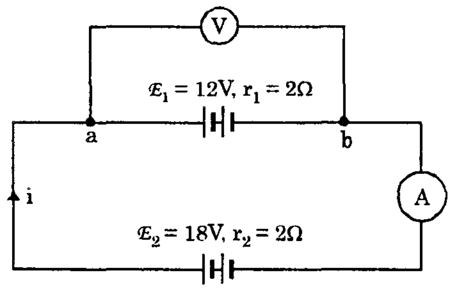


Fig. 8.39. Example 23.

Consider the clockwise direction as the positive direction of current.

$$\mathcal{E}_1 = -12V, \mathcal{E}_2 = 18V.$$

$$\therefore \quad \Sigma\mathcal{E} = 18V - 12V = 6V.$$

Hence the current in the circuit

$$I = \Sigma\mathcal{E}/\Sigma R = 6/4 = 1.5 \text{ amp.}$$

The rate of energy conversion in source 1st is $\mathcal{E}_1 I = 12 \times 1.5 = 18$ watts. Within this source, electrical energy is converted to non-electrical energy. The rate of dissipation of energy in source 1st is $I^2 r_1 = (1.5)^2 \times 2 = 4.5$ watts.

$\therefore$ Total power input to source 1st = 18 + 4.5
= 22.5 watts.

The rate of energy conversion in source 2nd is $\mathcal{E}_2 I = 18 \times 1.5 = 27$ watts. Within this source, non-electrical energy is converted to electrical energy. The rate of dissipation of energy in source 2nd is $I^2 r_2 = 1.5^2 \times 2 = 4.5$ watts.

$$\therefore \quad \text{Power output} = 27 - 4.5 = 22.5 \text{ watts.}$$

In the second part of question, when the sources are connected with unlike terminals together, the current

$$I = \Sigma\mathcal{E}/\Sigma R = 30/4 = 7.5 \text{ amp.}$$

Non-electrical energy is converted to electrical energy in both sources. The rates of conversion are respectively

$$\mathcal{E}_1 I = 12 \times 7.5 = 90 \text{ watts and } \mathcal{E}_2 I = 18 \times 7.5 = 135 \text{ watts.}$$

The rate of energy dissipation within the resources are respectively $I^2 r_1 = 112.5$ watts and $I^2 r_2 = 112.5$ watts.

$\therefore$ Total rate of energy conversion = 225 watts.

Hence the power output of source 2nd is 135 – 112.5 = 22.5 W.

This equals the power input to source 1st. Hence we see that the power output or input remains same and does not change with the change in polarity of a source.

Example 24 : *An electric furnace of resistance $R = 16\ \Omega$ is connected to a generator with a terminal velocity $V = 130$ volts by 60 m aluminium conductor leads of cross sectional area 10 mm^2. Calculate the power output of the generator on its having an internal resistance $r_0 = 0.1$ ohm and also the electrical energy expended in three hours. Given $\rho_{Al} = 2.9 \times 10^{-8}$ ohm meter.*

$$\text{Resistance of the leads} \quad R_l = \rho\frac{l}{S} = \frac{2.9\times10^{-8}\times60\times2}{10\times10^{-6}}$$

$$= 0.348 \text{ ohm.}$$

$$\text{The current through the leads } I = \frac{V}{R} = \frac{130}{10+0.1+0.348} = 12.44 \text{ amps.}$$

The furnace power consumption $P_f = VI = 130 \times 12.44 = 1617.2$ watts.

The power loss in the leads $P_l = I^2 R_l = 12.44^2 \times 0.348 = 53.85$ watts.

The power loss in the generator $P_g = I^2 r_0 = 12.44^2 \times 0.1 = 15.47$ watts

$\therefore$ Total power consumption $P = P_f + P_l + P_g = 1686.52$ watts

Therefore the emf of the generator $\mathcal{E} = P/I = 135.57$ volts.

The electrical energy expended in three hours

$$W_f = P_f t = 1617.2 \times 3 = 4.85 \text{ kilowatt-hours.}$$

Example 25 : *Find the potential difference $V_A - V_B$ and the rate of production of heat in R_I in the network shown in Fig. 8.40.*

At the steady state, the capacitor branch *CDGH* acts as an open branch, *i.e.*, of infinite resistance with zero current in it.

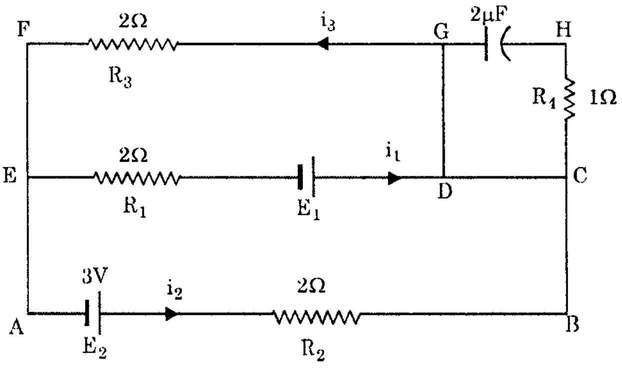


Fig. 8.40. Example 25.

If i_1, i_2 and i_3 be the currents in the *ED, AB* and *GF* branches respectively, then from the Kirchhoff's first law, we get at point *D*

$$i_1 + i_2 - i_3 = 0 \quad \text{or} \quad i_3 = i_1 + i_2.$$

Assuming the sources of the negligible resistance and applying Kirchhoff's second law to the meshes *ABCDEA* and *EDGFE*, we get,

$$i_2R_2 - i_1R_1 = \mathcal{E}_2 - \mathcal{E}_1$$

and $\quad i_1R_1 + (i_1 + i_2)R_3 = \mathcal{E}_1$

or $\quad i_2R_3 + i_1(R_1 + R_3) = \mathcal{E}_1.$

Solving these equations for i_1 and i_2, we get

$$i_1 = \frac{\mathcal{E}_1(R_2 + R_3) - \mathcal{E}_2R_3}{R_1R_2 + R_2R_3 + R_3R_1} \text{ and } i_2 = \frac{\mathcal{E}_2(R_1 + R_3) - \mathcal{E}_1R_3}{R_1R_2 + R_2R_3 + R_3R_1}$$

Substituting numerical values, we get

$$i_1 = \frac{12(2+2) - 3 \times 2}{2 \times 2 + 2 \times 2 + 2 \times 2} = 3.5 \text{ amp}$$

and $$i_2 = \frac{3(2+2) - 12 \times 2}{2 \times 2 + 2 \times 2 + 2 \times 2} = -1 \text{ amp.}$$

$$\therefore \quad V_A - V_B = -3 - 1 \times 2 = -5 \text{ volts.}$$

Rate of production of heat in $R_1 = i_1^2R_1 = (3.5)^2 \times 2 = 24.5$ joules.

Example 26 : *A system of twelve conducting wires, all of equal resistance, form the edges of a cube. Find the currents carried by the wires if a battery is connected without extra resistance to adjacent vertices of the cube.*

By symmetry and Kirchhoff's first law the distribution of currents in various parts of the circuit is as shown in Fig. 8.41. $\mathcal{E}$ is the emf, r the internal resistance of the battery and R the resistance of each edge of the cube.

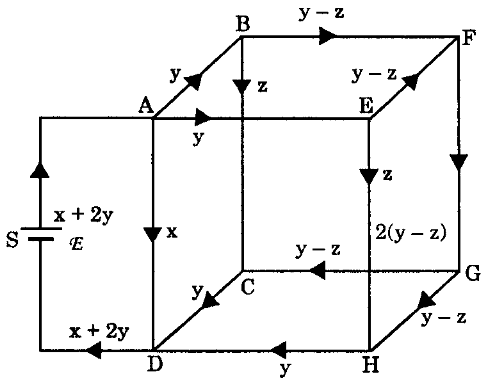


Fig. 8.41. Example 26.

On applying Kirchhoff's second law to the mesh *ASDA*, we get

$$(x + 2y)r + xR = \mathcal{E} \qquad ...(i)$$

For the closed loop *ABCDA*, we have,

$$yR + zR + yR - xR = 0,$$

or $$x = (2y + z) \qquad ...(ii)$$

For the closed loop *BFGCB*,

$$(y - z)R + 2(y - z)R + (y - z)R - zR = 0,$$

$$\therefore \quad 4(y - z) - z = 0 \text{ or } y = 5z/4. \qquad ...(iii)$$

Substituting this value in Eq. (*ii*), we get

$$x = 5z/2 + z = 7z/2. \qquad ...(iv)$$

Substituting the values of x and y in relation (i), we get,

$$\left(\tfrac{7}{2}z + 2\cdot\tfrac{5}{4}z\right)r + \tfrac{7}{2}zR = \mathcal{E}$$

or $$z = 2\mathcal{E}/(7R + 12r).$$

$$\therefore \quad x = \frac{7\mathcal{E}}{7R+12r} \text{ and } y = \frac{5\mathcal{E}}{14R+24r}.$$

Hence current along AB, AE, CD and $HD = y = 5\mathcal{E}/(14R + 24r)$.

Current along BC and EH $= z = 2\mathcal{E}/(7R + 12r)$.

Current along BF, EF, GC and GH $= y - z = \mathcal{E}/(14R + 24r)$.

Current along FG $= 2(y - z) = \mathcal{E}/(7R + 12r)$.

Current through the battery $= x + 2y = 12\mathcal{E}/(7R + 12r)$.

Example 27 : *Twelve equal wires, each of resistance R ohm are connected so as to form a skelton cube. An electric current enters the cube from one corner and leaves out at the diagonally opposite corner. Calculate the total resistance of this assembly.*

Let $ABCDEFGH$ be the skelton cube formed by twelve equal wires, each of resistance R ohm. Let the current entering at the corner A and leaving at the diagonally opposite corner G be I. Since the resistance of each wire is the same, the total current I is divided at a point A into three equal parts, *i.e.*, $I/3$ along AB, AE and AD arms. At points B, E and D, this current is further divided into two equal parts, so that the current in each of the arms BC, BF, EF, EH, DH and DC is $I/6$. At points C, F and H, the currents combine, so that the current in each of the arms CG, FG and HG is again $I/3$. These currents combine again at point G, such that the current leaving at point G is the current entering at point A, *i.e.*, I.

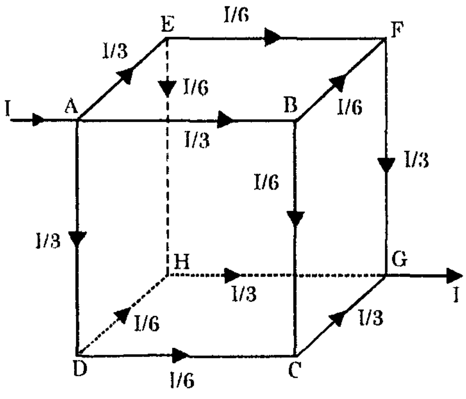


Fig. 8.42. Example 27.

Let $\mathcal{E}$ be the potential difference between the points A and G. Applying Kirchhoff's second law to any one of the paths between these points, *i.e.*, $ABCG$, we get,

$$\tfrac{1}{3}IR + \tfrac{1}{6}IR + \tfrac{1}{3}IR = \mathcal{E} = I\,R_{eq},$$

where R_{eq} is the equivalent resistance between the corners A and G.

$$\therefore \quad R_{eq} = \tfrac{1}{3}R + \tfrac{1}{6}R + \tfrac{1}{3}R = \tfrac{5}{6}R.$$

Example 28 : *In a Wheatstone's bridge, which is nearly balanced, the resistances, of the arms are 100, 10, 1000, 100.01 ohms. The cell has an emf of 2 volts and the negligible resistance. The galvanometer has a resistance of the 10 ohms and sensitivity 0.5 cm per 0.1 μA. Calculate values for the galvanometer deflections in the two possible arrangements of cell and galvanometer.*

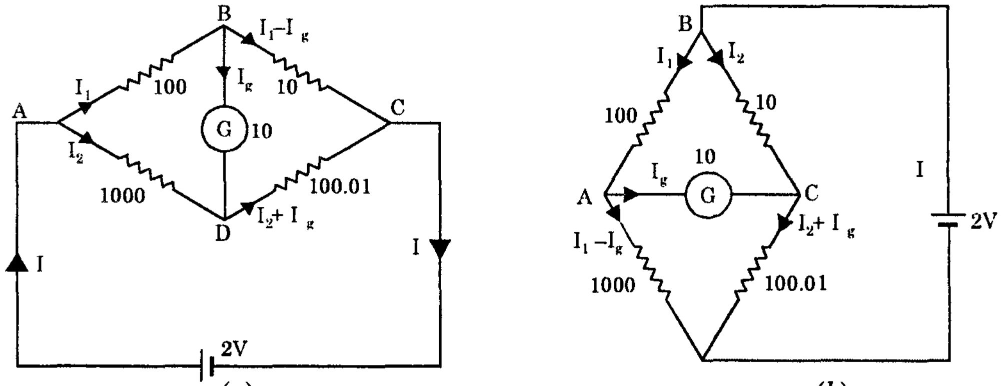
Let the bridge be arranged as shown in Fig. 8.43 (a). The bridge is nearly balanced, hence the current in the galvanometer arm may be assumed as zero. Therefore the effective resistance of the bridge is given by

$$\frac{1}{R_{eff}} = \frac{1}{100+10} + \frac{1}{1000+100.01} \text{ or } R_{\text{eff}} = 100 \text{ ohms.}$$

$$\therefore \quad \text{Current from the cell } I = \mathcal{E}/R = 2/100 = 0.02 \text{ amp.}$$

By Kirchhoff's first law, we get

$$I = I_1 + I_2 \text{ or } I_2 = 0.02 - I_1.$$



(a) (b)

Fig. 8.43. Example 28.

By applying Kirchhoff's second law to the loops *ABDA* and *BCDB*, we get

$$100I_1 + 10I_g - 1000(0.02 - I_1) = 0$$

and $10(I_1 - I_g) - 100.01(0.02 - I_1 + I_g) - 10I_g = 0.$

Solving these equations, we get

$$I_g = -1.49 \times 10^{-7} \text{ amp} = -0.149\ \mu A.$$

The negative sign indicates that the current is from *D* to *B* through the galvanometer.

∴ Deflection in galvanometer coil = (0.5/0.1) × 0.149 = 0.745 cm.

When the cell and galvanometer are interchanged, the current from cell is again 0.02 amp and the currents are as shown in Fig. 8.43 (*b*). By Kirchhoff's first law, we have

$$0.02 = I_1 + I_2 \text{ or } I_2 = 0.02 - I_1.$$

Applying Kirchhoff's second law to the meshes *ABCA* and *ACDA*

$$100I_1 + 10I_g - 10(0.02 - I_1) = 0.$$

and $1000(I_1 - I_g) - 100.01(0.02 - I_1 + I_g) = 0.$

Solving these equations, we get

$$I_g = -1.50 \times 10^{-7} \text{ amp} = -0.15\ \mu A.$$

Hence the current through the galvanometer flows from *C* to *A*.

∴ Deflection in galvanometer coil = (0.5/0.1) × 0.15 = 0.75 cm.

Thus we see that the current through the galvanometer is practically independent of the arrangement of the cell and the galvanometer. After calculating I_2 in both the arrangements, it can also be noted that in the first instance the current through 100.01 Ω (unknown resistance) is much less than in the second arrangement. Thus a correction for the effect of joule heating will have to be made, if this resistance has a large resistance temperature coefficient, in the second arrangement.

Example 29 : *A tetrahedron frame ABCD is formed by six wires, the resistances of opposite edges being equal. Prove that the whole resistance of the frame for a current entering at A and leaving at D is* $(r_1 r_3 + 2r_1 r_2 + r_2 r_3)\, r_3 / 2\, (r_1 + r_3)(r_2 + r_3)$, *where* r_1 *is the resistance of AB or CD,* r_2 *that of AC or BD and* r_3 *that of AD or BC.*

By the help of Kirchhoff's first law the distribution of current is as shown in the Fig. 8.44. By applying Kirchhoff's second law to the mesh *ABDA*, we have

$$r_1 I_1 + r_2(I_1 + I_3) - r_3(I - I_1 - I_2) = 0,$$

For the mesh *ACBA*

$$r_2 I_2 + r_3 I_3 - r_1 I_1 = 0,$$

For the mesh *CDBC*

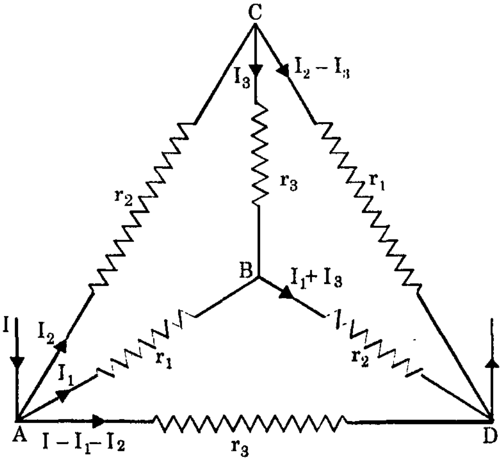


Fig. 8.44. Example 29.

$$r_1(I_2 - I_3) - r_2(I_1 + I_3) - r_3 I_3 = 0.$$

On rearranging, we get

$$(r_1 + r_2 + r_3) I_1 + r_3 I_2 + r_2 I_3 = r_3 I,$$

$$-r_1 I_1 + r_2 I_2 + r_3 I_3 = 0,$$

and $$-r_2 I_1 + r_1 I_2 - (r_1 + r_2 + r_3) I_3 = 0.$$

Solving these equations, we get current through AD

$$= I - I_1 - I_2 = \frac{r_2(r_1 + r_3) + r_1(r_2 + r_3)}{2(r_1 + r_3)(r_2 + r_3)} I.$$

If R is the whole resistance of the frame for a current entering at A and leaving at D, then

$$IR = (I - I_1 - I_2)\, r_3 = \frac{r_2(r_1 + r_3) + r_1(r_2 + r_3)}{2(r_1 + r_3)(r_2 + r_3)} I r_3$$

or $$R = (r_1 r_3 + r_2 r_3 + 2 r_1 r_2)\, r_3 / 2(r_1 + r_3)(r_2 + r_3).$$

Example 30 : *A Wheatstone bridge has three arms of 10 Ω each and one of 11 Ω. Find the current in the various arms of the bridge, taking the applied potential as 2 volts, the galvanometer resistance as 100 Ω and the battery resistance as negligible.*

The given resistances, 10, 10, 10, and 11Ω form the four arms of a Wheatstone bridge. There are two possible arrangements of galvanometer and battery. Let us solve the problem using an arrangement shown in Fig. 8.45.

If I be the current in the battery arm, i_1 along AB, and i_g along BD, then Kirchhoff's first law gives the current in various arms as shown in Fig. 8.45. Now applying Kirchhoff's second law for the mesh $ABDA$, we get

$$i_1 \times 10 + i_g \times 100 - (I - i_1) \times 10 = 0$$

or $$20 i_1 + 100 i_g - 10 I = 0 \qquad ..(i)$$

and for the mesh $BCDB$

$$(i_1 - i_g) \times 10 - (I - i_1 + i_g) \times 11 - i_g \times 100 = 0$$

or $$21\, i_1 - 121\, i_g - 11 I = 0 \qquad ..(ii)$$

and for the mesh $AECDA$

$$(I - i_1) \times 10 + (I - i_1 + i_g) \times 11 = 2$$

or $$-21\, i_1 + 11\, i_g + 21 I = 2 \qquad ..(iii)$$

Solving these equations, we get

$$i_1 = \frac{231}{2315},\ i_g = -\frac{1}{2315} \text{ and } I = \frac{452}{2315} \text{ amp.}$$

Therefore the currents through the arms AB, BC, AD, DC, DB and EA are

$$\frac{231}{2315}, \frac{232}{2315}, \frac{221}{2315}, \frac{220}{2315}, \frac{1}{2315} \text{ and } \frac{452}{2315} \text{ amp respectively.}$$

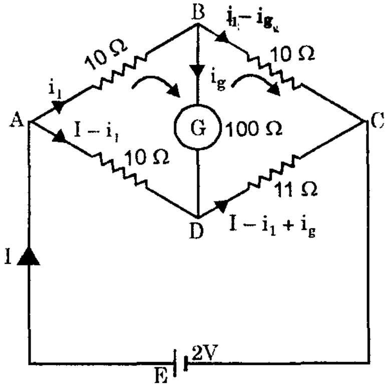


Fig. 8.45. Example 30.

Example 31 : *In the network shown in Fig. 8.46, find I, I_1 and the charge on the 7 μF capacitor after equilibrium conditions have been reached.*

After equilibrium conditions have been reached, no current passes through the capacitor branch. Thus at this stage, no current will flow in AB and DB branches. Therefore $I = I_1$ = current in DC branch.

Thus by applying Kirchhoff's second law for the mesh $ADCEA$, we get

$$2I_1 + 3I_1 = 6 - 6 = 0$$

or

$$I_1 = 0.$$

No current passes in the loop *ABCEA*, therefore there is no potential fall across the resistances in *AB* and *BC* arms. Thus the potential difference of 6*V* is applied across the plates of the 7 μF capacitor.

$\therefore$ Charge on the capacitor = $CV = 7\mu F \times 6V = 42\mu C$.

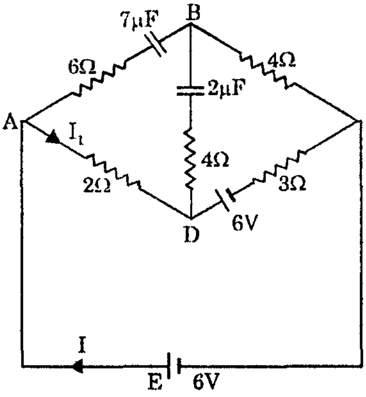


Fig. 8.46. Example 31.

Example 32 : *An accumulator having a steady emf of 2 volts is connected across the potentiometer wire of length 10 meters. The potentiometer wire is of manganin having resistance of 11.5 ohm/meter. A cell gives a null point at 690 cm. If a resistance of 5 Ω is put in series with the potentiometer wire, find the new position of the null point.*

Length of the potentiometer wire = 10 meters.

$\therefore$ Fall in potential per meter $= \rho = 2/10 = 0.2\ V/m$.

As the cell gives a null point at 690 cm, hence its emf

$$= \rho l = 0.2 \times 6.9 = 1.38 \text{ volts.}$$

Resistance of the potentiometer wire = 11.5 × 10 = 115 ohms.

When a resistance of 5 Ω is connected, total resistance becomes 115 + 5 = 120 ohms and the current $i = V/R = 2/120$ amp.

$$\therefore \quad \text{Potential fall per meter} = \frac{115}{120} \times 2 \times \frac{1}{10} = \frac{23}{120} V/m.$$

If l'be the position of null point on the potentiometer wire, then

$$\frac{23}{120} l' = 1.38$$

or

$$l' = \frac{1.38 \times 120}{23} = 7.2 \text{ meters.}$$

ORAL QUESTIONS

1. A lightning conductor installed in a building (*a*) does not allow the lightning to fall on the building at all, (*b*) forces the lightning to fall on other building near that building, (*c*) drives away the charged clouds, or (*d*) none of these.
2. During a thunderstorm you are travelling in a car. What will you do to protect yourself from the lightning? (*a*) remain in the car, (*b*) touch the nearest electric pole, (*c*) abandon the car and take shelter under a nearby tree or (*d*) get out of the car and lie flat on the ground.
3. At low temperatures approaching zero absolute some substances lose their electrical resistances. These substances then become (*a*) insulators, (*b*) good conductors, (*c*) superconductors, or (*d*) semiconductors.
4. The cell that can be recharged by passing a direct current through it is (*a*) dry cell, (*b*) voltaic cell, (*c*) an accumulator, or a (*d*) Daniel cell.
5. Is it true that the resistivity of insulators is very high compared to that of metals ?
6. The S.I. unit of electric current is (*a*) volt, (*b*) ampere, (*c*) ohm, or (*d*) coulomb.
7. A rheostat is used (*a*) to increase the resistance, (*b*) to decrease the resistance, (*c*) as potential divider, or (*d*) all of these.
8. Which of the following wires of some dimensions offers the least resistance to the flow of current ? (*a*) Cu, (*b*) Al, (*c*) Ag, or (*d*) Nichrome.
9. The filament used in electric bulb is made of (*a*) Cu, (*b*) Al, (*c*) Nichrome, or (*d*) Tungsten.
10. The heating elements generally used in an electric heater/iron is made of (*a*) Cu, (*b*) Al, (*c*) Nichrome, or (*d*) Tungsten.

11. Is it true that an ammeter is a low resistance instrument while a voltmeter is a high resistance instrument ?
12. The commonly used safety fuse wire is made of (*a*) Cu, (*b*) lead; (*c*) nickel, or (*d*) alloy of tin and lead.
13. The function of a fuse wire is (*a*) to stabilize the voltage, (*b*) decrease the current, (*c*) prevent an unduly high current from passing through a circuit, or (*d*) to control the power loss.
14. In an electric circuit the fuse is connected (*a*) in the live wire, (*b*) in the earth wire, (*c*) in the neutral wire, or (*d*) anywhere.
15. Which is not an example of convection current ? (*a*) Electric current in a constantan wire, (*b*) Electron beam in a CRT, (*c*) Electron motion in a vacuum tube or (*d*) Moving charged belt.
16. The relaxation time of mica ($\sigma = 10^{-15}$ mhos/m, $\varepsilon_r = 6$) is.....
17. Define conductance and resistance.
18. Define current density.
19. Explain the dependence of resistance on temperature from the point of view of the electron theory.
20. State Kirchhoff's current and voltage laws.
21. Compare series and parallel connections of resistances.
22. Explain the use of three terminals of a rheostat.
23. Is the current a vector quantity ? Explain.
24. What is the distinction between drift and rms velocities ?
25. Find the effect on the electron drift velocity of (*a*) doubling V, (*b*) doubling l and (*c*) doubling d in a wire of length l, diameter d across which a potential difference V is applied.
26. What is the physical meaning of nondivergence of j ?
27. When current passes through a conductor, which of the two i or $\mathbf{j}$ remains constant across every sectional area ?
28. What do you mean by non ohmic resistor ?
29. A current i enters the top of a copper sphere and leaves at its diametrically opposite point. Whether the energy is dissipated equally from each point of the sphere ?
30. The Joule heating in a resistor known to vary according to the relations $P = i^2 R = V^2/R$. How do you explain this different effect of resistance, so appeared ?
31. Giving some examples, how can you explain that current is not used up as it flows through a resistor ? What is lost in this case ?
32. If two capacitors are charged to the same potential when one grasp the two wires leading from the two plates of a charged capacitor. Why the cause of sensation of shock given to the body is greater for a capacitor of higher capacitance ?
33. Why do you not use wire wound resistors in high frequency assembly ?
34. What is the most important point which must remain in your mind while using Post Office box for Mance's and Thomson's methods ?
35. What is a transducer ?
36. A letter A is constructed of a uniform wire with a resistance 1.2 Ω per cm. The sides of the letter are 30 cm and the cross piece in the middle is 15 cm long. The apex angle is 60°. The resistance between the ends of the legs is.......... (*a*) 90 Ω, (*b*) 30 Ω, or (*c*) 48 Ω.
37. A good conductor is connected between a point of an electric circuit and the ground. Which one of the following is correct ? (*a*) Current flows from the point to the earth, (*b*) The potential of the point is the same as that of the earth, (*c*) When the current flows through the conductor to the earth, *d*) The current through the circuit remains unchanged.
38. Three identical resistances each of 10 Ω are connected such that two are in parallel and the combination is in series with the third. The maximum power that can be consumed by individual resistor is 20 watt. Then the maximum power consumed by the combination is (*a*) 60 W, (*b*) 15 W, (*c*) 30 W.

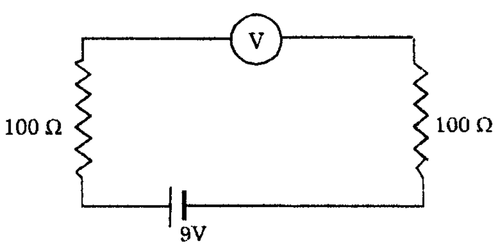


Fig. 8.47.

39. A voltmeter of resistance 9.8 kΩ has a least count of 0.2 V and range (0–10 V). Instead of connecting the voltmeter in parallel with one of the 100 Ω resistors by mistake, it is connected in series as show in figure 8.47. What will happen to the voltage ?

40. Kirchhoff's current law is (*a*) not applicable to accurrents, (*b*) applicable to instantaneous values of ac-currents, (*c*) applicable to rms value of ac-current.

Answers : 1. (*d*); 2. (*a*); 3. (*c*); 4. (*c*) 5. (yes); 6. ((*b*); 7. (*d*); 8. (*c*); 9. (*d*); 10. (*c*); 11. (yes); 12. (*d*); 13. (*c*); 14. (*a*); 15. (*a*); 16. (15 hrs); 36. (*c*); 37. (*b*); 38. (*c*); 39. (8.8 V); 40. (*b*).

PROBLEMS

1. Calculate the drift velocity for the copper wire of dia. 0.064 inch when a current of 10 amp passes through it. (3.6×10^{-4} m/sec)

2. The electron of a hydrogen atom is considered to be revolving round a proton in a circular orbit of radius $\hbar^2 / me^2$ with a velocity $e^2 / \hbar$. Find an expression for current, the circulating charge is equivalent to................. ($me^5 / 2\pi\hbar^3$)

3. A copper wire having a cross section of 0.01 cm^2 carries a current of 1.0 ampere. If the number of free electrons in copper is $8.0 \times 10^{22}/cm^3$, calculate the average drift velocity of free electrons. (0.78×10^{-4} m/sec)

4. In a neon gas discharge tube 2.9×10^{16} Ne^+ ions move to the right through a cross section of the tube each second, while 1.2×10^{18} electrons move to the left in the same time. Find the magnitude and the direction of the current. (2.384 amp, left to right)

5. A silver wire 1 mm in a diameter carries a charge of 90 coulombs in 75 min. Silver contains 5.8×10^{22} free electrons per cc. Calculate (*a*) the current in the wire and (*b*) the drift velocity of the electrons in wire in meters/sec. (0.02 *amp*, 2.74×10^{-6} m/sec)

6. In vacuum diode, the anode is parallel to the cathode at a distance 0.2 cm. If a potential of 100 volts is applied to the anode with respect to cathode, calculate the velocity of electrons at the points at the middle of the electrodes. (4.2×10^{-6} m/sec)

7. A current of 5 amp exists in a 10 ohm resistance for 4 min. Calculate the number of electrons passing in this time through any cross section of the resistor. (7.5×10^{31})

8. A steady beam of α-particles travelling with constant *K.E.* 20 MeV carries a current 2.5×10^{-7} amp. (*a*) How many α-particles strike the surface ⊥ to the direction of beam in 3 sec ? How many α-particles are there in a given 20 cm length of the beam at any instant ? (2.3×10^{12}, 5.0×10^{3})

9. A wire with a resistance of 20 ohms is drawn out so that its length becomes thrice the original length. Find the resistance of the streched wire. (18 ohms)

10. If a silver wire is stretched to make it 0.1% longer, what is the percentage change in its resistance ? (0.2%)

11. Show that if a straight wire lies along the x-axis, the rate of flow of charge is given by $dq/dt = -\sigma A\, dV/dx$, where A is the area of cross section of the wire, σ the electrical conductivity of the wire material and dV/dx is the potential gradient. Also make an analogy to heat conduction.

12. Calculate the mobility of the conduction electrons in aluminium. Given the conductivity $\sigma = 38.2$ MS/m and conduction electron density $n = 1.7 \times 10^{29}$ m^{-3}. (1.4×10^{-3} m^2/Vs)

13. Determine the number of conduction electrons per meter of Cu-wire. Given $\sigma = 58$ MS/m, $\mu = 3.2 \times 10^{-3}$ m^2/Vs, atomic weight = 63.54, density = 8.96×10^3 kg/m^3. Hence calculate the average number of electrons per atom. (1.13×10^{29}, 1.33)

14. A steady current i enters a highly conducting sphere of radius R through a thin wire. Find the resistance of the medium surrounding the sphere, infinite in size, of conductivity σ and permittivity ϵ_0. ($1/4\, \pi\sigma R$)

15. The space between two coaxial cylinders of radii a and b and of length l is filled with a medium of conductivity σ. Show that the resistance between the two cylinders is $(1/2\,\pi\,\sigma\,l)\log_e(b/a)$.

16. Find the leakage conductance per meter of a cylindrical coaxial cable whose inner conductor has a radius of 0.125 inch and whose outer conductor has an inner radius of 0.375 inch, if the space between is filled with material whose conductivity is 6.0×10^{-8} mho/m. $(3.3 \times 10^{-7}$ mho)

17. Show that if the arcs AC and BD in example 15 are taken to be two electrodes, the resistance of the conductor is changed to $(1/\sigma\alpha t)\log_e(b/a)$.

18. Find the resistance of a conic conductor of a circular cross section, whose side surfaces are the discs of radii a and b and l is the length of the conductor. The conductivity of the material is given as σ. $(l/\pi\sigma ab)$

19. A fault of given resistance develops in a telegraph wire. Show that with a given battery transmitting signals, the current received is least when the fault is at the middle of the wire.

20. Find an expression for the resistance of a conductor of length l if the cross section retains the same shape and the area increases linearly from A to $k\,A$ over length l. $[l/\sigma A\,(k-1)]\log_e k$.

21. A disc of thickness t has a radius b and a central hole of radius a. If the conductivity of the disc is σ, prove that the resistance between the hole and the rim of the disc is $\log_e(b/a)/2\,\pi\,t\,\sigma$.

22. A metal bar of conductivity σ is bent to form a flat 90° sector of inner radius a, outer radius b and thickness t. Show that the resistance of the bar between the vertical curved surfaces at $\rho = a$ and $\rho = b$ is $(2/\sigma\,\pi\,t)\log_e(b/a)$.

23. In the above problem find the resistance between the upper and lower surfaces, having spacing t. $4\,t/\sigma\,\pi\,(b^2 - a^2)$.

24. A telegraph wire joining two places A and B drops from one of its supports at a place C and rests on another wire which is earthed at both ends. If λ is the ratio of the current strength at A to that at B when the current in AB is sent from A and μ is the ratio when the current is sent from B, show that C divides AB in the ratio $(1/\mu - 1) : (\lambda - 1)$.

25. Express the rate of dissipation of energy in a resistor in terms of (1) potential difference and resistance. (2) resistance and current, (3) potential difference and resistance. Energy is dissipated in a resistor at the rate of 40 watts when the *pd* between its terminal is 60 V. What is its resistance ? (90 Ω)

26. Five points are connected by ten wires, each pair being joined by a wire of the same resistance R. Show that the resistance to a current entering at one of the points and leaving at any other point is 2R/5.

27. A power of 36 watts is to be expended in a resistor connected across the terminals of a battery having an emf of 20 volts and an internal resistance of 1 ohms. (*i*) What values of resistance will satisfy this condition. (*ii*) What will be the total power expenditure in each case. (9 and 1/9 ohm; 40 and 360 volts)

28. A source of emf $\mathcal{E}$ volts and internal resistance r ohms, supplies current to a heating coil. Calculate the resistance R of the heating coil in terms of r, so that (*i*) heating takes place most rapidly, (*ii*) three fourth of the total energy delivered by the source goes into the heating coil. (r, $3r$)

29. Eleven equal wires each of resistance r ohm form the edges of an incomplete cube. Find the total resistance from one end of the vacant edge of the cube to the other. (1.4 r ohm)

30. Forty identical wires, each of resistance R form a square net consisting of 16 meshes. Find the resistance between the diagonal points of the network. (47 R/22)

31. $ABCD$ is a uniform circular wire of resistance 2Ω. AOC and BOD are two wires each of resistance 1Ω placed at right angles to each other and in contact with each other at the centre of the circle. If a battery is placed in between A and D, find the resistance of the network. (15/14 Ω).

32. The sides of a hexagon $ABCDEF$ are formed of wires each of resistance R and wires each of the same resistance R join BF and BE. Show that the equivalent resistance of the conductors between A and D is 40 R/29.

33. The resistance of three wires BC, CA, and AB of the same uniform cross section and material are a, b and c respectively. Another wire from A of constant resistance d can make a sliding contact with BC. If a current enters at A and leaves at the point of contact with BC, show that the maximum resistance of the network is $(a + b + c)\, d/(a + b + c + 4d)$.

34. A wire forms a regular hexagon and the angular points are joined to the centre by wires each of which has a resistance 1/n of the resistance of a side of the hexagon. Show that the resistance to current entering at one angular point of the hexagon and leaving it by the opposite point is $2(n + 3)/(n + 1)(n + 4)$ times the resistance of a side of the hexagon.

35. Suppose a voltmeter reads the voltage of a very old cell to be 1. 40V, while a potentiometer reads its voltage to be 1.55V. What is (a) the emf of cell, (b) its internal resistance and (c) the current it would supply to a 5 Ω resistor ? Assume the voltmeter resistance to be 280 Ω. (1.55 V, 30 Ω, 0.044 A)

36. A nichrome heater dissipates 500 watts when the applied potential difference is 110 volts and the wire temperature is 800°C. If the wire is immersed in a bath of cooling oil of temp. 200°C, calculate the power dissipated. The applied pd remains the same and α for nichrome is about 4×10^{-4}/°C. (620 watts)

37. A galvanometer having resistance 100 Ω needs a current of 0.1 mA for full scale deflection. How can it be used as a voltmeter that can measure the voltage upto 15 V? (149.9 kΩ)

38. Each of the resistances in the network shown in Fig. 8.48 is R. Calculate the effective resistance between the terminals P and Q. (R)

39. A copper and iron wires, each of equal area of cross section A and lengths l_1 and l_2 are joined end to end. The copper end is maintained at a potential V_1 and the iron end at a lower potential V_2. If σ_1 and σ_2 are the conductivities of Cu and Fe respectively. Find the potential of the junction. $(\sigma_1 V_1/l_1 + \sigma_2 V_2/l_2)/(\sigma_1/l_1 + \sigma_2/l_2)$

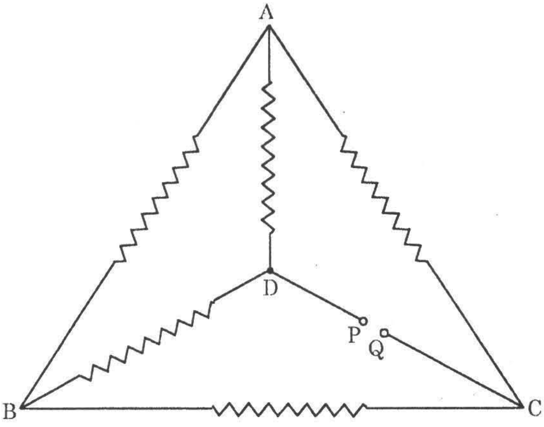


Fig. 8.48. Problem 38.

40. Four identical parallel coaxial rings are equally separated. The axial electric potential gradient is uniform and negative between the first two. It is equal and opposite between the third and fourth and it is zero between the second and third. A positive ion enters at first ring along the axis with a velocity v and emerges at the fourth one. Represent graphically the correct motion of ion.

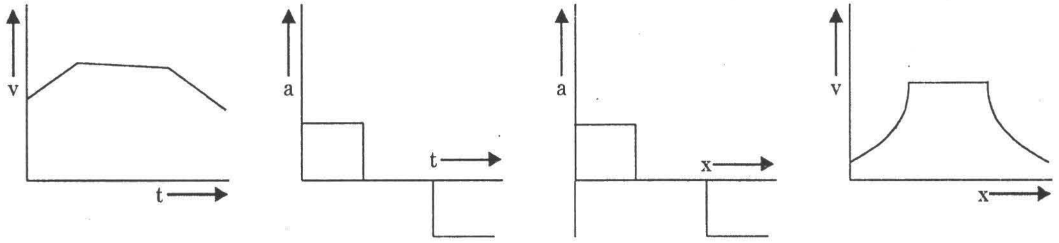


Fig. 8.49. Problem 40.

9 THERMO-ELECTRICITY

9.1 SEEBECK EFFECT

In 1822, Thomson J. Seebeck discovered that an emf could be produced by thermal means alone, in a circuit composed of two different metals with their junctions at different temperatures. This effect is called *Seebeck effect,* after his name. The two metals constitute a thermocouple, the emf produced in a circuit is called thermo emf (***thermo-electric motive force***) or a *Seebeck emf,* and the current in the circuit due to this emf is known as *thermo-electric current.*

In *a copper-iron couple*, the current flows from copper to iron through the hot junction and from the iron to copper through the cold junction. Similarly current flows from bismuth to antimony at the hot junction. A series of metals was given by Seebeck in the order that, when a circuit is made up of any two metals in the series, the current flows across the hot junction from the earlier occurring metal in the series to the one occurring latter. It is called *Seebeck's series,* and is *Bi, Ni, Co, Pd, Pt, U, Cu, Mn, Ti, Hg, Pb, Sn, Cr, Mo, Rh, Ir, Au, Ag, Zn, W, Cd, Fe, As, Sb, Te*. The separation of the metals in this series is the representative of the magnitude of the emf to be expected with a given temperature difference between the junctions. Thus we see that the emf produced between the junction of a *Bi–Sb* couple is greater than that of *Cu – Fe* couple. The metals to the left of *Pb* are called thermo-electrically negative and those to its right are thermoelectrically positive. The amount of emf produced for different thermocouples, taking lead as one of the two metals and cold junction at 0°C, is given below :

Other metal	*Sb*	*Fe*	*Rhodium*	*Cu*	*Pt*	*Ni*	*Constantan*	*Bi*
$\times 10^{-6}$ *V/°C*	*31*	*14*	*2*	*–3*	*–4*	*–18*	*–39*	*–69*

Thus emf for *Sb – Bi* couple = 31 – (–69) = 100 micro volts/°C and for *Cu–Fe* couple = – 3 – (+ 14) = –17 micro volts/°C.

For Cu–Fe couple, it is useful to remember Hot-coffee indicating that current flows from Cu to Fe at the *hot* junction. For Bi–Sb couple the word ABC is useful to remember. It indicates that current flows from antimony (*A*) to bismuth (*B*) at cold junction (*C*).

Let us consider a thermocouple of two metals, *Fe* and *Cu* (say). Their junctions are kept cold and hot. If a galvanometer *G* is connected in the circuit, the galvanometer deflection is observed which corresponds to thermoelectric current. If the temperature of the hot junction be continuously increased, thermoelectric current increases until maximum value is reached. The temperature of the hot junction at which maximum current flows is constant for a given thermocouple. It is independent of the temperature of cold junction and is called the *neutral temperature* for the couple. On further raising the temperature of hot junction the current decreases to zero and is then reversed. The temperature at which the current is zero and its reversal begins is called the *temperature of inversion or inversion temperature.* It depends on the temperature of cold junction and the chosen couple. It is always as much above the neutral temperature as the cold junction is below it, *i.e.*, $T_{\text{inversion}} - t_{\text{neutral}} = t_{\text{neutral}} - t_{\text{cold}}$. The graph between the emf and the temperature difference of the two junctions, as shown in Fig. 9.2, is of the parabolic form for majority of couples. Thus emf can be given as

$$\mathcal{E} = a + bt + ct^2 \qquad \text{...(1)}$$

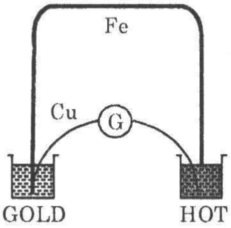


Fig. 9.1. Thermocouple.

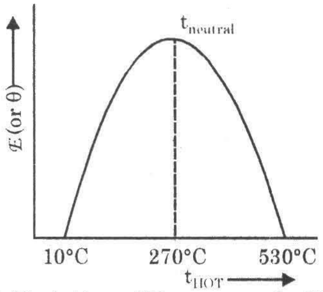


Fig. 9.2. Variation of thermo-emf with temp.

An Atomic View. The production of the emf at junction of two metals can be explained by *Modern theory*, which gives us knowledge of the structure of conducting substances. It suggests that all metals contain large number of free electrons. These electrons behave like the molecules of a gas and are referred often as *electronic gas*. The density of free electrons is different in different metals and depends upon the temperature. When two different metals are connected to form two junctions and these junctions are kept at different temperatures, free electrons will diffuse into one another metal at the junctions. Net flow is from higher electron pressure to the metal of lower electron pressure. If the junctions are at the same temperature the emf at one junction will be equal and opposite to that at the other junction. If these junctions are at different temperatures the electron diffusion takes place at different rates at the junctions and there is a net motion of the electrons. This sets emf between the junctions.

The Seebeck emf is the line integral of the field, which drives the electrons, around the thermocouple. Thus

$$\mathcal{E}_{BA(T_1,T_2)} = \frac{1}{e}\int_{T_1}^{T_2}(S_A{}^* - S_B{}^*)\,dT, \qquad \text{...(2)}$$

where $S_A{}^*$ is the property of the metal A, a function of temperature, a positive quantity and is known as *entropy transport per electron.*

If $S_A{}^*$ is greater than $S_B{}^*$, electrons are transferred more rapidly along metal *A* toward lower temperatures than along metal *B* and the positive current flows from *A* to *B* at the hot junction. The symbol $\mathcal{E}_{BA(T_1,T_2)}$ can be read as *emf from B to A at temperature T_1* less than T_2.

Contact Potential : As shown in Fig. 9.3, consider two metals *A* and *B* possessing different work functions. Let the metal *B* has the smaller work function and hence its free electrons are upto the higher energy level than that those of metal A. If these metals are brought in contact, electrons of higher energy level in metal B will pass over to metal A. Thus the energy level of B will be lowered and that of A will be raised, until both energy levels become equal.

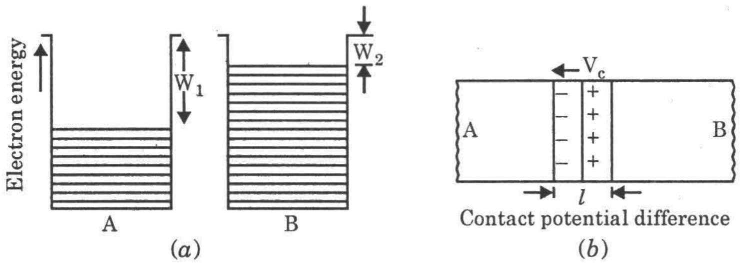


Fig. 9.3 (*a*) Electron energy levels of metals A and B, (*b*) the contact potential difference.

At a result, metal B becomes more positive and metal A more negative at the contact surface. This produces a potential difference V_C between two metals, known as *contact potential* of the metals,

$$V_C = (W_1 - W_2)/e. \qquad ...(3)$$

The electric field is thus produced which tends to prevent further transfer of electrons across the junction.

9.2 PELTIER EFFECT

In 1834, Jean C.A. Peltier discovered that if a current be allowed to pass through the junction of two different metals, the heat is either evolved or absorbed at the junctions, *i.e.*, the junction gets either heated or cooled. This effect is known as *Peltier effect* and the heat evolved or absorbed as *Peltier heat*, after the discoverer. Experiment has shown that the Peltier heat transferred at any junction is proportional to the quantity of electricity crossing the junction and changes its sign (cold junction becomes hot and hot becomes cold) when reversing the direction of current. Thus we see that the Peltier effect is *reversible*. The Peltier effect must not be confused with the Joule's effect. The main differences are :

Peltier's Effect	*Joule's Effect*
1. It is reversible effect.	1. It is an irreversible effect.
2. It takes place only at the junctions.	2. It takes place at all points of the circuit and arises from its resistance.
3. It may be heating or a cooling effect.	3. It is always a heating effect.
4. It is proportional to current only.	4. It is proportional to the square of current and to the resistance.
5. It depends upon the current direction.	5. It is independent of the sign and direction of current.

The Peltier effect may be explained as : metals contain free electrons and can be considered as moving like the gas molecules. Since the number of electrons per unit volume differs in different metals, there will be a flow of electrons from one metal to the other when they are in contact. This flow continues until sufficient charge has been carried at the junction to build up a potential difference of a value high enough to prevent any further movement at the junction. Work must be done for the passage of current. The energy absorbed (or evolved) at the junction when one coulomb of electricity flows is called the *Peltier coefficient*. It is denoted by Π. It is negative when heat is evolved. It varies as the absolute temperature of the junction and also depends on the metals used. Its unit is joules/coulomb, *i.e.*, volts and is given by

$$\Pi_{AB} = (T/e)\,[S_B^*(T) - S_A^*(T)] \qquad ...(4)$$

If the temperatures of the two junctions are the same the Peltier coefficients have the same value. Let the junctions are at temperatures T_2 and T_1, where $T_2 > T_1$. Then the Peltier coefficients under these conditions are $[\Pi_{AB}]_{T_2}$ and $[\Pi_{AB}]_{T_1}$. Let dq amount of charge moves round the circuit in a clockwise direction. Hence in one complete round, energy gained by the charge dq

$$dW = \text{Heat absorbed at the hot junction} - \text{Heat liberated at the cold junction}$$
$$= \{[\Pi_{AB}]_{T_2} - [\Pi_{AB}]_{T_1}\}dq.$$

If the Peltier effect is the only thermoelectric effect in the circuit, then the electromotive force, known as Peltier *emf* $\mathcal{E}$ is given as

$$\mathcal{E} = dW/dq = [\Pi_{AB}]_{T_2} - [\Pi_{AB}]_{T_1}.$$

This Peltier *emf* must not be confused with the constant potential difference between the metals. The latter depends upon the work function ϕ, the work required to remove free electrons from the metal. The distinction is clear on considering an open circuit formed by two metals A and B (Fig. 9.4). The contact potential difference V_{AB} between the metals A and B is defined as the pd between a point P just outside metal A and a point Q just outside metal B. If ϕ_A and ϕ_B be the work functions of metals A and B respectively and let Π_{AB} be the Peltier coefficient at the junction R. The potential difference

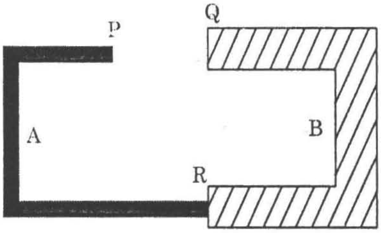


Fig. 9.4. An open circuit.

$$V_{AB} = -\phi_A/e - \Pi_{AB} + \phi_B/e = (\phi_B - \phi_A)/e - \Pi_{AB}$$
$$= (\phi_B - \phi_A)/e \text{ (Approx).} \qquad ...(5)$$

Here the second term is neglected as the *thermo emf* is of the order of a millivolt, while the contact potential is of the order of 1 volt.

In experiments, the Joule heating effect is dominant, hence it is very difficult to measure the Peltier coefficient by direct determining of heating and cooling at the junction. An attempt for direct measurement was made by Le Roux in 1867. A $Cu - Fe$ junction was immersed in a water bath and a current I was allowed to pass for time t in the direction iron to copper. The heat produced H_1 was given as

$$JH_1 = I^2Rt + \Pi It. \qquad ...(6)$$

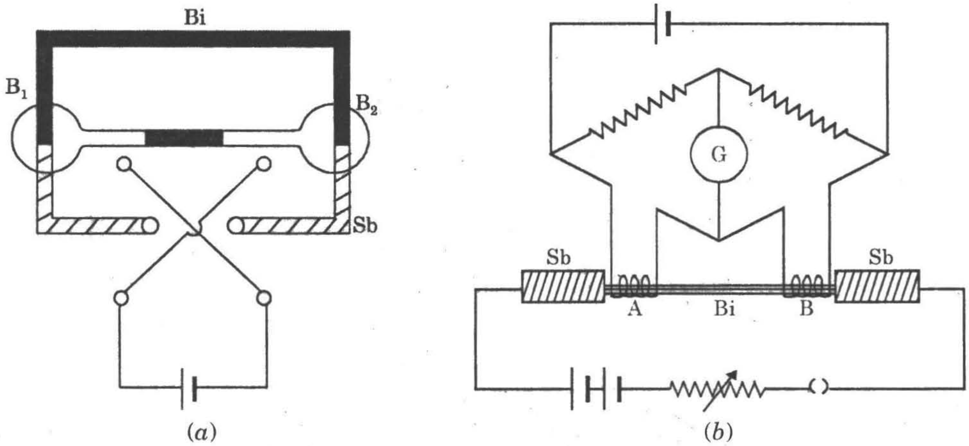
where R was the total resistance of the immersed metals. If the direction of current was reversed, the heat produced H_2 in this case was given as

$$JH_2 = I^2Rt - \Pi It, \qquad ...(7)$$

$$\therefore \quad \Pi = J(H_1 - H_2)/2It. \qquad ...(8)$$

The existence of the Peltier effect is confirmed if H_1 is not equal to H_2. Peltier's original arrangement was as is shown in Fig. 9.5 (*a*). Thick bars of antimony and bismuth were soldered at junctions and the junctions were enclosed in evacuated glass bulbs B_1 and B_2 joined by a thin capillary tube. When the current was allowed to pass from a battery in one direction, mercury thread in the capillary shifted in one direction and when the direction of current was reversed, the thread shifted in the opposite direction. As *Bi* and *Sb* were taken in the form of thick rods, the Joule heating effect was negligible and the shift in thread of mercury was due to Peltier effect only.

Another arrangement for the experimentally demonstration of Peltier effect is shown in Fig. 9.5 (*b*).



(a) (b)

Fig. 9.5. Peltier's Apparatus.

A compound bar is constructed by connecting a bar of bismuth between two indentical bars of antimony (*Sb*) with good contacts. At the junctions, *A* and *B* the two identical coils of fine insulated copper wire are wound. These coils are connected to the two arms of the Wheatstones bridge. The compound rod is placed in a glass envelope to ensure thermal isolation. The bridge is balanced with no current through the compound bar. Now the current is passed through the bar, the bridge becomes unbalanced. This is same as if coil *A* is heated and coil *B* is cooled. If the current in the compound bar is reversed, the direction of deflection in galvanometer of the Wheatstone's bridge is also reversed. This demonstrates the Peltier effect.

Thermodynamics of Peltier effect : Since the Peltier effect is a reversible effect and the heat is absorbed at the hot junction and evolved at the cold junction in a thermocouple. Lord Kelvin suggested that like in a reversible heat engine in thermocouple arrangement the ratio of the heat absorbed at temperature t_1 to that given up at t_2 ($t_2 < t_1$) should be the same as that of T$_1$ to T$_2$, where T$_1$ and T$_2$ are absolute temperatures. In other words *we can write that for all the sources and sinks in a reversible cycle the sum of the quantities dH/T is zero, i.e.,*

$$\oint dH/T = 0. \quad ...(9)$$

This relation is only correct when the irreversible processes occurring in the circuit produce negligible effects. The Joule effect is reduced by using metals in the form of thick bars and by passing very small current in the circuit.

If Π_1 and Π_2 are the Peltier coefficients at temperatures T_1 and T_2 respectively. From the definition of Peltier coefficient, heat absorbed at the hot junction is $\Pi_1 q$ and the heat given out to the cold junction is $\Pi_2 q$, when the charge q moves round the circuit at infinitely small speed. Hence,

$$\frac{\Pi_1 q}{\Pi_2 q} = \frac{T_1}{T_2} \text{ or } \Pi_1 - \Pi_2 = \frac{\Pi_2}{T_2}(T_1 - T_2).$$

The left hand side represents the value of Peltier effect and can be written as $\mathcal{E}$, the Peltier emf. Thus

$$\mathcal{E} = \Pi_1 - \Pi_2 = \frac{\Pi_2}{T_2}(T_1 - T_2). \quad ...(10)$$

If one junction is maintained at constant temperature T_2 then Π_2 is constant and $\mathcal{E} \propto (T_1 - T_2)$. But this is contrary to experience, experiments show that this emf is a parabolic function of the temperature difference. Lord Kelvin (Prof. W. Thomson) concluded in 1851 that the Peltier emf's at the junctions were not the only emfs acting in circuit. He predicted, from theoretical considerations, the existence of reversible heat effects in metals due to the difference of temperature across their ends. This effect is known as *Thomson effect*, after the discoverer.

9.3 THOMSON EFFECT

If the temperature at any two points in a conductor is different, the density of the electron gas must differ, as a result potential gradient is developed along the conductor between these points. This effect was first predicted by Thomson and is known as *Thomson effect*. Thus *Thomson effect is the absorption or evolution of heat due to flow of current in an unequally heated conductor*. In the thermocouple heat is absorbed or evolved at the junctions due to Peltier effect and along one or both the conductors forming this thermocouple due to Thomson effect. Thomson effect is also a reversible effect and is of similar origin to the Peltier effect, as both are due to non-uniform distribution of electrons in metal. Although the former is due to a change of temperature while the latter is due to a change of metal. The emf produced across the conductor is called the *Thomson emf*. It is positive from cold to hot, as the electron density is greater at lower temperatures. The *Thomson emf* between two points on a conductor which differ in temperature by dT may be expressed in the form $\sigma\, dT$, where σ is known as *Thomson coefficient* or the *specific heat of electricity*. σ is a function of temperature (changes from point to point) and also depends upon the material of the conductor. If the ends of a conductor are at temperatures T_1 and T_2 then the total energy absorbed or evolved when unit charge flows through it is the Thomson *emf* and is given by

$$\mathcal{E} = \int_{T_2}^{T_1} d\mathcal{E} = \int_{T_2}^{T_1} \sigma\, dT. \qquad ...(11)$$

If a current is passed along a copper bar AC, heated at the middle B, the heated point B will be at higher potential and thus the heat will be absorbed in the part AB and evolved in the part BC. It can be shown by selecting two equidistant points M and N from point B. The N is found to be at a higher temperature than M because of Thomson effect. Thus we see that in copper *the heat is transferred due to current flow in the direction of current. Hence σ for Cu is positive*. Thomson effect and thus Thomson coefficient σ is positive also for *Cd, Zn, Ag* and *Sb*. In the case of iron, the colder parts are at higher potentials. If the current passes along *ABC* in the wire, heat will be evolved in *AB* and absorbed in *BC*. Among two equidistant points M and N, N will be at lower temperature than M because of the Thomson effect. Thus in iron, *heat is transferred due to current flow in the direction opposite to the direction of current* flow and hence is said to have negative *Thomson effect*. Iron like other metals are *Bi, Co, Pt* and *Ni*. The Thomson effect is zero in lead.

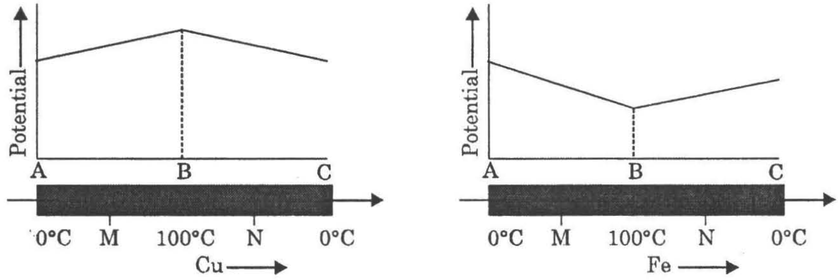


Fig. 9.6. Thomson effect in *Cu* and *Fe*.

The idea of the specific heat of electricity was given by Lord Kelvin as the Thomson phenomenon in *Cu* is analogous to the one which occurs when heat is absorbed by a liquid flowing in a tube, heated at one end. The amount of heat absorbed or evolved by unit mass of liquid in passing from one point to another, having temperature difference dT, is sdT. The amount of heat absorbed or evolved when a unit quantity of electricity passes through the conductor having temperature difference dT between its ends is σdT. Thus we see that σ is analogous to s, the *specific heat*. Metals for which σ is positive have a positive specific heat of electricity and are analogous with a real liquid as regards absorption or evolution of heat. Metals

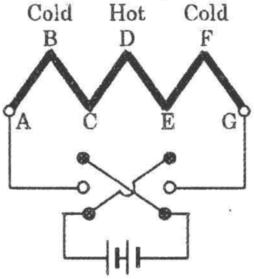


Fig. 9.7. Kelvin's experiment.

having negative specific heat of electricity behave like a hypothetical liquid which absorbs heat on cooling and evolves on heating. Comparing with a liquid, it can be shown that the *Peltier coefficient Π is analogous with a latent heat L of liquid.*

The existence of Thomson effect can also be explained on the electron theory, similar to that of Peltier effect. However, the electron theory can explain the existence of the positive Thomson effect but can not explain the possibilities of zero and negative values of σ.

Demonstrations of the Thomson effect : For this purpose, Lord Kelvin used a thick strip of iron bent in zig-zag manner. The free ends of this strip were connected to a battery through a reversing key. The corners *B* and *F* were kept cold and the corner *D* was kept hot by heating it to a higher temperature. When no current was passing, the corners *C* and *E* were found at equal temperatures on account of thermal conduction. When a current was allowed to pass through this strip, the thermometer at *C* indicated a higher temperature than the point *E*. The thermometer readings were reversed when the direction of current was reversed.

Another demonstration of the Thomson effect is due to S. G. Starling. In this experiment a large thick iron rod is bent into a U-shape, the bent is made red hot and the ends, kept cool in mercury baths *A* and *B*, are connected to a source of heavy current. The resistance coils R_1 and R_2 are wound very close together on the limbs and packed round with asbestos wool. These are placed in gaps of a meter bridge. The balance is first obtained without passing any current through the rod. Now a heavy current is passed through this rod, it causes a disturbance in the balance. This shift in the balancing point indicates the change in resistance of the coils due to change of temperature on account of absorption or evolution of heat along the limbs of the U-shaped iron rod. It thus confirms the existence of Thomson effect.

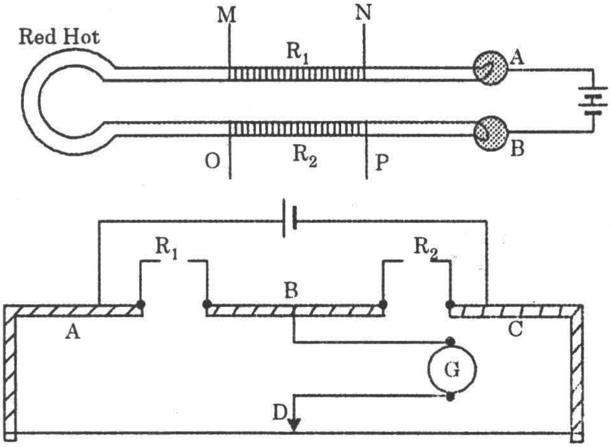


Fig. 9.8. Starling's experiment.

9.4 TOTAL EMF IN A THERMOCOUPLE (Kelvin's Treatment)

Let us consider a thermocouple, composed of conductors *X* and *Y*. Let the current be positive from *X* to *Y* at the hot junction and the temperatures of the hot and cold junctions be T_1 and T_2 respectively. The total emf responsible for the thermocurrent must be the sum of potentials due to Peltier and Thomson effects. Let Π_1 and Π_2 be the Peltier coefficients at the junctions at T_1 and T_2 respectively and σ_X and σ_Y be the Thomson coefficients for the metals *X* and *Y* respectively. When the electric charge flows in the thermocouple, the heat is absorbed and evolved at the junctions due to Peltier effect and all along the conductors due to Thomson effect.

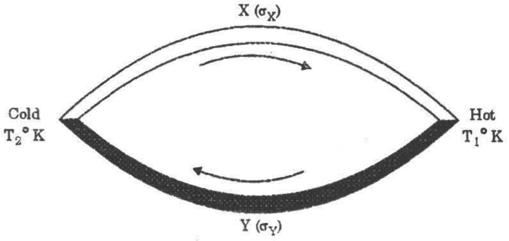


Fig. 9.9. Thermocouple.

The total emf $\mathcal{E}$ round the circuit is the energy gained by unit charge when passing round the circuit.

$$\therefore \mathcal{E} = \text{Sum of } pds. \text{ across various parts of the circuit}$$

$$= \Pi_1 - \Pi_2 + \int_{T_2}^{T_1} \sigma_X \, dT - \int_{T_2}^{T_1} \sigma_Y \, dT = \Pi_1 - \Pi_2 + \int_{T_2}^{T_1} (\sigma_X - \sigma_Y) dT. \quad \text{...(11)}$$

It is equal to the total Seebeck emf available in the circuit.

Thermodynamics of Thermo emf : Let us now compare thermoelectric circuit to the reversible heat engine, as the Peltier and Thomson effects are reversible and Joule heating effect is negligible. Let the two junctions of a thermocouple be at temperatures $T + dT$ and T and the corresponding values of Peltier coefficients are $\Pi + d\Pi$ and Π. From second law of thermodynamics we know that the total change of entropy in traversing one complete cycle of a perfectly reversible engine is zero. Hence we have

$$\frac{\Pi + d\Pi}{T + dT} - \frac{\Pi}{T} + \frac{(\sigma_X - \sigma_Y) dT}{T} = 0, \text{ or } \frac{T d\Pi - \Pi dT}{T(T + dT)} + \frac{(\sigma_X - \sigma_Y) dT}{T} = 0. \quad \text{...(12)}$$

As dT is the small change in T, hence $T(T + dT)$ in the denominator of the first term can be written as T^2. Thus we have

$$\frac{\Pi}{T} = \frac{d\Pi}{dT} + (\sigma_X - \sigma_Y). \quad \text{...(13)}$$

Considering the temperature of cold junction T as constant, the temperature of hot junction is varied. The variation of emf with the temperature of hot junction may be obtained by differentiating relation (12) with respect to T, as

$$\frac{d\mathcal{E}}{dT} = \frac{d\Pi}{dT} + (\sigma_X - \sigma_Y). \quad \text{...(14)}$$

This quantity is called the *Thermoelectric power or relative Seebeck coefficient* for the two metals and is denoted as P_{XY}.

From relations (13) and (14), we get

$$\Pi/T = d\mathcal{E}/dT \text{ or } \Pi = T\,(d\mathcal{E}/dT). \quad \text{...(15)}$$

Thus for any two metals the Peltier coefficient at a given temperature is given by the product of the thermoelectric power $(d\mathcal{E}/dT)$ at that temperature and the absolute temperature of the junction.

Differentiating Eq. (15) w.r. to T, we get

$$\frac{d\Pi}{dT} = T\frac{d^2\mathcal{E}}{dT^2} + \frac{d\mathcal{E}}{dT}. \quad \text{...(16)}$$

Combining this with Eq. (14), we get

$$\sigma_X - \sigma_Y = -T\,(d^2\mathcal{E}/dT^2). \quad \text{...(17)}$$

It gives a relationship for the difference between the two Thomson coefficients and second derivative of the graph of $\mathcal{E}$ against T. Hence by measuring Seebeck *emf* against the temperature, $(\sigma_X - \sigma_Y)$ can be calculated.

σ is zero for lead, positive for *Cu*-like metals and negative for *Fe*-like metals. Hence σ can be obtained by using one metal as lead and other the unknown metal. Thus

$$\sigma_{Cu} = -T\,(d^2\mathcal{E}/dT^2) \text{ and } \sigma_{\text{Fe}} = T\,(d^2\mathcal{E}/dT^2). \quad \text{...(18)}$$

The drawback of Kelvin's theory given above is that it neglects the irreversible flow of heat due to temperature gradient and the irreversible Joule heating.

9.5 ONSAGER'S IRREVERSIBLE COUPLED FLOW

The drawback of Kelvin's theory was removed in the irreversible coupled flow method of Onsager. The coupled flow means the flow of electric charge together with the heat. A temperature difference of ΔT between the two points of a conductor causes a heat current I_h from the higher temperature $(T + \Delta T)$ to the lower temperature T. The hot end loses an entropy $I_h/(T + \Delta T)$ to the conductor per second whereas the cold end gains an entropy I_h/T per second from the conductor. Therefore the rate of entropy production

$$\begin{aligned}\frac{dS}{dt} &= \frac{I_h}{T} - \frac{I_h}{T+\Delta T} \simeq I_h \frac{\Delta T}{T^2} \\ &= I_S \, \Delta T/T, \qquad \text{...(19)}\end{aligned}$$

where $I_S = I_h/T$ = the entropy current.

If an electric current I_E simultaneously flows through the conductor under the action of potential difference $\Delta\mathcal{E}$ between the two points considered then the rate of energy dissipation in the conductor = $I\Delta\mathcal{E}$. Thus an additional entropy $I_E\,\Delta\mathcal{E}/T$ is produced per second due to the flow of electric current. Hence the net rate of production of entropy due to irreversible coupled flow is

$$\frac{dS}{dt} = I_S \frac{\Delta T}{T} + I_E \frac{\Delta\mathcal{E}}{T}. \qquad \text{...(20)}$$

For small values of ΔT and $\Delta\mathcal{E}$, the entropy current and the electric current are linear functions of $\Delta T/T$ and $\Delta\mathcal{E}/T$. Thus we may write

$$I_S = a_{11}\,(\Delta T/T) + a_{12}\,(\Delta\mathcal{E}/T) \qquad \text{...(21)}$$

and

$$I_E = a_{21}\,(\Delta T/T) + a_{22}\,(\Delta\mathcal{E}/T), \qquad \text{...(22)}$$

where a_{11} and a_{22} are the coefficients related to the thermal and electric conductivities. The coefficients a_{12} and a_{21} are the coupling coefficients, the former accounts for the influence of the potential difference on the entropy current while the latter accounts for the influence of the temperature difference on the electric current. Onsager has shown $a_{12} = a_{21}$, the reciprocal relationship. At constant temperature $(\Delta T = 0)$, above equations give

$$I_S/I_E = a_{12}/a_{22} = \tau, \qquad \text{...(23)}$$

where τ is called *entropy transport parameter.*

For $I_E = 0$, equation (22) reduces to

$$\Delta\mathcal{E} = -(a_{21}/a_{22})\Delta T = -(a_{12}/a_{22})\,\Delta T = -\tau\Delta T. \qquad \text{...(24)}$$

For a thermocouple with two metals X and Y, let T_1 and T_2 the temperatures of cold and hot junctions respectively. The integration of equation (24) around the whole circuit gives

$$\mathcal{E} = \Sigma(-\tau\,\Delta T) = \int_{T_1}^{T_2} (\tau_X - \tau_Y)\,dT, \qquad \text{...(25)}$$

where $\mathcal{E}$ the Seeback emf.

When the electric current I_E flows is the circuit, at the hot junction, heat current I_{hX} flows into the junction and I_{hY} flows out of the junction. Thus to maintain the temperature constant, the heat withdrawn per second from the junction is

$$H = (I_{hX})_{\Delta T=0} - (T_{hY})_{\Delta T=0} + I_E^2 R, \qquad \text{...(26)}$$

where R is the resistance of the junction.

$\therefore$ Peltier heat per second $\Pi_2 I_E = H - I_E^2 R = (I_{hX})_{\Delta T=0} - (I_{hY})_{\Delta T=0}$ ...(27)

The heat current is related with the entropy current as :

$$I_{hX} = T_2\, I_{sX} \text{ and } I_{hY} = T_2\, I_{sY},$$

where I_{sX} and I_{sY} are the entropy currents. Using equations (23), we also have

$$I_{sX} = \tau_X I_E \quad \text{and} \quad I_{sY} = \tau_Y I_E.$$

Hence equation (27) reduces to

$$\Pi_2 = T_2 (\tau_X - \tau_Y) \qquad ...(28)$$

In terms of τ, the equation (22) reduces to

$$I_E/a_{22} = \tau\, (\Delta T/T) + \Delta\mathcal{E}/T \qquad ...(29)$$

If $\Delta T = 0$, this equation reduces to ohm's law, as

$$I_E/a_{22} = \Delta\mathcal{E}/T = I_E \;\Delta R/T.$$

ΔR being the resistance or $a_{22} = T/\Delta R$. Therefore equation (29) reduces to

$$I_E\, \Delta R = \tau\, \Delta T + \Delta\mathcal{E}. \qquad ...(30)$$

When an electric field I_E flows, the heat current $(I_h)_{T+\Delta T}$ flowing into the conductor at a point with temperature $T + \Delta T$ is different from that flowing out of a conductor at a point with temperature T, *i.e.*, $(I_h)_T$. Thus the net heat production in the conductor per second

$$H' = I_E\, \Delta\mathcal{E} + (I_h)_{T+\Delta T} - (I_h)_T.$$

At a uniform temperature T, we have $I_h = TI_s = T\tau\, I_E$. If the temperature is not uniform, τ will be different at different points in the conductor. Hence

$$(I_h)_{T+\Delta T} = (T + \Delta T)\left(\tau + \frac{\partial \tau}{\partial T}\Delta T\right) I_E.$$

$$\therefore \qquad H' = I_E\left[\Delta\mathcal{E} + (T + \Delta T)\left(\tau + \frac{\partial \tau}{\partial T}\Delta T\right) - T\tau\right] \simeq I_E\left(\Delta\mathcal{E} + T\frac{\partial \tau}{\partial T}\Delta T + \tau\,\Delta T\right). \qquad ...(31)$$

Substituting the value of $\Delta\mathcal{E}$ from eqn. (30), we get,

$$H' = I_E^2 \Delta R + I_E\, T\frac{\partial \tau}{\partial T}\Delta T. \qquad ...(32)$$

In terms of Thomson coefficient σ, the Thomson heat per second is give by

$$\sigma I_E\, \Delta T = -\left(H' - I_E^2\, \Delta R\right) = -I_E T\frac{\partial \tau}{\partial T}\Delta T$$

$$\therefore \qquad \sigma = -\,T\, \partial\tau/dT. \qquad ...(33)$$

For the thermocouple X, Y, we therefore get

$$\sigma_X - \sigma_Y = -T_2\frac{d}{dT_2}\left(\tau_X - \tau_Y\right). \qquad ...(34)$$

Differentiation of equation (25) with respect to T_2 with T_1 = const., gives

$$d\mathcal{E}/dT_2 = \tau_X - \tau_Y. \qquad ...(35)$$

Therefore equation (28) reduces to

$$\Pi_2 = T_2\, d\mathcal{E}/dT_2. \qquad ...(36)$$

This is same as obtain by Kelvin's thermodynamic treatment, equation (15).

Using equation (35) in equation (34), we get

$$\sigma_X - \sigma_Y = -\,T_2\, d^2\mathcal{E}/dT_2^2. \qquad ...(37)$$

9.6 EXPERIMENTAL LAWS OF THERMOELECTRIC CIRCUITS

There are two simple and experimentally established laws.

(1) Law of successive temperatures (*Law of intermediate temperatures*) : This states that for a given thermocouple, the emf for any specified temperature difference is equal to the sum of the emf's corresponding to any smaller intervals into which the given range of temperature may be subdivided. Thus

$$\mathcal{E}_{T_1}^{T_n} = \mathcal{E}_{T_1}^{T_2} + \mathcal{E}_{T_2}^{T_3} + \mathcal{E}_{T_3}^{T_4} + \ldots \mathcal{E}_{T_{n-1}}^{T_n} \qquad \ldots(38)$$

It can be proved as under : Let us chose any temperature T intermediate between T_1 and T_2. If $\mathcal{E}_1$ and $\mathcal{E}_2$ be the emf's for the temperature intervals $T_1 - T$ and $T - T_2$ respectively, we have

$$\mathcal{E}_1 = \Pi_{T_1} - \Pi_T + \int_T^{T_1} (\sigma_X - \sigma_Y)\, dT \qquad \ldots(39)$$

and

$$\mathcal{E}_2 = \Pi_T - \Pi_{T_2} + \int_{T_2}^{T} (\sigma_X - \sigma_Y)\, dT. \qquad \ldots(40)$$

Adding these

$$\mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 = \Pi_{T_1} - \Pi_{T_2} + \int_{T_2}^{T_1} (\sigma_X - \sigma_Y)\, dT. \qquad \ldots(41)$$

Which is the emf for the temperature difference $T_1 - T_2$, we may therefore divide $T_1 - T_2$ into any number of intervals with the same result.

(2) Law of intermediate metals : This states that the insertion of an additional metal or metals into any thermoelectric circuit does not alter the effective emf of the circuit, provided that the both ends of each such conductor are at the same temperature. For this let us include a third metal Z in the thermocouple consisting of two metals X and Y. The total emf is thus given by

$$\mathcal{E} = [\Pi_{XY}]_{T_1} + \int_{T_1}^{T_2} \sigma_Y\, dT + [\Pi_{YZ}]_{T_2} + \int_{T_2}^{T_2} \sigma_Z\, dT + [\Pi_{ZX}]_{T_2} + \int_{T_2}^{T_1} \sigma_X\, dT.$$

or

$$\mathcal{E} = [\Pi_{XY}]_{T_1} + [\Pi_{YZ} + \Pi_{ZX}]_{T_2} + \int_{T_2}^{T_1} (\sigma_X - \sigma_Y)\, dT. \qquad \ldots(42)$$

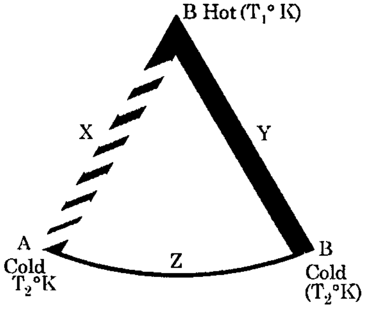


Fig. 9.10. Third metal Z in the thermocouple.

For a closed circuit composed of three metals which are at the same temperature

$$\Pi_{XY} + \Pi_{YZ} + \Pi_{ZX} = 0 \text{ or } \Pi_{YZ} + \Pi_{ZX} = -\Pi_{XY}.$$

$$\therefore \mathcal{E} = [\Pi_{XY}]_{T_1} - [\Pi_{XY}]_{T_2} + \int_{T_2}^{T_1} (\sigma_X - \sigma_Y)\, dT. \qquad \ldots(43)$$

= emf of the circuit composed of two metals X and Y.

This may be generalized to any number of intermediate metals, whose ends are at the same temperature. Thus we see that the thermocouple is unaffected by the introduction of other metals under same conditions. This is the most important property of thermocouple circuits, *as the introduction of measuring instruments, Cu-connecting wires and other accessories required for the measurement-purposes will not effect the emf.*

9.7 THERMOELECTRIC DIAGRAMS

Experiment shows that when the cold junction of a thermocouple is at constant temperature and the temperature of the hot junction is varied, the Seebeck emf is almost a parabolic function of the temperature, *i.e.*,

$$\mathcal{E} = \alpha t + \tfrac{1}{2}\beta t^2, \qquad \ldots(44)$$

where t is the temperature difference between the junctions of a thermocouple in degrees Celsius (°C) and α and β are constants, known as *thermoelectric constants*. These constants are characteristic of the metals forming thermocouple. Such parabolic curves, showing variation of thermo emf with temperature difference, are called *thermo-electric curves.*

The rate of change of $\mathcal{E}$ with the temperature T, $d\mathcal{E}/dT$ is numerically equal to $d\mathcal{E}/dt$ and is given by

$$d\mathcal{E}/dt = \alpha + \beta t. \qquad \ldots(45)$$

This quantity is called the *thermoelectric power* or the relative Seebeck coefficient and is denoted by the symbol P. Thermoelectric power of metal X with respect to metal Y at any temperature T_1 is obtained by the measurement of slope of the thermoelectric curve at temperature T_1. From relation (45) it is clear that the graph between $d\mathcal{E}/dt$ and t will be a straight line, the slope of which gives the constant β and the intercept at $t = 0$ gives the constant α. The values of these constants for some metals and alloys relative to lead are given in Table 9.1.

Table 9.1. Thermoelectric constants

Metal		$\alpha\,(\mu V/°C)$	$\beta\,(\mu V/°C^2)$
Aluminium (*Al*)		–0.76	0.0039
Antimony (*Sb*)		35.58	0.146
Bismuth (commercial) *Bi*		–43.7	–0.465
Cadmium (*Cd*)		3.06	0.029
Constantan (60% *Cu* + 40% *Ni*)		–38.1	–0.089
Copper (*Cu*)		2.76	0.012
Gold (*Au*)		2.90	0.009
Iron (soft) *Fe*		16.65	–0.030
Manganin (84% *Cu*, 12% *Mn*, 4% *Ni*)		1.366	0.0008
Mercury (*Hg*)		–8.81	–0.033
Molybdenum (*Mo*)		5.89	0.043
Nickel (*Ni*)		–19.1	–0.030
Platinum (*Pt*)		–3.04	–0.033
Platinum-iridium	85% *Pt*, 15% *Ir*	14.08	0.0106
	90% *Pt*, 10% *Ir*	13.21	0.0075
Platinum rhodium	85% *Pt*, 15% *Rh*	6.69	0.0107
	90% *Pt*, 10% *Rh*	7.01	0.0064
Silver (*Ag*)		2.50	0.012
Steel		10.8	–0.016
Tungston (*W*)		1.594	0.0341
Zinc (*Zn*)		3.05	–0.01

These values should be regarded as indicative values only, as the effect of the state of purity, method of preparation and previous history of the metals count much. Here $\mathcal{E}_{\text{metal, lead}}$ is assumed positive when current flows at the hot junction from lead to the metal listed.

To find the values of α and β for the thermocouple consisting two metals X and Y, we can make use of relations

$$\alpha_{XY} = \alpha_{X,\text{lead}} - \alpha_{Y,\text{lead}} \text{ and } \beta_{XY} = \beta_{X,\text{lead}} - \beta_{Y,\text{lead}}.$$

As the thermoelectric power is independent of the temperature of the cold junction, hence it is not necessary to specify this temperature. The procedure of finding total emf of a thermocouple from thermoelectric curves is more tedious.

Tait has shown that it is possible to compute the emf produced in a thermocouple by constructing diagrams, known as *Tait diagrams* or *thermoelectric diagrams*. These are the curves between the rate of change of emf with temperature $d\mathcal{E}/dT$ and the temperature of the hot junction T for the thermocouple, consisting of lead and another metal. One metal is taken lead since for this Thomson effect is either zero, or negligibly small. The Tait diagram is a straight or a slightly curved line, as shown in Fig. 9.11.

Let us consider a small strip $abIJ$ of thickness dT at temperature T. If the thickness is assumed very small, the thermoelectric power P can be represented by aI for all the points over ab. Thus the emf $d\mathcal{E}$ between T and $T + dT$ is given by

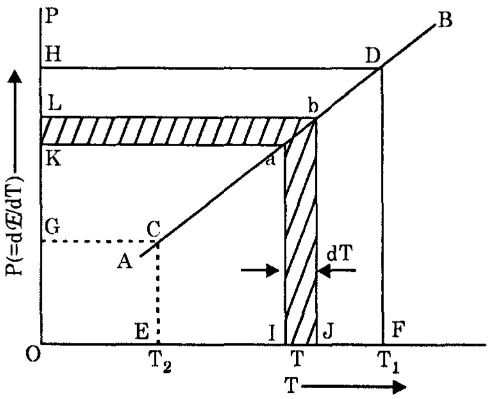


Fig. 9.11. Tait diagram.

$$d\mathcal{E} = (d\mathcal{E}/dT)\,dT = aI \times IJ$$

$$= \text{Area of the strip } abJI.$$

∴ Total emf between temperatures T_2 and T_1 is

$$\mathcal{E} = \int_{T_2}^{T_1} d\mathcal{E} = \int_{T_2}^{T_1} \frac{d\mathcal{E}}{dT}\,dT = \text{Area } CDFE.$$

$$= FE \times \tfrac{1}{2}(CE + DF) = (T_1 - T_2) \times \tfrac{1}{2}(P_2 + P_1)$$

= *Difference of temperatures × Average value of thermoelectric powers at these temperatures.* ...(46)

The Peltier coefficient or the Peltier emf at temperature T_1 is given by

$$\Pi_1 = T_1(d\mathcal{E}/dT)_{T_1} = T_1 P_1 = \text{area } OFDH. \quad ...(47)$$

$$\text{Similarly } \Pi_2 = T_2(d\mathcal{E}/dT)_{T_2} = T_2 P_2 = \text{Area } OECG. \quad ...(48)$$

$$\therefore \quad \Pi_1 - \Pi_2 = \text{Area } OFDH - \text{Area } OECG.$$

We know that $\sigma_{\text{lead}} - \sigma_{\text{metal}} = -T\,(d^2\mathcal{E}/dT^2)$.

$$\therefore \quad \sigma_{\text{metal}} = T\frac{d^2\mathcal{E}}{dT^2} = T\frac{d}{dT}\left(\frac{d\mathcal{E}}{dT}\right) = T\frac{dP}{dT} \qquad (\because \sigma_{\text{lead}} = 0)$$

or $\sigma_{\text{metal}}\,dT = TdP = \text{Area of the strip } abLK$

$$\therefore \quad \int_{T_2}^{T_1} \sigma_{\text{metal}}\,dT = \text{Area } CDHG. \quad ...(49)$$

Hence the total emf is given by

$$\left[\mathcal{E}_{\text{lead, metal}}\right]_{T_2}^{T_1} = \Pi_1 - \Pi_2 - \int_{T_2}^{T_1}(\sigma_{\text{lead}} - \sigma_{\text{metal}})\,dT$$

$$= \text{Area } OFDH - \text{Area } OECG - \text{Area } CDHG$$

$$= \text{Area } CDFE. \quad ...(50)$$

This is the same result as given by Eq. (46). This emf is such that the current flows from lead to metal across the hot junction.

If we require total thermo emf generated in a thermocouple constructed of two metals *a* and *b* (or a special case *Cu* and *Fe*). Two thermoelectric lines are constructed on the Tait diagram, one for *Cu* and other for *Fe*.

$$\left[\mathcal{E}_{\text{Pb, Cu}}\right]_{T_2}^{T_1} = \text{Area } BEFC \quad ...(51)$$

$$\left[\mathcal{E}_{\text{Pb, Fe}}\right]_{T_2}^{T_1} = \text{Area } AEFD \quad ...(52)$$

$$\therefore \quad \mathcal{E}_{\text{Cu, Fe}} = \mathcal{E}_{\text{Cu, Pb}} + \mathcal{E}_{\text{Pb, Fe}} = -\mathcal{E}_{\text{Pb, Cu}} + \mathcal{E}_{\text{Pb, Fe}}$$

$$= \text{Area } ABCD \quad ...(53)$$

The direction of emf is indicated by the anticlockwise direction, *i.e.*, at hot from *Cu* to *Fe* and at cold from *Fe* to *Cu*.

Peltier coefficient at the hot junction for *Cu* – *Fe* is given by

$$\left[\Pi_{\text{Cu, Fe}}\right]_{T_1} =$$

$$\left[\Pi_{\text{Cu, Pb}}\right]_{T_1} + \left[\Pi_{Pb,\text{ Fe}}\right]_{T_1} = \left[\Pi_{Pb,\text{ Fe}}\right]_{T_1} - \left[\Pi_{Pb,\text{ Cu}}\right]_{T_1}$$

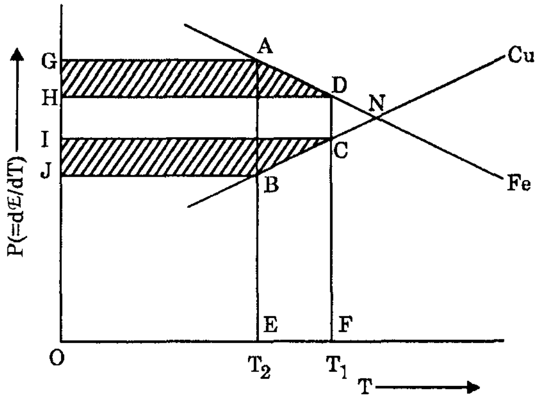


Fig. 9.12. Tait diagram for a thermocouple of two metals.

$$= T_1\left(\frac{d\mathcal{E}}{dT}\right)_{T_1,\,\mathrm{Fe}} - T_1\left(\frac{d\mathcal{E}}{dT}\right)_{T_1,\,\mathrm{Cu}} = T_1\,FD - T_1\,FC$$

$$= T_1\,CD = \text{Area } HICD. \text{ (Energy absorbed)} \qquad \text{...(54)}$$

Similarly $\left[\Pi_{\mathrm{Cu,\,Fe}}\right]_{T_2} = T_2\,AB = \text{Area } GJBA$ (Energy given out) ...(55)

Since $\mathcal{E}_{Cu,\,Fe} = \mathcal{E}_{Pb,\,Fe} - \mathcal{E}_{Pb,\,Cu}$

$\therefore$ $P_{Cu,\,Fe} = d\mathcal{E}/dT = P_{\mathrm{Pb,\,Fe}} - P_{\mathrm{Pb,\,Cu}}$. ...(56)

Thus the thermoelectric power of two metals with respect to each other at a given temperature is simply given by the difference of the respective ordinates for the thermoelectric lines of the two metals, each with respect to the standard metal (lead taken generally). Since $\sigma_{\mathrm{lead}} = 0$, hence

$$\int_{T_2}^{T_1} \sigma_{\mathrm{Cu}}\,dT = \int_{T_2}^{T_1} T\,dP = \text{Area } IJBC \text{ (Energy absorbed)}$$

Similarly $\int_{T_2}^{T_1} \sigma_{\mathrm{Fe}}\,dT = \int_{T_2}^{T_1} T\,dP = \text{Area } GHDA$ (Energy absorbed)

$\therefore$ Total emf $= \text{Area } (HICD + IJBC + GHDA - GJBA)$

$= \text{Area } ABCD.$ ...(57)

The neutral temperature of a thermocouple is the temperature corresponding to the point at which two thermoelectric lines intersect. At this temperature the thermoelectric power ($d\mathcal{E}/dT$) of the two metals is zero, or the emf $\mathcal{E}$ is maximum. Hence Peltier coefficient $\Pi = 0$ and the emf is wholly contributed by the Thomson effect.

For elements, having negative values of Thomson coefficient, thermoelectric lines slope downwards and the area representing Thomson effect is negative and reverse is the case for elements having +ve values of σ. For the thermocouple consisted of two metals having opposite signs of σ, the Thomson emf due to one increases and due to other decreases with the temperature. At the neutral temperature, two emf's are just equal and the total emf in the circuit is thus zero. Beyond this temperature the decreasing factor dominates and the net emf decreases with temperature. If the temperature is increased, the total emf becomes zero at the temperature of inversion.

9.8 MEASUREMENT OF THERMO-EMF

Thermocouples are of great value for the measurement of temperatures upto 1600°C. Copper and constantan thermocouples are satisfactory upto 300°C. Iron-constantan serves upto 800°C. For higher temperatures platinum and some platinum alloy must be used. Since thermo-emf to be measured is small, a potentiometer wire having a small potential drop per unit length along it is required. An ordinary potentiometer is modified for this purpose by connecting it in series with a resistance box. Experimental procedure is given below.

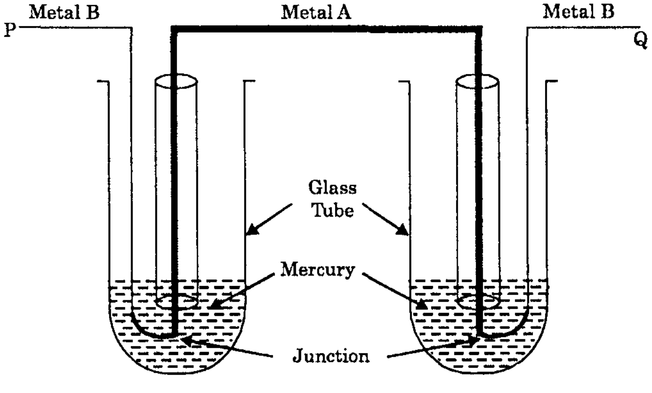


Fig. 9.13. Thermocouple.

The thermocouple is first constructed by passing one end of metal (A) wire down a thin glass tube and joining to a metal (B) wire. To ensure good contact at the junction the combination is placed in a test tube containing a small amount of mercury and the junction is dipped in the mercury. The other end of metal (*A*) wire is allowed to pass through glass tube and is connected with the one end of metal (*B*) wire, as shown in Fig. 9.13. Free ends (*P* and *Q*) of the metal (*B*) wire are connected to the measuring device. One junction is placed in air and other in an oil bath or in a sand bath.

A high resistance *R* is connected in series with the potentiometer wire *AB* and the combination is connected to a battery E_1 in series with a rheostat R_1 and key K_1. The current enters in the combination at *A′* and leaves at *B*. The +ve pole of the standard cell E_2 is connected through a protective resistance R_2 (10 K Ω) to the higher potential terminal *A′*, the –ve pole of this cell is connected through a two way key to the one terminal of the galvanometer, whose second terminal end is connected to the variable terminal *C* of the potentiometer. The higher potential terminal of the thermocouple is connected to the one terminal of the two way key and the low potential terminal of which is connected to the wire through the sliding terminal (jockey) *J*.

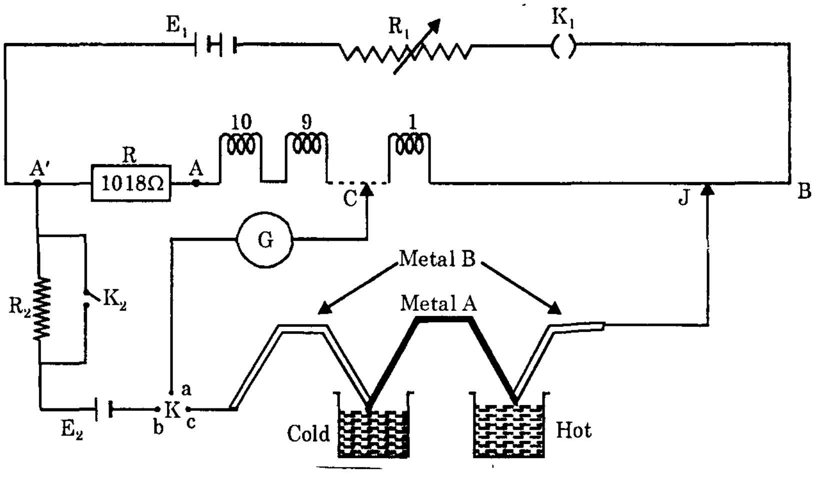


Fig. 9.14. Measurement of thermo emf.

The resistance box *R* is adjusted to about 1000 ohms and the standard cell is connected to the galvanometer by putting plug in *ab* gap of two way key *K*. The variable contact *C* is adjusted to end *A* (or dial reading ten) and the rheostat R_1 is so adjusted that the galvanometer shows no deflection. The final adjustment is made by removing shunt from the galvanometer and by closing key K_2. At this time the *pd* across *R* is balanced by the emf $\mathcal{E}$ of the standard cell and the current in the main circuit is $\mathcal{E}/R$.

The emf across the thermocouple is now balanced by keeping R_1 unchanged by putting plug in the gap *ac* of two way key by adjusting *C* and *J* in such a way that there is no deflection in galvanometer. At this time the thermo emf is equal to the potential drop across the potentiometer wire between *C* and *J*. If ρ is the resistance per unit length of the wire and *l* is the length of the wire between *C* and *J*, then

$$\text{Thermo emf} = \rho l I = \rho l \mathcal{E}/R. \qquad \text{...(58)}$$

In practice standard cadmium cell is used, for this *R* is taken 1018 ohms as $\mathcal{E}$ = 1.018 volts for this cell. The potentiometer wires are generally having value of ρ as 0.01 ohm/cm. If *l* is in cms, the

$$\text{emf} = 10^{-5} \times l \text{ volts.}$$

To investigate the variations of thermo emf with the temperature, keeping the cold junction at constant temperature, the hot junction is heated to different temperatures. A graph is drawn between emf and the temperature of the hot junction. From this curve the neutral temperature, temperature of inversion and the constants α and β of the equation of this curve are obtained.

9.9 APPLICATIONS OF THERMO EMF

Thermo-electric effect is very useful in measuring several physical quantities in addition to temperature.

(*i*) **Thermopile :** One of the best known applications is the detection of small amounts of radiant heat by means of the device known as *thermopile*. To increase the sensitivity many identical thermocouples are connected in series. To reduce the resistance the metals are used in the form of thick bars. Antimony and bismuth bars are used, as these metals are widely separated in the thermoelectric series and a comparatively large emf is produced for a small difference of temperature between the junctions. The free ends of the thermopile are connected to a galvanometer of which the resistance is preferably equal to the resistance of the thermopile. One set of alternate junctions is exposed to the radiation and the other set of the junctions is covered with a brass cap to maintain the temperature of these junctions constant. The galvanometer deflection will be due to thermo-emf developed which is approximately proportional to the radiant energy falling on the thermopile.

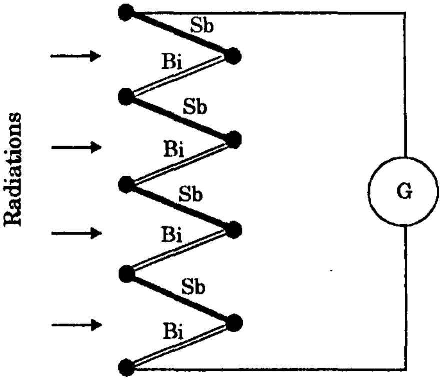


Fig. 9.15. Thermopile.

A relation between the rate at which radiant energy is being received and the thermoelectric current produced, is obtained by calibration. From the rate of reception of energy the rate of emission of energy can be calculated, if the distance and area of the source are known.

(*ii*) **Bolometer :** The principle of the Bolometer is similar, but it consists of a piece of resistance wire forming one arm of a bridge. When it receives energy its temperature rises, consequently its resistance rises and the balance of the bridge is destroyed. The current therefore flows through the galvanometer and a relation between this current and the rate of reception of energy can be found by calibration. The Bolometer is calibrated by passing current of different values in the main circuit and observing change of corresponding resistance of the Bolometer wire. Thus a relation between absorption of energy and change of resistance of the Bolometer wire can be obtained.

(*iii*) **Boy's radio micrometer :** It is more sensitive arrangement, and is the combination of the thermocouple and the galvanometer. A moving coil consists of a single loop of *Cu* suspended between the poles of a strong magnet by a quartz fibre. The ends of this loop are soldered to short rods of antimony and bismuth which are soldered at the other ends to a very thin blackened copper disc, as shown in Fig. 9.16. When the radiation to be measured is directed on to the disc, the junction of antimony and bismuth is heated and the thermocurrent is produced in the loop. As the loop is hanging in a magnetic field hence experiences a couple. This couple deflects the coil and the deflection is measured by the lamp and scale arrangement. This deflection is a measure of thermocurrent and hence of the radiation falling over the disc. Once the system is calibrated with the known values of radiations, the unknown radiation can be measured quite accurately.

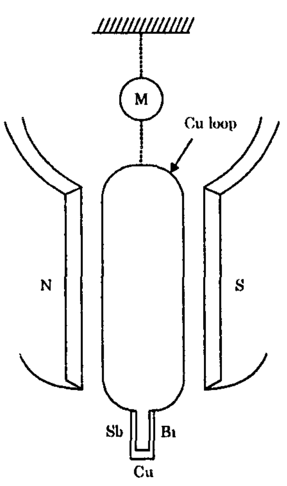


Fig. 9.16. Boy's radio micrometer.

(*iv*) **Duddell thermo-galvanometer :** It is a modification of Boy's radio-micrometer for the measurement of small alternating or varying currents. A loop of silver wire hangs by a quartz fibre in between the poles *N*, *S* of a permanent magnet. The lower ends of the loop are attached to little pieces of bismuth and antimony respectively. These pieces are in contact at the bottom, as shown in Fig. 9.17. Below this junction is situated the heater consisting of a straight filament

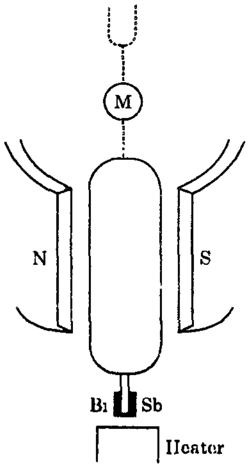


Fig. 9.17. Duddell thermo galvanometer.

about 3 mm long with two straight legs leading to the terminals through which the current to be measured is allowed to pass in the heater. The heater element is taken straight so that its inductance and capacitance can be assumed as negligible and the instrument can be used for high frequency current. When the current passes through the heater, part of the heat developed is radiated to the *Bi–Sb* junction, the result is that thermo-current is set up in the loop and the loop is deflected as it is suspended between the poles *N,S*. The deflection is read by lamp and scale arrangement and is proportional to the square of the current passing through the heater.

(*v*) Thermoelectric pyrometer : We know that some of the thermoelectric lines, for example those of *Cu* and *Ag*, are nearly parallel. If they are actually parallel, the emf round one of these couples would be proportional to the difference of the temperature of the junctions. This fact is made use of in the case of pyrometers constructed for measurement of high temperatures. Couple is usually of pure platinum and an alloy of platinum and iridium, or platinum and rhodium. In the Fery radiation pyrometer, the radiations are received on a concave mirror, which brings them to focus at the point where a sensitive couple is situated. The thermo-emf generated is read by a millivolmeter connected to the free ends of the couple.

(*vi*) Thermomilliammeter : Sir J.A. Fleming devised a sensitive form of ammeter for the measurement of small alternating or continuous currents in which the heating produced by the current passing through a fine constantan wire *AB* warms the junction *C* of a tellurium-bismuth couple. The wire *AB* and the couple are in vacuum. A sensitive galvanometer *G* is connected in series with the thermocouple. The deflection in the galvanometer is proportional to the current passing in wire *AB*.

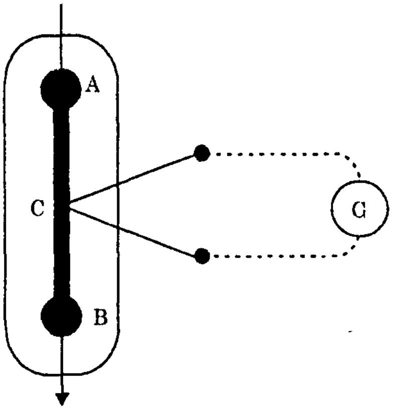


Fig. 9.18. Thermomilliammeter.

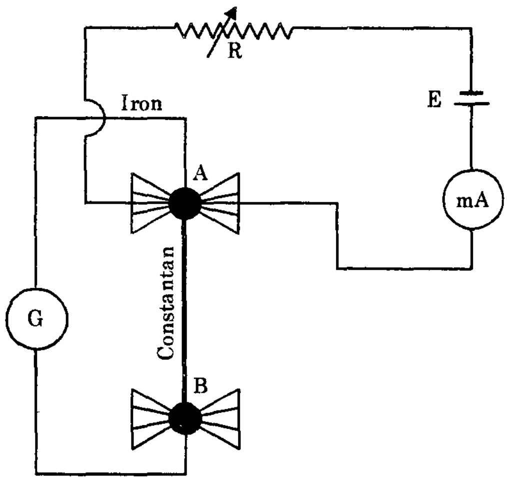
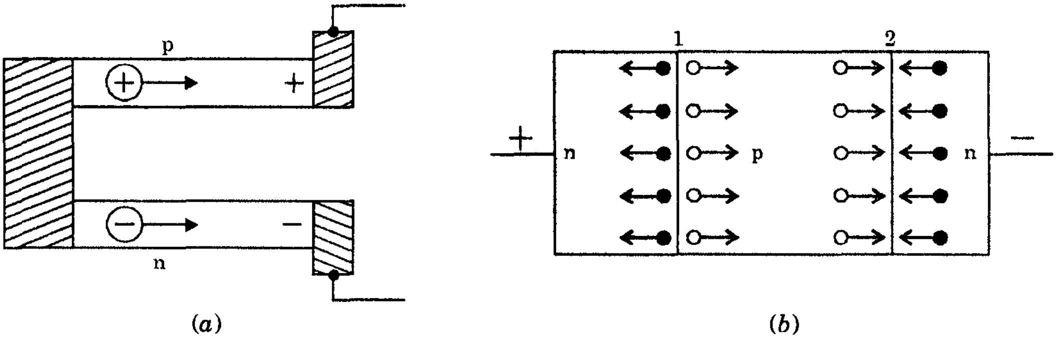


Fig. 9.19. Ratio balance.

(*vii*) Ratio-balance : The idea that the absorption of heat at a thermal junction, when the direction of the current is the same as that of the Peltier emf has been employed by Prof. Callendar in 1910 for the measurement of the radiant heat. Two similar discs *A* and *B* each supported by four stout iron and four constantan wires, are the junctions of an iron constantan couple. The discs also form the junction of the single wire iron and constantan circuit in which the galvanometer *G* is included. Another circuit contains a battery, an adjustable resistance *R* and a milliammeter mA, as shown in Fig. 9.19. When radiation falls on the disc (say *A*), a thermo current flows and the galvanometer is deflected. Resistance *R* is adjusted until the two discs remain at the same temperature as indicated by the zero deflection in galvanometer. At this time the radiant energy absorbed will be equal to Peltier cooling. Knowing the current i and Peltier coefficient Π, the rate of absorption of radiant energy is calculated. The radiation may also be allowed to fall on the disc B instead of A, for the purpose of making a control measurement. In the later design the discs are replaced by small cups.

***(viii)* Semiconductor Thermo-electric Generator** : Semiconductors differ from metals in that the concentration of free electrons and holes increases sharply with increasing temperature. Thus much higher thermo-emf and higher efficiency can be developed between the junctions, than those developed in thermocouples of dissimilar metals. A semiconductor thermo-electric generator works on the following principle :

A semiconductor thermo-electric generator consists of n-type and p-type semiconductor rods joined by a strip of metal. When the metal strip is heated, it acts as a hot junction. As a result, the charge carriers begin to diffuse from the hot junction to the cold ends of the semiconductors. The holes in the p-type semiconductor flow from the hot junction and thus its cold end becomes positively charged. The electrons in the n-type semiconductor flow from the hot junction and thus the cold end becomes negatively charged. Therefore a potential difference is developed across the cold ends of p- and n-type semiconductors, which will oppose further diffusion of carriers. Eventually a state of equilibrium is reached at which force due to electrostatic field is balanced by the force responsible for diffusion (*i.e.*, non-electrostatic field). The potential difference at the equilibrium stage is called the semiconductor thermo-emf. A higher voltage can be obtained by pilling up such semiconductors. This is called semiconductor thermoelectric generator. Such generators are employed as power supplies for radio transmitters in rockets and satellites.



(*a*) (*b*)

Fig. 9.20. Semiconductor thermoelectric generator.

A strong Peltier effect is found at the boundary/junction between a p-type and n-type semiconductors and is applied in *thermoelectric coolers*. When a direct current is allowed to pass through a slab consisting of n, p and n type of semiconductors with two junctions n-p and p-n, one junction is heated and other is cooled. The electrons in the n-side and the holes in the p-side move away from junction 1, new electron-hole pairs are thermally generated. The band gap energy absorbed in this process is supplied by the lattice. Thus the temperature of the junction 1 decreases. On the other hand, there is a recombination of electron-hole pairs at the junction 2. The energy equal to the band gap energy is transformed into heat and released at the junction 2 and thus increases the temperature. The temperature difference between the junctions 1 and 2 is considerable when the thermal conductivity of the semiconductor is low. The semiconductor thermoelectric coolers are widely employed in thermostat, biological research and vacuum techniques.

A plasma thermocouple, consisting of a metal or an alloy in an evacuated container carrying liquid cerium, has also used in industries. The partly evaporation and dissociation of cerium form a plasma of charged particles. The thermocurrent from a plasma thermocouple is much greater than that from the usually used thermocouples.

EXERCISES

Example 1 : *Find the emf of a Cu-Fe thermocouple when temperature of hot is 100°C and that of cold is 0°C.*

We know that

$$\mathcal{E}_{Cu,\,Fe} = \alpha t + \tfrac{1}{2}\beta t^2.$$

By the law of intermediate metals, we have

$$\begin{aligned}\mathcal{E}_{Cu,Fe} &= \mathcal{E}_{Cu,Pb} + \mathcal{E}_{Pb,Fe} = \mathcal{E}_{Cu,Pb} - \mathcal{E}_{Fe,Pb}\\ &= \left(\alpha_{Cu}\,t + \tfrac{1}{2}\beta_{Cu}\,t^2\right) - \left(a_{Fe}\,t + \tfrac{1}{2}\beta_{Fe}\,t^2\right) = (\alpha_{Cu} - \alpha_{Fe})t + \tfrac{1}{2}(\beta_{Cu} - \beta_{Fe})t^2\\ &= (2.76 - 16.65)100\ \mu V + \tfrac{1}{2}(0.012 + 0.030)100^2\ \mu V = -\,1.179\ mV,\end{aligned}$$

where the minus sign indicates that the direction of the current is from iron to copper at the hot junction.

Example 2 : *The emf in a thermocouple, one junction of which is kept at 0°C is given by* $\mathcal{E} = at + \tfrac{1}{2}\beta t^2$. *Find the neutral temperature and the Peltier and Thomson coefficients.*

In absolute scale of temperature, the relation can be written as

$$\begin{aligned}\mathcal{E} &= \alpha(T - 273) + \tfrac{1}{2}\beta(T - 273)^2\\ \therefore\qquad d\mathcal{E}/dT &= \alpha + \beta(T - 273) \text{ and } d^2\,\mathcal{E}/dT^2 = \beta\end{aligned}$$

At the neutral temperature $d\mathcal{E}/dT = 0$

or

$$\begin{aligned}(T_n - 273) &= t_n = -\,\alpha/\beta.\\ \text{Peltier coefficient } \Pi &= T(d\mathcal{E}/dT) = T[\alpha + \beta(T - 273)] = T\,(\alpha + \beta t).\\ \text{Thomson coefficient } \sigma &= T(d^2\mathcal{E}/dT^2) = \beta T.\end{aligned}$$

Example 3 : *The thermoelectric power* $d\mathcal{E}/dT$ *against lead for iron at a temperature t°C is given by 1734 – 4.87 t μV/°C and that for copper by 136 + 0.95 t μV/°C. Find the emf of a Cu-Fe couple with junctions at 0°C and 100°C.*

From the law of intermediate metals, for three metals *Cu*, *Fe* and lead, we have the thermo emf for the group of two metals.

$$\mathcal{E}_{Cu,\,Fe} = \mathcal{E}_{Cu,\,Pb} + \mathcal{E}_{Pb,\,Fe} = \mathcal{E}_{Cu,\,Pb} - \mathcal{E}_{Fe,\,Pb}$$

Therefore the differentiation with respect to *T* gives

$$\begin{aligned}\frac{d\mathcal{E}_{Cu,Fe}}{dT} &= \frac{d\mathcal{E}_{Cu,Pb}}{dT} - \frac{d\mathcal{E}_{Fe,Pb}}{dT} = (136 + 0.95\,t) - (1734 - 4.87\,t)\\ &= (-1598 + 5.82\,t)\ \mu V/{}^\circ C.\\ \therefore\qquad \left[\mathcal{E}_{Cu,Fe}\right]_{0^\circ}^{100^\circ} &= \left[-1598t + \tfrac{1}{2} \times 5.82t^2\right]_{0^\circ}^{100^\circ}\\ &= -\,159800 + 2.91 \times 100^2 = -0.1307\ V.\end{aligned}$$

Example 4 : *A certain thermocouple, with one junction at 0°C has an emf given by* $\mathcal{E} = [80\ t - 15000\ \{1 - \exp\,(-\,0.0025t)\}]\ \mu V$. *Determine the Peltier coefficient at 0°C and show that the difference between the Thomson coefficients is practically stationary in a small temperature range in the neighbourhood of 127°C.*

Peltier coefficient at 0°C (=273°K) $= T(d\mathcal{E}/dT)_{273}$

$$= 273[80 - 15000 \times 0.0025 \exp\,(0)]\ \mu V = 11.6\ mV.$$

Difference between the Thomson coefficients

$$= \sigma_X - \sigma_Y = -\,T(d^2\mathcal{E}/dT^2) = -(273 + t)\,(d^2\mathcal{E}/dt^2)$$

But $d^2\mathcal{E}/dT^2 = 15000 \times (0.0025)^2 \exp\,(0.0025t) = 93.75 \times 10^{-3} \exp(-0.0025t)$,

$$\therefore\qquad \sigma_X - \sigma_Y = -\,(273 + t)\,[93.75 \times 10^{-3} \exp\,(-0.0025t)].$$

For the stationary value of $\sigma_X - \sigma_Y$, its first derivative must be zero.

$$\frac{d}{dt}[\sigma_X - \sigma_Y] = 0 = -[93.75 \times 10^{-3} \exp(-0.0025t)] + (273 + t)[93.75 \times 10^{-3} (0.0025) \exp(-0.0025t)]$$

$$\therefore \quad 1 - (273 + t)(0.0025) = 0 \text{ or } t = 127°C.$$

Example 5 : *For a certain thermocouple, $\mathcal{E} = a_1 \theta + b_1 \theta^2$, where θ°C is the temperature of the hot junction, the cold junction being at 0°C. Given $a_1 = 20\ \mu V/°C$, $b_1 = -1/40\ \mu V/°C^2$. (i) Calculate the temperature at which $\mathcal{E} = 0$. (ii) Find the temperature at which the Peltier coefficient is a maximum and hence obtain the difference of the Thomson coefficients for the pair of metals of the thermocouple.*

For the given thermocouple $\mathcal{E} = a_1\theta + b_1\theta^2$, with $a_1 = 20\ \mu V/°C$ and $b_1 = -1/40\ \mu V/°C^2$.

$$\therefore \quad \mathcal{E} = 0 = (20\ \mu V)\,\theta - (1/40\ \mu V)\,\theta^2 \text{ or } \theta = 0°C \text{ and } 800°C.$$

Peltier coefficient $= T(d\mathcal{E}/dT) = T\,(a_1 + 2b_1\,\theta)$ will be maximum if its first derivative is zero, *i.e.*,

$$a_1 + 2b_1\,\theta + T(2b_1) = 0 \text{ or } a_1 + 2b_1(\theta + 273 + \theta) = 0$$

$$\text{or} \quad \theta = \tfrac{1}{2}[-a_1/2b_1 - 273] = \tfrac{1}{2}[20 \times 20 - 273] = 63.5°C.$$

The difference of Thomson coefficients $= \sigma_X - \sigma_Y = -T\,(d^2\mathcal{E}/dT^2)$

$$\text{Since } d^2\mathcal{E}/dT^2 = d^2\mathcal{E}/d\theta^2 = 2b_1$$

$$\therefore \quad \sigma_X - \sigma_Y = -(273 + \theta)(2b_1) = -(273 + 63.5)(-1/20)$$

$$= 18.825\ \mu V/°C.$$

Example 6 : *Calculate the Peltier and Thomson coefficients, the neutral temperature, the temperature of inversion and the total emf for a Cu – Fe thermocouple whose junctions are at 0°C and 100°C.*

By the law of intermediate metals, we have

$$\alpha_{Cu,\,Fe} = \alpha_{Cu,\,Pb} - \alpha_{Fe,\,Pb} = 2.76 - 16.65 = -13.89\ \mu V/°C$$

$$\beta_{Cu,\,Fe} = \beta_{Cu,\,Pb} - \beta_{Fe,\,Pb} = 0.012 + 0.030 = 0.042\ \mu V/°C^2$$

$$\text{Neutral temperature } t_n = -\alpha/\beta = 330.7°C.$$

$$\therefore \quad \text{Temperature of inversion} = 2(t_n - t_{cold}) = 661.4°C.$$

$$\text{Total emf } \mathcal{E} = \alpha t + \tfrac{1}{2}\beta t^2 = -13.89 \times 100 + \tfrac{1}{2} \times 0.042 \times 100^2 = -1179\ \mu V$$

$$\text{Thermoelectric power } P = d\mathcal{E}/dT = d\mathcal{E}/dt = \alpha + \beta t.$$

$$\therefore \quad \text{Peltier coefficient } \Pi = T(d\mathcal{E}/dT) = T\,(\alpha + \beta t)$$

$$\text{at } 100°C\ \Pi_{373} = 373\,(-13.89 + 0.042 \times 100) = -3614.37\ \mu V$$

$$\text{at } 0°C\ \Pi_{273} = 273\,(-13.89) = -3791.97\ \mu V.$$

Difference in Thomson coefficients $\sigma_{Cu} - \sigma_{Fe} = -T d^2\mathcal{E}/dT^2$

$$\therefore \text{Total Thomson emf} = \int_{T_1}^{T_2} (\sigma_{Cu} - \sigma_{Fe})\,dT = \int_{T_1}^{T_2} \left(-T\frac{d^2\mathcal{E}}{dT^2}\right) dT = \int_{T_1}^{T_2} -T\beta\,dT = -\frac{\beta}{2}\left[T_2^2 - T_1^2\right]$$

$$= -\tfrac{1}{2} \times 0.042[373^2 - 273^2] = -1356.6\ \mu V.$$

$$\text{Hence the total emf} = -3614.37 - (-3791.97) - 1356.6 = -1179\ \mu V.$$

which is the same as calculated by direct method.

Example 7 : *A thermocouple is constructed of gold and iron whose thermoelectric powers are 2.90 + 0.02 t and 16.65 – 0.059 t micro volts per °C respectively. Calculate the neutral temperature and the maximum emf obtainable with this couple.*

The thermoelectric power $P = d\mathcal{E}/dT$

∴The thermoelectric power of gold-iron thermocouple

$$P = \frac{d\mathcal{E}}{dT} = \frac{d}{dT}[\mathcal{E}_1 - \mathcal{E}_2] = \left(\frac{d\mathcal{E}}{dT}\right)_{\text{gold}} - \left(\frac{d\mathcal{E}}{dT}\right)_{\text{iron}}$$

$$= (2.90 + 0.02t) - (16.65 - 0.059\,t) = -13.75 + 0.079t$$

At neutral temperature $P = 0$, therefore

$$0 = -13.75 + 0.079t \text{ or } t = t_{\text{neutral}} = 13.75/0.079 = 174.05°C$$

We have $P = d\mathcal{E}/dT$ and hence $d\mathcal{E} = PdT$, therefore the maximum emf is given by

$$\mathcal{E}_{\max} = \int_0^{t_n} PdT = \int_0^{t_n}(-13.75 + 0.079t)dT$$

$$= (-13.75t + 0.0395t^2)_0^{174.05} = -1196.6\,\mu V$$

$$\simeq 1.2\ mV.$$

Example 8 : *Find the relation for Peltier coefficient in terms of electron density, using classical electron theory.*

Consider two metals X and Y in contact. The junction is maintained at temperature T. The electron densities are n_1 and n_2 in the metals X and Y respectively. The higher electron pressure in X will tend to drive electrons from X to Y. These electrons will set up an electric field and an equilibrium will be achieved when the electric field pressure is equal to the diffusion pressure of electrons, which corresponds to a *pd* between X and Y or the Peltier emf.

The Peltier coefficients Π, defined as the work required to transport unit quantity of charge across the junction, will be given by

$$\Pi = \int_{V_1}^{V_2} PdV,$$

where V_1 is the volume occupied by the electrons in unit-charge in metal X and V_2 in metal Y.

Assuming the electrons as the molecules of a perfect gas,

$$P = \tfrac{1}{3}mnC^2 = \tfrac{2}{3}nkT.$$

If volume V occupied unit charge, then the number of electrons in a unit volume

$$n = 1/eV$$

and hence

$$P = 2\,kT/3\,eV.$$

∴

$$\Pi = \int_{V_1}^{V_2} \frac{2\,kT}{3\,eV}dV = \frac{2\,kT}{3\,e}\int_{V_1}^{V_2}\frac{dV}{V}$$

$$= \frac{2\,kT}{3\,e}\log_e\left(\frac{V_2}{V_1}\right) = \frac{2\,kT}{3\,e}\log_e\frac{n_1}{n_2}.$$

ORAL QUESTIONS

1. What will be the direction of current at the hot junction of *Ni–Fe* thermocouple ? Whether the emf produced between the junctions is greater or smaller than that in *Cu–Fe* thermocouple ?
2. The neutral temperature of a thermocouple depends on materials forming the couple only. The inversion temperature depends on which factors ?
3. Distinguish the effects – (*i*) Peltier's effect, (*ii*) Joule's effect, and (*iii*) Thomson's effect.
4. What do you mean by +ve and –ve Thomson's effects ?
5. How can you use *Cu* – leads for connecting galvanometer in the thermocouple circuit of materials other than *Cu* ?
6. What do you conclude from two lines constructed on the Tait diagram ?
7. Why do you use a resistance of 1018 ohms in series of a potentiometer ?
8. What do you expect if no balance is obtained while balancing the thermo-emf across the potentiometer ?
9. What is a contact potential ?

10. The emf developed by a thermocouple is measured with the help of a potentiometer and not by a moving coil millivoltmeter because (*a*) the potentiometer is more sensitive than millivoltmeter, (*b*) measurement using a potentiometer is made without any draw of current from the thermocouple, (*c*) the potentiometer makes measurement simple.
11. Compare the thermocouple with a reversible heat engine.
12. A *Cu–Fe* thermocouple is not used to measure temperature above 250°*C*. Why ?
13. Thermopile is the device (*a*) used for the detection of small amount of radiant heat, (*b*) consists of large number of thermocouples connected in series, (*c*) connected in parallel.
14. Thermo-milliammeter is used for the measurement of small (*a*) *dc*, (*b*) *ac*, (*c*) both of these.
15. Thermo-electric generator consists of (*a*) two-dissimilar metals, (*b*) dynamo, (*c*) *n*- and *p*-type semiconductor rods joined by a strip of metal.
16. Which materials are used in a plasma thermocouple ?

ANSWERS : 10(*b*); 13(*a, b*); 14(*c*); 15(*c*); 16 cerium.

PROBLEMS

1. Compute for a *Cu – Ni* thermocouple at temperature 100°*C* (*a*) thermal emf, (*b*) $d\mathcal{E}/dt$, (*c*) the Peltier emf, (*d*) Peltier heat transferred at the hot junction if a current of 1 *mV* is maintained for 1 hr ? [–1.42 *mV*, –12. 1 μ*V*/°*C*, –4.51 *mV*, -3.88×10^{-3} *cal.*]
2. Calculate Π_{ab} and $(\sigma_a - \sigma_b)$ for a bismuth – silver thermocouple at 0°*C*. [2.12×10^{-2} *V*, 6.6 μ*V*/°*C*]
3. If θ is the temperature of the hot junction of a thermocouple and the cold junction is at 0°*C*, then its emf $\mathcal{E} = \alpha\theta + \frac{1}{2}\beta\theta^2$.

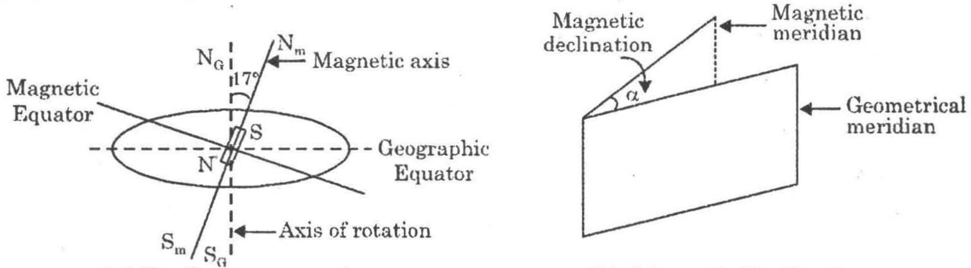
   Discuss whether or not the emf of such a thermocouple may be represented by $\mathcal{E} = p + q\,\theta + r\theta^2$, where *p, q* and *r* are constants and the cold junction is not at 0°*C*.
4. Two semicircular rings, one of *Cu* and one of constantan are joined together to form a ring whose mean diameter may be taken as 14 cm. Each metal has a cross-sectional area of 2 cm^2. Calculate the thermoelectric current through this ring if the junctions are at 0°*C* and 200°*C*. (17.4 *amp*)
5. Calculate the emf of a *Ag – Fe* thermocouple with junctions at 20°C and 80°C. The thermo-electric powers of *Ag* and *Fe* are 3.34 + 0.008 *t* and 16.65 – 0.03 *t* respectively. (685 μ*V*)
6. For a chromel-alumel thermocouple, when cold junction is at 0°C, the emf is 18.37 μ*V* and 44.24 μ*V* when the hot junction is at 444.6°C and 1083°C respectively. Calculate (*a*) Themo emf, (*b*) the Peltier coefficient, (*c*) the thermo electric power and (*d*) the Thomson coefficient at 600°C. (24.71 μ*V*, 0.04076 μ*V*/ °C, $\sigma_X - \sigma_Y = 6.43 \times 10^{-4}$ μ*V*/°C)
7. The emf for a *Cu – Ni* thermocouple is given by the equation $\mathcal{E}$(in μ*V*) = $16.34\,t - 0.021\,t^2$, where *t* is in °C. Calculate the thermoelectric power and the Peltier coefficient at 100°C. (12.14 μ*V*/°C, 4.528 m*V*)
8. A thermoelectric circuit consists of two bars of *Cu* and *Fe* with the ends joined and the junctions are maintained at 19° and 20°C respectively. Compare the heat dissipated due to Joule effect with that conducted along the bars. Given the thermo-electric power at 20°*C* as 14 μ*V*/°*C*, the resistivities of *Cu* and *Fe* as 1.6 and 9.8 μ*V*/cm and the thermal conductivities as 0.9 and 0.2 *cal* cm^{-1} sec^{-1} respectively. (7.4×10^{-6})
9. In a thermocouple constructed from lead and a semiconductor, the thermoelectric power is given by $d\mathcal{E}/dt = a + b/(t + 273)$, where *t* is the temperature of one junction, the other being kept at 0°C. Calculate rate of absorption of energy in the semiconductor when a current *I* flows throgh it and the temperature difference between its ends is 1°C. [$bI/(276 + t)$]
10. The thermoelectric power of iron is 17.5 μ*V*/°C at 0°C and zero at 360°C, that of copper is 5 μ*V*/°C at 450°C and zero at –50°C. Draw the thermoelectric power lines for these metals and calculate the end of the *Cu – Fe* thermocouple with junctions at 0°C and the neutral temperature. (–2.465 m*V*)

10

MAGNETOSTATICS

10.1 MAGNETISM

Few substances have the power of attracting iron, but not materials such as wood, amber, etc. Such bodies are called *magnets* or the *magnetized* substances. The attracting force is called *magnetic force.* The first magnetic phenomena to be observed were those associated with so called, *natural magnets,* the fragments of an ore of iron known as *lodestone.* The word magnetism comes from the district of *Magnesia in Asia Minor,* which is one of the places at which the lodestones were found. If a lodestone is put in iron filings, the filings are found to cluster more thickly about two regions on the stone than anywhere else. These regions are named as "*poles*". When it is suspended freely, it stays along north-south line, the one end pointing north is called the *north pole* of the lodestone and the other the *south pole*. Similar poles repel and dissimilar poles attract each other. The earth itself is a great magnet having south polarity somewhere near its geographical north pole and north polarity somewhere near its geographical south. The locations of the geographic north (N_G) and south (S_G) poles (which lie on the earth's *rotation axis*) do not coincide with the magnetic north (N_m) and south (S_m) poles. The line joining magnetic poles is known as *magnetic axis.* The angle between the rotation axis and magnetic axis is about 17°. Thus the geographic and magnetic equators are not identical.



(*a*) Earth as a magnet (*b*) Magnetic Declination.

Fig. 10.1.

The *magnetic meridian* at any place is a vertical plane containing the magnetic axis of a freely suspended magnet at rest in the earth's magnetic field. The *geographical meridian* at a place is a plane containing the place and the earth's axis of rotation. The angle between these meridians is called the magnetic declination (α).

The magnetic field is not parallel to the earth's surface except at the magnetic equator. At other locations, a magnetic needle suspended freely at its centre of gravity to move in a horizontal and vertical plane would point down as well as toward the north in the northern hemisphere. *The angle the needle makes with the horizontal plane is called the dip angle or magnetic inclination.* The needle sets vertically at the poles and horizontally at the magnetic equator. The N or S pole of the needle dips according as the needle is north or south of the magnetic equator.

The magnetic needle points the resultant magnetic field. If I is the total intensity (flux density) of the earth's magnetic field and θ is the angle of dip, then

$$\tan\theta = \frac{\text{Vertical component } V}{\text{Horizontal component } H}. \qquad ...(1)$$

Now-a-days electricity and magnetism are related by *electromagnetism*. The laws of attraction or repulsion are quite similar. There are certain fundamental differences and certain resemblances between electric charges and magnetic poles (or charged bodies and magnets).

1. A magnet attracts pieces of magnetic material only; *a charged body attracts any light object.*
2. A freely suspended magnet orients itself in a particular direction (geographical north–south); *a charged body does not.*
3. A magnet has polarity, northward force on the north pole is just balanced by the southward force on the south pole; *a body charged inductively has polarity + ve and – ve, while a body with a free charge has not.*
4. If a magnet is cut in two, we find two complete magnets each of which has equal north and south poles. Thus it is impossible to isolate north and south poles or different poles, *it can be done in the case of electric charges of opposite signs.*
5. A magnet magnetizes a piece of magnetic material when placed in its neighbourhood, the induced magnetism disappears (for soft iron) when the magnet is removed; *a charged body has exactly corresponding properties, but will induce a charge on any insulated conductor instead of an particular ones only.*
6. There is no effect on a magnet when touched with the hand; *the charged body is discharged.*
7. If we bring two magnets together, nothing happens except that they exert a force on each other; *if two oppositely charged bodies are brought together a spark will pass between them.*
8. No substance is known in which magnetic poles move under the influence of magnetic force with the *freedom that electric charges move in a conductor when subject to electric forces.*
9. No force is exerted by an electric charge on a magnetic pole when both are at rest relative to the observer.
10. The magnetic field lines have neither a beginning nor an end. The field lines of a bar magnet or electromagnet do not begin at the N-pole and end at the S-pole, but extend through the interior of the bar or core with termination. *The electric field lines begin at the positive charge and terminate at the negative charge.* Thus the magnetic monopole does not exist.

10.2 MAGNETIC FIELD

Upto 1820, no connection was known between electrical and magnetic phenomena. In 1820, Danish Scientist Hans Christian Oersted observed the magnetic effects of current and thus showed a connection between electricity and magnetism. Twelve years later, Faraday found that a momentary current existed in a closed circuit while the current in a nearby circuit was being started or stopped. The same effect was also observed by the motion of a magnet toward or away from the circuit. Thus we see that *the work of Oersted demonstrated the magnetic effects by moving electric charges and that of Faraday the electric effects (production of current) by moving magnets. Now-a-days it is believed that all so-called magnetic phenomena result from forces between electrostatic charges in motion and the magnetism can not be considered as a separate entity.* The possibility that the magnetic properties of matter were the result of tiny atomic currents (due to continuous rotation of electrons in atoms about the axis passing through the atomic nuclei) was first suggested by Ampere.

We define the space around a current carrying conductor (or a magnet) as the site of a magnetic field, as we define the space near a charged rod as the site of an electric field. The magnetic field around a moving charge exists in addition to the electrostatic field which surrounds the charge whether the charge is in motion or not. If a second charged particle is placed near the moving charge it will experience a *force due to the electric field whether it is in motion or at rest and a force due to the magnetic field only if it is in motion.* The magnetic field, like the electric field, is a vector field and its magnitude and direction at any point are specified by a vector **B** called the *magnetic induction* or *magnetic flux density.*

Magnetic field can be represented by lines, called *lines of induction*, just as the electric field is represented by electric lines of force. B is related to its line of induction in the following ways :

(i) *The tangent drawn at any point to a line of induction gives the direction of* **B** *at that point.*

(ii) *The number of lines per unit area gives an idea about the magnitude of the magnetic field vector* **B**. *This field is large if lines are closer and is small if they are far apart.*

In a uniform magnetic field, where the **B**-vector has the same magnitude and direction at all points, the lines of induction are equally spaced, parallel and straight. The lines of induction simply give a graphical representation of the variation of **B** throughout a certain region of space.

The magnetic flux Φ_B across a surface is defined in the same way as the electric flux Φ_E for the electric field. Thus the flux $d\Phi_B$ across an area $d\mathbf{A}$ is,

$$d\Phi_B = B_n dA = \mathbf{B}\cdot d\mathbf{A}.$$

$\therefore$ Total flux across an finite area $= \Phi_B = \int B_n dA = \int \mathbf{B}\cdot d\mathbf{A}$. ...(2)

If **B** is uniform and normal to a finite area **A**, then the flux across the area is $\Phi_B = BA$.

10.3 THE DEFINITION OF B

An unknown electric field **E** can be determined by magnitude and direction of the force **F** on a test charge q_0 at rest. To explore an unknown magnetic field **B** we must measure the magnitude and direction of the force **F** on a moving test charge q_0, if assumed to be placed in this field. Let us assume that there is no electric field present.

Consider a positive test charge q_0 moving with velocity v through any point P in space. If a side ways deflecting force F acts on it, one can say that a magnetic field **B** is present at P, which can be defined in terms of F and other measured quantities. If the direction of **v** is varied through point P, keeping its magnitude constant, we find that although **F** remains ⊥ to **v**, its magnitude changes and there is one direction of **v** in which a magnetic field exerts no force on the moving charge. *This direction is defined as the direction of the* **B**-*vector.*

Having determined the direction of **B**, let us now keep the velocity of the moving charge perpendicular to the magnetic field. The force is found perpendicular to both the **B** and **v**. If the velocity is not perpendicular to the direction of the magnetic field, but makes an angle ϕ with the field, the force acting on the moving charge is also perpendicular to both the magnetic field **B** and to **v** and has a magnitude proportional to **v** sin ϕ (the component of **v** perpendicular to B). The magnitude of the magnetic field vector **B** at any point P is given as

$$B = F/q_0 v \sin\phi \text{ or } F = q_0 vB \sin\phi. \quad ...(3)$$

In vector form, it can be written as

$$\mathbf{F} = q_0(\mathbf{v} \times \mathbf{B}). \quad ...(4)$$

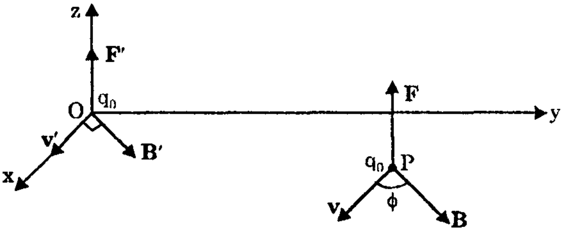


Fig. 10.2. Force on a charge moving in a magnetic field.

From this relation we see that *(a)* **F** *is perpendicular to the plane formed by* **v** *and* **B** *and thus always is a sideways force, (b)* **F** *vanishes as (i)* $v \to 0$, *(ii)* $\mathbf{B} = 0$, *or (iii)* **v** *and* **B** *are either parallel or antiparallel, (c)* $\mathbf{v} \perp \mathbf{B}$, *the magnetic force* **F** *has its maximum value, given as* $F = q_0 v B$ *and (d) the work done by the force in a static magnetic field is zero, as the direction of* **F** *and the direction of motion of charge are always at right angles i.e., K. E. of a moving charge remains unaltered in a static magnetic field.*

The definition of **B** is similar in spirit to the definition of the electric field **E**, which is defined as $\mathbf{F} = q_0\mathbf{E}$. In this case the only characteristic direction is that of **F**, but in defining **B**, two characteristic directions those of **v** and **F**, appear. The unit of magnetic field **B** can be obtained from relation (3) and is the (newton/coulomb)/(meter/sec.). This is given the special name weber/meter^2 or *tesla* in *SI* units. The unit of the magnetic flux is *weber*. The *gauss* is the unit of magnetic field strength in the *CGS* unit. 1 Tesla (T) = 10^4 gauss (G).

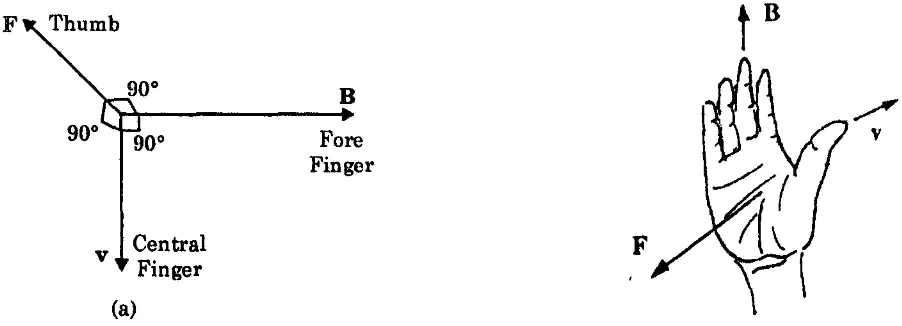
If a charged particle moves through a region in which both an electric field and a magnetic field are present, the resultant force

$$\mathbf{F} = q_0\mathbf{E} + q_0(\mathbf{v} \times \mathbf{B}). \quad ...(5)$$

This relation is known as *Lorentz relation* and **F** as *Lorentz force*.

The direction of **F** is ⊥ to the plane containing **v** and **B** and is given by :

(*i*) *Fleming's left hand rule :* According to this rule, the forefinger, the central finger and thumb of the left hand are stretched in such a way that they are mutually perpendicular to each other. If the central finger shows the direction of current (or **v**) and the forefinger shows the direction of magnetic field or pointing magnetic field vector **B**, then the thumb will give the direction of deflecting force **F**, as shown in Fig. 10.3 (*a*).



(*a*) (*b*)

Fig. 10.3. Direction of **B**.

(*ii*) *Right hand palm rule* : If the thumb of the right hand points in the direction of **v** and the fingers point in the direction of **B**, then the palm points in the direction of the force **F**, as shown in Fig. 10.3 (*b*).

(*iii*) *Right hand screw rule* : The direction of force **F** is the direction of the linear motion of a right hand screw when rotated in the sense that moves the vector $\boldsymbol{v}$ toward the vector $\boldsymbol{B}$.

If the direction of $\boldsymbol{B}$ is perpendicular to the paper, symbols • and × are used to represent it. The symbol *dot* or a dot in a circle shows that the field line is coming toward the reader (*i.e.*, outside the paper) and the symbol *cross* or a cross in a circle shows that the field line is going away from the reader (*i.e.*, inward the paper). The symbols *dot* and *cross* are supposed to represent the tip and tail of an arrow.

Instead of a point charge, if a charge is distributed in a volume with charge density $\rho(\mathbf{r})$ and moving with a velocity $\mathbf{v}(\mathbf{r})$. Then the force acting upon the mobile charge contained in a volume element dV at position $\mathbf{r}$ is given by

$$d\mathbf{F}(\mathbf{r}) = dq\,(\mathbf{r})\,[\mathbf{v}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})] = \rho(\mathbf{r})\,dV\,[\mathbf{v}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]$$

$$= [\mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})]\,dV \qquad \text{as } \rho(\mathbf{r})\,\mathbf{v}(\mathbf{r}) = \mathbf{j}(\mathbf{r})$$

or

$$\mathbf{F} = \int_V \mathbf{j}(\mathbf{r}) \times \mathbf{B}(\mathbf{r})\,dV. \qquad \text{...(6)}$$

10.4 FORCE ON A CURRENT CARRYING CONDUCTOR

Let us consider a conducting wire of length (l) and cross-sectional area A, in which the current i is passing. If this wire is placed in a magnetic field **B**, which is perpendicular to the plane of the diagram pointing in downward direction, shown by sign × in Fig. 10.4. The current in a metal wire is carried by the negative charges (free, or conduction electrons) and the positive charges (positive ions). If the positive charge q_1 is moving with a drift velocity $\mathbf{v}_1$ it is acted on by a force $\mathbf{F}_1$, given as

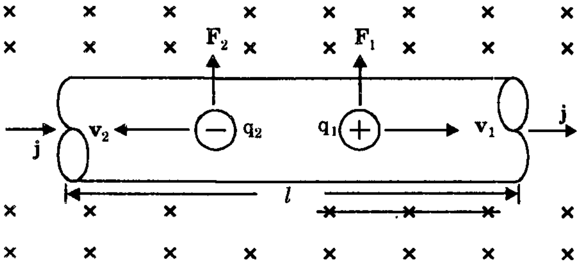


Fig. 10.4. Force on a current carrying conductor.

$$\mathbf{F}_1 = q_1(\mathbf{v}_1 \times \mathbf{B}).$$

The direction of $\mathbf{F}_1$ is upward, if the direction of the current density **j** is from left to right.

The negative charge q_2 moving in opposite direction is acted on by a force

$$\mathbf{F}_2 = -q_2(\mathbf{v}_2 \times \mathbf{B}).$$

It is also in upward direction as the velocity $\mathbf{v}_2$ is opposite to $\mathbf{v}_1$. If n_1 is the number of +*ve* charge carriers per unit volume and n_2 that of –*ve* charge carriers per unit volume, then the total force **F** on all charge carriers of the given conductor

$$\mathbf{F} = (n_1 Al)\,q_1(\mathbf{v}_1 \times \mathbf{B}) - (n_2 Al)\,q_2(\mathbf{v}_2 \times \mathbf{B}) = \Sigma Al n_i\, q_i(\mathbf{v}_i \times \mathbf{B})$$

But we know that the vector current density $\mathbf{j} = \Sigma n_i\, q_i \mathbf{v}_i$, hence

$$\mathbf{F} = Al\,(\mathbf{j} \times \mathbf{B}) = V(\mathbf{j} \times \mathbf{B}).$$

This relation is called the *Ampere's law* and defines the force (known as *Ampere force*) acting on an element of electric current of density j enclosed in a volume V.

As $i = Aj$ and the direction of the current is along the length or displacement vector **l**, hence the above relation can the expressed as

$$\mathbf{F} = i\,(\mathbf{l} \times \mathbf{B}). \qquad \text{...(7)}$$

This relation is very useful when the conductor is of any shape and makes any arbitrary angle with the field. Actually it is equivalent to the relation (4) which is for a single carrier.

If we consider a differential element of a conductor of length dl, then the force $d\mathbf{F}$ is given by

$$d\mathbf{F} = i(d\mathbf{l} \times \mathbf{B}). \qquad ...(8)$$

The equations (7) and (8) describe both the magnitude and direction of the force. The direction of $\mathbf{F}$ (or $d\mathbf{F}$) is $\perp$ to the plane containing $\mathbf{j}$ (or $\mathbf{l}$) and $\mathbf{B}$.

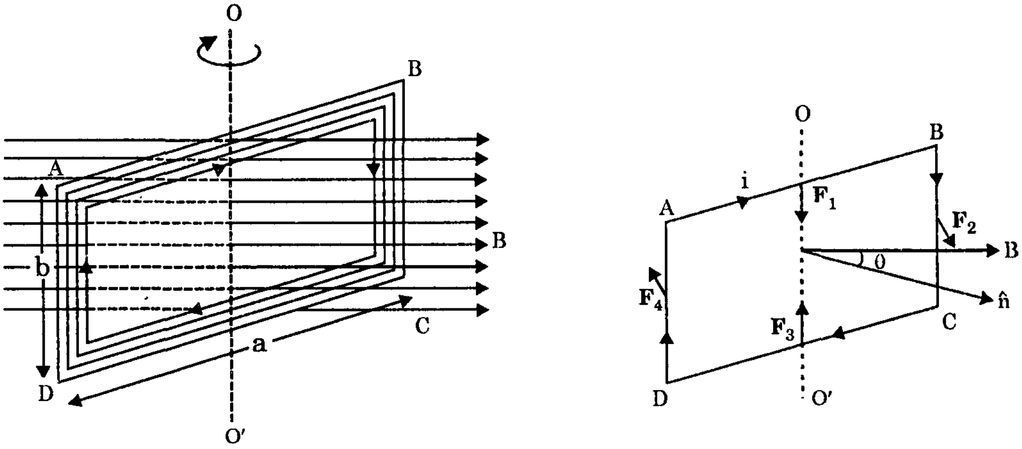
10.5 FORCE AND TORQUE ON A CURRENT LOOP IN A UNIFORM MAGNETIC FIELD

Let us consider a very simple shape of the loop, *a rectangular loop,* as shown in Fig. 10.5. Its sides are of lengths a and b. It is free to rotate about the axis through its centre of mass, *i.e.,* axis OO' which is $\perp$ to the direction of the uniform magnetic field $\mathbf{B}$. Let the normal to the plane of the loop makes an angle θ with the field direction $\mathbf{B}$.

If the current i is allowed to pass through the loop, say in clockwise direction when viewed from the right of the Fig. The net force on the loop is the forces on its four sides. Using the resultant of the force Eq. (7) applied to the top side AB, we find

$$\mathbf{F}_1 = i(\mathbf{l} \times \mathbf{B}) \text{ or } F_1 = iaB \sin(90° - \theta) = ia\,B \cos\theta.$$

The vector cross product indicates that this force is in the downward direction. One can show very easily that the force $\mathbf{F}_3$ on the side CD has the same magnitude but points in the upward direction. As these forces $\mathbf{F}_1$ and $\mathbf{F}_3$ have the same line of action, hence the net force on the loop from these elements AB and CD is zero and the net torque due to these forces is zero.



(*a*) (*b*)

Fig. 10.5. Current loop in a magnetic field.

The force $\mathbf{F}_2$ on the side BC is $ib\,B$. It is along the outward normal to the plane of paper. The force $\mathbf{F}_4$ on the side AD has the same magnitude $ib\,B$ but is along the inward normal to the plane of paper. These forces $\mathbf{F}_2$ and $\mathbf{F}_4$ are thus oppositely directed. These do not have the same line of action and thus constitute a couple. The net torque tends to rotate the loop in the direction of the torque. Using Fig. 10.6, the view looking downward on the loop, we find for the total torque or the moment of the couple $\tau = ib\,B \times a \cos(90 - \theta) = iBA \sin\theta$, where A is the area of the loop $= ab$.

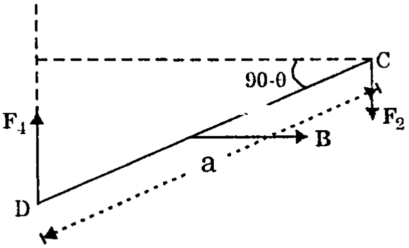


Fig. 10.6. Torque on a loop.

In the vector form, it can be written as

$$\tau = i(\mathbf{A} \times \mathbf{B}), \qquad \text{...(9)}$$

where **A** is the vector area of the loop, having direction given by the right hand screw rule as : *If the current loop is rotated in the direction of the current, then A points in the direction of travel of a right hand screw subject to such a rotation.*

Eq. (9) also holds good for all plane loops of any shape having an area **A** (to be proved in example 11).

If a coil consists N turns, the torque acting on the entire coil is

$$\tau' = N\tau = iNAB \sin\theta. \qquad \text{...(10)}$$

The coil is rotated by this torque until the equilibrium position reaches in which area vector **A** is in the direction of field **B**.

Let us now compare the relation (10) with the expression for the torque acting on the electric dipole of dipole moment **p** placed in an external electric field **E**, *i.e.*, $\tau = \mathbf{p} \times \mathbf{E}$ or $\tau = pE \sin\theta$. We thus see that the coil has a magnetic dipole moment $\mathbf{p}_m$ (= $iN\mathbf{A}$) analogous to the electric dipole moment of an electric dipole and thus acts as a *magnetic dipole*. It is also represented as μ or m. Hence relation (9) can be written as

$$\tau = \mathbf{p}_m \times \mathbf{B}. \qquad \text{...(11)}$$

The direction of $\mathbf{p}_m$ is the direction of thumb, *if the right hand encircles the current loop or coil so that fingers point in the direction of the current.*

When the current loop or a magnetic dipole is placed in a magnetic field **B**, work must be done by an external agent to change the orientation of the dipole. This work done is the potential energy of the magnetic dipole. By analogy with the assumption made for electric dipoles, magnetic energy is assumed zero when $\mathbf{p}_m$ is ⊥ to **B**. If the magnetic dipole is turned through an angle θ from the zero energy position ($\theta = \pi/2$), then

$$\begin{aligned} U &= \int_{90^\circ}^{\theta} \tau d\theta = p_m B \int_{90^\circ}^{\theta} \sin\theta \, d\theta \\ &= -p_m B \cos\theta = -\mathbf{p}_m \cdot \mathbf{B}. \qquad \text{...(12)} \end{aligned}$$

It is same as for energy of an electric dipole $\mathbf{p}_E$ in an electric field **E**, *i.e.*, $U = -\mathbf{p}_E \cdot \mathbf{E}$.

10.6 THE BIOT SAVART LAW

Shortly after Oersted's discovery in 1820 that a current produces a magnetic field, Biot and Savart (1820) first and Ampere (1820 – 25) through experiments, established the basic experimental laws relating the magnetic induction **B** to the currents. Ampere discovered the law of interaction between current elements. Biot and Savart formulated the law for the force with which the current element idl acts on a magnetic pole at a distance.

Biot Savart's law is the relationship defining the differential magnetic field strength $d\mathbf{B}$ produced by a differential current element idl. This law states that the magnetic field $d\mathbf{B}$ produced at a point P by a differential current element idl is proportional to the product idl and the sine of the angle θ between the element idl and the line joining P to the element and is inversely proportional to the square of the distance r between P and the element, *i.e.*,

$$\begin{aligned} dB &\propto idl \sin\theta / r^2 \\ &= \frac{\mu_0}{4\pi} \frac{idl \sin\theta}{r^2}, \end{aligned}$$

where the constant of proportionality in *SI* units is $\mu_0/4\pi$. The μ_0 is called the *permeability of the free space*.

In terms of the vector form the above relation reduces to

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{id\mathbf{l}\times\mathbf{r}}{r^3}. \qquad ...(13)$$

The direction of $d\mathbf{B}$ will be perpendicular to the plane containing $d\mathbf{l}$ and $\mathbf{r}$ and may be given by any of the following rules.

(*a*) *Maxwell's cork-screw rule :* If the direction of the current through the conductor is represented by the linear motion of the cork–screw motion, then the direction of the magnetic field can be represented by the direction of rotation of the cork–screw.

(*b*) *Right hand claps rule :* If a conductor carrying a current is clapsed in the right hand so that the thumb indicates the current direction, then the direction of the magnetic field is represented by the finger tips round the wire.

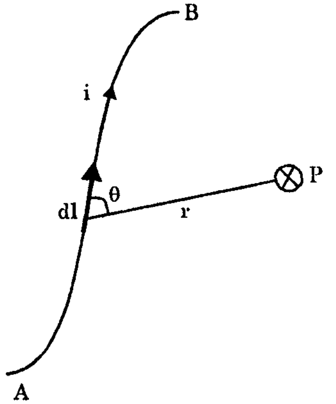


Fig. 10.7. Magnetic field due to current element.

The current elements have no separate existence. All elements making up the complete current contribute to **B**. Therefore we have

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\frac{id\mathbf{l}\times\mathbf{r}}{r^3}. \qquad ...(14)$$

where line integral is required to include the contribution of all current elements.

The current contributions may be either the line current, surface current and volume current. If we define

$$idl = KdS = jdV, \qquad ...(15)$$

where j = current density = current/area = volume current density

and K = current/distance = surface current density

$$\mathbf{B} = \int_L \frac{\mu_0}{4\pi}\frac{id\mathbf{l}\times\mathbf{r}}{r^3} \qquad \text{(Line current)}$$

$$\mathbf{B} = \int_S \frac{\mu_0}{4\pi}\frac{\mathbf{K}\times\mathbf{r}\,dS}{r^3} \qquad \text{(Surface current)}$$

$$\mathbf{B} = \int_V \frac{\mu_0}{4\pi}\frac{\mathbf{j}\times\mathbf{r}}{r^3}dV \qquad \text{(Volume current)} \qquad ...(16)$$

Relation (13) can not be verified experimentally, however **B**, given by Eq. (14), can be measured experimentally. The differential relation for the magnetic induction due to current element was established by Pierre Simon Laplace. It is expressed in modern language as given by Eq. (13) and is thus known as *Laplace law*. This is widely known as *Biot–Savart law*. Although there has been controversy over its naming. Some authors have also given the names Ampere's theorem, Biot law, etc.

To find the relation (13) in terms of charge carriers, let us consider a small section dl of a wire of a cross section A carrying a current i. If n is the number of charge carriers per unit volume, each of charge q and moving with a drift velocity **v**, then

$$i = nq\mathbf{v}.\mathbf{A} = nqAv.$$

The magnetic field induction due to the current element dl at any point P at a distance **r** from this element is

$$dB = \frac{\mu_0}{4\pi}\frac{nqAvdl}{r^2} \qquad ...(17)$$

To define unit of current, *ampere*, let us consider two elements A and B of lengths $d\mathbf{l}_1$ and $d\mathbf{l}_2$ having currents i_1 and i_2 respectively. In calculating the force between these elements, the first step is to calculate the magnetic induction $d\mathbf{B}_2$ at $d\mathbf{l}_1$ due to the current in the element $d\mathbf{l}_2$

$$d\mathbf{B}_2(\mathbf{r}_1) = \frac{\mu_0}{4\pi} i_2 \frac{d\mathbf{l}_2 \times \mathbf{r}_{12}}{r_{12}^{\ 3}}.$$

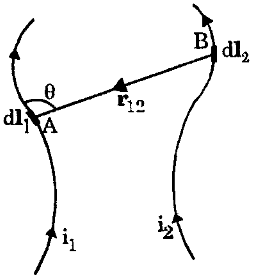


Fig. 10.8. Force between two current carrying wires.

In this case $d\mathbf{B}_2$ is $\perp$ to $d\mathbf{l}_2$ and $\mathbf{r}_{12}$ and is directed into the paper, as $\mathbf{r}_{12}$ points from **B** to **A**. Next the force equation is applied to the left hand element, giving*

$$d^2\mathbf{F}_1 = i_1 d\mathbf{l}_1 \times d\mathbf{B}_2 = \frac{\mu_0}{4\pi} \frac{i_1 i_2}{r_{12}^{\ 3}} d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12}). \qquad ...(18)$$

The total force on the first wire due to the current in the entire length of the second wire is obtained by integrating this equation over the whole lengths l_1 and l_2.

$$\therefore \qquad \mathbf{F} = \int_{l_1}\int_{l_2} d^2\mathbf{F} = \frac{\mu_0}{4\pi} i_1 i_2 \int_{l_1}\int_{l_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12})}{r_{12}^{\ 3}} \qquad ...(19)$$

This is known as *Ampere's law* of force between current loops.

This equation does not appear to be symmetrical in respect of $d\mathbf{l}_1$ and $d\mathbf{l}_2$, *i.e.*, the force on one current element due to the other current element does not appear to be the force on the other current element due to the first one. A use of vector algebra shows that this equation is symmetrical and thus not violating Newton's third law. This relation is equivalent of Coulomb's law equation for the force between static charges. If these current elements are each of unit length and at a unit distance apart and are so oriented that the factor $d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12})/r_{12}^{\ 3}$ is unity. Then the current in the elements is defined to be an ampere, if the resulting force is 10^{-7} newton. From this we get $\mu_0 = 4\pi \times 10^{-7}$ Web/amp. m. Thus we see that μ_0 is obtained from the definition *hence cannot be measured experimentally.* On the other hand ε_0 is a measured quantity and *can be found from Coulomb's law.*

10.7 MAGNETIC FIELD OF A MOVING POINT CHARGE

We know that a point charge q, at rest in the observer's inertial system, produces an electric field along the radius vector and is given as

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^3} \mathbf{r} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r}. \qquad ...(20)$$

If this charge is moving relative to the observer's inertial system, it produces a magnetic field as well as an electric field. The magnitude of the magnetic field is proportional to the speed of the charge relative to the observer (provided speed $v < c$, the velocity of light), An observer travelling with a moving charge detects no magnetic field. The magnetic field vector **B** at the point P at a distance **r** from the charge q moving with a velocity **v** is found by replacing $\mathbf{j}dV = q\mathbf{v}$ in equation (16) and is given by

$$\mathbf{B} = \frac{\mu_0 q}{4\pi r^3} (\mathbf{v} \times \mathbf{r}). \qquad ...(21)$$

The direction of **B** is thus perpendicular to the plane of **v** and **r** *and is the direction of advance of a right handed screw if rotated from* **v** to **r**. Its magnitude is

$$B = \frac{\mu_0}{4\pi} \frac{q\, v \sin\theta}{r^2}.$$

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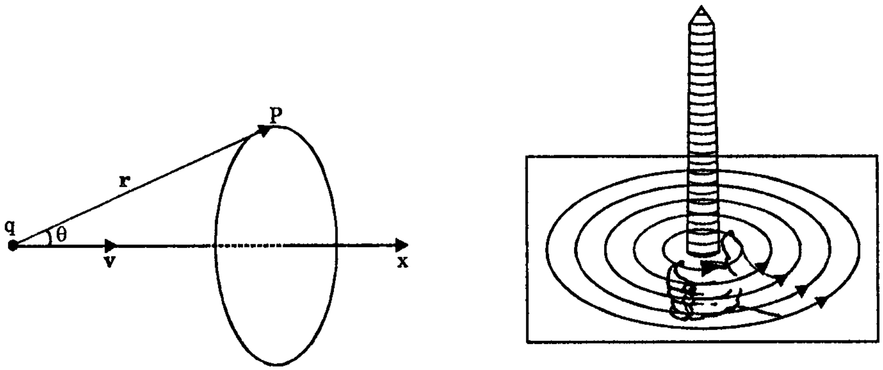
* The differential force $d^2\mathbf{F}_1$ is a second order differential because it involves the product of two first order differentials $d\mathbf{l}_1$ and $d\mathbf{l}_2$.



(a) (b)

Fig. 10.9. Magnetic field due to a moving charge.

The **B** is thus zero at all points on a line through the charge in the direction of **v**, as $\theta = 0$ or π, $\sin\theta = 0$, for all such points. **B** is maximum in the plane perpendicular to the **v** and through the charge q, as $\sin\theta = 1$ at all points in this plane. **B** remains unaltered in magnitude at all points on the circumference of circle passing through P and lying in a plane ⊥ to **v** with its centre on the **v** direction. Thus we see that unlike the **E** field of the charge which is radial, the lines of induction of **B** are circles in plane ⊥ to the velocity vector **v**, with centres on a line through this vector as shown in Fig. 10.9 (*b*).

The constant of proportionality is $\mu_0/4\pi$ and is having value 10^{-7} weber/amp m. Thus quantity

$$\mu_0 = 4\pi \times 10^{-7}\ WbA^{-1}\ m^{-1} \text{ (or newton sec}^2\text{/coulomb}^2\text{)}.$$

$$\therefore \quad \frac{\text{Constant of proportionality in electric field relation}}{\text{Constant of proportionality in magnetic field relation}} = \frac{1/4\pi\varepsilon_0}{\mu_0/4\pi}$$

$$= \frac{9.0 \times 10^9 \text{ Newton–meter}^2\text{/coul}^2}{10^{-7} \text{ Newton sec}^2\text{/coul}^2} = 9 \times 10^{16} \text{ metre}^2\text{/sec}^2,$$

which is equal to the square of the speed of light c. This suggests that there may be a *close relation between electricity, magnetism and light, i.e.,*

$$\varepsilon_0\mu_0 = 1/c^2. \qquad ...(22)$$

Using relation (4), the force acting on a charge q', moving with velocity $\mathbf{v}'$, in a magnetic field **B** is given by

$$\mathbf{F} = q'\mathbf{v}' \times \mathbf{B}.$$

If this field **B** is set up by a charge q moving with velocity **v**, then

$$\mathbf{F} = q'\mathbf{v}' \times \frac{\mu_0 q}{4\pi r^3}(\mathbf{v} \times \mathbf{r}) = \frac{\mu_0 q q'}{4\pi r^3}\mathbf{v}' \times (\mathbf{v} \times \mathbf{r}). \qquad ...(23)$$

This force is called the magnetic force between two moving charges and is denoted by $\mathbf{F}_m$ to distinguish with the coulomb force (electrostatic force) between static charges

$$\mathbf{F}_e = \frac{1}{4\pi\varepsilon_0}\frac{qq'}{r^3}\mathbf{r}. \qquad ...(24)$$

Comparison of Eqs. (23) and (24) shows that $\varepsilon_0\mu_0$ must have the dimensions of an inverse velocity squared. The same has been obtained in Eq. (22). Thus we see that

(*i*) $F_m/F_e \le vv'/c^2$, *i.e.,* for the charge particles moving with velocities much smaller than the velocity of light c, the magnetic interaction will be very much smaller than the electrostatic interaction. Although there are systems of particles where there is not so;

(*ii*) The direction of $\mathbf{F}_m$ is not along the line joining the charge particles (*i.e.*, not a central force) unless $\mathbf{v}'$ happens to be perpendicular to $\mathbf{r}$. The force is always in a plane of $\mathbf{r}$ and $\mathbf{v}$.

(*ii*) Since $\mathbf{F}_m$ is always perpendicular to $\mathbf{v}'$, $\mathbf{v}' . \mathbf{F}_m = 0$. Thus a magnetic force never does any work on a charged particle.

10.8 DIVERGENCE OF MAGNETOSTATIC FIELD

Divergence of magnetic field **B** is defined as the flux of **B** through the surface S enclosing per unit volume. It can be obtained directly from the Biot-Savart law, as

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}\times\mathbf{r}}{r^3}\,dV, \qquad \text{(as } i d\mathbf{l} = \mathbf{j}dV\text{)}$$

$$\therefore \qquad \operatorname{div}\mathbf{B} = \nabla\cdot\mathbf{B} = \nabla\cdot\frac{\mu_0}{4\pi}\int_V \frac{(\mathbf{j}\times\mathbf{r})}{r^3}\,dV. \qquad \text{...(25)}$$

To solve this equation, let us assume that the position vectors of the field point (at which **B** is measured) and source point (current element) be $\mathbf{r}_2$ and $\mathbf{r}_1$ respectively. Thus Eq. (25) may be written as,

$$\nabla\cdot\mathbf{B}(\mathbf{r}_2) = \nabla\cdot\frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}_1)\times(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}\,dV_1. \qquad \text{...(26)}$$

Since the del operator is a function of the field point (test point) but the independent variable under the integral sign is a function of the source point hence the del operator may be moved to the right of the integral sign. Therefore,

$$\nabla\cdot\mathbf{B}(\mathbf{r}_2) = \frac{\mu_0}{4\pi}\int_V \nabla\cdot\left[\frac{\mathbf{j}(\mathbf{r}_1)\times(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}\right]dV_1$$

Using the vector identity

$$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathrm{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$$

$$\text{with } \mathbf{A} = \mathbf{j}\,(\mathbf{r}_1) \text{ and } \mathbf{B} = (\mathbf{r}_2-\mathbf{r}_1)/|\mathbf{r}_2-\mathbf{r}_1|^3, \text{ we get}$$

$$\nabla\cdot\left[\mathbf{j}(\mathbf{r}_1)\times\frac{(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}\right] = \frac{(\mathbf{r}_2-\mathbf{r}_1)}{|\mathbf{r}_2-\mathbf{r}_1|^3}\cdot\nabla\times\mathbf{j}(\mathbf{r}_1) - \mathbf{j}(\mathbf{r}_1)\cdot\left[\nabla\times\frac{\mathbf{r}_2-\mathbf{r}_1}{|\mathbf{r}_2-\mathbf{r}_1|^3}\right]$$

The first term on the right is zero as $\mathbf{j}(\mathbf{r}_1)$ is a function of the source point $(\mathbf{r}_1)$ while the del operator derivates w.r.t. the field point $(\mathbf{r}_2)$.

$$\text{As} \qquad \nabla\times\frac{\mathbf{r}}{r^3} = \nabla\times\left[-\nabla\left(\frac{1}{r}\right)\right] = -\nabla\times\nabla\left(\frac{1}{r}\right).$$

Since the curl of the gradient of any function is identically zero hence the second term $\nabla\times[(\mathbf{r}_2-\mathbf{r}_1)/|\mathbf{r}_2-\mathbf{r}_1|^3]$ is also zero. Consequently the entire integrand vanishes and

$$\nabla\cdot\mathbf{B} = 0. \qquad \text{...(27)}$$

Thus the vector **B** is found to be *solenoidal*. We have found relation (27) for magnetostatics but this can be found to hold true in general. Comparing it to the analogous equation in electrostatics ($\nabla\cdot\mathbf{D} = \rho$), we can conclude that there is no magnetic analog of an electric charge. There are no *magnetic charges* from which lines of **B** can emerge. The magnetic flux

$$\Phi_M = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_V \nabla\cdot\mathbf{B}\,dV, \qquad \text{...(28)}$$

Since $\nabla\cdot\mathbf{B}$ is always zero, hence the flux over any closed surface is zero (*i.e., lines of* **B** *close into themselves and do not have sources or sinks of magnetic flux*). It shows that the magnetic charge known as the *magnetic monopole* can not exist. Thus the relation

$$\int_S \mathbf{B}.d\mathbf{S} = \int_S \mathbf{B}.\hat{n}\,dS = 0 \qquad \text{...(29)}$$

is known as *Gauss's law for magnetic fields. The relation (27) is known as the differential form of Gauss's law.* Despite of intensive research for the existence of monopoles no experimental evidence has yet been found. If any source or sink of magnetism is ever found the law of Biot-Savart will have to be altered.

10.9 CALCULATION OF B BY USING BIOT-SAVART LAW

(*a*) **Long straight wire :** A point P is chosen at a fixed distance R from the long straight wire AB, carrying a current i. Let us consider a small element dx at a distance x from point O, the foot of perpendicular from point P to the wire. Using Biot-Savart law for this element, we get

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i}{r^3}\, d\mathbf{x} \times \mathbf{r}.$$

The direction of $d\mathbf{B}$ is perpendicular to the plane of $d\mathbf{x}$ and $\mathbf{r}$ and is directed into the plane of paper. This is the direction for all current elements. Hence the total magnetic field

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0}{4\pi} i \int \frac{d\mathbf{x} \times \mathbf{r}}{r^3}, \text{ or } B = \frac{\mu_0}{4\pi} i \int_{r_1}^{r_2} \frac{dx \sin\theta}{r^2}.$$

From Fig. 10.10, $\sin(180 - \theta) = R/r$ or $r = R \operatorname{cosec}\theta$ and $x = -R \cot\theta$ or $dx = R \operatorname{cosec}^2\theta \, d\theta$.

$$\therefore\ B = \frac{\mu_0}{4\pi} i \int_{\theta_1}^{\theta_2} \frac{R \operatorname{cosec}^2\theta \sin\theta \, d\theta}{R^2 \operatorname{cosec}^2\theta} = \frac{\mu_0}{4\pi}\frac{i}{R}\int_{\theta_1}^{\theta_2} \sin\theta \, d\theta.$$

$$= \frac{\mu_0}{4\pi}\frac{i}{R}\cdot(\cos\theta_1 - \cos\theta_2). \qquad \text{...(30)}$$

For a wire of infinite length, $\theta_1 = 0$ and $\theta_2 = \pi$, Thus we have

$$B = \frac{\mu_0}{4\pi}\frac{i}{R}(1+1) = \frac{\mu_0}{4\pi}\cdot\frac{2i}{R} = \frac{\mu_0}{2\pi}\frac{i}{R}. \qquad \text{...(31)}$$

This relation indicates that the value of B is same for all points at the same distance from the long straight wire, thus the lines of magnetic induction are circles concentric with wire and lying in planes ⊥ to it. This relation is analogous to the expression for **E** due to a long charged rod $(1/4\pi\varepsilon_0)(2\lambda/r)$ and thus shows its *electrostatic equivalent nature.*

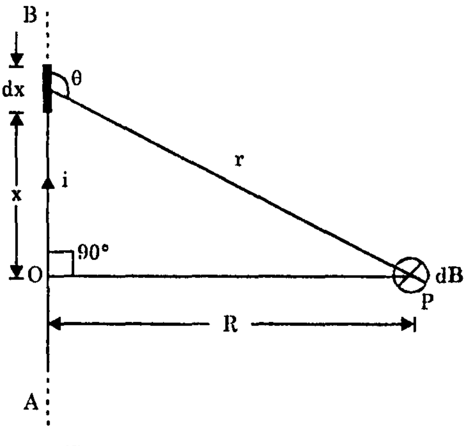


Fig. 10.10. Long straight wire.

(*b*) **Circular current loop :** Let us consider a circular loop of radius a, having turns n and carrying a current i. The magnetic field $d\mathbf{B}$ at point P on its axis at a distance x from its centre, due to a current element AB of length $d\mathbf{l}$ is

$$d\mathbf{B} = \frac{\mu_0}{4\pi} i \frac{d\mathbf{l} \times \mathbf{r}}{r^3}.$$

If the plane of the loop or coil is ⊥ to the plane of the paper, the vector $d\mathbf{l}$ for a current element at the top of the loop points perpendicularly out of the page. Since $\mathbf{r}$ is perpendicular for all elements around the loop, $\mathbf{r} \perp d\mathbf{l}$ hence $d\mathbf{B}$ is always ⊥ to the plane consisting of $\mathbf{r}$ and $d\mathbf{l}$. Thus $d\mathbf{B}$ must lie in the plane of paper, as shown in Fig. 10.11. It can be resolved into two components, one $d\mathbf{B}_{\parallel}$ (or $dB \sin\alpha$) along the axis of the loop and the other $d\mathbf{B}_{\perp}$ (or $dB \cos\alpha$) at right angles to the axis. By symmetry the component of $d\mathbf{B}$ perpendicular to the axis vanishes for the complete loop as the component due to the top element AB is cancelled by the field component due to bottom element $A'B'$ and so on and therefore the resultant B is in the direction of the axis and is equal to $\int dB \sin\alpha$, where α is the angle between $\mathbf{r}$ and the axis of the loop and $\sin\alpha = a/r$. Hence

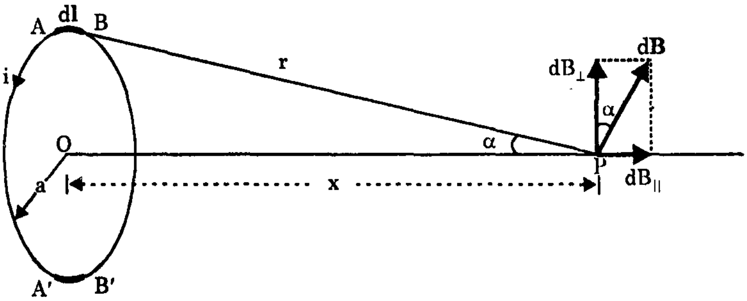


Fig. 10.11. Circular current loop.

$$B = \int \frac{\mu_0}{4\pi} i \frac{dl}{r^2} \sin 90° \sin \alpha = \frac{\mu_0}{4\pi} \frac{i}{r^2} \sin \alpha \int dl,$$

As α and r are constants, so they are taken outside the integral. The integral of dl around the loop is $2\pi a$. If r and α are expressed in terms of the constants a and x, we then get

$$B = \frac{\mu_0}{4\pi} \cdot \frac{i}{r^2} \frac{a}{r} \cdot 2\pi a = \frac{\mu_0}{2} \frac{i\,a^2}{r^3} = \frac{\mu_0}{2} \frac{ia^2}{(a^2 + x^2)^{3/2}} \, web/m^2. \qquad ...(32)$$

At the center of the loop, $x = 0$ and $B = \mu_0 i/2a$, pointing along the axis of the loop in a direction depending on the direction of current flow. **B** is assumed positive when it is directed toward the viewer and the current is assumed positive when it is directed counterclockwise as we view it. If the coil has N closely packed turns, the result is multiplied by N. Thus

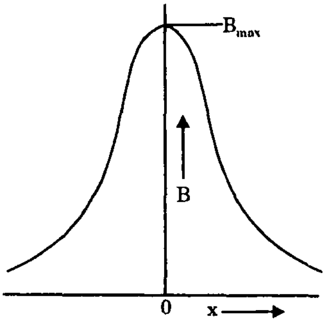


Fig. 10.12. Variation of B along the axis of a circular coil.

$$B = \frac{\mu_0}{2} \frac{iNa^2}{(a^2 + x^2)^{3/2}}. \qquad ...(33)$$

We have not considered the effect of current leads to the loop, their contribution to the field can be made vanishingly small by keeping them close together. As the lines of force are concentric circles around the wires, these are in opposite directions for the two wires carrying currents in opposite directions, they cancel when the wires are close together as in the lead wires.

As $\pi a^2 = A$, the area of the loop, and $r = x$ approximately for the points for which $x >> a$, we therefore get

$$B = (\mu_0/4\pi)\,(2\,iNA/x^3). \qquad ...(34)$$

Thus the axial B – field of the loop at large distances is like the E–field of an electric dipole [$E = (1/4\pi\varepsilon_0)\,(2\,p_E/x^3)$] on the axis of an electric dipole and the term (iNA) corresponds to electric dipole moment p_E of the dipole. Therefore the term iNA is called the *magnetic dipole moment of the loop* and is represented by p_m (or m). Thus the current loop may be regarded as a *magnetic dipole* generating its own magnetic field, given by

$$\mathbf{B} = \left(\frac{\mu_0}{4\pi}\right) \frac{2\mathbf{p}_m}{x^3} = \frac{\mu_0}{4\pi} \cdot \frac{2\mathbf{m}}{x^3}. \qquad ...(35)$$

In section 2.4 (a), we found that the electric field at large distances from an electric dipole is given as

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \left[\frac{3(\mathbf{p} \cdot \mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3} \right], \qquad ...(36)$$

Since except for the different roles played by ε_0 and μ_0, the magnetic field along the axis of a current loop has the same behaviour as the electric field along the axis of an electric dipole. Thus the magnetic field at large distances from a current loop may also be written similar to Eq. (36), as

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right] \quad ...(37)$$

It was pointed out in article 10.5 that a current loop is acted on by a torque if placed in an external magnetic field. The loop is in stable equilibrium when its plane is at right angles to the external field. But we have just proved that this coil has its own magnetic field along its axis, *i.e.*, ⊥ to its plane, hence the equilibrium position is such that within the area enclosed by the loop, the loop's own field is in the same direction as that of the external field. The lines of force due to the current through the coil with in space nearer to it is shown in Fig. 10.13. It shows that at the centre the field is nearly constant, *i.e.*, the lines of force are approximately parallel equidistant straight lines.

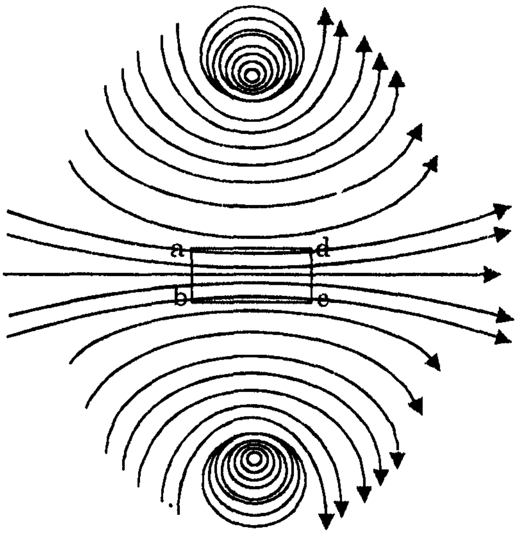


Fig. 10.13. Lines of force due to current carrying coil.

(*c*) Field midway between two similar coaxial circular coils (Helmholtz Galvanometer) : In the preceding article we have seen that the field **B** varies with the distance x. It is maximum at the centre of the coil and decreases with x on both sides of the coil. Thus if two similar coils are connected in series, wound in the same sense and are placed exactly parallel to each other, then the magnetic field at any point on the axis (or the line joining centres of the coils) will be additive.

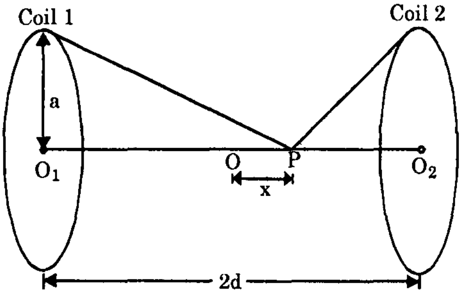


Fig. 10.14. Two similar coaxial coils.

Let the coils each having N turns and carrying current i are placed a distance $2d$ apart. Let B be the magnetic field at point P on the common axis, which is at a short distance x from the midway point O, Fig. 10.14. Assuming the current as flowing in the same direction in each coil and the radius of each coil as a, we get

$$B = \frac{\mu_0}{2} ia^2 N\left[\frac{1}{[(d+x)^2+a^2]^{3/2}} + \frac{1}{[(d-x)^2+a^2]^{3/2}}\right]$$

The field will be uniform (or constant) if $dB/dx = 0$, *i.e.*

$$\frac{\mu_0}{2} ia^2 N\left[\frac{-3(d+x)}{[(d+x)^2+a^2]^{5/2}} + \frac{3(d-x)}{[(d-x)^2+a^2]^{5/2}}\right] = 0$$

or $\quad (d+x)\,[(d-x)^2+a^2]^{5/2} = (d-x)\,[(d+x)^2+a^2]^{5/2}. \quad ...(38)$

Now $\quad [(d+x)^2+a^2]^{5/2} = (d^2+a^2+2xd)^{5/2} \quad (x^2 < d^2)$

$$= (d^2+a^2)^{5/2}\left[1+\frac{2xd}{d^2+a^2}\right]^{5/2} = (d^2+a^2)^{5/2}\left[1+\frac{5xd}{d^2+a^2}\right]$$

[As $2xd/(d^2+a^2) << 1$]

Similarly $[(d-x)^2+a^2]^{5/2} = (d^2+a^2)^{5/2}\left[1-\frac{5xd}{d^2+a^2}\right]$

Substituting these values in Eq. (38), we get

$$(d+x)\left[1-\frac{5xd}{d^2+a^2}\right] = (d-x)\left[1+\frac{5xd}{d^2+a^2}\right]$$

or $\quad 5d^2/(d^2+a^2) = 1$, therefore $d = a/2$ or $2d = a$.

Thus we see that, to obtain a very uniform field in the central region between the coils, the distance apart $2d$ must be equal to a, the radius of each coil. The field B at point O due to current i in the coils

$$B = 2\frac{\mu_0}{2} ia^2N\frac{1}{[(a/2)^2+a^2]^{3/2}} = \frac{8\mu_0 iN}{5\sqrt{5}a}. \qquad \text{...(39)}$$

The resultant field in the space between the coils is the sum of the fields due to the each coil individually, and may be traced out graphically by erecting ordinates at a number of points equal to the sum of the ordinates at those points. The graph of the resultant is a straight line for a considerable distance on either side of the mid-point O. It is due to the fact that on either side of O, field due to one coil increases while the field due to other coil decreases and the rates of increase and decrease are equal at points close to O. This is shown graphically in Fig. 10.15. The dotted curves show the fields for the separate coils and the full line curve, the resultant field and the horizontal part indicates uniformity of field. Fig. 10.16 shows the lines of force between the coils and indicates that there is a region, in the middle of the field, where the field is practically uniform as the lines of force are nearly horizontal within this region.

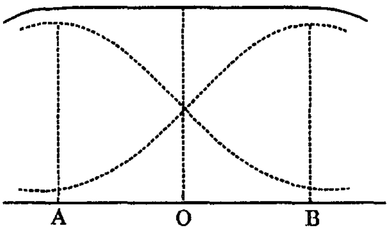


Fig. 10.15. Fields due to two similar coaxial coils.

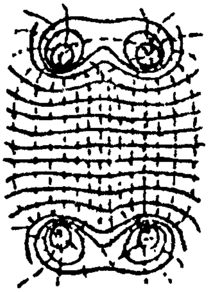


Fig. 10.16. Lines of force between two coaxial coils.

Helmholtz Galvanometer : Helmholtz galvanometer is the improved form of the tangent galvanometer. It consists of two identical circular coils (same radius a and same number of turns N) with their planes parallel and at a distance a apart. The coils are connected in series and so wound that the current passes through these coils in the same direction.

As discussed earlier, the magnetic field is uniform over a considerable region at the midway point O, as the field due to one coil decreases while that due to another coil increases. As the coils are identical and carrying the same current, the decrease in field is nearly balanced by an equal increase in field. If a magnetic compass is placed at this mid point, it will move in a region of uniform magnetic field. A suspended or a pivoted magnet, usually a deflection magnetometer, is placed at the mid-point O on the common axis. As the field is uniform around this point O, the error due to the finite size of the needle in tangent galvanometer is avoided in this case.

When there is no current in the coils. The coils are set in the magnetic meridian, *i.e.* in the direction of horizontal component of the earth's magnetic field B_H or the magnetic needle is parallel to the planes of the coils. If a current i is passed through the coils, the magnetic field is produced along the axis of the coils or perpendicular to the magnetic meridian or B_H. If θ is the deflection in the compass, then the tangent law gives

$$B = B_H \tan\theta,$$

$$\text{i.e.,} \quad \frac{8\mu_0 N i}{5\sqrt{5}a} = B_H \tan\theta$$

$$\text{or} \quad i = \frac{5\sqrt{5}aB_H}{8\mu_0 N}\tan\theta, \qquad \text{...(40)}$$

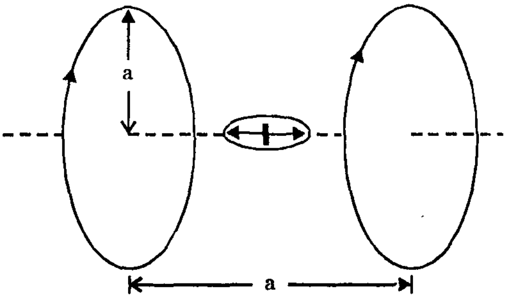


Fig. 10.17. Helmholtz galvanometer.

Thus the current is measured accurately using Helmholtz galvanometer in terms of tan θ.

(*d*) **Field on the axis of a spinning charged disc :** Let us consider a uniformly charged thin disc of charge density σ, radius R rotating with an angular velocity ω about the axis of symmetry (x-axis), as shown in Fig. 10.18. If the disc is assumed to be divided into large number of concentric rings. The ring-shaped portion of the disc of radius a and radial thickness da thus constitutes a current ring of current $di = (2\pi\, a\, da\, \sigma)\, v = 2\pi\, a\, da\, \sigma\, (\omega/2\pi) = \sigma\omega a da$.

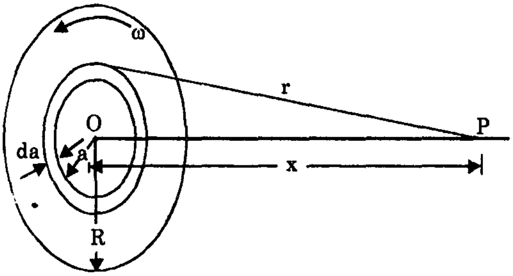


Fig. 10.18. Spinning charged disc.

The magnetic field at the axial point P due to this ring

$$dB = \frac{\mu_0}{2}\frac{(\sigma\omega\, ada)a^2}{r^3}$$

$$\therefore \quad B = \int dB = \frac{\mu_0}{2}\sigma\omega\int_0^R \frac{a^3 da}{(a^2+x^2)^{3/2}}.$$

Let $a^2 + x^2 = t^2$. Since x is constant, therefore $ada = tdt$.

$\therefore$

$$B = \frac{\mu_0\sigma\omega}{2}\int_x^{\sqrt{R^2+x^2}}\frac{(t^2-x^2)\,t\,dt}{t^3} = \frac{\mu_0\sigma\omega}{2}\int\left(1-\frac{x^2}{t^2}\right)dt$$

$$= \frac{\mu_0\sigma\omega}{2}\left[\frac{R^2+2x^2}{\sqrt{R^2+x^2}} - 2|x|\right]. \qquad \text{...(41)}$$

(*e*) **Field along the axis of a solenoid :** The name solenoid was first given by Ampere to a wire wound in a closely spaced spiral over a hollow cylindrical non-conducting core, as he thought the cylindrical coil as a canal confining lines of magnetic field in it. The name solenoid is from the Greek word for canal. If n is the number of turns per unit length, each carries a current i, uniformly wound round a cylinder of radius a, the number of turns per length dx of the solenoid is ndx. Thus the magnetic induction field at the axial point O due to this element dx is

$$dB = \frac{\mu_0}{2}\frac{in\,dx\,a^2}{(a^2+x^2)^{3/2}}.$$

Its direction is along the axis of the solenoid and in the sense of advance of a right handed screw. From geometry of the Fig. 10.19, we know $x = a\cot\theta$ and $dx = -a\,\text{cosec}^2\theta\, d\theta$ and hence $dB = -\frac{1}{2}\mu_0 ni\sin\theta\, d\theta$.

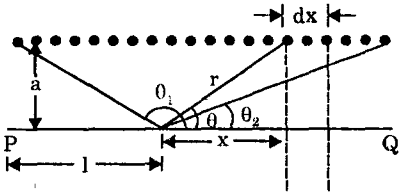


Fig. 10.19. Solenoid.

Total field B due to the entire solenoid is

$$B = \frac{1}{2}\mu_0 ni \int_{\theta_1}^{\theta_2} (-\sin\theta)\, d\theta = \frac{1}{2}\mu_0 ni\,[\cos\theta_2 - \cos\theta_1] \quad \text{...(42)}$$

or

$$B = \frac{\mu_0 ni}{2}\left[\frac{L-l}{[a^2+(L-l)^2]^{1/2}} + \frac{l}{[a^2+l^2]^{1/2}}\right], \quad \text{...(43)}$$

where L is the total length of the solenoid. The value of B can be easily determined from this equation at any point inside the solenoid on its axis.

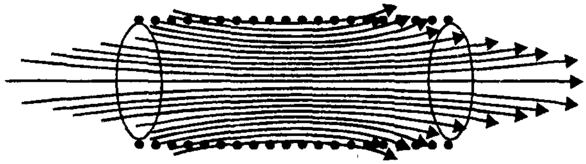


Fig. 10.20. Lines of force due to solenoid.

If the solenoid is of infinite length ($L >> a$) and the point O is chosen at the middle, *i.e.*, if $\theta_1 = 180°$, $\theta_2 = 0°$, then we get

$$B = \mu_0 ni. \quad \text{...(44)}$$

At the end of the solenoid, $\theta_2 = 0°$, $\theta_1 = 90°$, we get

$$B = \tfrac{1}{2}\mu_0 ni. \quad \text{...(45)}$$

Thus the field strength at the end of a long solenoid is just one half that at the centre and the half the lines of force passing through the central section of a long solenoid pass out through the sides before reaching the end. (Fig. 10.20).

The quantity ni, the number of amperes circulating around the coil per metre of length of coil, is known as *solenoidal current density* and is denoted as $\mathbf{j}^*$ (or λ) which is to be distinguished from the coventional current density $\mathbf{j}$.

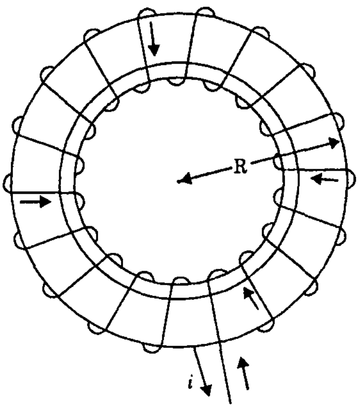


Fig. 10.21. Toroid.

Relation (44) may be applied also to a solenoid of small radius bent round to form a *toroid or anchor ring*. In an ideal toroid, the field is confined entirely within the core and is sensibly uniform. The value of the magnetic field at any point on the mean circumferential line is given by

$$B = \mu_0 ni = \mu_0 (N/2\pi R)\, i$$
$$= \mu_0 Ni/2\pi R, \quad \text{...(46)}$$

where *N is the total number of turns and R* is the mean radius of ring.

***(f)* Infinite current sheet :** To find the magnetic field due to the infinite plane current sheet, the plane is regarded as a superposition of straight filamentary currents. Let the sheet of thickness t be placed in x-y plane. If the current density inside the sheet is j *amp/m*^{2}, then $\mathbf{j}\, t$ *amp/m* is called the *surface current density or sheet current density*. It is represented by $\mathbf{K}$. If the charge moves in the direction of y, then $\mathbf{K} = K\hat{y}$.

As shown in Fig. 10.22(*b*), PL is the normal from the field point P on the surface of current density $\mathbf{K}$, which is perpendicular to the plane of the paper in inward direction. The current through an elementary strip QR of width dx at a distance x from L is Kdx. This strip may be treated as an infinitely long current wire that produces a magnetic field dB at point P, given by

$$dB = \frac{\mu_0}{2\pi}\frac{Kdx}{PQ}.$$

It is inclined at an angle θ to PS and perpendicular to PQ. Its component along PS will be $dB\cos\theta$. If we take another elementary strip of width dx on the left side of L at a distance x,

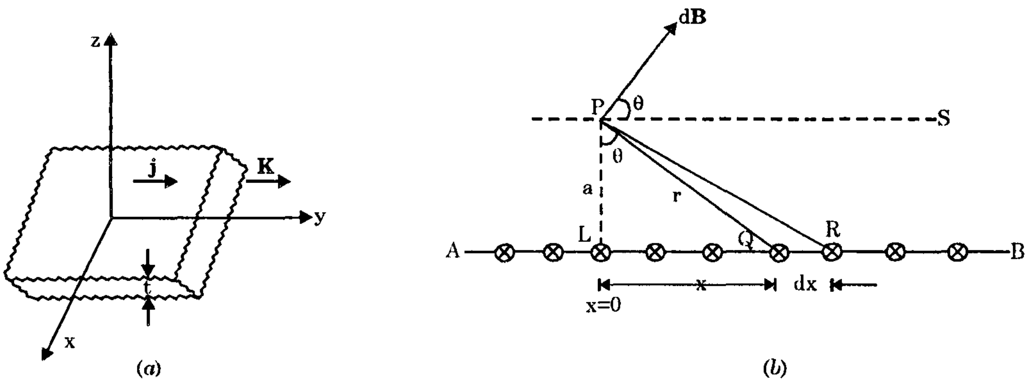


Fig. 10.22. Current sheet.

it will also produce the magnetic field of same magnitude but inclined at an angle – θ to *PS*. The resultant of the fields due to such pairs of strips will be along *PS*. Thus we have

$$B = 2\int dB\cos\theta = 2\int_0^{\infty}\frac{\mu_0}{2\pi}K\frac{dx}{r}\cos\theta$$

$$= 2\int_0^{\pi/2}\frac{\mu_0 K}{2\pi}\frac{a\sec^2\theta\, d\theta}{a\sec\theta}\cos\theta = \frac{\mu_0 K}{\pi}\int_0^{\pi/2}d\theta = \frac{1}{2}\mu_0 K \qquad ...(47)$$

(From $\Delta\, PLQ$, $x = a\tan\theta$, $dx = a\sec^2\theta\, d\theta$, $r = a\sec\theta$)

If $\hat{n}$ be the unit vector normal to the sheet and pointing towards *P*, then Eq. (47) may be written as

$$\mathbf{B} = \tfrac{1}{2}\mu_0\,\mathbf{K}\times\hat{n}. \qquad ...(48)$$

10.10 AMPERE'S CIRCUITAL LAW

We know that the lines of magnetic field are continuous and do not arise from any source in the way as lines of electric force originate on charges. These thus form loops without any beginning or end. This property may be used to go through the nature of magnetic field and for calculating the magnetic field in certain situations of high symmetry.

Let us use expression for the field round a long wire carrying a current *i*, *i.e.*, $B = \mu_0 i/2\pi r$. The magnetic lines are concentric around the wire in the plane of the paper if the wire is ⊥ to this plane. The line integral around the path of radius *r*, starting at any point and returning to the same point, is

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int B\,dl = \int\frac{\mu_0 i}{2\pi r}dl = \frac{\mu_0 i}{2\pi r}\int dl$$

$$= (\mu_0 i/2\pi r)2\pi r = \mu_0 i. \qquad ...(49)$$

We have taken *r* outside the integral as it is constant for the particular circuital path. This relation is also true, whatever be the nature of the closed path. For this let us consider the diagram shown in Fig. 10.23. The field **B** at any point *P* is in the plane of the paper and is ⊥ to the radius vector **r** from the wire at *O*. This **B**-vector makes an angle θ with an element *PP′* (= $d\mathbf{l}$) of a closed path *PQR* encircling the wire, its component along $d\mathbf{l}$ is $B_l = B\cos\theta$. If *PS* be the normal from *P* on *OP′*, then from the triangle *PP′S*, we get

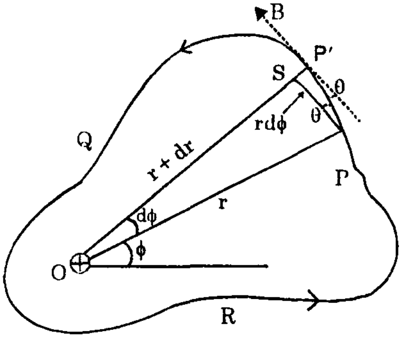


Fig. 10.23. Line integral of *B* round a closed path.

$$r\,d\phi = dl\cos\theta \text{ or } dl = r\,d\phi/\cos\theta.$$

$$\therefore \quad \mathbf{B} \cdot d\mathbf{l} = B_t dl = B \cos \theta \, dl = Brd\phi$$

$$= \frac{\mu_0 i}{2\pi r} rd\phi = \frac{\mu_0}{2\pi} id\phi.$$

Hence the line integral of **B** around the closed path is

$$\oint \mathbf{B} \cdot d\mathbf{l} = \frac{\mu_0 i}{2\pi} \int d\phi = \mu_0 i. \qquad \text{...(50)}$$

Thus we see that *the line integral does not depend on the shape of the path or on the position of the wire with in it.* If the current is in opposite sign, the line integral would have the opposite sign. If any number of long straight conductors pass through the surface bounded by the path, *the line integral equals μ_0 times the algebraic sum of the currents.* For the path, through which no current flows is chosen (Fig. 10.24), the line integral is zero, because the angle ϕ then has the same value at the start and finish of any round trip. Thus the conductor may contribute to the value of **B** but the line integral of such a field around the path not consisting this conductor is always zero.

Above results can be summarized as : *The line integral of the magnetic induction* **B** *around any closed path is equal to* μ_0 *times the net current across the area bounded by this path. It is called Ampere's circuital law,* also named as *Ampere's law. It plays the same role in magnetostatics that Gauss's law ($\int \mathbf{E} \cdot d\mathbf{S} = q/\varepsilon_0$) played in electrostatics.*

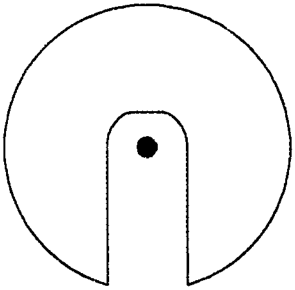


Fig. 10.24. Closed path excluding wire.

Differential form of Ampere's law : In a region of distributed current flow, the total current $i = \int \mathbf{j} \cdot d\mathbf{S}$, where **j** is the current density in an element $d\mathbf{S}$ of the surface bounded by the closed path. Hence we get

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 \int \mathbf{j} \cdot d\mathbf{S}.$$

The line integral of the left hand side may be changed into surface integral by using Stokes theorem and we have

$$\int \text{curl}\, \mathbf{B} \cdot d\mathbf{S} = \mu_0 \int \mathbf{j} \cdot d\mathbf{S} = \int (\mu_0 \mathbf{j}).d\mathbf{S}.$$

On both the sides we have surface integrals, therefore the integrands are also equal, *i.e.,*

$$\text{curl}\, \mathbf{B} = \mu_0 \mathbf{j}. \qquad \text{...(51)}$$

This is called *differential form of Ampere's law.* This establishes a local relation between current density **j** and magnetic induction **B** at the same point. Here we have a new kind of situation which is quite different from electrostatics, where we had curl **E** = 0, or $\oint \mathbf{E} \cdot d\mathbf{l} = 0$. The relation (49) is the integral form of the Ampere's law. It is a special case of Biot-Savart's law, as the former is derived from the latter.

Unlike the electrostatic field the magnetostatic field does have rotation hence cannot be conservative. At any point where **j** = 0, the value of $\nabla \times \mathbf{B} = 0$. If a field exists at all, **j** can not be zero everywhere and $\nabla \times \mathbf{B} \neq 0$ everywhere. For the simply connected volumes which do not link any currents, it is possible to treat **B** as conservative in this restricted volume. One may then proceed to treat **B** in a manner analogous to electrostatics. This is the case with permanent magnets with no conventional currents.

10.11 APPLICATIONS OF AMPERE'S LAW

At first glance it appears that the Ampere's law is used to determine the current i by an integration. Instead, the current is usually known and the law provides a method of finding the magnetic field at any point. This is quite similar to the Gauss's law in electrostatics where

E or **D** is found using the given charge distribution. The usefulness of Ampere's law is restricted to situations where the symmetry of the given current is great so that $\oint \mathbf{B}\cdot d\mathbf{l}$ can be readily evaluated. Two conditions must be met :

(1) At each point of the closed path **B** is either tangential or normal to the path.

(2) **B** has the same value at all points of the path where **B** is tangential.

(*a*) Long cylindrical wire : Let us consider a straight wire *AB* of radius *R*, along which a current *i* is flowing. The lines of magnetic induction are concentric circles centered in the wire in planes ⊥ to the wire. Hence the field *B* at point *P* at a distance *r* from the axis of the wire [Fig. 10.25 (*a*)] is given by Ampere's law as

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i, \quad \text{or} \quad \oint B\,dl = \mu_0 i$$

$$\therefore \qquad B\cdot 2\pi r = \mu_0 i \quad \text{or} \quad B = \mu_0 i/2\pi r.$$

$$\text{Vectorially} \qquad \mathbf{B} = \mu_0 R^2 \mathbf{j} \times \mathbf{r}/2r^2, \qquad \text{...(52)}$$

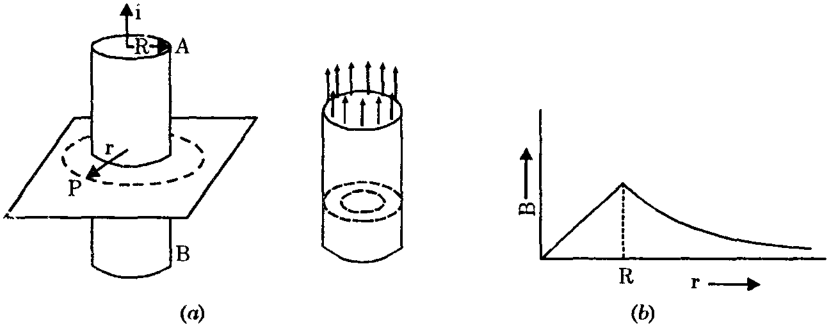
where **j** is the uniform current density in the wire.

If $r < R$, *i.e.*, the point *P* is inside the wire. Symmetry suggests that **B** is tangent to the path of concentric circles. If the current is steady and therefore distributed uniformly through the cross section of the wire, the current surrounded the path of radius *r* is ir^2/R^2. Thus from Ampere's law

$$\oint \mathbf{B}\cdot d\mathbf{l} = \oint B\,dl = \mu_0 i',$$

$$\therefore \qquad B(2\pi r) = \mu_0 ir^2/R^2 \quad \text{or} \quad B = \mu_0 ir/2\pi R^2.$$

$$\text{Vectorially} \qquad \mathbf{B} = \mu_0 \mathbf{j} \times \mathbf{r}/2. \qquad \text{...(53)}$$



(*a*) (*b*)

Fig. 10.25. Magnetic field due to a straight current wire.

At the surface of the wire, $r = R$ and we get the same relation as obtained from Eq. (52). The graph between *B* and *r* is shown in Fig. 10.25 (*b*), which shows that for a wire with uniform distribution of current, the magnetic field is zero at the centre, maximum at the surface and zero at the infinity.

(*b*) Long solenoid : Let us consider a long straight solenoid ($L >> R$) having *n* turns per unit length. If the solenoid wire is closely spaced and carrying a current i_0. The field of a solenoid is uniform on its cross-section in the middle region of the solenoid. Ampere's law can thus easily be applied for the value of **B** inside the solenoid with in the middle region.

For the field at point inside the solenoid, let us consider a rectangular path *ABCDA* as the path of integration. This path is particularly simple as :

(*a*) Outside the solenoid, along *CD*, we can assume field to be zero, as it is many times weaker than that inside.

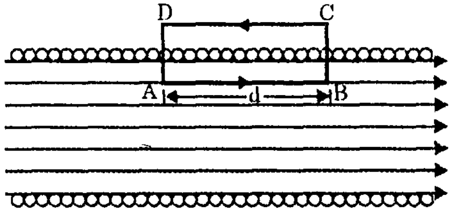


Fig. 10.26. Long solenoid.

$$\therefore \qquad \int_C^D \mathbf{B}\cdot d\mathbf{l} = 0.$$

(*b*) Along *BC* and *DA*, **B** is either zero (for outside parts of *BC* and *DA*) or perpendicular to **B** (for the parts inside the solenoid).

$$\therefore \qquad \int_B^C \mathbf{B}\cdot d\mathbf{l} = \int_D^A \mathbf{B}\cdot d\mathbf{l} = 0.$$

(*c*) Inside the solenoid, **B** is constant and is along *AB*.

$$\therefore \qquad \int_A^B \mathbf{B}\cdot d\mathbf{l} = B(AB) = Bd.$$

Let us now apply Ampere's law for *ABCDA*, we get

$$\oint \mathbf{B}\cdot d\mathbf{l} = \int_A^B \mathbf{B}\cdot d\mathbf{l} + \int_B^C \mathbf{B}\cdot d\mathbf{l} + \int_C^D \mathbf{B}\cdot d\mathbf{l} + \int_D^A \mathbf{B}\cdot d\mathbf{l} = \mu_0 i$$

$$\therefore \qquad Bd + 0 + 0 + 0 = \mu_0(nd)i_0 \text{ or } B = \mu_0 n i_0. \qquad \text{...(54)}$$

This is same as obtained by Biot-Savart law (Eq. 44).

(c) Toroid : Let us consider an endless, solenoid in the form of a ring also called *toroid*, wound uniformly with N turns. The lines of force are circles in planes parallel to the paper with centres on the axis through *O*. For the path 1 of radius r_1, **B** at all the points will be tangent to the path and the current enclosed is zero. Hence,

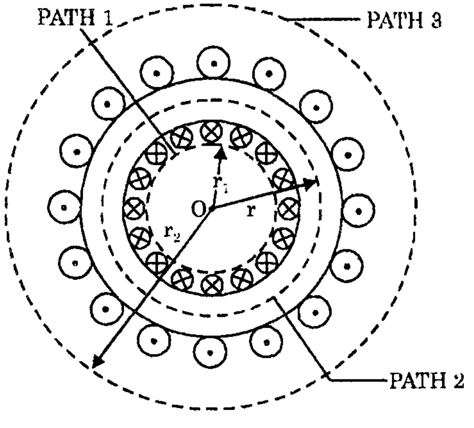


Fig. 10.27. Toroid.

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi r_1 = \mu_0 i = 0. \qquad \text{...(55)}$$

The line integral and thus $B = 0$ at all points of the path. The line integral and thus **B** is also zero at all points of the path 3, (*i.e.*, outside the toroid), as each turn of the winding passes twice through the area bounded by this path carrying equal currents in opposite directions and the net current *i* through the area is therefore zero.

Thus the field of the toroid is confined wholly to the space enclosed by the windings and the toroid will work as a solenoid bent into a circle. For the path 2, we have

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\,2\pi r = \mu_0 N i \text{ or } B = \mu_0 N i/2\pi r. \qquad \text{...(56)}$$

If the cross section of the toroid is sufficiently small, the variations in r can be neglected and $N/2\pi r$ can be written as n, the number of turns per unit length, then $B = \mu_0 n i$, which is as for a long solenoid.

(*d*) Infinitely long coaxial transmission line : Consider an infinitely long coaxial transmission line which consists of two coaxial cylinders with their axes along *z*-axis and inner and outer radii as *a* and *b* respectively. The inner conductor ($r = a$) carries current *i*, while the outer conductor (inner radius *b* and thickness *t*) carries return current $-i$. Assuming the current to be distributed uniformly in both the conductors. Therefore Ampere's law may be applied to four paths : 1 (inside the inner cylinder $r \leq a$), 2 (outside the inner cylinder $a \leq r \leq b$), 3 (inside the outer cylinder $b \leq r \leq b + t$) and 4 outside the outer cylinder $r \geq b + t$).

For path 1 ($0 \leq r \leq a$)

$$\oint_1 \mathbf{B}\cdot d\mathbf{l} = \mu_0 i_1,$$

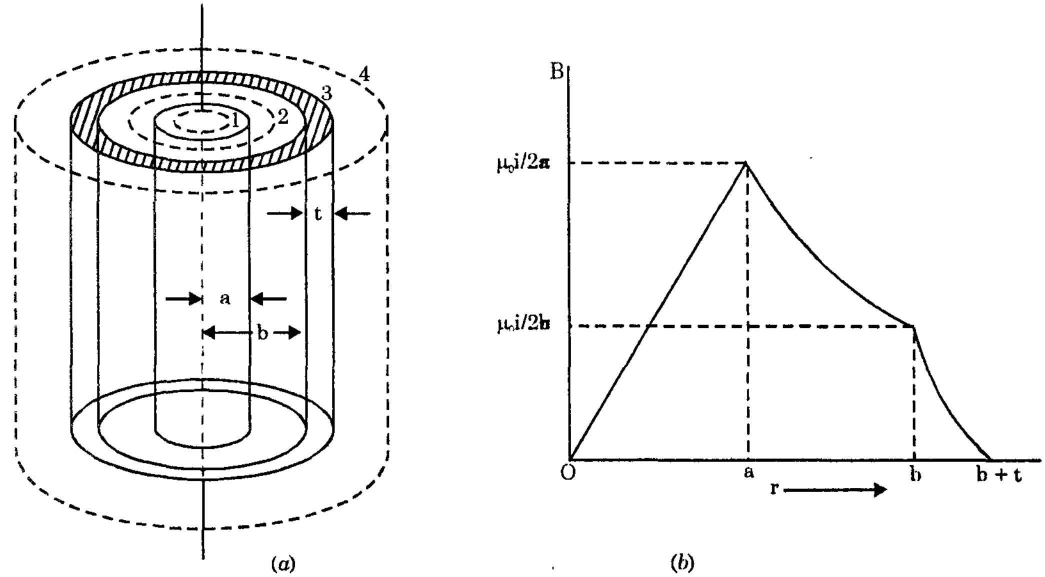
$$B.2\pi r = \mu_0\, i\pi r^2/\pi a^2 \quad \text{or} \quad B = \mu_0 i r/2\pi a^2. \qquad \text{...(57)}$$

For path 2 ($a \leq r \leq b$)

$$\oint_2 \mathbf{B}\cdot d\mathbf{l} = \mu_0 i_2, \quad B.2\pi r = \mu_0 i$$

or

$$B = \mu_0 i/2\pi r. \qquad \text{...(58)}$$



(a) (b)

Fig. 10.28(a). Coaxial transmission line. (b). Variation of **B** with r.

For path 3 ($b \le r \le b + t$) $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i_3$, $B.2\pi r = \mu_0(i - i')$

where $i' = 65\dfrac{\pi r^2 - \pi b^2}{\pi(b+t)^2 - \pi b^2}i = \dfrac{r^2 - b^2}{t^2 + 2bt}i$

$$\therefore \qquad B = \frac{\mu_0 i}{2\pi r}\left[1 - \frac{r^2 - b^2}{t^2 + 2bt}\right] \qquad \text{...(59)}$$

For path 4 ($r \ge b + t$) $\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i_4$ or $B.2\pi r = i - i = 0$

$$\therefore \qquad B = 0. \qquad \text{...(60)}$$

10.12 MAGNETIC VECTOR POTENTIAL A

The vector identity div curl ≡ 0 shows that the solution of the equation div **B** = 0 can be represented in the form

$$\mathbf{B} = \text{curl}\,\mathbf{A}. \qquad \text{...(61)}$$

The vector field **A**, the curl of which is equal to the magnetic field **B** is known as *vector potential of a magnetic field* **B**.

Ambiguity of vector potential : Since the curl of a gradient of scalar is always zero, therefore the vector **A** is not uniquely defined via Eq. (61). A field $\mathbf{A}' = \mathbf{A} + \nabla\varphi$ also satisfies $\mathbf{B} = \nabla \times \mathbf{A}'$, where φ is any scalar field.

$$\nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times (\nabla\varphi) = \nabla \times \mathbf{A} = \mathbf{B}.$$

The ambiguity of vector potential is similar to the ambiguity of the scalar potential in the electrostatic field. The only difference is that the scalar potential is defined accurately to an arbitrary constant, while the vector potential is defined to within an arbitrary function of a certain class.

Lorentz Gauge : Since the potential is chosen ambiguously, **A** will be specified uniquely only if its divergence as well as its curl is given. We choose

$$\text{div}\,\mathbf{A} = 0. \qquad \text{...(62)}$$

This choice is called *Lorentz gauge* (the gauging condition for the potential). The arbitrariness in the choice of the vector potential indicates that the vector potential plays only an auxiliary role and can not be measured experimentally.

Equation for vector potential : From $\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$, we get

$$\nabla \times (\nabla \times \mathbf{A}) = \mu_0 \mathbf{j}.$$

Using the vector identity $\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - (\nabla \cdot \nabla)\mathbf{A}$; we get

$$\text{grad div } \mathbf{A} - \nabla^2 \mathbf{A} = \mu_0 \mathbf{j}.$$

Using the gauge condition (62), the above relation becomes

$$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{j}. \qquad \text{...(63)}$$

Thus the cartesian components of **A** will satisfy the equations :

$$\nabla^2 A_x = -\mu_0 j_x,\ \nabla^2 A_y = -\mu_0 j_y \text{ and } \nabla^2 A_z = -\mu_0 j_z, \qquad \text{...(64)}$$

Each component of the vector potential thus satisfies Poisson's equation ($\nabla^2 \varphi = -\rho/\varepsilon_0$), which has the solution

$$\varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0} \int_v \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} dv'. \qquad \text{...(65)}$$

If all currents are concentrated in a finite region of space, then by analogy with Eq. (65), the solution of equations (64) can be written as

$$A_i(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_v \frac{j_i(\mathbf{r}')\, dv'}{|\mathbf{r} - \mathbf{r}'|}, \qquad \text{...(66)}$$

where i stands for x, y and z. In vector form, we have

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_v \frac{\mathbf{j}(\mathbf{r}')\, dv'}{|\mathbf{r} - \mathbf{r}'|}. \qquad \text{...(67)}$$

In a case of a filamentary current i through a differential length dl' along the wire, we have $j\, dv' = (i/S)(S\, dl') = i\, dl'$. Thus Eq. (67) may be written as

$$d\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{i(\mathbf{r}')\, d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|} \qquad \text{...(68)}$$

Summing up over all volume elements of the filament, we get,

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi} \int_v \frac{i(\mathbf{r}')\, d\mathbf{l}'}{|\mathbf{r} - \mathbf{r}'|}. \qquad \text{...(69)}$$

The components of **A** vary as $1/r$, like the electric potential V, which does not diverge with in a charge distribution. As divergence of a curl of a vector is always zero and div **B** = 0 can be written as a curl of a vector and thus **A** is a vector. Due to these reasons **A** is known as **vector potential.**

10.13 APPLICATIONS OF VECTOR POTENTIAL

The integral for **A** is easier to evaluate than the original equation for **B**. Hence **B** is obtained by calculating first **A** and then making curl operation. Few applications are given below :

(1) **Biot Savart Law :** Since **B** = curl **A**; hence the contribution of δ**A** to the magnetic field **B** is δ**B**, given as

$$\begin{aligned}
\delta\mathbf{B} &= \nabla \times \delta\mathbf{A} = \nabla \times \frac{\mu_0}{4\pi} i \frac{\delta\mathbf{l}}{r} = \frac{\mu_0}{4\pi} i\, \nabla \times \left(\frac{\delta\mathbf{l}}{r}\right) \\
&= \frac{\mu_0}{4\pi} i \left[\frac{1}{r} \nabla \times \delta\mathbf{l} + \nabla\left(\frac{1}{r}\right) \times \delta\mathbf{l}\right] = \frac{\mu_0}{4\pi} i\, \nabla\left(\frac{1}{r}\right) \times \delta\mathbf{l} \qquad (\because\ \nabla \times \delta\mathbf{l} = 0) \\
&= -\frac{\mu_0}{4\pi} i\, \delta\mathbf{l} \times \left(-\frac{\mathbf{r}}{r^3}\right) = \frac{\mu_0}{4\pi} i \frac{\delta\mathbf{l} \times \mathbf{r}}{r^3}. \qquad \text{...(70)}
\end{aligned}$$

It is the Biot–Savart law.

(2) **Magnetic field B of a straight wire or a line conductor :** To determine the vector potential **A** at point P due to a long filamentary wire of length $2L$, let us choose cylindrical co-ordinates with the z-axis along the wire length and the origin of the co-ordinates at the middle

of wire, as shown in Fig. 10.29. Since the current flows in the direction of the z-axis, hence $A_x = A_y = 0$ and **A** must be in the z-direction. Thus

$$A = A_z = \frac{\mu_0}{4\pi}\int_{-L}^{L}\frac{i\,dz'}{r}$$

$$= \frac{\mu_0 i}{4\pi}\int_{-L}^{L}\frac{dz'}{[(z-z')^2+\rho^2]^{1/2}}$$

$$= \frac{\mu_0 i}{4\pi}\left[\log_e\left\{-(z-z')+\sqrt{(z-z')^2+\rho^2}\right\}\right]_{-L}^{L}$$

$$= \frac{\mu_0 i}{4\pi}\log_e\left[\frac{-z+L+\sqrt{(z-L)^2+\rho^2}}{-z-L+\sqrt{(z+L)^2+\rho^2}}\right] \qquad ...(71)$$

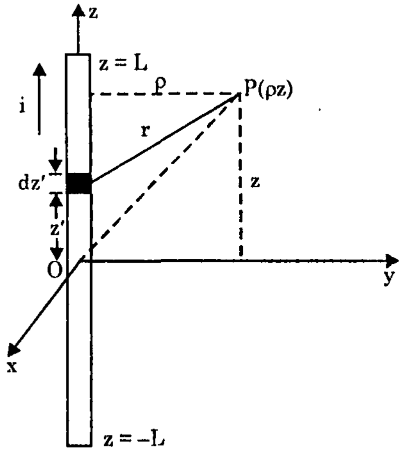


Fig. 10.29. Straight filamentary current.

If we calculate **B** by the formula **B** = curl **A**, we should use cylindrical co-ordinates. The only non zero component of magnetic induction **B** is B_ϕ, where ϕ is the axial angle of the cylindrical system of co-ordinates. Thus we have

$$B = B_\phi = -\frac{\partial A_z}{\partial \rho} = \frac{\mu_0 i}{4\pi\rho}\left[\frac{-z+L}{[\rho^2+(z-L)^2]^{1/2}}+\frac{z+L}{[\rho^2+(z+L)^2]^{1/2}}\right] \qquad ...(72)$$

For a very long straight conductor $\rho << L$ or $L \to \infty$, we find from above equations

$$A = A_z\,(L\to\infty) = -\frac{\mu_0 i}{2\pi}\log_e\rho + \text{const} \qquad ...(73)$$

and $$B = B_\phi\,(L\to\infty) = \mu_0 i/2\pi\rho. \qquad ...(74)$$

The constant in Eq (73) disappears upon differentiation to calculate **B**.

(3). Magnetic field of a small current loop or the magnetic dipole : Consider a circular loop of wire of radius a and carrying a current i and placed in the $x-y$ plane. To find **B**, first calculate **A** at a point P on the $x-z$ plane. The vector potential $d\mathbf{A}$ at this point due to an element $d\mathbf{l}$ of the loop

$$d\mathbf{A} = \frac{\mu_0}{4\pi}i\frac{d\mathbf{l}}{r'}.$$

Since $d\mathbf{l}$ is in x–y plane, hence $d\mathbf{A}$ is in x–y plane. For any given value of r', we can consider two symmetrical $d\mathbf{l}$ elements whose y–components add and whose x–components cancel. Thus total vector potential due to the whole loop

$$A = \frac{\mu_0}{4\pi}i\int_0^{2\pi}\frac{dl}{r'}\cos\phi = \frac{\mu_0}{4\pi}i\int_0^{2\pi}\frac{(a\,d\phi)\cos\phi}{r'} \qquad ...(75)$$

From the geometry of the figure

$$r'^2 = r^2 + a^2 - 2ar\cos\psi.$$

or $$\frac{1}{r'} = \frac{1}{r[1+a^2/r^2-(2a/r)\cos\psi]^{1/2}} \approx \frac{1}{r}\left[1-\frac{a^2}{2r^2}+\frac{a}{r}\cos\psi\right].$$

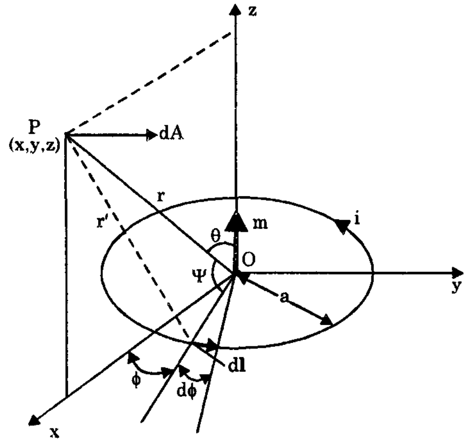


Fig. 10.30. Current loop.

We have assumed here that $a << r$.

To replace $\cos\psi$, let us write **r.a** in two ways and compare

$$(x\mathbf{i}+z\mathbf{k})\cdot(a\cos\phi\,\mathbf{i}+a\sin\phi\,\mathbf{j}) = ra\cos\psi \text{ or } x\cos\phi = r\cos\psi.$$

$$\therefore \qquad \frac{1}{r'} = \frac{1}{r}\left[1-\frac{a^2}{2r^2}+\frac{ax}{r^2}\cos\phi\right]$$

and
$$A = \frac{\mu_0 i}{4\pi}\int_0^{2\pi}\frac{a\cos\phi}{r}\left[1-\frac{a^2}{2r^2}+\frac{ax\cos\phi}{r^2}\right]d\phi$$

$$= \frac{\mu_0}{4\pi}\frac{i\pi a^2}{r^3}x = \frac{\mu_0}{4\pi}\frac{i\pi a^2}{r^2}\sin\theta = \frac{\mu_0}{4\pi}\frac{m}{r^2}\sin\theta$$

or
$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi}\nabla\times\frac{\mathbf{m}}{r}, \qquad \text{...(76)}$$

where **m** is the dipole moment, given by $\mathbf{m} = i(\pi a^2)\,\mathbf{k} = i\mathbf{S}$. It is along the motion of the right hand screw if is rotated along the current flow.

The magnetic field **B** is given by

$$\mathbf{B} = \nabla\times\mathbf{A} = \frac{\mu_0}{4\pi}\nabla\times\left(\nabla\times\frac{\mathbf{m}}{r}\right) = \frac{\mu_0}{4\pi}\left[\nabla\left(\nabla\cdot\frac{\mathbf{m}}{r}\right)-\nabla^2\left(\frac{\mathbf{m}}{r}\right)\right]$$

$$= \frac{\mu_0}{4\pi}\nabla\left[\mathbf{m}\cdot\nabla\left(\frac{1}{r}\right)\right]. \qquad \text{...(77)}$$

Here we have used relations $\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ and $\nabla^2(1/r) = 0$ for $r \neq 0$.

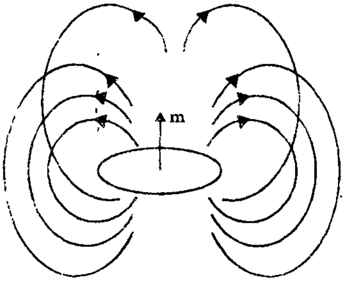


Fig. 10.31. Magnetic lines of force due to a small loop of current.

The relation (77) is of the same form as the expression for the electric field due to an electric dipole of dipole moment **p**,

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\nabla\left[\mathbf{p}\cdot\nabla\left(\frac{1}{r}\right)\right]$$

Thus the electric and magnetic dipoles give rise to similar fields. If we draw lines of force due to these dipoles, we find a fundamental difference–*The electric lines of flux leave and terminate on charges while the magnetic lines of flux are continuous closed loops, as shown in Fig. 10.31.*

10.14 MAGNETIC MOMENT

For an arbitrary current loop the magnetic dipole moment is defined as $\mathbf{m} = i\mathbf{S}$. The positive direction of current flow is related to the positive direction of the surface by the usual right hand rule. To generalize the magnetic dipole moment to a volume distribution of stationary current, we first express the vector area. Let **r** be a vector from some convenient origin to a point on the arbitrary current loop C. The vector area of the shaded region due to an arc of length dl is

$$d\mathbf{S} = \tfrac{1}{2}\mathbf{r}\times d\mathbf{l}.$$

The area subtended by the contour C is

$$\mathbf{S} = \oint_C d\mathbf{S} = \tfrac{1}{2}\oint_C \mathbf{r}\times d\mathbf{l}.$$

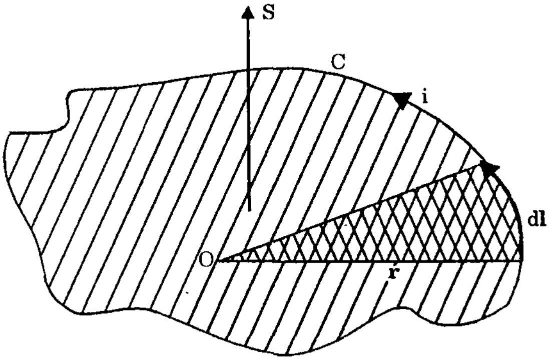


Fig. 10.32. Current loop.

The magnetic dipole moment of an arbitrary shaped loop is therefore given as

$$\mathbf{m} = \tfrac{1}{2} i \oint_C \mathbf{r} \times d\mathbf{l}. \qquad ...(78)$$

For a volume distribution of current, current can be separated into a large number of infinitesimal closed current flow tubes. For any one flow tube, current $di = jdS$, where j is the current density and dS is the cross sectional area of this flow tube. A single flow tube thus contributes a dipole moment given as

$$d\mathbf{m} = \tfrac{1}{2} di \oint_C \mathbf{r} \times d\mathbf{l} = \oint_C \left(\tfrac{1}{2} dS\right) \mathbf{r} \times \mathbf{j} dl.$$

Since $dSdl$ is the volume element dv, hence by summing over all current flow tubes (integrating over dS), we get the general relation for the magnetic dipole moment as

$$\mathbf{m} = \tfrac{1}{2} \oint \mathbf{r} \times \mathbf{j}\, dv. \qquad ...(79)$$

The magnetic moment of a distribution of charge elements dq ($\mathbf{r}$) moving at $\mathbf{r}$ with average velocity $\mathbf{v}(\mathbf{r})$ is given by

$$\mathbf{m} = \tfrac{1}{2} \int \mathbf{r} \times \mathbf{v}(\mathbf{r})\, dq\,(\mathbf{r}), \qquad ...(80)$$

where the sum is over all the charge elements of the distribution.

Torque on a magnetic dipole : If an electric dipole **p** is placed in an uniform electrostatic field **E**, it will experience a torque $\tau = \mathbf{p} \times \mathbf{E}$, but no translational force. Similar to it a magnetic dipole **m** experiences a torque $\tau = \mathbf{m} \times \mathbf{B}$, when placed in magnetic field **B**. The torque is such that it tends to align the dipole with the field. This relation has also been proved for rectangular loop dipole in article 10.5. The generalization to an arbitrary current loop is obtained by considering the loop as made up of a large number of infinitesimal square loops, each with a dipole moment $d\mathbf{m}$. The torque on each square loop dipole is $d\mathbf{m} \times \mathbf{B}$. Its integration over all the infinitesimal dipoles gives the relation for the total torque.

$$\tau = \int d\mathbf{m} \times \mathbf{B} = \int i(d\mathbf{S} \times \mathbf{B}). \qquad ...(81)$$

10.15 MAGNETIC SCALAR POTENTIAL

The differential form of Ampere's law ($\nabla \times \mathbf{B} = \mu_0 \mathbf{j}$) shows that **B** is not a conservative field and thus cannot be written as a gradient of a scalar potential. There are certain regions of space, called simply connected (regions outside of the current sources in which there is no possibility of any closed path being linked with a current), where magnetic field **B** can be described as a conservative field and can be expressed as the gradient of a scalar potential function,

$$\mathbf{B} = -\mu_0 \nabla \varphi_m, \qquad ...(82)$$

where φ_m is called the *magnetic scalar potential.*

However, the divergence of **B** is also zero, which gives

$$\nabla.\mathbf{B} = -\mu_0 \nabla^2 \varphi_m = 0$$

or

$$\nabla^2 \varphi_m = 0. \qquad ...(83)$$

This is just Laplace's equation. We can find unique solutions for it using boundary conditions as was done in electrostatics.

Let us now find an expression for φ_m in terms of the current distribution outside the volume τ (in which there are no currents) using Biot-Savart law. Consider a single current loop C, the space outside of which is not simply connected, but we can make it so by eracting an imaginary surface to span the loop.

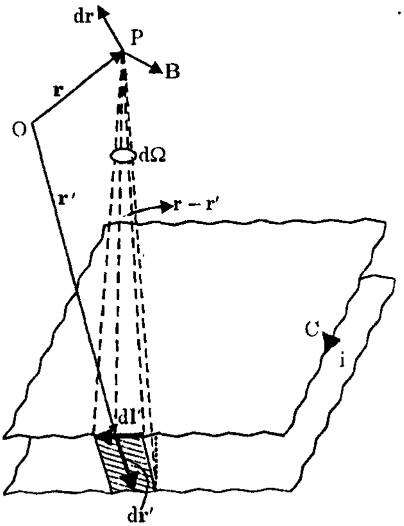


Fig. 10.33. Magnetic scalar potential of a current loop.

If φ_m is to be the magnetic scalar potential function in τ, then

$$d\varphi_m \equiv \tilde{N}\, \varphi_m \cdot d\mathbf{r} = -\mathbf{B} \cdot d\mathbf{r}/\mu_0, \qquad \text{...(84)}$$

where $d\mathbf{r}$ is an infinitesimal displacement of the field point P, which brings about a change in potential $d\varphi_m$.

From Biot-Savart law, the magnetic field $\mathbf{B}$ at point P due to a current loop C is given by

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} i \oint_C \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad \text{...(85)}$$

$$\therefore \qquad d\varphi_m(\mathbf{r}) = -\frac{i}{4\pi}\left[\oint_C \frac{d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\right] . d\mathbf{r} = -\frac{i}{4\pi} \oint_C \frac{[d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')].d\mathbf{r}}{|\mathbf{r} - \mathbf{r}'|^3}.$$

The change in $d\varphi_m$ by a displacement of P by $d\mathbf{r}$ is the same as the change in φ_m affected by displacing the loop rigidly by $d\mathbf{r}' = -d\mathbf{r}$. Thus we have

$$d\varphi_m(\mathbf{r}) = \frac{i}{4\pi} \oint_C \frac{[d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')].d\mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}$$

Since $\quad [d\mathbf{l} \times (\mathbf{r} - \mathbf{r}')].d\mathbf{r}' = (d\mathbf{r}' \times d\mathbf{l}).(\mathbf{r} - \mathbf{r}') = d\mathbf{S}'.(\mathbf{r} - \mathbf{r}')$

$$\therefore \qquad d\varphi_m(\mathbf{r}) = \frac{i}{4\pi} \oint_C \frac{d\mathbf{S}'.(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{i}{4\pi} d\Omega, \qquad \text{...(86)}$$

where $d\Omega$ is the change in solid angle as observed at P, when the whole current loop is rigidly displaced by $d\mathbf{r}'$. Thus we may write the magnetic scalar potential at point P as

$$\varphi_m = (i/4\pi)\, \Omega, \qquad \text{...(87)}$$

where Ω is the solid angle subtended by the loop C at point P.

$$\therefore \qquad \mathbf{B} = -\mu_0 \nabla\varphi_m = -(\mu_0 i/4\pi)\, \nabla\Omega. \qquad \text{...(88)}$$

The electrostatic potential is single valued. Since the solid angle Ω is not single valued, the magnetic scalar potential φ_m is not a single valued. Each time the path links the circuit, the solid angle along the path changes by 4π and the magnetic scalar potential changes by

$$\frac{i}{4\pi}(\Omega + 4\pi - \Omega) = i.$$

Unlike the electrostatic potential, the difference between the scalar potential between two points is not well defined unless the path followed between these points is specified.

10.16 MAGNETIC SHELL

A magnetic shell is a thin sheet of a material magnetized in the direction perpendicular to its surface. It is assumed that a large number of dipoles form a magnetic shell. These dipoles have their moments parallel to the positive normal drawn on elements of surface.

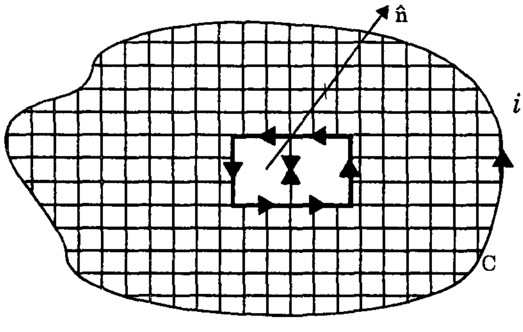


Fig. 10.34. Magnetic shell.

Magnetic shell and its equivalence with a current loop : Consider a closed circuit C enclosing an area S and carrying a current i. The surface area may be divided into many small loops by means of a mesh, as shown in Fig. 10.34. If each small loop formed by the mesh carries the same current i as originally was carried by the circuit C. Because of the cancellation of currents in the common branch of adjacent loops, all currents in the interior will cancel out and there will be the current only in the circuit C. Small loops of currents may be assumed as magnetic dipoles, be equivalent to a closed current loop.

Strength of magnetic shell : The strength of a magnetic shell may be defined as the magnetic moment per unit area.

Thus the potential due to a magnetic shell will be the same as due to a closed loop carrying a current equal to the strength of the magnetic shell.

$$\varphi_m = (i/4\pi)\,\Omega.$$

At a *point very close to the surface,* the solid angle $\Omega = 2\pi$, hence

$$\varphi_m = \tfrac{1}{2}i. \qquad \text{...(89)}$$

At a point inside a closed surface, the solid angle $\Omega = 4\pi$, hence

$$\varphi_m = i. \qquad \text{...(90)}$$

10.17 GALVANOMETERS

Instruments in which the magnetic effects are utilised for the detection and measurement of current, charge, potential difference, resistance and power, are termed as *galvanometers.* Ammeter, voltmeter, ohm-meters and watt-meters are the modified forms of the galvanometers, *e.g.,* an ammeter is a galvanometer in which a small resistance is connected in parallel with its coil. The main types of galvanometers are : *Moving coil dead beat or ballistic, Moving magnet, Unipivot, Vibration, String or Einthoven.*

Moving coil ballistic galvanometers are used for the measurement of charge. Unipivots are used in ammeters, voltmeters, and other portable instruments. Vibration galvanometers are used for the detection and measurement of small low-frequency alternating currents. String galvanometers are used for the detection of very small intermittent rushes of current and recording them, particularly in physiological work.

The properties of a galvanometer to be considered, when it is being chosen, for a special purpose are as follows : (*a*) *Periodic time,* (*b*) *Damping,* (*c*) *Zero-keeping qualities,* (*d*) *Resistance,* (*e*) *Optical system,* (*f*) *Independence of its motion by external magnetic fields and* (*g*) *Sensitivity.*

10.18 MOVING COIL GALVANOMETER

This type of galvanometer was first devised by Lord Kelvin in 1890 and was modified for laboratory use by D' Arsonval. The principle of a moving coil galvanometer is that when a current is passed through a coil, suspended freely in a magnetic field, it experiences forces in directions given by Fleming's left hand rule. These forces rotate the coil. The deflection in coil is measured, which gives us the strength of current.

A moving coil galvanometer consists of a rectangular flat coil (10 to 20 turns) of insulated fine Cu-wire wound on a light brass or aluminium frame. It is suspended between the poles of a permanent horse-shoe-shaped magnet by a fine conducting wire or thin flat strip of phosphor bronze, which also serves to conduct the current to and from the coil. The lower end of the wire is attached to a small spring (loosely wound helix) and the upper through the suspension to the torsion head at the top of the instrument. The axis of rotation of the coil coincides with the principal axis of symmetry of the magnetic field. To increase the magnetic flux in the gap, a cylindrical soft iron core is arranged within the coil.

The pole pieces of the magnet are cylindrically concave in shape so that the lines of force are radial in the narrow annular gap and the magnetic field **B** is perpendicular to the coil surface, in its any position. Either a pointer is attached to the coil or mirror is attached to the suspension wire. The whole instrument is enclosed to a non-magnetic metallic cover to protect it from air droughts. This cover has a central glass window to allow the light to go to and from the mirror, and three levelling screws at the bottom surface (base). Before using the galvanometer the coil is first released and the galvanometer is levelled properly, such that the coil swings freely between the pole pieces. The moving system is completely insulated from the metal case. The arrangement is shown in Fig. 10.35.

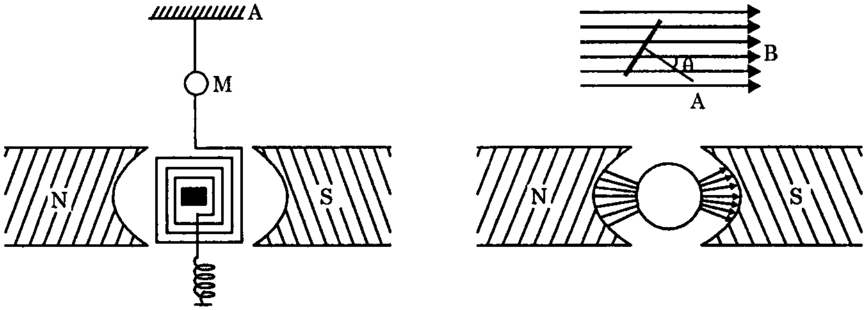


Fig. 10.35. Moving coil galvanometer.

When a current i enters the coil of N turns through the upper end of suspension A and leaves through the lower end B of the spring, horizontal and oppositely directed side forces are exerted on its vertical sides. These forces produce a couple about a vertical axis through its centre given by the equation

$$\tau = Ni\,(\mathbf{A} \times \mathbf{B}) \text{ or } \tau = Ni\,AB \sin\theta, \qquad ...(91)$$

where A is the area of the coil suspended in a magnetic induction B and θ is the angle between the normal to the plane of the coil and the field direction.

The coil rotates in the direction of this couple and comes to rest in such a position that the restoring torque exerted by the suspension wire equals the deflecting torque due to the side forces,

$$\textit{Restoring torque } c\,\theta = \textit{Deflecting couple } NiAB \sin\theta$$

or

$$i = \frac{c}{NAB}\frac{\theta}{\sin\theta}, \qquad ...(92)$$

where c is known as torque constant of the material of suspension wire. As the field is made radial by making the pole pieces cylindrical, the plane of the coil always remains parallel to the magnetic field **B** or $\mathbf{A} \perp \mathbf{B}$, thus we have

$$i = (c/NAB)\,\theta = K\theta, \qquad ...(93)$$

where K is called the *galvanometer constant.* Thus the deflection angle of the galvanometer is proportional to the current through it.

The deflection angle is observed with the aid of a beam of light reflected from a small mirror M cemented to the upper suspension, which is two times an angle θ through which the mirror is rotated. Reflected light is observed on a scale placed at a distance one metre from the galvanometer. If d is the deflection in millimeters of a light beam reflected by the galvanometer mirror on a scale 1 metre away, then,

$$i = K\,(d/2 \times 10^3) = K'd. \qquad ...(94)$$

where K' is called the figure of merit of the galvanometer. Currents of the order of 10^{-8} amp can be measured by a most sensitive galvanometer of this type.

Sensitivity of a galvanometer : A galvanometer is said to be sensitive if for a given small current, the deflection θ is sufficiently large and is defined as $d\theta/di$. Using Eq. (93), we get

$$d\theta/di = NAB/c. \qquad ...(95)$$

Thus we see that the higher sensitivity is achieved by increasing N, A and B and by decreasing c.

(i) No of turns N : The higher sensitivity is achieved by winding more turns on the coil, this increases the resistance of the coil and is undesirable in many uses. This also increases the mass of the coil which is limited by the tensile strength of the suspension. More turns are allowed by employing oxidised aluminium wire in place of Cu-wire in some galvanometers.

(ii) Area A : To increase galvanometer sensitity, the area cannot be increased indefinitely as it is limited by the separation between the poles, without decreasing B. Only due to this reason, rectangular coil of the galvanometer is always suspended in such a way that its smaller side is along the magnetic field direction.

(iii) Field B : The magnetic field induction **B** is increased by using a powerful laminated magnet of cobalt steel and by placing a soft iron cylinder within the coil. This field **B** must be so large that the deflection does not change by outer magnetic fields, such as horizontal component of each magnetic field, due to any small magnet placed in the neighbourhood of the coil.

(iv) Torque constant c : The torque constant c is decreased by using a thin and long suspension ($c = \pi\eta r^4/2l$). The suspension should be of such a material that it can support the coil and must have a great tensile strength. c is further decreased by using the suspension in the form of rectangular strip instead of the wire of circular cross section. The rectangular strip possesses an additional advantage of offering a comparatively large surface area for the dissipation of heat produced by the current.

Moving coils of copper or aluminium have a resistance temperature coefficient of about 0.4% per 1°C. To reduce it, a swamp resistance, in the form of a short fixed spiral of eureka wire of low temperature coefficient, is connected in series with the coil inside the instrument case. This swamp resistance is of greater value than that of moving coil, hence the change in the coil resistance with temperature produces small change in total resistance.

The advantages of these galvanometers are :

(i) The deflection of the coil is independent of direction and strengths of the external magnetic fields, hence does not require any adjustment in magnetic meridian and may be set up in any convenient position.

(ii) The coil moves in the radial field hence results the deflection directly proportional to the strength of the current passing through it. Thus an evenly divided scale can be used for measuring the current.

(iii) It can be made dead-beat or ballistic easily.

(iv) Its sensitivity can be increased without sacrificing accuracy.

(v) It is most useful for the detection of very small current of the order of 10^{-8} ampere.

(vi) It can be used as an ammeter and voltmeter.

10.19 AMMETERS AND VOLTMETERS

The ordinary ammeter and voltmeters are merely pivot galvanometers which have their sensitivities reduced in the proper manner. In pivot galvanometers the moving coil, instead of being suspended, is pivoted between two jewel bearings. These galvanometers are very useful as may be used in any position and are conveniently portable.

When the galvanometer is used as a current measuring instrument (ammeter), it is desirable that its resistance be very low compared with the total resistance in the circuit, so that the current in the circuit does not alter when the galvanometer is placed in series with source, etc., in the circuit. For this purpose the galvanometers are wound with a few turns of heavy wire to keep the resistance R_g as low as possible and a low resistance S is connected in parallel with the galvanometer. This small resistance in parallel with the galvanometer is known as *shunt*. The value of this shunt will depend upon the sensitivity of the galvanometer and thus upon the range I of an ammeter. If I_g is the total current through the galvanometer which can produce full scale deflection, then the resistance S of the shunt should be adjusted in such a way that the current $I - I_g$ passes through the shunt. At this stage

Potential across the shunt = Potential across the galvanometer coil

$$S(I - I_g) = I_g R_g$$

or $$S = [I_g/(I - I_g)] R_g. \qquad ...(96)$$

When a galvanometer is to be used as a voltmeter (voltage measuring instrument), its resistance must be much larger than any resistance across which it is to be placed. Otherwise the voltmeter will decrease the current flow in the circuit and the *p.d.* measured will be smaller than its actual value *Ir*. Galvanometers used for this purpose are wound with many turns of fine wire. The total resistance is increased by connecting a high resistance R in its series. The value of this resistance will depend upon the range of the voltmeter, such that the potential across the system of galvanometer and resistance in series

$$I_g (R + R_g) = V \text{ or } R = V/I_g - R_g. \qquad .(97)$$

In ideal case, $S \to 0$ and $R \to \infty$ for the ammeter and voltmeter respectively.

10.20 DEAD-BEAT AND BALLISTIC

The galvanometers may be classified according to their use as *dead beat and ballistic types*. For the measurement of current, a steady deflection proportional to the current is required. The spot of light on the scale should not oscillate about its final position before coming to rest. A device, introduced to prevent this undesirable oscillation, is known as *damping* and the instrument as *dead-beat*. The damping is produced by (*a*) viscosity of air and (*b*) eddy currents set up when the coil is wound on a metallic frame. On the other hand, ballistic galvanometers measure *charge*, (*i.e.*, sudden discharge and not a continuous current). In this case it is essential that the whole of the charge be passed through the coil before it begins to move, *i.e.*, the time of passage of charge must be smaller than the time occupied by the deflection. The coil therefore receives an impulse and a throw is observed. To achieve this a coil of high time period, at least 6 sec, is required. As time period $T = 2\pi \; (I/c)^{1/2}$, hence moving part should have a large moment of inertia I and the supension should be as of smaller torque constant c. The damping should also be small as possible so that coil can swing freely.

However a moving coil ballistic galvanometer can be damped by connecting a resistance across its terminals, as the motion of the coil in a magnetic field will induce a current which will oppose the motion of the coil. The damping of a galvanometer can be varied by altering the resistance R of the closed circuit in which it is placed. Critical damping is obtained by choosing R so that, the galvanomeer is just dead-beat and comes to rest without over-swing in a very short time.

Let us now investigate the condition, when a moving coil galvanometer can be used as dead-beat or a ballistic. Let θ be the deflection at any instant t when an instantaneous current i is passing through the coil at that instant. Different following couples act on the coil and keep the coil in equilibrium position.

(i) $I \, (d^2\theta/dt^2)$, *the retarding couple due to the moment of inertia I of the coil. (ii)* $a \, (d\theta/dt)$, *the retarding couple due to damping,* which is assumed to be proportional to the angular velocity $(d\theta/dt)$ of the coil, *(iii)* $NAB \, (d\theta/dt)$ $(= G \, d\theta/dt)$ the *electromagnetic damping* due to induced current, *(iv)* $L \, (di/dt)$, the *opposite emf due to the self inductance* L of the galvanometer and its circuit, *(v)* $NABi = Gi$, the *displacement couple* due to the current i. *(vi)* $c \, \theta$ the *restoring couple* due to the suspension fibre.

Assuming the factor due to self-inductance as negligible, the current due to the emf $\mathcal{E}$ applied to the circuit terminals.

$$i = \frac{1}{R}\left[\mathcal{E} - G\frac{d\theta}{dt}\right], \qquad \text{...(98)}$$

where R is the total resistance, *i.e.* the galvanometer resistance + external circuit resistance. Thus from Newton's law, we get

$$I\frac{d^2\theta}{dt^2} + a\left(\frac{d\theta}{dt}\right) + c\,\theta = Gi = \frac{G}{R}\left[\mathcal{E} - G\frac{d\theta}{dt}\right]$$

$$I\frac{d^2\theta}{dt^2} + \left(a + \frac{G^2}{R}\right)\frac{d\theta}{dt} + c\theta = \frac{G\mathcal{E}}{R}$$

or

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + k^2\theta = f. \qquad \text{...(99)}$$

where

$$2b = \frac{1}{I}\left(a + \frac{G^2}{R}\right), k^2 = \frac{c}{I} \text{ and } f = \frac{G\mathcal{E}}{IR}.$$

When the galvanometer is used for the measurement of steady current, then $G\mathcal{E}/R$ will be applied continuously and final deflection θ_0 will be given by Eq. (93). When the galvanometer is used ballistically for charge measurement, the emf $\mathcal{E} = 0$ and hence solution will be same as the complementary function of Eq. (99).

Let us now find the general solution of equation which is the sum of the complementary function and the particular integral. The complementary function is the solution of the equation

$$\frac{d^2\theta}{dt^2} + 2b\frac{d\theta}{dt} + k^2\theta = 0. \qquad \text{...(100)}$$

Let a solution of this equation be $\theta = A_0e^{\alpha t}$. Substituting the value of θ and its derivatives in the equation, we get.

$$A_0e^{\alpha t}\,[\alpha^2 + 2b\alpha + k^2] = 0.$$

As $A_0e^{\alpha t}$ cannot be zero, hence $\alpha^2 + 2b\alpha + k^2 = 0$

or

$$\alpha = -b \pm (b^2 - k^2)^{1/2} = -b \pm r$$

$\therefore$

$$\theta = A_1e^{(-b+r)t} + A_2e^{(-b-r)t}, \qquad \text{...(101)}$$

where A_1 and A_2 are constants and $r = (b^2 - k^2)^{1/2}$.

The particular integral will be $\theta = f/k^2$ as Eq. (99) is satisfied by the substitution of $\theta = f/k^2$, $d\theta/dt = 0$, $d^2\theta/dt^2 = 0$ in it. Hence the total solution now becomes

$$\theta = e^{-bt}[A_1e^{rt} + A_2e^{-rt}] + f/k^2. \qquad \text{...(102)}$$

When the instrument is used to measure steady current, the final deflection θ is obtained for large time t and the equation (102) thus gives $\theta = \theta_0 = f/k^2$. To find constants A_1 and A_2, consider that when $t = 0$, $\theta = 0$ and $d\theta/dt = 0$, hence Eq. (102) with $f/k^2 = \theta_0$ gives.

$$0 = A_1 + A_2 + \theta_0$$

and

$$d\theta/dt = 0 = -b\,[A_1 + A_2] + r\,[A_1 - A_2].$$

Solving these equations, we get

$$A_1 = -(r + b)\,\theta_0/2r \text{ and } A_2 = -(r - b)\,\theta_0/2r.$$

Thus equation (102) becomes

$$\theta = -e^{-bt}\left[\frac{r+b}{2r}e^{rt} + \frac{r-b}{2r}e^{-rt}\right]\theta_0 + \theta_0. \qquad \text{...(103)}$$

As $r = (b^2 - k^2)^{1/2}$. Hence there are three possible cases :

(a) **$b^2 > k^2$ or r is real.** Thus Eq. (103) indicates that the galvanometer deflection θ increases gradually from 0 to the steady value θ_0 and motion is *non-oscillatory or dead-beat* and the galvanometer is known as *overdamped (b, the damping term is large).*

(b) **$b^2 = k^2$ or r is zero.** The galvanometer deflection returns to rest point in the minimum time and the velocity never changes in direction. The motion is just non-oscillatory and the galvanometer is known as *critical damped.* In this case

$$\left[\frac{1}{2I}\left(a + \frac{G^2}{R}\right)\right]^2 = \frac{c}{I}.$$

If the circuit resistance is responsible for the appreciable damping, then a can be neglected and we get $c = G^4/4IR^2$.

(c) **$b^2 < k^2$ or r is imaginary.** Putting $r = (b^2 - k^2)^{1/2} = j\sqrt{(k^2 - b^2)} = jp$ in Eq. (102), we have

$$\begin{aligned}\theta &= e^{-bt}[A_1 e^{jpt} + A_2 e^{-jpt}] + \theta_0 \\ &= e^{-bt}(A_1 + A_2)\cos pt + j(A_1 - A_2)\sin pt] + \theta_0 \\ &= Ce^{-bt}\sin(pt + \phi) + \theta_0. \qquad \text{...(104)}\end{aligned}$$

where $C = (A_1 + A_2)^2 - (A_1 - A_2)^2]^{1/2}$ and $\phi = \tan^{-1}(A_1 + A_2)/j(A_1 - A_2)$.

Eq. (104) shows that the motion of the galvanometer is simple harmonic, having time period

$$T = \frac{2\pi}{p} = \frac{2\pi}{\sqrt{(k^2 - b^2)}} = 2\pi\left[\frac{c}{I} - \frac{(a + G^2/R)^2}{4I^2}\right]^{-1/2} \qquad \text{...(105)}$$

and of which amplitude dies away, leaving $\theta = \theta_0$. For the galvanometers used for charge measurement there is no continuous source of emf, *i.e.*,

$$\begin{aligned}\mathcal{E} &= 0 \text{ and thus } \theta_0 = 0. \text{ Hence} \\ \theta &= C\,e^{-bt}\sin\left[(k^2 - b^2)^{1/2}\,t + \phi\right]. \qquad \text{...(106)}\end{aligned}$$

The galvanometer deflection oscillates about its zero position and these oscillations are known as *under-damped oscillations* and the galvanometer is known as *oscillatory* or **ballistic**. Thus for ballistic motion of a galvanometer $b^2 < k^2$, *i.e.*,

$$\frac{1}{2I}\left[a + \frac{G^2}{R}\right] < \left(\frac{c}{I}\right)^{1/2} \qquad \text{...(107)}$$

or *I*, *R* and *c* should be large and *a* and G^2 should be smaller.

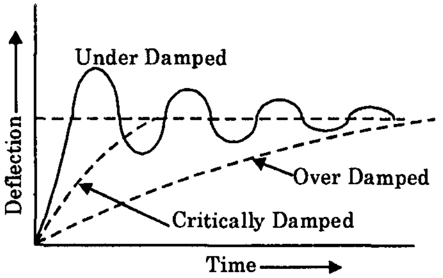


Fig. 10.36. Motions of galvanometer for various damping constants.

The motions of the galvanometer coil in above mentioned three cases are shown in Fig. 10.36.

10.21 BALLISTIC GALVANOMETER

Ballistic galvanometer is the instrument for the mesurement of the charge sent through it in a short interval of time. For the correct measurement of charge, whole discharge should pass before the galvanometer has moved appreciably, time of swing [$T = 2\pi\sqrt{(I/c)}$] should be greater than the time of discharge. The damping should be small for oscillatory motion. The basic conditions with the B.G. are discussed in the preceding article.

Both moving magnet and moving coil types of ballistic galvanometers are used, but the latter are more common to accept. The principle and construction of such galvanometers are the same as those of dead-beat moving coil galvanometers, except that :

(*i*) The time period T is large and hence the moment of inertia I of the coil is large and torsional couple c is small. To increase I, copper wire coil is used and to decrease c, a thin and long phosphor strip suspension is used.

(*ii*) The total damping is very small. The damping due to viscosity of air is very small. When coil circuit is closed, induced currents are set up in the coil circuit and the frame, which will damp the oscillations of the coil. This type of damping is called *electromagnetic damping*. It can be reduced to zero by using non-conducting frame, such as of ivory, celluloid or bamboo.

Consider a galvanometer coil of turns N, area A, and moment of inertia I suspended from a torsion wire of torque constant c in a magnetic induction field $\mathbf{B}$. Suppose a current pulse is sent through the coil. The current is at first zero, then rises to maximum and then comes to zero in such a short time Δt that the coil does not move through an appreciable angle while the current pulse exists but rotates only after current has ceased. During the current pulse, the torque, at any instant when current is i, is given by $\tau = Ni\mathbf{A} \times \mathbf{B}$ which is $Ni\,AB$ when the plane of coil is parallel to the magnetic field, *i.e.*, $\mathbf{A} \perp \mathbf{B}$.

The angular impulse given to the galvanometer coil is equal to the change in angular momentum and is given by

$$NAB \int_0^{\Delta t} i\,dt = NABq = I\omega, \qquad \text{...(108)}$$

where ω is the angular velocity with which the coil begins to rotate.

The initial angular momentum $I\omega$ corresponds to an initial kinetic energy of the coil and is converted to elastic potential energy of the torsion fibre which is the energy spent in twisting the suspension wire. Hence

$$K.E. = \frac{1}{2} I\omega^2 = P.E. = \int_0^{\theta_0} c\theta\, d\theta = \frac{1}{2} c\theta_0^{\,2}$$

$$\text{or} \qquad I\omega^2 = c\theta_0^{\,2}. \qquad \text{...(109)}$$

As we know that $T = 2\pi\sqrt{(I/c)}$ or $I = cT^2/4\pi^2$, hence by eliminating ω from Eqs. (108) and (109) and then substituting this value of I, we get.

$$N^2A^2B^2q^2 = c^2T^2\,\theta_0^{\,2}/4\pi^2$$

$$\text{or} \qquad q = \frac{T}{2\pi}\frac{c}{NAB}\theta_0 = \frac{T}{2\pi}K\theta_0 = K'\theta_0, \qquad \text{...(110)}$$

where K' is a constant, known as *ballistic constant or ballistic reduction factor and* θ_0 is the first throw or ballistic throw. If θ_0 is measured in mm on the scale with the lamp and scale arrangement, then K' is defined as the charge required to produce one mm deflection of light reflected from the mirror attached with the suspension wire on the scale placed at a distance of 1 meter. It is also known as *figure of merit* of galvanometer. Thus

Figure of merit of ballistic galvanometer = $(T/2\pi)$ × *Figure of merit of dead beat galvanometer.*

Correction of damping. In deducing the relation (110) we have assumed that the whole of the K.E. is used in twisting the suspension of the coil. But we know that there are damping forces (frictional damping and electromagnetic damping), in overcoming of which some part of K.E. is used. Thus the swings get progressively shorter and shorter. If θ_1, θ_2, θ_3 ... be the successive swings to left and right, it is found experimentally that the ratio of the consecutive deflections is a constant quantity, *i.e.*

$$\theta_1/\theta_2 = \theta_2/\theta_3 = \theta_3/\theta_4 = \dots = d \text{ (const.)} \qquad \text{...(111)}$$

This constant is known as *decrement* and its logarithm as *logarithmic decrement* which is denoted as λ. Hence

$$\theta_1/\theta_2 = \theta_2/\theta_3 = \theta_3/\theta_4 = \ldots e^{\lambda} \; [As \; \log_e d = \lambda \; or \; d = e^{\lambda}]$$

For a complete vibration,

$$\frac{\theta_1}{\theta_3} = \frac{\theta_1}{\theta_2} \times \frac{\theta_2}{\theta_3} = d^2 = e^{2\lambda}.$$

Thus the value of decrement when coil completes a quarter of a vibration will be $e^{\lambda/2}$ and hence the first true deflection θ_0 is given by

$$\theta_0/\theta_1 = e^{\lambda/2} \text{ or } \theta_0 = \theta_1 e^{\lambda/2} = \theta_1 (1 + \lambda/2).$$

Here we have neglected higher order terms, as λ is very small.

Hence the relation for charge with damping correction is

$$q = \frac{T}{2\pi} \frac{c}{NAB} \theta_1 \left(1 + \frac{\lambda}{2}\right). \qquad \ldots(112)$$

Measurement of ballistic reduction factor. The ballistic galvanometer, K' of which is to be determined, is connected with a battery and a capacitor of known capacity C with the help of Morse key, as shown in Fig. 10.37. The capacitor is first charged by pressing knob A of the Morse Key. If the key is then released, the battery circuit is broken and the capacitor is discharged through the ballistic galvanometer. If the deflection is out of scale use a rheostat as a potential divider. If θ_1 is the first throw of the galvanometer with a potential difference V, then we get

$$q = CV = K'\theta_1 (1 + \lambda/2)$$

$$\text{or} \quad K' = CV/\theta_1 (1 + \lambda/2) \; Coulomb/radian. \qquad \ldots(113)$$

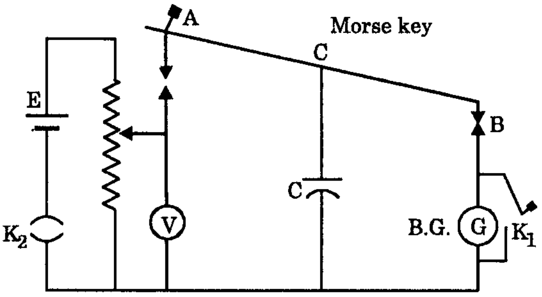


Fig. 10.37. Determination of K.

λ can be calculated by observing first and eleventh throws of ballistic galvanometer coil and by using the relation

$$\frac{\theta_1}{\theta_{11}} = \frac{\theta_1}{\theta_2} \times \frac{\theta_2}{\theta_3} \ldots \frac{\theta_9}{\theta_{10}} \times \frac{\theta_{10}}{\theta_{11}} = e^{10\lambda}$$

$$\text{or} \quad \lambda = \frac{1}{10} \log_e \frac{\theta_1}{\theta_{11}} = \frac{2.3026}{10} \log_{10} \frac{\theta_1}{\theta_{11}} = \frac{2.3026}{10} \log_{10} \frac{d_1}{d_{11}}, \qquad \ldots(114)$$

where d_1 and d_{11} are the throws of the spot of light on the scale. Hence by substituting numerical values of C, V, λ and θ_1 (= $d_1/2D$), we can calculate ballistic reduction factor K'.

Uses of Ballistic Galvanometer

(a) Comparision of capacitances : The circuit is connected as shown in Fig. 10.38, C_1 and C_2 are the capacitances to be compared. The plug in K_2 is first put to the left and the capacitor C_1 is charged with emf $\mathcal{E}$ by pressing knob A of Morse key. After releasing the knob, the capacitor C_1 is discharged through the ballistic galvanometer. If θ_1 is the first throw, then

$$q_1 = C_1\mathcal{E} = K'\theta_1 (1 + \lambda/2).$$

Similarly the experiment is repeated with capacitor C_2 by pressing Morse key with the plug on the right only in key K_2 and then releasing. If θ_2 is the first throw in this case, then

$$q_2 = C_2\mathcal{E} = K' \theta_2 (1 + \lambda/2).$$

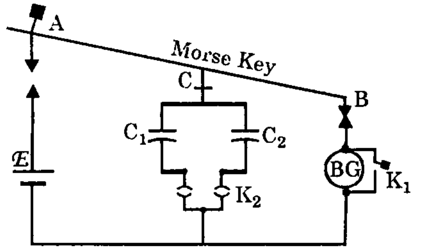


Fig. 10.38. Comparision of capacitors.

$$\therefore \quad C_1/C_2 = \theta_1/\theta_2 = d_1/d_2, \qquad ...(115)$$

where d_1 and d_2 are the first throws of the light spot on the scale with capacitors C_1 and C_2 respectively.

(*b*) ***Measurement of capacitance of a capacitor*** : The capacitance of a given capacitor may be measured, if a capacitor of known capacitance is also available, using the method discussed above. Otherwise the following method is used. The connections are made as shown in Fig. 10.39, C is the unknown capacitor, R is a high resistance (~ 10,000 Ω) and r is a standard resistance of 1Ω. A steady current i flows through these resistances.

Initially key K is kept open and the knob A of the Morse key is pressed down. The capacitor C is now charged to a potential difference iR. This charge will be

$$q = C(iR).$$

If the knob A is released the charge of the capacitor will go to the ballistic galvanometer. If θ_1 is the first throw in the coil, then

$$q = CiR = \frac{T}{2\pi}\cdot\frac{c}{NAB}\theta_1\left(1+\frac{\lambda}{2}\right)$$

or
$$C = \frac{T}{2\pi}\cdot\frac{c}{NAB}\frac{1}{iR}\theta_1\left(1+\frac{\lambda}{2}\right) \qquad ...(116)$$

For the unknown values of c, N, A and B, a steady current i_g is allowed to pass through the galvanometer, by closing key K. If ϕ be the steady deflection, then

$$i_g = (c/NAB)\,\phi \text{ or } c/NAB = i_g/\phi.$$

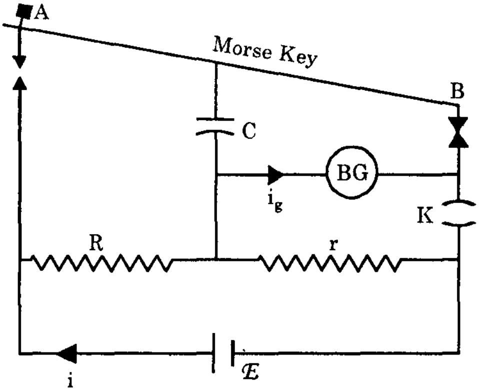


Fig. 10.39. Measurement of capacitance.

If G be the galvanometer resistance, which is shunted by the resistance r, the current through the galvanometer is given by

$$i_g = [r/(r+G)]\,i = (r/G)\,i. \qquad (\text{As } r < G)$$

$$\therefore \quad c/NAB = ri/G\phi.$$

Substituting this value of c/NAB in Eq. (116), we get

$$C = \frac{T}{2\pi}\frac{1}{iR}\frac{ri}{G\phi}\theta_1\left(1+\frac{\lambda}{2}\right) = \frac{T}{2\pi}\frac{r}{RG}\frac{\theta_1}{\phi}\left(1+\frac{\lambda}{2}\right). \qquad ...(117)$$

Hence C can be calculated by measuring these quantities.

10.22 SENSITIVITY OF GALVANOMETER

(1) Current sensitivity : The current sensitivity of a galvanometer is defined as the deflection in millimeters produced on a scale placed at a distance of one metre from the mirror, of a dead beat galvanometer, when a current of 10^{-6} ampere passes through the galvanometer coil. Thus,

$$\text{current sensitivity } S_i = d\theta/di = NAB/c. \qquad ...(118)$$

(2) Charge sensitivity : It is defined as the deflection in millimeters produced on a scale placed at a distance of one metre from the mirror of a ballistic galvanometer, when a charge of 10^{-6} coulomb is allowed to pass through the galvanometer coil. Thus

$$\text{Charge sensitivity } S_q = \frac{d\theta}{dq} = \frac{NAB}{c}\frac{2\pi}{T}\frac{1}{(1+\lambda/2)} = \frac{NAB}{c}\frac{2\pi}{T}$$

$$= (2\pi/T) \times \text{current sensitivity } S_i. \qquad ...(119)$$

Here $\lambda/2$ is neglected in comparison to 1 as λ is very small.

(3) Voltage sensitivity : It is defined as the deflection in millimeters produced on a scale at a distance of one metre from the mirror by the potential of one microvolt. Thus

$$\textit{Voltage sensitivity } S_v = d\theta/dV = d\theta/Rdi = S_i/R. \qquad \text{...(120)}$$

Hence S_v is the current sensitivity divided by the circuit resistance.

(4) The meg-ohm sensitivity : It is the resistance in meg-ohms placed in series with the galvanometer in order to get a deflection of 1 mm on the scale of one metre apart by a potential difference of 1 volt.

(5) Factor of merit : When the sensitivity has been reduced to allow for time period T and resistance R_g of the galvanometer, it is called *normal sensitivity or factor of merit.*

If the number of turns is increased to N by keeping area A constant, the length of the winding will increase N times but the cross sectional area of the wire must be necessarily reduced by N. Thus the resistance R_g of the coil will correspondingly increase by N^2 ($\because R = \rho l/A$). As

$$S_i = \frac{NAB}{c} = \frac{NABT^2}{4\pi^2 I} = NT^2 \times \text{const.}$$

$$\propto R_g^{1/2}\, T^2.$$

$$\therefore \qquad \text{Factor of merit} = 100\, D/T^2 R_g^{1/2}, \qquad \text{...(121)}$$

where D is the deflection in millimeters on a scale at one metre from the coil, for current sensitivity due to 1m A current.

EXERCISES

Example 1. *The earth's magnetic field at some place on the earth has a magnitude of 56,900 γ with a dip angle of 71° and a declination of 6° W. Find the force on an electron that moves horizontally and due south in the earth's magnetic field with a velocity of 10^7 m/sec.*

Let the geometrical north is along the x-axis and west along the y-axis. Thus the components of earth's magnetic field in rectangular coordinates

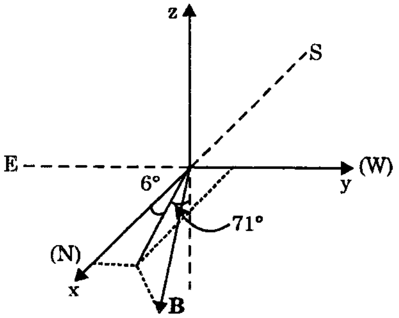


Fig. 10.40. Example 1.

$$\begin{aligned} B_x &= (56900\,\gamma)\cos(-71^\circ)\cos 6^\circ \\ &= 18400\,\gamma = 1.84 \times 10^{-5}\,T \\ B_y &= (56900\,\gamma)\cos(-71^\circ)\sin 6^\circ \\ &= 1940\,\gamma = 1.94 \times 10^{-6}\,T \\ B_z &= (56900\,\gamma)\sin(-71^\circ) \\ &= -53800\,\gamma = -5.38 \times 10^{-5}\,T. \end{aligned}$$

Thus we may write **B** as

$$\mathbf{B} = (18.4\,\mathbf{i} + 1.94\,\mathbf{j} - 53.8\,\mathbf{k}) \times 10^{-6} \text{ tesla}$$

Charge of the electron $q = -e = -1.6 \times 10^{-19}\,C$,

Velocity of the electron

$$\mathbf{v} = -10^7\,\mathbf{i} \text{ m/s}.$$

Since the force on the electron

$$\mathbf{F} = -e\mathbf{v} \times \mathbf{B}$$

$$\begin{aligned} \therefore \qquad \mathbf{F} &= (-1.6 \times 10^{-19})(-10^7\,\mathbf{i}) \times (18.4\,\mathbf{i} + 1.94\,\mathbf{j} - 53.8\,\mathbf{k}) \times 10^{-6} \\ &= (86.1\,\mathbf{j} + 3.1\,\mathbf{k}) \times 10^{-18}\,N. \end{aligned}$$

Example 2. *Calculate the force between two parallel circular coaxial coils of nearly the same size being separated by a small distance in free space. For what distance between the coils is this force a maximum.*

The force between two equal parallel circular coaxial coils, separated by a distance d (< R), and carrying currents i_1 and i_2 is given by

$$F_0 = \mu_0 i_1 i_2 / 2\pi d,$$

where F_0 is the force per unit length of the one coil due to the other coil. The radius of the coil is R, then net force

$$F = 2\pi R F_0 = \mu_0\, i_1 i_2 R/d.$$

If the coils have radii R_1 and R_2, Fig. 10.41, ($R_1 \sim R_2$). The force per unit length of coil B is

$$F_0 = \mu_0\, i_1 i_2/2\pi l,$$

where l is the distance along AB. By symmetry, the component of F_0 normal to the x-axis will cancel out when the entire coil B is considered. Thus the force on the whole coil B due to coil A is

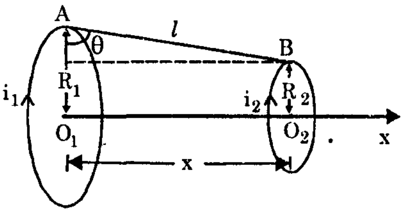


Fig. 10.41. Example 2.

$$F = F_0 \sin\theta .\, 2\pi R_2$$

$$= \frac{\mu_0 i_1 i_2}{2\pi l}\cdot\frac{x}{l}\cdot 2\pi R_2 = \frac{\mu_0 i_1 i_2 R_2 x}{x^2 + (R_1 - R_2)^2}$$

The force will be maximum, when $dF/dx = 0$

$$\text{or} \qquad \frac{d}{dx}\left[\frac{x}{x^2 + (R_1 - R_2)^2}\right] = 0 \quad \text{or} \quad x^2 = (R_1 - R_2)^2$$

$$\text{or} \qquad x = R_1 \sim R_2.$$

i.e., the force will be maximum when the distance between the centres of the coils is equal to the difference of their radii.

Example 3. *A 5.0 MeV proton is falling vertically downward in uniform field of magnetic induction 1.5 webers/m², pointing horizontally from south to north. Find the force exerted on it.*

We know that the mass of proton = 1.7×10^{-27} kg and charge on it = 1.6×10^{-19} coul. Given that the kinetic energy of the proton

$$K.E. = \tfrac{1}{2}\, mv^2 = 5\, MeV = 5 \times 1.6 \times 10^{-13}\,\text{joule}$$

$$\therefore \qquad \text{velocity } v \text{ of the proton} = \sqrt{\left(\frac{2K.E.}{m}\right)} = \sqrt{\frac{2 \times 8 \times 10^{-13}}{1.7 \times 10^{-27}}}$$

$$= 3.07 \times 10^7 \text{ m/sec.}$$

The magnetic force on the proton $\mathbf{F} = q\mathbf{v} \times \mathbf{B}$ and its magnitude

$$F = qvB \sin\theta = (1.6 \times 10^{-19}) \cdot (3.07 \times 10^7)(1.5) \sin 90^\circ$$

$$= 7.37 \times 10^{-12}\, N.$$

The direction of $\mathbf{F}$ will be $\perp$ to $\mathbf{v}$ and $\mathbf{B}$ both. As $\mathbf{v}$ is in downward direction and $\mathbf{B}$ is from south to north horizontally, hence $\mathbf{F}$ will be in horizontal plane in west-east direction.

Example 4. *An electron is moving with a velocity 2i + 3j m/sec in an electric field of intensity 3i + 6j + 2k and a magnetic field of 2j + 3k tesla. Find the magnitude and direction of the Lorentz force acting on the electron.*

Lorentz force $\mathbf{F} = q\,[\mathbf{E} + \mathbf{v} \times \mathbf{B})$.

In our problem, $q = -1.6 \times 10^{-19}$ coul, $\mathbf{E} = 3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}$ *and* $\mathbf{B} = 2\mathbf{j} + 3\mathbf{k}$. Therefore

$$\mathbf{F} = -1.6 \times 10^{-19}\,[3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} + (2\mathbf{i} + 3\mathbf{j}) \times (2\mathbf{j} + 3\mathbf{k})]$$

$$= -1.6 \times 10^{-19}\,[3\mathbf{i} + 6\mathbf{j} + 2\mathbf{k} + 9\mathbf{i} - 6\mathbf{j} + 4\mathbf{k}]$$

$$= -1.6 \times 10^{-19}\,[12\mathbf{i} + 6\mathbf{k}] = -9.6 \times 10^{-19}\,[2\mathbf{i} + \mathbf{k}].$$

Its magnitude $F = 9.6 \times 10^{-19}\sqrt{4+1} = 2.15 \times 10^{-18}$ newton.

It is in the $x-z$ plane, making an angle θ with the x-axis, where $\theta = \cos^{-1}\left(-2/\sqrt{5}\right)$.

Example 5. *Find the force on the portion of the current carrying wire shown in Fig. 10.42.*

The forces on the parts *PQ* and *ST* of the current carrying wire are zero as these are parallel to the direction of **B**. Situation is different along the arc *QRS*. Consider the small element of arc $d\mathbf{l}$. The force on this element is

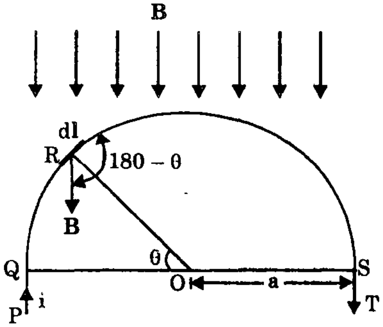


Fig. 10.42. Example 5.

$$d\mathbf{F} = i d\mathbf{l} \times \mathbf{B},$$

or $$dF = i\,dl\,B \sin(180-\theta) = i\,dl\,B \sin\theta.$$

Its direction must be ⊥ to the plane containing $d\mathbf{l}$ and **B**, *i.e.*, into the page. Each little element of arc will contribute a force in the same direction hence the total force

$$F = \int dF = \int iB \sin\theta\,dl = iB \int \sin\theta\,dl.$$

If $d\theta$ be the angle subtended by the element at *O*, then $dl = a\,d\theta$.

$$\therefore\ F = iB \int \sin\theta a\,d\theta = iBa \int_0^{\pi} \sin\theta\,d\theta = 2iBa.$$

This is the same force that a straight wire *QS* would experience.

Example 6. *A current strip 2.0 cm wide carrying a current of 10 amp is placed along the x-axis. Find the force on the strip per unit length if the uniform magnetic field B = 0.2 T is applied along the y-axis.*

Force on a current carrying conductor is given by the relation

$$d\mathbf{F} = i\,d\mathbf{l} \times \mathbf{B}.$$

The filament current $i d\mathbf{l}$ may be replaced by the sheet current $\mathbf{K}dS$. Thus for the strip

$$d\mathbf{F} = (\mathbf{K}\,dS) \times \mathbf{B}.$$

Here $\mathbf{K} = (10/0.02)\,\hat{x}$, $\mathbf{B} = 0.2\ \hat{y}$ and $dS = dxdy$,

$$\therefore\quad d\mathbf{F} = (10/0.02)\,(0.2)\,dxdy\,\hat{z}$$

or $$\mathbf{F} = \int_{-0.01}^{0.01} \int_0^{l} 100\,dxdy\,\hat{z} = 2l\hat{z}\ \text{N/m}$$

or $$\text{Force/length } F/l = 2\hat{z}\ \text{N/m}.$$

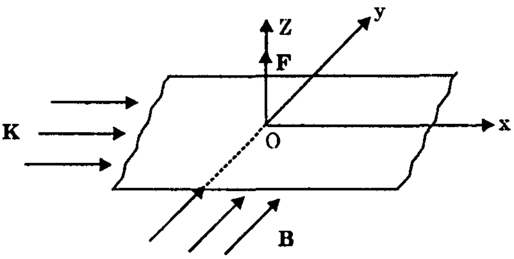


Fig. 10.43. Example 6.

Example 7. *Find the total force on the circuit (Fig. 10.44) due to the magnetic field acting in downward direction ⊥ to the paper plane.*

The force on the straight wire portion *PR* is $F_1 = i(PR)B = iLB$. It is directed downward in the plane of the page. Consider a small element $d\mathbf{l}$ of the arc. The force on this element $dF_2 = iBdl$. It is directed along *OQ*. Since $d\mathbf{F}_2$ for various elements are in different directions. The horizontal component of $d\mathbf{F}_2$ will be cancelled by an equal and opposite component due to a symmetric element on the other side of the arc. Therefore the total force

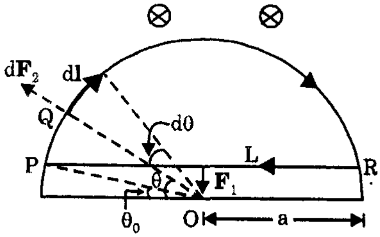


Fig. 10.44. Example 7.

$$F_2 = F_{2y} = \int dF_2 \sin\theta = \int iB \sin\theta\,dl.$$

As dl can be written as $a\,d\theta$, hence

$$F_{2y} = iaB \int \sin\theta\,d\theta = iaB \int_{\theta_0}^{180-\theta_0} \sin\theta\,d\theta$$

$$= 2ia\,B \cos\theta_0 = iBL. \qquad (\text{as } a\cos\theta_0 = L/2)$$

Thus, we see that F_{2y} is equal and opposite to F_1 and the total force is hence zero. It is true for any closed circuit in a uniform magnetic field.

Example 8. *Find the expression for the force acting between two current carrying coplanar parallel wires.*

Let us consider two coplanar parallel wires each of length L and separated by a distance x.

If the currents in these wires flow in the same direction then the magnetic field at point O due to the current element $d\mathbf{l}_2$ of the second wire is given as

$$d\mathbf{B}_2 = \frac{\mu_0}{4\pi} i_2 \frac{d\mathbf{l}_2 \times \mathbf{r}_{12}}{r_{12}^3}.$$

In this magnetic field current element $d\mathbf{l}_1$ experiences a force

$$d^2\mathbf{F} = \frac{\mu_0}{4\pi} i_1 i_2 \frac{d\mathbf{l}_1 \times [d\mathbf{l}_2 \times \mathbf{r}_{12}]}{r_{12}^3}.$$

Therefore, the total force on the first wire due to the current in the entire length of the second wire

$$\mathbf{F} = \frac{\mu_0}{4\pi} i_1 i_2 \iint \frac{d\mathbf{l}_1 \times [d\mathbf{l}_2 \times \mathbf{r}_{12}]}{r_{12}^3}.$$

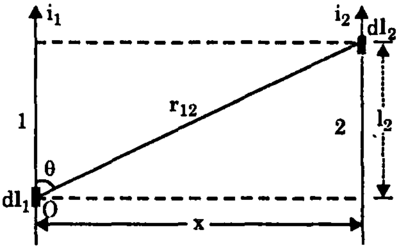


Fig. 10.45. Example 8.

Using the geometry of the figure, the vector product $d\mathbf{l}_2 \times \mathbf{r}_{12}$ has a magnitude $dl_2\,(x^2 + l_2^2)^{1/2} \sin\theta$ and direction normal to the page in outward direction. Therefore $d\mathbf{F}$ must be perpendicular to $d\mathbf{l}_1$ and in the plane of paper towards the second wire.

$$F = \frac{\mu_0}{4\pi} i_1 i_2 \int dl_1 \int \frac{dl_2 (x^2 + l_2^2)^{1/2} \sin\theta}{(x^2 + l_2^2)^{3/2}}.$$

Force on a unit length of first wire due to the entire length of second wire

$$f = \frac{\mu_0 i_1 i_2}{4\pi} x \int_{-L/2}^{L/2} \frac{dl_2}{(x^2 + l_2^2)^{3/2}} = \frac{\mu_0 i_1 i_2}{2\pi x [1 + 4\,(x/L)^2]^{1/2}}$$

If these wires are of infinite length, then we have

$$f = \mu_0\, i_1 i_2 / 2\pi x.$$

Example 9. *A dc-current of 5 amp is flowing through a hollow cylindrical straight conductor (pipe) of diameter one meter. Calculate the mechanical pressure acting normally inward on the surface because of parallel current.*

The wall circumference of the cylindrical conductor may be considered as a large number of parallel current carrying conductors which attract each other. Let I be current passing along the pipe of radius R, the current per unit width is $I/2\pi R$. Therefore the current along the width dx at point A will be $Idx/2\pi R$. Consider an element of width dl at point P. As clear from the geometry of Fig. 10.46,

$$dl \cos\theta = r d\theta \text{ or } dl = r d\theta / \cos\theta.$$

$\therefore$ Current through the element $dl = (r\, d\theta/\cos\theta)\,(I/2\pi R)$

Therefore the force per unit length between two parallel currents through the elements dx and dl is given by

$$dF = \frac{\mu_0}{2\pi} \frac{Idx}{2\pi R} \cdot \frac{Ird\theta}{2\pi R \cos\theta} \cdot \frac{1}{r}$$

Inward component of this force will be $dF \cos\theta$. The total force on the element of width dx due to the whole circumferential elements

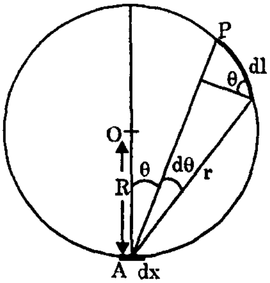


Fig. 10.46. Example 9.

$$F = \int_{-\pi/2}^{\pi/2} \frac{\mu_0}{2\pi} \frac{I^2}{4\pi^2 R^2} dx \frac{r d\theta . \cos\theta}{\cos\theta \, r}$$

or force per unit area of cylinder is

$$F/dx \times 1 = F/dx = \int_0^{\pi/2} 2 . \frac{\mu_0}{2\pi} \frac{I^2}{4\pi^2 R^2} d\theta = \frac{\mu_0 I^2}{8\pi^2 R^2}$$

or $$\text{Pressure } P = \frac{\mu_0 I^2}{8\pi^2 R^2} = \frac{4\pi \times 10^{-7} \times 5^2}{8\pi^2 \times (0.5)^2} = 1.59 \times 10^{-9} \text{ newton/meter}^2.$$

Example 10. *Prove that Newton's third law is obeyed in the interaction between any pair of current carrying circuits, although it is not valid when applied to the forces between two differential current elements.*

Consider two differential current elements, one in each loop. The magnetic field at the current element $i_1 \, d\mathbf{l}_1$ due to $i_2 \, d\mathbf{l}_2$ is given by Biot Savart law as

$$d\mathbf{B}_2 = \frac{\mu_0}{4\pi} \frac{i_2 \, d\mathbf{l}_2 \times \mathbf{r}_{12}}{r_{12}^3} .$$

Thus the force on the current element $i_1 \, d\mathbf{l}_1$ is given by

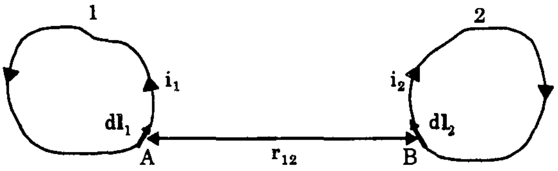


Fig. 10.47. Example 10.

$$d^2\mathbf{F}_{12} = i_1 \, d\mathbf{l}_1 \times d\mathbf{B}_2 = \frac{\mu_0}{4\pi} i_1 i_2 \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12})}{r_{12}^3} .$$

Similarly we can find the expression for the force on current element $i_2 \, d\mathbf{l}_2$ due to $i_1 \, d\mathbf{l}_1$ as

$$d^2\mathbf{F}_{21} = \frac{\mu_0}{4\pi} i_1 i_2 \frac{d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{r}_{21})}{r_{21}^3}$$

Since $r_{12} = r_{21}$, but $d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12}) \neq - d\mathbf{l}_2 \times (d\mathbf{l}_1 \times \mathbf{r}_{21})$, therefore $d^2\mathbf{F}_{12} \neq - d^2\mathbf{F}_{21}$, and thus the Newton's third law is not valid in this situation. As isolated current elements do not exist, therefore to make a proper test of Newton's third law, we should calculate the mutual forces between two current loops, not simply between differential current elements.

The total force $\mathbf{F}_{12}$ that acts on loop 1 due to the magnetic field produced by loop 2 is given by

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} i_1 i_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12})}{r_{12}^3} .$$

Using the vector identity $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C}) \, \mathbf{B} - (\mathbf{A} \cdot \mathbf{B}) \, \mathbf{C}$, the integrand may be written as $d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{r}_{12})/r_{12}^3 = [(d\mathbf{l}_1 \cdot \mathbf{r}_{12}) \, d\mathbf{l}_2 - (d\mathbf{l}_1 \cdot d\mathbf{l}_2) \, \mathbf{r}_{12}]/r_{12}^3$

The first term gives rise to the integral

$$\oint_1 \oint_2 \frac{(d\mathbf{l}_1 \cdot \mathbf{r}_{12}) d\mathbf{l}_2}{r_{12}^3} = \oint_2 d\mathbf{l}_2 \left(\oint_1 \frac{\mathbf{r}_{12} . d\mathbf{l}_1}{r_{12}^3} \right)$$

The integral over loop 1 has the form of a closed line integral of a time independent inverse square law field $(\mathbf{r}_{12}/r_{12}^3)$. Such field is conservative and thus the closed line integral vanishes.

$$\therefore \qquad \mathbf{F}_{12} = - \frac{\mu_0 i_1 i_2}{4\pi} \oint_1 \oint_2 \frac{\mathbf{r}_{12} \, (d\mathbf{l}_1 \cdot d\mathbf{l}_2)}{r_{21}^3}$$

Similarly the force acting on the loop 2 due to the magnetic field produced by loop 1 is given by

$$\mathbf{F}_{21} = - \frac{\mu_0 i_1 i_2}{4\pi} \oint_1 \oint_2 \frac{\mathbf{r}_{21} \, (d\mathbf{l}_2 \cdot d\mathbf{l}_1)}{r_{21}^3} .$$

Since $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = d\mathbf{l}_2 \cdot d\mathbf{l}_1$ and $\mathbf{r}_{12} = -\mathbf{r}_{21}$, therefore $\mathbf{F}_{12} = -\mathbf{F}_{21}$. Thus we see that the Newton's third law is valid for the physically realizable condition of complete current loops.

Example 11. *Prove that the torque acting on the flat current loop placed in a magnetic field is independent of the shape of the loop.*

Consider a flat loop of wire in xy-plane. Magnetic field $\mathbf{B}$ is acting in yz–plane and making an angle θ with the z-axis. Let us find the force on the small length ds_1 making an angle ϕ_1 with the y-axis. The force F_{1z} acting on this element due to field component B_y is

$$F_{1z} = iB_y\, ds_1 \sin\phi_1.$$

The force acting due to component B_z is

$$F_{1y} = iB_z\, ds_1.$$

Similarly forces on the element ds_2 due to field components B_y and B_z are

$$F_{2z} = iB_y\, ds_2 \sin\phi_2 \text{ and } F_{2y} = iB_z\, ds_2.$$

As elements ds_1 and ds_2 are very small, hence the lever arm of the forces F_{1y} and F_{2y} is very small and no torque will act due to these forces.

The forces F_{1z} and F_{2z} are equal as $ds_1 \sin\phi_1 = ds_2 \sin\phi_2 = ds$ and directed in the $-z$ and $+z$ directions respectively and hence constitute a couple. The moment of the couple or the torque

$$d\tau = iB_y\, ds\,(y_1 - y_2)$$

$$= iB_y \times \text{area of the strip in the loop.}$$

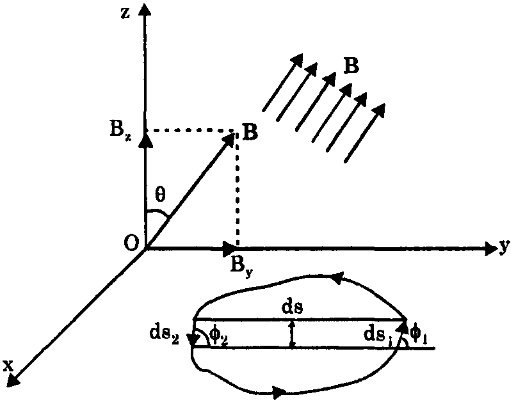


Fig. 10.48. Example 11.

It is along x-axis as area of the strip is having vector direction along z-axis. If we divide the whole loop into large number of elements and consider oppositely placed pairs and find $d\tau$ for each pair, then the total torque acting on the loop.

$$\tau = iB_y \text{ (area of the loop } A) = iBA \sin\theta.$$

or $$\tau = i\,(\mathbf{A} \times \mathbf{B}).$$

It is same as obtained for a rectangular loop, hence the statement is correct.

Example 12. *A bar magnet of length 2l having magnetic moment of magnitude m is dropped through a thick conducting, non magnetic and infinitely long cylinder and falls along the axis of the cylinder. Derive an expression for the emf induced in the ring PP' of radius r (a < r < b), where a and b are the inner and outer radii respectively of the cylinder.*

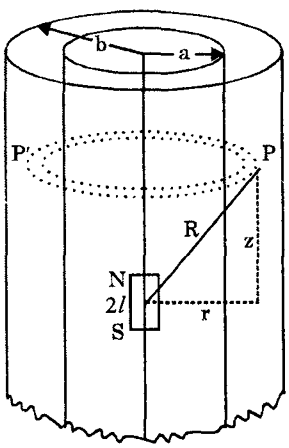


Fig. 10.49. Example 12.

The magnetic field due to the falling dipole at point P is given by

$$\mathbf{B} = B_r\hat{r} + B_z\hat{z}.$$

By symmetry the net radial component is zero, therefore the useful component B_z is given by

$$B_z = \frac{\mu_0}{4\pi}\frac{m}{2l}\left[\frac{z-l}{R_1^3} - \frac{z+l}{R_2^3}\right]$$

where $R_1^2 = r^2 + (z-l)^2$ and $R_2^2 = r^2 + (z+l)^2$.

The magnetic flux through a ring of radius r at a distance z from the centre of the magnet along the axis of the cylinder is

$$\Phi = \int_0^r B_z\, 2\pi r\, dr = \int_0^r \frac{\mu_0 m}{8\pi l}\left[\frac{z-l}{R_1^3} - \frac{z+l}{R_2^3}\right] 2\pi r dr$$

$$= \frac{\mu_0 m}{4l}\left[\int_0^r \frac{z-l}{R_1^3}\, r dr - \int_0^r \frac{z+l}{R_2^3}\, r dr\right]$$

Let $(z-l)^2 + r^2 = p, \quad \therefore \quad 2rdr = dp$

and $(z+l)^2 + r^2 = q, \quad \therefore \quad 2rdr = dq$

$$\Phi = \frac{\mu_0 m}{4l}\left[\int \frac{(z-l)\, dp}{2p^{3/2}} + \int \frac{(z+l)\, dq}{2q^{3/2}}\right]$$

$$= \frac{\mu_0 m}{4l}\left[\frac{(z-l)}{2}\left(\frac{p^{-1/2}}{-1/2}\right)_{(z-l)^2}^{(z-l)^2+r^2} - \frac{z+l}{2}\left(\frac{q^{-1/2}}{-1/2}\right)_{(z+l)^2}^{(z+l)^2+r^2}\right]$$

$$= \frac{\mu_0 m}{4l}\left[\frac{z+l}{R_2} - \frac{z-l}{R_1}\right]$$

The induced emf in the ring of radius r is given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = -v\frac{d\Phi}{dz} = -\frac{\mu_0 mv}{4l}\frac{d}{dz}\left[\frac{z+l}{R_2} - \frac{z-l}{R_1}\right]$$

$$= -\frac{\mu_0 mv}{4l} r^2 \left[\frac{1}{R_2^3} - \frac{1}{R_1^3}\right]$$

Here v is the velocity of the magnet at this instant.

Example 13. *The flat dielectric disc of radius a carries an excess charge on its surface of σ coul/m^2. Consider the disc to rotate around the axis passing through its centre and ⊥ to its plane with angular speed ω rad/sec. If a magnetic field **B** is directed ⊥ to the rotation axis, show that the torque which acts on the disc is σωπBa^4/4.*

Let the disc be placed in the x-y plane with the magnetic field along y-axis and is rotated about the z-axis. Consider an annular ring of radius r and of thickness dr on this disc. Charge with in this ring = $\sigma\, 2\, \pi r\, dr$.

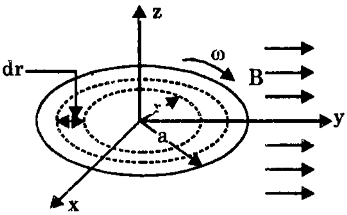


Fig. 10.50. Example 13.

As ring rotates with angular velocity ω, the current

$$i = dq/dt = \sigma\, 2\, \pi\, r\, dr\, (\omega/2\pi)$$
$$= \sigma\, \omega\, r dr.$$

As the torque acting on the current loop of area d**A** placed in a magnetic field **B** is

$$d\tau = i(d\mathbf{A}) \times \mathbf{B} = i\, dA\, (-\mathbf{k}) \times B(\mathbf{j}) = i\, dA\, B\, \mathbf{i}.$$

Hence the torque on this annular ring $d\tau = \sigma\omega r dr \pi r^2 B$

and $$\tau = \int_0^\pi \sigma\pi\omega B r^3 dr = \frac{\sigma\pi\omega B a^4}{4}.$$

It will be along x-direction.

Example 14. *A long straight vertical wire carries a current of 30 amperes directed upwards. Calculate the position of neutral point, if the horizontal component of earth's magnetic field is 2 × 10^{-5} web / m^2.*

We know that the magnetic field **B** due to a long straight wire carrying a current i at a distance r *from* the wire is given as

$$B = \mu_0 i/2\pi r.$$

The direction of this field is perpendicular to the plane consisted of the wire and the distance vector **r**. Thus for a vertical wire, the field will be in horizontal direction. According to this problem, this is balanced by the horizontal component of earths magnetic field B_H, Therefore

$$B_H = \mu_0 i/2\pi r \quad \text{or} \quad r = \mu_0 i/2\pi B_H.$$

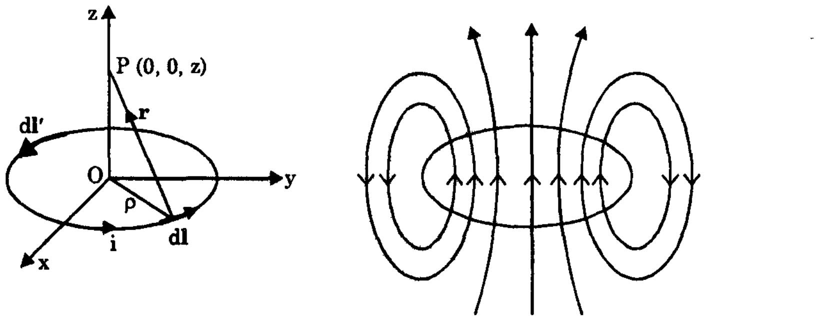
Substituting the numerical values, we get

$$r = 4\pi \times 10^{-7} \times 30/2\,\pi \times 2 \times 10^{-5}$$
$$= 0.30 \text{ meter} = 30 \text{ cm.}$$

Example 15. *A circular loop located on $x^2 + y^2 = 16$, $z = 0$ carries a direct current of 5 amp along $\mathbf{a}_\phi$. Determine* **B** *at (0, 0, 3) and (0, 0, –3). Draw the magnetic lines of forces due to this loop.*

The magnetic field $d\mathbf{B}$ at a point P (0, 0, z) due to the current element idl is given by Biot-Savart law, as

$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{id\mathbf{l} \times \mathbf{r}}{r^3}$$



(*a*) Circular current loop (*b*) Magnetic lines of forces due to this current loop

Fig. 10.51. Example 15.

In cylindrical coordinates (ρ, φ, z),

$$d\mathbf{l} = \rho d\phi\, \mathbf{a}_\phi,\ \mathbf{r} = -\rho\, \mathbf{a}_\rho + z\, \mathbf{a}_z \text{ and } x^2 + y^2 = \rho^2,$$

where $\mathbf{a}_\rho$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$ are the unit vectors along the ρ, φ and z-directions respectively.

$$\therefore \qquad d\mathbf{l} \times \mathbf{r} = \begin{vmatrix} \mathbf{a}_\rho & \mathbf{a}_\phi & \mathbf{a}_z \\ 0 & \rho d\phi & 0 \\ -\rho & 0 & z \end{vmatrix} = \rho z d\phi \mathbf{a}_\rho + \rho^2 d\phi \mathbf{a}_z.$$

Therefore
$$d\mathbf{B} = \frac{\mu_0}{4\pi}\frac{i}{(\rho^2 + z^2)^{3/2}}(\rho z d\phi \mathbf{a}_\rho + \rho^2 d\phi \mathbf{a}_z)$$

By symmetry the contribution along $\mathbf{a}_\rho$ due to an element $d\mathbf{l}$ is balanced by due to an oppositely placed element $d\mathbf{l}'$. Thus net contribution along $\mathbf{a}_\rho$ is zero. Thus

$$\mathbf{B} = \int d\mathbf{B} = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{i\rho^2 d\phi}{(\rho^2 + z^2)^{3/2}}\mathbf{a}_z = \frac{\mu_0 i\rho^2}{2(\rho^2 + z^2)^{3/2}}\mathbf{a}_z$$

Substituting the numerical values, we get

$$\mathbf{B} \text{ at } (0, 0, 3) = \frac{4\pi \times 10^{-7} \times 5 \times 4^2}{2\,(4^2 + 3^2)^{3/2}} = 1.28\pi \times 10^{-7} \text{ Web/m}^2.$$
$$\mathbf{B} \text{ at } (0, 0, -3) = 1.28\pi \times 10^{-7} \text{ Web/m}^2$$

The magnetic lines of forces are sketched in Fig. 10.51(*b*).

Example 16. *A straight segment OC of length l meter of a circuit carrying a current i amp is placed along the x-axis. Two infinitely long straight wires A and B are placed symmetrically on the y-axis at a and –a meter respectively. If the each wire carries a current i amp into the plane of the paper, find the force acting on the segment OC. What will happen if the current in the wire B is reversed ?*

Consider a current element of length dx at a distance x from the origin O at point P. The magnetic field due to the current i in the wire A at this point P

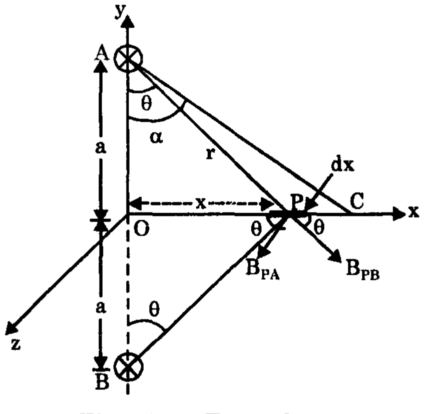


Fig. 10.52. Example 16.

$$\mathbf{B}_{PA} = \frac{\mu_0}{4\pi}\frac{2i}{r} \text{ in } xy \text{ plane } \perp \text{ to } \mathbf{r}.$$

Its horizontal (x–) component will cancel with the horizontal component of the magnetic field B_{PB} at point P due to the current in wire B. Thus the net magnetic field will be in the $-y$ direction.

$$\mathbf{B}_P = \frac{\mu_0}{2\pi}\frac{2i}{r}\sin\theta\,(-\mathbf{j}) = \frac{\mu_0}{\pi}\frac{i}{a}\cos\theta\sin\theta\,(-\mathbf{j}).$$

The force due to this field on the current element dx at point P

$$d\mathbf{F} = i\,d\mathbf{x}\times\mathbf{B}_P = \frac{\mu_0}{\pi}\frac{i^2}{a}\cos\theta\sin\theta\,dx\,(-\mathbf{k}).$$

Since $x = a\tan\theta$, $\therefore\ dx = a\sec^2\theta\,d\theta$.

Total force acting on segment OC

$$\mathbf{F} = \frac{\mu_0}{\pi}i^2(-\mathbf{k})\int\tan\theta d\theta = \frac{\mu_0}{\pi}i^2(-\mathbf{k})[\log\sec\theta]_0^\alpha$$

or

$$\mathbf{F} = \frac{\mu_0}{\pi}i^2\log[\sec\alpha](-\mathbf{k}) = \frac{\mu_0}{2\pi}i^2\log\left(\frac{a^2+l^2}{a^2}\right)(-\mathbf{k}).$$

If the current in wire B is reversed, the net magnetic field at point P will be in $-x$ direction. Thus the force on the segment due to magnetic field will be zero.

Example 17. *Two straight wires carry currents i_1 and i_2 enter into the page and are at a distance d apart. Find magnetic field at any point R and the force between these wires.*

The fields $\mathbf{B}_1$ and $\mathbf{B}_2$ due to the wires P and Q are in the plane of the paper as wires are $\perp$ to it and are $\perp$ to the radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively.

$$B_1 = \mu_0 i_1/2\pi r_1 \quad\text{and}\quad B_2 = \mu_0 i_2/2\pi r_2.$$

Taking x- and y-components, we get

$$B_x = B_{1x} + B_{2x} = \frac{\mu_0}{2\pi}\left(\frac{i_1}{r_1}\sin\theta_1 + \frac{i_2}{r_2}\sin\theta_2\right)$$

and

$$B_y = B_{1y} + B_{2y} = \frac{\mu_0}{2\pi}\left(-\frac{i_1}{r_1}\cos\theta_1 + \frac{i_2}{r_2}\cos\theta_2\right)$$

The resultant magnetic field will be of magnitude $B = (B_x^2 + B_y^2)^{1/2}$ and making an angle θ, with the x-axis, given by the relation $\theta = \tan^{-1} B_y/B_x$.

$$B = \frac{\mu_0}{2\pi}\left[\frac{i_1^2}{r_1^2} + \frac{i_2^2}{r_2^2} - \frac{2i_1i_2}{r_1r_2}\cos(\theta_1+\theta_2)\right]^{1/2}$$

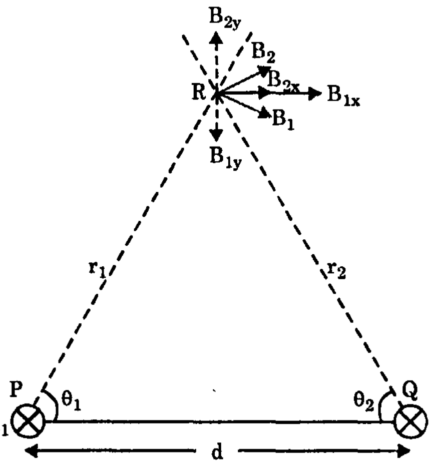


Fig. 10.53. Example 17.

The magnetic field at Q due to the current in wire P is

$$B = \mu_0 i_1/2\pi d.$$

As $\mathbf{B}$ is $\perp$ to wire Q, the force on a length L of wire Q is

$$\mathbf{F} = i_2\,\mathbf{L}\times\mathbf{B} \text{ or } F = i_2 L\mu_0 i_1/2\pi d,$$

or the force per unit length $F/L = \mu_0 i_1 i_2/2\pi d$.

Example 18. *Given 3 straight, infinitely long, equally spaced thin wires, each carrying a current i in the same direction. Calculate the location of two zeros in the magnetic field. If the middle wire is displaced rigidly a very small distance x upward while the other wires are held fixed, describe the motion of the middle wire.*

Assuming the given wires as coplaner, the points of zero magnetic field must also be located in the same plane. Let the distance of such a point P from the middle wire be x.

Thus for a point P of zero magnetic field, the Ampere's circuital law gives

$$\frac{\mu_0 i}{2\pi(d+x)}+\frac{\mu_0 i}{2\pi x} = \frac{\mu_0 i}{2\pi(d-x)}$$

$$\therefore \qquad x = \pm\ d/\sqrt{3}$$

Thus there are two points of zero magnetic field which are at a distance $d/\sqrt{3}$ from the central wire on the either side of it.

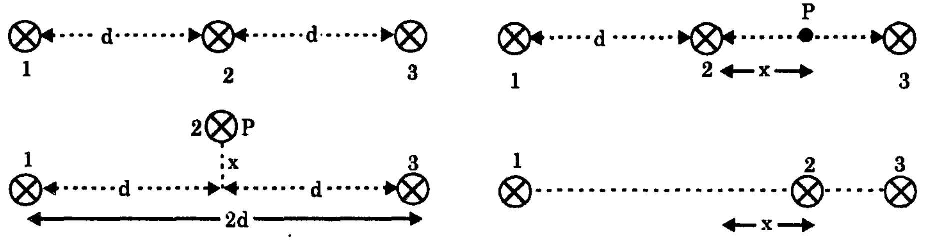


Fig. 10.54. Example 18.

When the middle wire is displaced a small distance x in the same plane, a resultant force per unit length on the wire is

$$f = \frac{\mu_0 i^2}{2\pi\,(d+x)}-\frac{\mu_0 i^2}{2\pi\,(d-x)} = \frac{\mu_0 i^2 (2x)}{2\pi\,(d^2-x^2)}$$

As $x < d$, this force is approximately given as

$$f \simeq \frac{\mu_0 i^2 x}{\pi d^2} \quad \text{or} \quad f \propto x.$$

As the force is proportional to the displacement, hence the motion is simple harmonic about the equilibrium position, with a periodic time $T = 2\pi\,(\pi m/\mu_0)^{1/2}\,d/i$.

The problem can also be solved by assuming the displacement in the normal direction and will have

$$f = -\frac{\mu_0 i^2 x}{\pi d^2} \text{ and same value of } T.$$

Example 19. *In the Bohr model of the hydrogen atom the electron circulates around the nucleus in a path of radius 5.1 × 10^{-11} metre at a frequency ν of 6.8 × 10^{15} rev./sec. (i) What value of B is set up at the centre of the orbit? (ii) What is the equivalent magnetic dipole moment ?*

We know that the current is defined as the rate of flow of charge, therefore in terms of frequency (ν) current

$$i \;=\; e\nu = 1.6\times10^{-19}\times6.8\times10^{15} = 1.09\times10^{-3}\ \text{amp}.$$

The electron circulating around the nucleus is equivalent to a current carrying circular loop, where the loop radius is equal to the radius of the Bohr orbit. Thus the magnetic field B at the centre of the orbit is given by

$$B = \frac{\mu_0 i a^2}{2(x^2+a^2)^{3/2}} = \frac{\mu_0 i}{2a} \qquad (\because x = 0)$$

$$= 4\pi \times 10^{-7} \times 1.09 \times 10^{-3}/2 \times 5.1 \times 10^{-11} = 13.4 \text{ weber/meter}^2.$$

The current carrying circular loop is equivalent to a magnetic dipole, with magnetic dipole moment

$$p_m = NiA = 1 \times 1.09 \times 10^{-3} \times 3.14 \times (5.1 \times 10^{-11})^2$$

$$= 8.9 \times 10^{-24} \text{ amp. meter}^2.$$

Example 20. *A wire shaped to a regular haxagon of side 2 cm carries a current of 2 amp. Find the magnetic induction at the centre of the hexagon.*

Let each side of the hexagon be b and the corners are at a distance a from the centre. The magnetic field at O due to element AB is thus given by the relation

$$\mathbf{B} = \frac{\mu_0 i}{4\pi}\sum \frac{d\mathbf{l} \times \mathbf{r}}{r^3}.$$

It is clear from the vector product that **B** points away from the reader in the direction normal to the plane of paper (inwards). It is the case for all six elements. Hence resultant field will be 6 times the magnetic field due to single current element. Therefore, the total field

$$B = 6\frac{\mu_0 i}{4\pi}\sum \frac{dl \sin\theta}{r^2} = 6\frac{\mu_0 i}{4\pi R}(\sin\theta_1 + \sin\theta_2).$$

Here $\theta_1 = \theta_2 = 30°$, $R = GO = a\sqrt{3/2}$.

$$\therefore \quad B = 6\frac{\mu_0 i}{4\pi a\sqrt{3}/2}\left[\frac{1}{2}+\frac{1}{2}\right] = \frac{\sqrt{3}\,\mu_0 i}{\pi a}$$

$$= \frac{1.732 \times 4\pi \times 10^{-7} \times 2}{\pi \times 2 \times 10^{-2}}$$

$$= 6.93 \times 10^{-5} \text{ wb/m}^2.$$

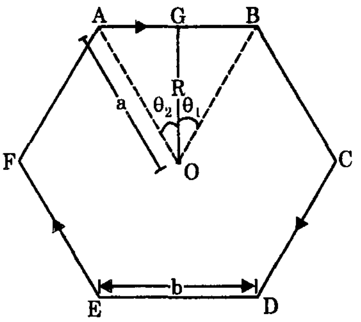


Fig. 10.55. Example 20.

Example 21. *A wire loop carrying a current i is placed in the x-y plane as shown in Fig. 10.56. If a particle with charge q and mass m is placed at the centre 0 and given a velocity v along* **NO**, *find its instantaneous acceleration. (b) If an external uniform magnetic induction field* **B** = B**i** *is applied, find the force and the torque acting on the loop due to this field.*

In $\Delta\, MON$, given $\angle MON = 120°$, $MO = NO$ therefore

$$\angle OMN = \angle ONM = 30°.$$

The length of perpendicular from O to $MN = a \sin 30° = a/2$.

The magnetic field $\mathbf{B}_1$ at O due to the current i in the chord NM is given as

$$\mathbf{B}_1 = \frac{\mu_0}{4\pi}\frac{i(\cos 30° + \cos 30°)}{a/2}(-\mathbf{k}) = \frac{\mu_0 i}{4\pi}\frac{2\sqrt{3}}{a}(-\mathbf{k}).$$

The field at O due to arc MN

$$\mathbf{B}_2 = \frac{\mu_0}{4\pi}\frac{i}{a}\cdot\frac{2\pi}{3}\mathbf{k}.$$

$\therefore$ Net field $\mathbf{B} = \mathbf{B}_1 + \mathbf{B}_2$

$$= \frac{\mu_0}{4\pi}\frac{i}{a}\left[-2\sqrt{3} + \frac{2\pi}{3}\right]\mathbf{k} = \frac{\mu_0}{4\pi}\frac{i}{a}(1.369)(-\mathbf{k})$$

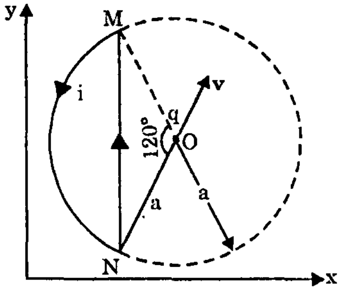


Fig. 10.56. Example 21.

The force acting on a charge q moving with a velocity v along NO.

$$\mathbf{F} = q\mathbf{v}\times\mathbf{B} = q\,v\,B\,\hat{n},$$

where $\hat{n}$, is the unit vector perpendicular to **NO** and **B**, *i.e.,* in the xy-plane away from the loop. The acceleration of the particle

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{qvB}{m}\hat{n} = \frac{1.369\times10^{-7}\,qvi}{ma}\hat{n}.$$

When the magnetic field **B** is applied in the x-direction, the loop placed in x-y plane will experience zero net force. It will however experience a torque τ which is given by

$$\tau = i\,\mathbf{A}\times\mathbf{B} = i\left(\tfrac{1}{3}\pi a^2 - \tfrac{1}{2}\times 2a\cos30^\circ\times a\sin30^\circ\right)\mathbf{k}\times B\,\mathbf{i}$$

$$= ia^2\left(\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right)B\,\mathbf{j}.$$

Example 22. *A wire carrying a current i is in the shape of a plane curve $r = f(\theta)$, (r, θ) being polar coordinates. Find the magnetic field at the pole. Deduce also the expression for the field at the focus of an elliptical wire carrying a current i.*

We know that

$$\mathbf{B} = \frac{\mu_0}{4\pi}i\int\frac{d\mathbf{l}\times\mathbf{r}}{r^3} = \frac{\mu_0}{4\pi}i\int\frac{dl\sin\phi}{r^2}\hat{n},$$

where ϕ is the angle between $d\mathbf{l}$ and $\mathbf{r}$.

As $\sin\phi = rd\theta/dl$, therefore

$$B = \frac{\mu_0}{4\pi}i\int\frac{rd\theta}{r^2} = \frac{\mu_0 i}{4\pi}\int\frac{d\theta}{r}.$$

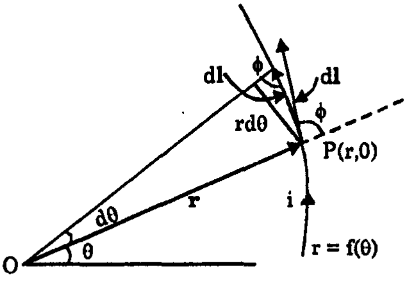


Fig. 10.57. Example 22.

The integration being extended over the length of the wire. In the second part of question current carrying wire is in the elliptical form, hence

$$r = l/(1 + e\cos\theta).$$

$$\therefore \qquad B = \frac{\mu_0}{4\pi}\frac{i}{l}\int_0^{2\pi}(1+e\cos\theta)\,d\theta = \frac{\mu_0 i}{2l}.$$

Example 23. *Find the magnetic field intensity at a point P, which is at a distance b from the centre of a circular conductor of radius a and carrying a current i.*

By Biot-Savart law, we have

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int\frac{d\mathbf{l}\times\mathbf{r}}{r^3}.$$

But the circle is a plane curve, hence by the result of example (22) we have

$$B = \frac{\mu_0}{4\pi}i\int_0^{2\pi}\frac{d\theta}{r}.$$

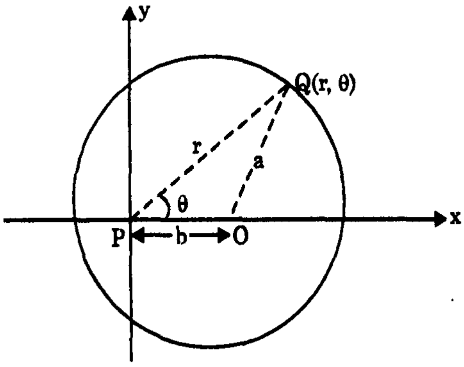


Fig. 10.58. Example 23.

Here point P is the origin. PO is along x-axis and Q is the point with polar coordinates r, θ on the circle. Thus from geometry of figure 10.58, we have

$$a^2 = r^2 + b^2 - 2br\cos\theta$$

or $$r^2 - 2br\cos\theta + b^2 - a^2 = 0.$$

or $$\frac{1}{r} = \frac{1}{b\cos\theta + \sqrt{(a^2 - b^2\sin^2\theta)}}$$

$$= \frac{\sqrt{(a^2 - b^2\sin^2\theta)} - b\cos\theta}{a^2 - b^2}$$

$\therefore$

$$B = \frac{\mu_0}{4\pi} i \int_0^{2\pi} \frac{\sqrt{(a^2 - b^2 \sin^2\theta)} - b\cos\theta}{a^2 - b^2} d\theta = \frac{\mu_0}{4\pi}\frac{a}{(a^2-b^2)} i \int_0^{2\pi} \left(1 - \frac{b^2\sin^2\theta}{a^2}\right)^{1/2} d\theta$$

$$= \frac{\mu_0}{4\pi}\frac{a}{a^2-b^2} i f_e\left(\frac{b}{a}, 2\pi\right),$$

where f_e is an elliptic function.

If $b/a << 1$, then

$$\left(1 - \frac{b^2\sin^2\theta}{a^2}\right)^{1/2} = 1 - \frac{b^2\sin^2\theta}{2a^2} = 1 - \frac{b^2}{4a^2}(1-\cos 2\theta)$$

$$\therefore \quad B = \frac{\mu_0}{4\pi}\frac{a}{a^2-b^2} i \int_0^{2\pi}\left(1 - \frac{b^2}{4a^2}\right) d\theta = \frac{\mu_0}{8}\frac{(4a^2-b^2)}{a(a^2-b^2)} i.$$

Example 24. *An infinitely long circular cylinder of radius 3 a has an infinitely long cylindrical hole of radius a such that its centre is at a distance a from the centre of the big cylinder. The solid part of the cylinder carries a current i, distributed uniformly over the cross section and out from the plane of the paper. Find (a) the magnetic field at all points on the plane P containing the axes of the cylinders. (b) The magnetic field throughout the hole.*

Let us consider the long cylinder along the z-axis. Let $\mathbf{j}_1$ and $\mathbf{j}_2$ be the current densities due to the currents i_1 and i_2 in the small and large cylinders respectively.

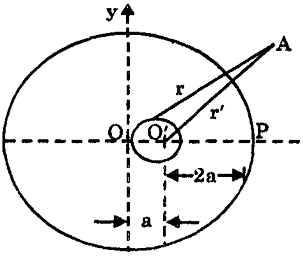


Fig. 10.59. Example 24.

$$\therefore \quad i = i_2 - i_1 = 9\pi a^2 j - \pi a^2 j = 8\pi a^2 j$$

(Here $\mathbf{j}_1 = -\mathbf{j}$ and $\mathbf{j}_2 = \mathbf{j}$.)

$$\therefore \quad j = i/8\pi a^2 \text{ and therefore } i_1 = \pi a^2 j = i/8$$

$$\text{and} \quad i_2 = 9\pi a^2 j = (9/8)\, i.$$

The axes of the cylinders are assumed in the z-direction. If their centres are assumed along the x-axis, then the plane P will be a xz-plane. If r and r' be the distances of the field point from the cylinder and hole respectively, then

$$r = (x^2 + y^2)^{1/2} \text{ and } r' = [(x-a)^2 + y^2]^{1/2}$$

We know that the magnetic field B inside the solid cylinder is given by

$$B(r) = \mu_0 Ir/2\pi R^2, \qquad (r \le R)$$

$$\therefore \quad B_{2x} = \frac{-\mu_0 i_2 y}{2\pi(3a)^2} = -\frac{\mu_0 iy}{16\pi a^2}, B_{2y} = \frac{\mu_0 ix}{16\pi a^2}$$

Outside the cylinder, $B(r) = \mu_0 I/2\pi r = \mu_0 Ir/2\pi r^2$,

$$\therefore \quad B_{2x} = -\frac{9\mu_0 iy}{16\pi(x^2+y^2)} \text{ and } B_{2y} = \frac{9\mu_0 ix}{16\pi(x^2+y^2)} \qquad (r > 3a)$$

Similarly for the hollow cylinder, we get

$$B_{1x} = -\frac{\mu_0 iy}{16\pi a^2}, B_{1y} = \frac{\mu_0 i(x-a)}{16\pi a^2} \qquad (r \le a)$$

$$B_{1x} = -\frac{\mu_0 iy}{16\pi[(x-a)^2 + y^2]}, B_{1y} = \frac{\mu_0 i(x-a)}{16\pi[(x-a)^2+y^2]} \qquad (r > a)$$

On the plane P, $y = 0$ and $B_{2x} = B_{1x} = 0$, hence

$$B_x = 0 \text{ and } B_y = B_{2y} - B_{1y}.$$

We therefore have

(*i*) Inside the solid part ($2a \le x \le 3a$, and $-3a \le x \le 0$)

$$B_y = \frac{\mu_0 ix}{16\pi a^2} - \frac{\mu_0 i(x-a)}{16\pi\left[(x-a)^2+y^2\right]} = \frac{\mu_0 i(x^2-ax-a^2)}{16\pi a^2(x-a)}$$

(*ii*) Inside the hole ($0 < x < 2a$)

$$B_y = \frac{\mu_0 ix}{16\pi a^2} - \frac{\mu_0 i(x-a)}{16\pi a^2} = \frac{\mu_0 i}{16\pi a}.$$

(*iii*) Outside the cylinder ($|x| > 3a$)

$$B_y = \frac{9\mu_0 ix}{16\pi(x^2+y^2)} - \frac{\mu_0 i(x-a)}{16\pi[(x-a)^2+y^2]} = \frac{\mu_0(8x-9a)i}{16\pi x(x-a)}.$$

(*b*) The magnetic field at all points inside the hole ($r \le a$) is

$$B_x = -\frac{\mu_0 iy}{16\pi a^2} + \frac{\mu_0 iy}{16\pi a^2} = 0$$

and

$$B_y = \frac{\mu_0 ix}{16\pi a^2} - \frac{\mu_0 i(x-a)}{16\pi a^2} = \frac{\mu_0 i}{16\pi a}.$$

This field is uniform inside the hole and is along the positive y-direction.

Example 25. *A long cylindrical conductor of radius a has a cylindrical hole of radius b (<a), the axis of the hole is parallel to the axis of the conductor. If the remaining part of the conductor has a uniform current density j parallel to the axis of the conductor. Show that the magnetic field induction* ***B*** *inside the hole is uniform.*

There is no current density inside the hole, while it is **j** *outside* the hole but inside the conductor. This may be explained as a superposition of **j** into the paper and **j** out of the paper. Thus the magnetic field at any point P inside the hole of radius b and at a distance r ($r < a$) from the centre O of the conductor

$$\mathbf{B}_P = \mathbf{B}'_P + \mathbf{B}''_P$$

where $\mathbf{B}'_P$ is due to entire conductor with current density **j** and $\mathbf{B}''_P$ is due to the current density (**–j**) inside the hole (cavity)

$$\therefore \quad \mathbf{B}_P = \left(\tfrac{1}{2}\mu_0\mathbf{j}\times\mathbf{OP}\right) + \tfrac{1}{2}\mu_0(-\mathbf{j})\times\mathbf{O'P}$$

$$= \tfrac{1}{2}\mu_0\mathbf{j}\times(\mathbf{OP}-\mathbf{O'P}) = \tfrac{1}{2}\mu_0\mathbf{j}\times(\mathbf{OP}+\mathbf{PO'})$$

$$= \tfrac{1}{2}\mu_0\mathbf{j}\times\mathbf{OO'}.$$

Since **OO′**, the distance between the axis of the conductor and the axis of the hole is constant, therefore the magnetic field **B** inside the hole is uniform and is normal of **OO′** and **j** both. This problem may also be solved as example 24.

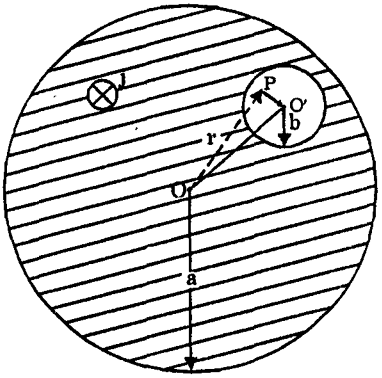


Fig. 10.60. Example 25.

Example 26. *Two short solenoids, each of radius 6 cm, and length 13 cm, uniformly wound with 30 turns of wire per cm are placed coaxially with a gap of 6 cm between them. If these are connected in series. Find the magnetic field at the centre of the gap due to the current of 1 ampere through the combination.*

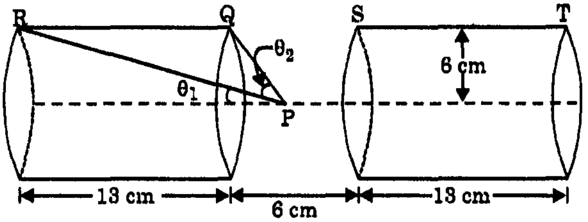


Fig. 10.61. Example 26.

Magnetic field at any point P on the axis of the solenoid is given as

$$B_1 = \tfrac{1}{2}\mu_0 ni(\cos\theta_1 - \cos\theta_2).$$

The field is in the direction along the

symmetry axis. The field will be same due to both the solenoids as they are identical, have been placed symmetrically and the equal current is passing through them.

$$\therefore \quad B = B_1 + B_2 = 2B_1 = \mu_0 ni\,(\cos\theta_1 - \cos\theta_2)$$

$$= 4\times 3.14\times 10^{-7}\times 3000\times 1\left[\frac{16}{\sqrt{(16^2+6^2)}} - \frac{3}{\sqrt{(3^2+6^2)}}\right]$$

$$= 1.84\times 10^{-3}\ \text{web/m}^2.$$

Example 27. *A thin dielectric rod is bent into the form of a circle of radius a. If charge per unit length on the rod is λ and the circle is rotated round its axis at a speed of n rotations per second, find the value of **B** at point P on the axis at a distance b from the centre.*

The moving charge caused by the rotation of the circle constitutes the current. By the definition of current the flow of charge per sec through any area of cross section

$$i = (2\pi an)\lambda.$$

This problem reduces to : **B** at a point on the axis of a circular coil carrying current i. We know from article 10.9 (b) that

$$B = \frac{\mu_0}{2}\frac{ia^2}{(a^2+b^2)^{3/2}} = \frac{\mu_0(2\pi an\lambda)a^2}{2(a^2+b^2)^{3/2}}$$

$$= \mu_0\pi a^3 n\lambda/(a^2+b^2)^{3/2}.$$

Example 28. *Derive an expression for **B** at a distance x from the axis of a long cylindrical wire of radius a and carrying a current i_0.*

The cross section of the wire, ⊥ to its length, is shown in Fig. 10.62. Assuming that the current i_0 is distributed uniformly over the cross section of the wire. Now consider a circular path of radius x round the axis of wire. By symmetry, we see that the **B** at any point over this path is tangential to it every where. Thus by Ampere's law

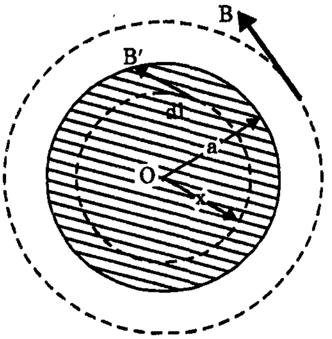


Fig. 10.62. Example 28.

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0 i,$$

$$\text{or} \quad \oint B\,dl = B2\pi x = \mu_0\ \text{(current enclosed by the path)}$$

There are three possible cases—

(i) $x > a$, current i = total current i_0 passing through the whole wire.

$$B2\pi x = \mu_0 i_0,\ \text{or}\ B = \mu_0 i_0/2\pi x.$$

(ii) $x < a$, the fraction of the current that passes through the enclosed path contributes to **B**.

$$B2\pi x = \mu_0 i = \mu_0\,(i_0/\pi a^2)\,\pi x^2,\ \text{or}\ B = \mu_0 i_0 x/2\pi a^2.$$

(iii) $x = a$, $i = i_0$ therefore $B = \mu_0 i_0 a/2\pi a^2 = \mu_0 i_0/2\pi a.$

Example 29. *Find the flux crossing the portion of the plane $\phi = \pi/4$ defined by $0.02 > r < 0.05$ m and $0 < z < 1.5$ m. A current filament of 2.5 amp is along the z-axis.*

For the current carrying filament along the z-axis, the magnetic field is given by

$$\mathbf{B} = (\mu_0 i/2\pi r)\,\mathbf{a}_\varphi$$

$$\text{Since} \quad d\mathbf{S} = dr\,dz\,\mathbf{a}_\varphi$$

$$\therefore\ \text{Flux} \quad \Phi =$$

$$\int \mathbf{B}.d\mathbf{S} = \int_0^{1.5}\int_{0.02}^{0.05}\frac{\mu_0 i}{2\pi r}\mathbf{a}_\varphi . dr\,dz\,\mathbf{a}_\varphi$$

$$= \int_0^{1.5}\int_{0.02}^{0.05}\frac{\mu_0 i}{2\pi r}dr\,dz = \frac{1.5\mu_0 i}{2\pi}\log_e\frac{0.05}{0.02}$$

$$= 6.87\times 10^{-7}\ Wb.$$

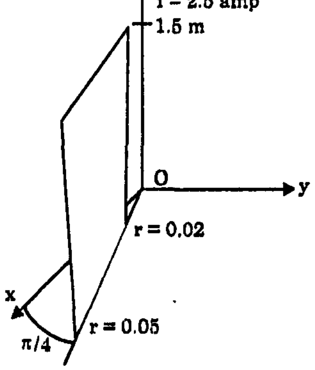


Fig. 10.63. Example 29.

Example 30. *An infinite current sheet lies in the $z = 0$ plane with $K = K\,\mathbf{a}_y$. Find the magnetic vector potential* **A** *for this sheet.*

It is clear from the Biot-Savart law and conservations of symmetry that **B** has only an x-component.

Let us consider a square loop $PQRS$ with sides $2\,a$ in the x-z plane with its centre at the origin. Applying the Ampere's law to this loop, we get

$$\oint \mathbf{B}.d\mathbf{l} = \mu_0 i$$

or $B\,2a + 0 + B\,2a + 0 = \mu_0 K\,2a$

$$\therefore \quad B = \tfrac{1}{2}\mu_0 K.$$

Thus for $z > 0$, $\mathbf{B} = (\mu_0 K/2)\mathbf{a}_x$

For an arbitrary orientation of the current sheet,

$$\mathbf{B} = \tfrac{1}{2}\mu_0 \mathbf{K}\times\hat{n},$$

which is independent of the distance from the sheet.

We know that **B** = curl **A**, therefore

$$\text{curl} \quad \mathbf{A} = \tfrac{1}{2}\,\mu_0 K \mathbf{a}_x$$

or $(\partial A_z/\partial y - \partial A_y/\partial z) = \tfrac{1}{2}\,\mu_0 K$

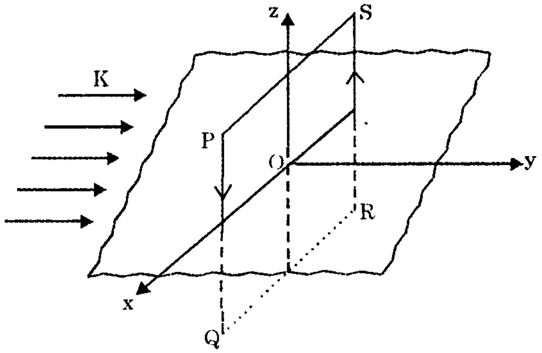


Fig. 10.64. Example 30.

As **A** must be independent of x and y, therefore

$$-\partial A_y/\partial z = \tfrac{1}{2}\mu_0 K \quad \text{or} \quad A_y = -\tfrac{1}{2}\,\mu_0 K(z - z_0)$$

For $z > 0$, $\quad \mathbf{A} = -\tfrac{1}{2}\,\mu_0 K(z - z_0)\mathbf{a}_y = -\tfrac{1}{2}\,\mu_0 (z - z_0)\mathbf{K}$

For $z < 0$, $\quad \mathbf{A} = \tfrac{1}{2}\mu_0 (z - z_0)\mathbf{K}.$

Example 31. *The vector potential* **A** *in a certain region is given by* $\mathbf{A} = x\,\mathbf{j} - y\,\mathbf{i}$. *Explain how will the lines of force look like? What is the direction of magnetic field* **B** *in the given space?*

The magnetic field in a region where the vector potential **A** is given by

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y & x & 0 \end{vmatrix} = 2\mathbf{k}.$$

Thus the magnetic lines of force will be straight lines parallel to the direction of z (or **k**) and perpendicular to $x - y$ plane.

Example 32. *Show that the magnetic energy stored per unit length of a straight conductor carrying a current of 10 amp is independent of the radius of the conductor.*

We know that the magnetic field inside a current carrying long cylindrical wire of radius R is given by

$$B = \mu_0\, ir/2\pi R^2,$$

where i is the current.

The magnetic energy per unit volume

$$u = B^2/2\mu_0.$$

The magnetic energy stored with a hollow cylinder of radii r and $r+dr$ and length l

$$dU = (B^2/2\mu_0)\,2\pi r\,drl,$$

$$\therefore \quad U = \int_0^R \frac{1}{2\mu_0}\left(\frac{\mu_0 i r}{2\pi R^2}\right)^2 .2\pi r l\,dr$$

$$\text{or} \quad \frac{U}{l} = \frac{\mu_0 i^2}{4\pi R^4}\int_0^R r^3 dr = \frac{\mu_0 i^2}{16\pi}.$$

Example 33. *A long horizontal rigidly supported wire carries a current i_a of 100 amperes. Directly above it and parallel to it, there is a fine wire that carries a current i_b of 20 amperes and weighs 0.073 newton per meter. How far above the lower wire should the second wire be kept if we wish to support it by magnetic repulsion.*

We know that the force between the two parallel current carrying wires is attractive if the currents are in the same direction and is repulsive if these are in the opposite directions. In the present problem, the repulsive force is balanced by the weight of the upper wire. Thus the magnetic repulsive force per unit length

$$\frac{F}{l} = \frac{\mu_0}{2\pi}\frac{i_1 i_2}{R} = \frac{4\pi\times10^{-7}}{2\pi}\times\frac{100\times20}{R}$$

Since the weight per unit length $= F/l = 0.073\ \text{Nm}^{-1}$

$$\therefore \qquad R = 4\times10^{-4}/0.073 = 5.48\times10^{-3}\ \text{meter} = 5.48\ \text{mm}.$$

Example 34. *Copper wire is wound around a cylindrical solenoid of 6 cm diameter and 30 cm length. There are 5 turns per cm. The wire has a diameter of 0.163 cm and has a resistance of 0.01 ohm per meter. If this solenoid is connected across a 24 volt battery. What will be the magnetic field strength within the solenoid?*

The solenoid is long hence the magnetic field strength within the solenoid is given by $B = \mu_0 ni$ weber/meter^2, where n is the number of turns per meter and i is the current in amperes flowing in the solenoid of radius r and length L. In this problem

$$n = 5\times100 = 500\ \text{turns/meter},$$

Total length l of the wire $= 2\pi rnL = 2\pi\times0.03\times500\times0.3 = 9\pi$

and the resistance R of this wire $= 9\pi\times0.01 = 0.09\pi$ ohm.

$$\therefore \qquad \text{Current } i = V/R = 24/0.09\pi = 800/3\pi\ \text{amp}.$$

Substituting numerical values in the above relation, we get

$$B = (4\pi\times10^{-7})\times500\times(800/3\pi) = 5.33\times10^{-2}\ \text{weber/m}^2.$$

Example 35. *A capacitor charged upto 2 volts is discharged through a ballistic galvanometer having time period of 12 seconds and current sensitivity 2.2 × 10^{-8} amp/cm. If the first and eleventh throws of galvanometer are 9.6 cm and 8.0 cm respectively, calculate the capacitance of this capacitor.*

The deflection θ in the ballistic galvanometer due to a charge q is given by the relation

$$q = \frac{T}{2\pi}\frac{c}{NAB}\theta(1+\lambda/2).$$

where the logarithmic decrement λ is given as

$$\lambda = \frac{2.3026}{10}\log_{10}\frac{\theta_1}{\theta_{11}} = \frac{2.3026}{10}\log_{10}\frac{9.6}{8.0} = 0.018.$$

Current sensitivity $= c/NAB = 2.2\times10^{-8}$ amp/cm

and the time period $T = 12$ seconds.

Substituting numerical values, we get

$$q = \frac{12}{2\times3.14}\times2.2\times10^{-8}\times9.6\left(1+\frac{0.018}{2}\right)$$
$$= 4.072\times10^{-7}\ \text{Coulomb}.$$

Since $q = CV$, hence the capacitance of the capacitor

$$C = q/V = 4.072\times10^{-7}/2 = 2.036\times10^{-7}\ \text{farad}.$$

ORAL QUESTIONS

1. In the relation **F** = q**v** × **B**, which pairs (**F**, **v**; **F**, **B**; **v**, **B**) are always perpendicular to each other ?
2. If a beam of electrons travels in a straight line in a certain region. Can we say that there is no electric field in the region, or no magnetic field ?
3. A beam of electrons is deflected side ways in passing through a certain region. Is it due to (*a*) magnetic field. (*b*) electric field or (*c*) both of these ?
4. What type of motions do you expect, when the proton moves (*a*) along **B**, (*b*) ⊥ to **B** or (*c*) along a line at an angle of θ^0 to the direction of **B** ?
5. What is the value of the net force acting on a current carrying (*a*) rectangular or (*b*) circular loop placed in a uniform magnetic field ? What do you expect about the torque in each case ?
6. Does the work required to turn a current loop end for end in an external field constant and independent of the original orientation of the loop ?
7. Compare Coulomb's law and Biot-Savart law.
8. The sum of the tangential components of the magnetic fields around a closed circle is zero. What it represents ?
9. Do magnetic lines of force have starts and finishes ?
10. Can a current carrying coil be used to point to the magnetic poles of the earth ?
11. Why do you use Helmholtz galvanometer in place of tangent galvanometer ?
12. What determines which method one should use in calculating **B**. Ampere's law or the Biot-Savart law.
13. What do you mean by the relations : curl **B** ≠ 0 and div **B** = 0 ?
14. In electronics, the wires carrying equal and opposite currents are often twisted together to reduce magnetic field effects. What is the basis for this procedure.
15. The direction of the magnetic field of a long straight wire carrying current is (*a*) in the direction of current, (*b*) radially outward, (*c*) along line circling the current. (*c*)
16. A current element idl is located at the origin, the current is in the direction of the z-axis. The x-component of the field at a point $P(x, y, z)$ is (*a*) zero, (*b*) $-iy\, dl\, (x^2 + y^2 + z^2)^{3/2}$, (*c*) infinity. (*b*)
17. A uniformly charged spherical metal ball of mass M, radius R and charge q is related with an angular velocity ω about one of its diameters. The ratio of its magnetic dipole moment to the mechanical moment (angular momentum) is (*a*) $q/2MR$, (*b*) $q\,\omega/MR$ or (*c*) $q/2M$. (*c*)

   [For a uniformly charged spherical ball magnetic moment $\mu = q\omega R^2/5$ *and mechanical moment (angular momentum* J*)* $= 2M\omega R^2/5$*].*
18. Which one of the following is not a source of magnetostatic field : (*a*) a dc current in a wire (*b*) an accelerated charge, (*c*) a permanent magnet. (*b*)
19. The z-axis carries filamentary current of 5 amp along $\hat{z}$, which one of the following is incorrect (*a*) $\mathbf{B} = -(\mu_0/2\pi)\ \hat{x}\ A/m$ at (0, 5, 0), (*b*) $\mathbf{B} = -(\mu_0/2\pi)\ (0.8\hat{x} + 0.6\hat{y})$ at (– 3, 4, 0), (*c*) $\mathbf{B} = (\mu_0/2\pi)\ \hat{\phi}\ A/m$ at (5, π/4, 0). (*b*)
20. which is not the characteristic of a static magnetic field (*a*) it is conservative, (*b*) it is solenoidal, (*c*) it has no sources or sinks. (*a*)
21. The same current i passes through two identical coaxial circular coils in opposite directions. The magnetic field at a point on the axis midway between the coils is (*a*) the same as that produced by one coil, (*b*) zero, (*c*) the double of that produced by one coil. (*b*)
22. How do you distinguish over-damped, critically-damped and under damped oscillations ?
23. How can one use ballistic galvanometer as dead beat galvanometer ?
24. How is the figure of merit of the ballistic galvanometer related with that of the dead beat galvanometer.
25. What do you mean by the logarithmic decrement of the B.G.

ANSWERS : 1. F, v and F, B; 2. No; 3. *c*; 15. *c*; 16. *b*; 17. *c*; 18. *b*; 19. *b*; 20. *a*; 21. *b*.

PROBLEMS

1. Current i flows in a long straight conductor having the shape of a groove with a cross section in the form of a thin half ring of radius a. The current is directed into the plane of the paper. Find the magnetic field **B** on the axis of the conductor. $(\mu_0 i/\pi^2 a)$
2. Three very long parallel wires are arranged as to form a isoceles triangle. They carry equal currents of 2.5 A with the current in C pointing into the page while the currents in A and B point out of the page. If $AB = BC = 10$ cm, find the magnetic field at P, the midpoint between A and C. Also find the force per unit length on the wire at B. $(1.58 \times 10^{-5}\ T, 1{,}77 \times 10^{-5}N/m)$
3. The conducting triangular loop OAB in Fig. 10.65 carries a current of 5 amp. Find **H** at (0, 0, 5) due to sides OB and AO of the loop. $(-29.55\ \mathbf{a}_y\ mA/m,\ -15.31\ \mathbf{a}_x + 15.31\ \mathbf{a}_y\ mA/m)$.

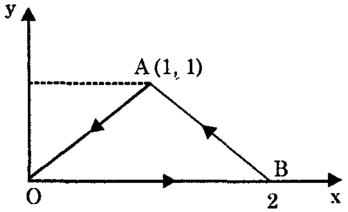


Fig. 10.65. Problem 3.

4. A rectangular loop of sides b and c carries a current i. Find the magnetic flux density at the point of intersection of the loop diagonals. $[2\mu_0 i\,(b^2 + c^2)^{1/2}/\pi\,bc]$
5. A long non-magnetic cylindrical conductor with inner radius a and outer radius b carries a current I. The current density in the conductor is uniform. Find the magnetic field set up by this current inside the hollow space $(r < a)$, within the conductor $(a < r < b)$ and outside the conductor $(r > b)$. $[0, (\mu_0 i/2\pi r)\,(r^2 - a^2)/(b^2 - a^2), \mu_0 i/2\pi r]$
6. A thin ring of radius 5 cm is placed on plane z = 1cm so that its centre is at (0, 0, 1 cm). If it carries a current of 3 A along a_φ, find **B** at (0, 0, – 1cm) $(3 \times 10^{-5}\ \mathbf{a}_z$ Web/m$^2)$
7. Calculate the current which, flowing through a Helmholtz galvanometer having coils each of 25 turns and 10 cm radius, produces a deflection of 45°. The earth's horizontal magnetic intensity is 0.3 oersted. [0.18 amp]
8. A current i flows through a wire bent as shown in Fig. 10.66. Calculate the force exerted on the wire, if it is placed in a uniform magnetic field B. The field is acting ⊥ to the plane of wire and away from the reader. $[2i\,B\,(L + R)]$

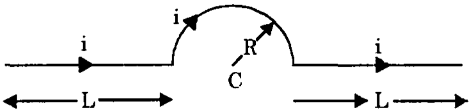


Fig. 10.66. Problem 8.

9. Find the resultant force in the are AC subtending an angle $3\pi/2$ at the centre carrying a current i and is placed in a uniform magnetic field B directed into the page. (zero)
10. In the above problem, find the torque on the loop as a result of B. Repeat if the field is in the plane of page directed from left to right. Find the direction of the torque vector.

$$\left[0, \tfrac{3}{4}\pi b^2 iB \,\text{in}\, y-\text{direction}\right]$$

11. A toroid (radius 10 cm) of circular cross section (*dia* 2 cm) whose centre is at the origin and axis along z-axis has 1000 turns. If the toroid carries a current of 100 mA, find **B** at (3 cm – 4 cm, 0) and (6 cm, 9 cm, 0). $(0, 1.85 \times 10^{-4}$ Web/m$^2)$
12. An infinitely long conductor is bent into an L-shape in $x-y$ plane. If a direct current of 4 amp flows in the conductor, find the magnetic filed **B** at point (2, 2, 0). $(8.5 \times 10^{-7}\ \mathbf{a}_z$ Web/m$^2)$
13. A current carrying loop is bent to form a regular polygon of n-sides. Show that at the centre of the polygon $B = (ni/2\pi r)\sin(\pi/n)$, where i is the current in the loop and r is the radius of the circle circumscribed by the polygon.
14. Given that $\mathbf{B} = 2.5 \sin(\pi x/2)\, e^{-2y}\, \mathbf{a}_z$. Find the total magnetic flux crossing the strip $z = 0, y \geq 0, 0 \leq x \leq 2$m. (1.59 Web)
15. The magnetic-induction **B** in a certain region is 2 Web/m^2 and along + ve x– direction. If $abcdef$ prism is placed in this field in such a way that surface bcde is in $x - y$ plane and $abcf$ in $y - z$

plane. The sides *bc* and *ed* along x-axis and each of length 30 cm., *ab* and *ef* along z – axis and each of length 40 cm. If the height $af = be = cd = 30$ cm. Find the magnetic flux across each surface.

16. Two infinite current sheets, each of constant density $\mathbf{K}_0$ are parallel and having currents oppositely directed. Find the force per unit area on the sheets. $\left(\frac{1}{2}\mu_0 K_0^2 \text{ repulsive}\right)$

17. Find the power required to turn a cylindrical set of n-conductors against the field at N revolutions per minute, if $\mathbf{B} = B_0 \sin 2\phi\, \boldsymbol{a}_r$ and the currents change direction in each quadrants where the sign of **B** changes. If i is the current in the each conductor and cylinder has a length l and radius r. ($B_0\, n\, i\, l\, r\, N/60$ watt)

18. An electron in a magnetic field $B = 4 \times 10^{-3}$ Web/m^2 has a circular path of radius 0.35×10^{-10} m and a maximum torque of 7.85×10^{-26} Nm. Find the angular velocity. (2×10^{16} rad/sec)

19. A current sheet $\mathbf{K} = 3\mathbf{a}_x$ A/m lies in the $z = 0$ plane and a current filament is located at $y = 0$, $z = 4$m. Determine the magnitude and direction of the current in the filament if $\mathbf{B} = 0$ at (0, 0, 1.5 m.). ($23.5\, \boldsymbol{a}_x A$)

20. The dielectric sphere of radius b spins with angular speed ω about the y-axis. *If* it carries a uniform surface charge σ coul/m^2 and a magnetic field B exists in the x-direction. Show that the torque acting on it is $\tau = \frac{4}{3}\pi\omega b^4\, B$ along z-direction.

21. If a current i flows through a circuit in the shape of a regular hexagon of side a. Find the induction at the centre of the hexagon. $\left(\sqrt{3}\,\mu_0 i/\pi a\right)$

22. Find the expression for the magnetic induction at the centre of an elliptical wire carrying a current i. ($\mu_0 i\, L/4A$, *where L is the perimeter and A the area of ellipse)*

23. Two infinite cylindrical conductors are placed parallel to one another at a distance $2a$ apart. They carry equal and opposite currents. Show that in the equatorial plane the gradient of the magnetic field is greatest at a distance $a/\sqrt{3}$ from the plane through the axes of the cylinders.

24. A solenoid of length 20 cm and radius 2 cm is closely wound with 200 turns. Calculate the magnetic field intensity at the centre of an end of the solenoid. The current in the winding is 5 amperes. [3.14×10^{-3} Web/m^2]

25. A large number n of closely spaced turns of fine wire are wound in a single layer upon the surface of a wooden sphere of radius R such that they cover it completely. The planes of turns are parallel. If the current i flows in the wire, find the magnetic field at the centre of the sphere.
($\mu_0\, ni/4R$)

26. A small coil with n turns and of area A is suspended inside a solenoid of length L and a total number of turns N on it. The axis of the coil makes an angle θ to the axis of the solenoid. Find the torque on the coil if the same current i passes through the coil and the solenoid. ($\mu_0 n N i^2 A/L$)

27. A long bar magnet has poles at its two ends. Magnetic field B is radial near the poles. A circular current loop is placed as shown in Fig. 10.67, find the magnitude and direction of the resultant force on the loop.

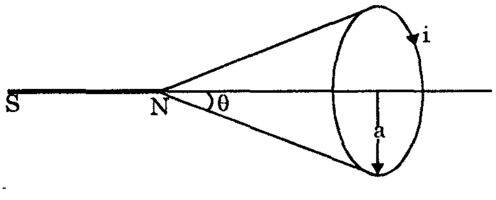


Fig. 10.67. Problem 27.

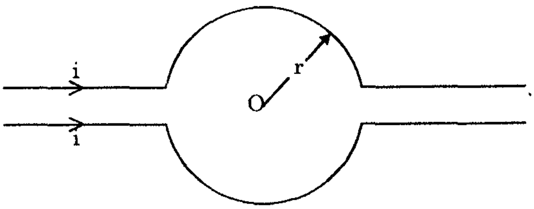


Fig. 10.68. Problem 30.

28. Show that the vector potential due to two parallel line currents flowing in opposite directions is $\mathbf{A} = \mu_0 i \hat{z} \log_e (d_2/d_1)/2\pi$, where d_1 and d_2 are the distances from the point of observation to the wires and $\hat{z}$ is a unit vector parallel to the wires.

29. A solenoid of square cross section has n turns per unit length and carries current i. The cross sectional dimension is a. If the solenoid is very long, find the axial magnetic induction at its centre. [$\mu_0 ni$].

30. A pair of infinitely long thin wires carry equal currents i. They are bent as shown in Fig. 10.68. Find the magnetic induction **B** at the centre O of the circular parts ?

31. A flat coil is wound so that it contains N number of turns per unit distance along its radius. Prove that the magnetic induction at the centre of such a coil of inside radius a outside radius b and carrying a current i is $\frac{1}{2}\mu_0 \, i \, N \log(b/a)$.

32. Two long straight parallel wires are kept in free space at a distance of 1.0 meter and 1.0 ampere current is flowing in the same direction in each wire. Calculate the direction and magnitude of the mutual force per meter between the two wires. What will be the difference if the currents flow in mutually opposite directions. (2×10^{-7} newton)

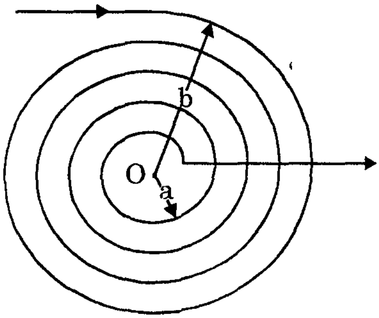


Fig. 10.69. Problem 31.

33.

Find the current density **j** for the magnetic vector potential $\mathbf{A} = (10/r^2)\, \boldsymbol{a}_z$ Web/m infree space. $(-40/\mu_0 r^4)\, \boldsymbol{a}_z$ A/m^2.

34. A current distribution produces a magnetic field whose vector potential $\mathbf{A} = x^2 y\, \boldsymbol{a}_x + y^2 x\, \boldsymbol{a}_y - 4xyz\, \mathbf{a}_z$, find the magnetic field **B** at $(-1, 2, 5)$. $(20\, \boldsymbol{a}_x + 40\, \boldsymbol{a}_y + 3\, \boldsymbol{a}_z)$

35. The resistance of a moving coil ballistic galvanometer is 125 ohms. A steady potential difference of 0.00025 volt deflects the spot of light on the scale through 15 cm. The suspended system has an undamped oscillation of periodic time 10 sec. When a condenser is discharged through the galvanometer, the corrected first throw is 5 cm. Find the quantity of charge discharged. If the condenser was charged to 1 volt, what is its capacity ? (1.06 μF)

36. A moving coil galvanometer has a current sensitivity of 50 mm/μA when the scale is at a distance 1 m from the scale. The time period of free oscillation is 3.142 s. If the galvanometer is dead beat when the total circuit resistance is $2k\Omega$, calculate the moment of inertia of the moving system about the axis of rotation, assuming the electromagnetic damping only.

(8×10^{-7} kg m^2)

11

CHARGED PARTICLES IN ELECTROMAGNETIC FIELDS

11.1 CHARGED PARTICLE EQUATION OF MOTION

The velocity v of a particle of mass m under the influence of an external force **F** is governed by the relation

$$\mathbf{F} = \frac{d}{dt}(m\mathbf{v}) = \frac{d}{dt}\left(m\frac{d\mathbf{r}}{dt}\right). \qquad \text{...(1)}$$

Under non-relativistic condition, m is constant and Eq. (1) reduces to

$$\mathbf{F} = m\frac{d\mathbf{v}}{dt} = m\dot{\mathbf{v}} = m\frac{d^2\mathbf{r}}{dt^2} = m\ddot{\mathbf{r}}. \qquad \text{...(2)}$$

In rectangular coordinates, the force, velocity and position may be expressed as

$$\mathbf{F} = F_x\,\mathbf{i} + F_y\,\mathbf{j} + F_z\,\mathbf{k} = F_x\,\mathbf{a}_x + \mathrm{F}_y\,\mathbf{a}_y + F_z\,\mathbf{a}_z,$$

$$\mathbf{v} = v_x\,\mathbf{i} + v_y\,\mathbf{j} + v_z\,\mathbf{k} = v_x\,\mathbf{a}_x + v_y\,\mathbf{a}_y + v_z\,\mathbf{a}_z$$

and

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = x\,\mathbf{a}_x + y\,\mathbf{a}_y + z\,\mathbf{a}_z. \qquad \text{...(3)}$$

$\therefore$

$$\mathbf{F} = m\left[\ddot{x}\,\mathbf{a}_x + \ddot{y}\,\mathbf{a}_y + \ddot{z}\,\mathbf{a}_z\right] \qquad \text{...(4)}$$

Here $\mathbf{a}_x$, $\mathbf{a}_y$ and $\mathbf{a}_z$ are the unit vectors along the x, y and z-axes respectively.

In cylindrical coordinates, the force, velocity and position may be expressed in terms of cylindrical coordinates (ρ, ϕ, z) with their unit vectors $\mathbf{a}_\rho$, $\mathbf{a}_\phi$ and $\mathbf{a}_z$, as

$$\mathbf{F} = F_\rho\,\mathbf{a}_\rho + F_\phi\,\mathbf{a}_\phi + F_z\,\mathbf{a}_z,\ \mathbf{v} = v_\rho\,\mathbf{a}_\rho + v_\phi\,\mathbf{a}_\phi + v_z\,\mathbf{a}_z \text{ and}$$

$$\mathbf{r} = \rho\,\mathbf{a}_\rho + z\,\mathbf{a}_z.$$

$\therefore$

$$\mathbf{v} = \frac{d}{dt}(\mathbf{r}) = \frac{d}{dt}(\rho\,\mathbf{a}_\rho + z\,\mathbf{a}_z) = \frac{d\rho}{dt}\mathbf{a}_\rho + \rho\frac{d\,\mathbf{a}_\rho}{dt} + \frac{dz}{dt}\mathbf{a}_z. \qquad \text{...(6)}$$

It is because the direction of $\mathbf{a}_\rho$ remains fixed as either ρ or z is varied. The change in $\mathbf{a}_\rho$ with $\phi = \partial\,\mathbf{a}_\rho/\partial\,\phi = \mathbf{a}_\phi$.

Therefore $d\mathbf{a}_\rho/dt = \mathbf{a}_\phi\, d\phi/dt = \mathbf{a}_\phi\, \dot\phi = \mathbf{a}_\phi\omega$.

Similarly $\dot{\mathrm{v}} = (\ddot\rho - \rho\omega^2)\,\mathbf{a}_\rho + \dfrac{1}{\rho}\dfrac{d}{dt}(\rho^2\omega)\, a_\phi + \ddot z\, \mathbf{a}_z$. ...(7)

It is because $d\mathbf{a}_\phi / dt = -\, \mathbf{a}_\rho\, d\phi/dt = -\, \mathbf{a}_\rho\omega$.

Thus we see that effects of rotational motion is seen in cylindrical coordinate expressions.

11.2 MOVING CHARGES IN ELECTRIC FIELD

We consider following cases :

(a) Motion in uniform electric field : The force **F** acting on an ion or charge q moving in a uniform electric field **E** is $q\mathbf{E}$. It experiences constant acceleration **f** in the direction of **F**.

$$\mathbf{F} = m\mathbf{f} = q\mathbf{E} \quad \text{or} \quad \mathbf{f} = (q/m)\,\mathbf{E}. \qquad ...(8)$$

On integration, we get

$$\mathbf{v} = d\mathbf{r}/dt = (q/m)\,\mathbf{E}t + C.$$

As at $t = 0$, $\boldsymbol{v} = \boldsymbol{u}$, hence $C = \boldsymbol{u}$ and $\boldsymbol{v} = (q/m)\,\mathbf{E}t + \boldsymbol{u}$.

On further integration we get $\mathbf{r} = \frac{1}{2}(q/m)\,\mathbf{E}t^2 + \boldsymbol{u}t + C'$.

As at $t = 0$, $\mathbf{r} = \mathbf{r}_0$ (initial position), hence

$$\mathbf{r} = \tfrac{1}{2}\,(q/m)\,\mathbf{E}t^2 + \boldsymbol{u}t + \mathbf{r}_0. \qquad ...(9)$$

There are two particular cases.

(i) Longitudinal field : When the electric field acts along the direction of motion of the particle, *i.e.*, the case of longitudinal field, the both **u** and $\mathbf{r}_0$ are directed along the electric field **E**. In this case the resultant position vector **r** is also along the direction of **E** and the problem acts as if it is one dimensional one.

If at the initial position $\mathbf{r}_0 = 0$ and $\mathbf{u} = 0$, then we have

$$\mathbf{v} = (q/m)\,\mathbf{E}t \text{ and } \mathbf{r} = \tfrac{1}{2}\,(q/m)\,\mathbf{E}t^2. \qquad ...(10)$$

Thus we again have one dimensional problem.

When the charged particle moves along the direction of electric field, it gains kinetic energy and therefore its velocity increases. If the electric field is produced by maintaining potential difference V, the increase in K.E. is given by

$$\Delta\,(\text{K.E.}) = \tfrac{1}{2}\,m\,(v_2{}^2 - v_1{}^2) = qV. \qquad ...(11)$$

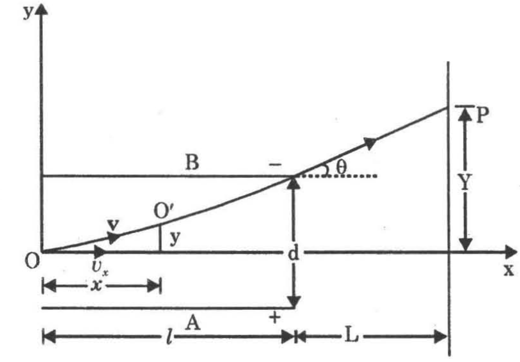


Fig. 11.1. Moving charges in a transverse electric field.

(ii) Transverse electric field : Consider a charged particle of mass m *and* charge q, which enters with the initial velocity v_x *along* x-axis between two parallel plates A and B. If the plates are of length l, separated by a distance d and kept across potential difference of V volts, the electric field $E = V/d$ will act along the positive y-axis. As there is no force along the x-axis, hence the time taken by the particle to pass through the plates $T = l/v_x$. As a force $q\mathbf{E}$ acts along y-axis, hence the acceleration produced in the particle will be along y-axis and given by $\mathbf{f} = q\mathbf{E}/m$. Since there was no component

of **v** *along* y-axis at the time of entrance (*i.e.*, $u_y = 0$), hence the velocity v_y along y-axis at any instant t is

$$v_y = f_y t = (qE/m)\,(x/v_x)$$

and the displacement $$y = \tfrac{1}{2}\, f_y t^2 = \tfrac{1}{2}\,(qE/m)\,(x/v_x)^2 = \tfrac{1}{2}\,(qE/mv_x^{\,2})\,x^2. \qquad ...(12)$$

Here x is the distance travelled along x-axis in time t (position O' on the path).

Thus the path of the charged particle in between the plates is a parabola in the x-y plane. When the particle leaves the plates, it will travel in a straight line along the tangent to the parabola at the point where the particle leaves the plates. The angle is given by

$$\tan\theta = (dy/dx)_{x=l} = qEl/mv_x^{\,2}.$$

The total y-displacement at a distance L from the ends of the plate is given by

$$Y = \text{displacement between plates} + \text{outside the plates}$$

$$= \frac{1}{2}\frac{qE}{m}\frac{l^2}{v_x^{\,2}} + L\tan\theta = \frac{qEl}{mv_x^{\,2}}\left(\frac{l}{2}+L\right). \qquad ...(13)$$

This principle is used in *cathode ray tubes,* where the electrons are allowed to pass through the parallel plates and the beam strikes the fluorescent screen.

(*b*) Motion in alternating electric field : If the electric field is an alternating, given by $E = E_0 \sin\omega t$, then the above relation reduces to

$$\mathbf{F} = m\mathbf{f} = m\,d\mathbf{v}/dt = md^2\mathbf{r}/dt^2 = q\mathbf{E}_0 \sin\omega t. \qquad ...(14)$$

On integrating with the condition $\mathbf{v} = 0$ at $t = 0$, we get

$$\mathbf{v} = -\frac{q\mathbf{E}_0}{m\omega}\cos\omega t + \frac{q\mathbf{E}_0}{m\omega} = \frac{q\mathbf{E}_0}{m\omega}[1-\cos\omega t].$$

On integrating with the condition $\mathbf{r} = 0$ at $t = 0$, we get

$$\mathbf{r} = -\frac{q\mathbf{E}_0}{m\omega^2}\sin\omega t + \frac{q\mathbf{E}_0}{m\omega}t. \qquad ...(15)$$

(*c*) Motion in non-uniform electric field : The motion of charged particle (electron) in nonuniform fields is of considerable interest in the field of *electron optics.* The electron is allowed to pass in a region of a variable electric field. The component of velocity parallel to the equipotential surface is unperturbed. The velocity changes in the direction of potential gradient and is given by the equation.

$$\Delta\mathbf{v} = F\Delta t/m = \nabla V(e/m)\,\Delta t. \qquad ...(16)$$

As will be seen in electrostatic focusing, the velocities are related as

$$\frac{v_1}{v_2} = \frac{\sin\theta_2}{\sin\theta_1} = \left(\frac{V_1}{V_2}\right)^{1/2}. \qquad ...(17)$$

11.3 MOVING CHARGES IN UNIFORM MAGNETIC FIELD

We know that the force experienced by a charged particle of charge q moving with velocity **v** in a magnetic field **B** is given by the relation

$$\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) \qquad ...(18)$$

Let us discuss the following cases :

(*a*) If a charged particle is moving along the line of magnetic field, it will move as it is, as it will experience no force ($\mathbf{v} \parallel \mathbf{B}$).

(*b*) If a particle of mass m and charge q is introduced with its initial velocity **v** in the plane normal to **B** then the force $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$ will act on it in this plane in the direction ⊥ to **v** and **B**. Since the direction of this force is and remains normal to the velocity, hence,

$$\mathbf{F}\cdot\mathbf{v} = q(\mathbf{v}\times\mathbf{B}).\,\mathbf{v} = 0$$

or
$$m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v} = \frac{d}{dt}\left(\frac{mv^2}{2}\right) = 0.$$

and therefore, $\frac{1}{2}\,mv^2 = \text{constant}.$ ...(19)

This equation shows that the magnetic field performs no work and produces no change in the kinetic energy of the particle. Since the force **F** due to magnetic field is always perpendicular to the direction of motion and **v** and **B** are constant in magnitude, hence force **F** will also be constant and the particle will move along a circular path of radius R given by

centripetal force mv^2/R = *Force* qvB.

or $R = mv/qB.$ ...(20)

Thus, the radius is proportional to the momentum of the particle. The work of the magnetic force is zero. It only changes the direction of motion.

The corresponding angular velocity

$$\omega = v/R = qB/m, \qquad \text{...(21)}$$

or the frequency $f = \omega/2\pi = qB/2\pi m.$ ...(22)

This relation shows that the charged particle revolves in a plane ⊥ to the magnetic field with the angular frequency independent of its linear velocity v. If the particle loses or gains energy continuously the path will be a spiral. The radius of curvature decreases in the former case while increases in the latter case.

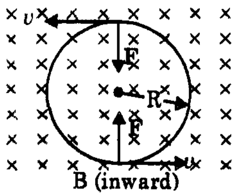
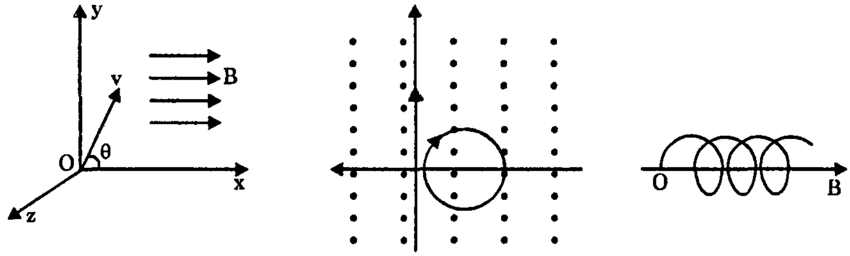


Fig. 11.2. Orbit of a charged particle in a uniform magnetic field.

(*c*) If the charged particle having charge q is neigher parallel nor perpendicular but making an angle θ with the direction of **B** (as shown in Fig. 11.3), then one can split the velocity vector into two components, v_x along **B** and v_y normal to **B**. Since $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$, therefore

$$\mathbf{F} = q\,(v_x\,\mathbf{a}_x + v_y\,\mathbf{a}_y)\times B\,\mathbf{a}_x = qB\,[-v_y\,\mathbf{a}_z] = -\,qBv_y\,\mathbf{a}_z. \qquad \text{...(23)}$$



(*a*) (*b*) (*c*)

Fig. 11.3. Helical path of a charged particle.

It shows that the force acting on the charged particle is in negative z-direction. The force is zero in the x-direction, therefore v_x is unaffected by the field. Thus the particle will move in the direction of **B** with a constant velocity v_x, As v_y is the velocity component in the direction ⊥ to **B**, the particle will travel in a circle, which is in *y-z plane,* of radius R $(= mv_y/qB)$. Thus the resultant motion will be the superposition of two motions, one constant velocity v_x *along* **B** and other circular motion ⊥ to **B**. Hence the resultant path will be a helix, with its axis along x-axis and radius in y-z plane. As the pitch of a helix is the distance travelled along its axis in one revolution, hence

pitch of $q = v_x \times$ time period $T = v_x \times 2\pi m/qB = 2\pi\, mv_x/qB.$ ...(24)

In general the velocity of the particle may be represented as

$$\mathbf{v} = \mathbf{v}_\perp + \mathbf{v}_\parallel, \qquad ...(25)$$

where $\mathbf{v}_\perp$ is the component of velocity perpendicular to the magnetic field and $\mathbf{v}_\parallel$ is the component of velocity in the direction of the magnetic field. Therefore the component of Lorentz force in the direction of magnetic field is zero, *i.e.,*

$$\mathbf{F}_\parallel = q\,(\mathbf{v} \times \mathbf{B})_\parallel = 0 \qquad ...(26)$$

$$\therefore \qquad m\frac{d\mathbf{v}_\parallel}{dt} = 0 \text{ and } \mathbf{v}_\parallel = \text{constant}. \qquad ...(27)$$

Substituting value of **v** in the equation of motion $m d\mathbf{v}/dt = q(\mathbf{v} \times \mathbf{B})$, we get

$$m\frac{d}{dt}(\mathbf{v}_\perp + \mathbf{v}_\parallel) = q\,(\mathbf{v}_\perp + \mathbf{v}_\parallel) \times \mathbf{B}.$$

$$\text{or} \qquad m\frac{d\mathbf{v}_\perp}{dt} = q\mathbf{v}_\perp \times \mathbf{B}. \qquad ...(28)$$

The angle between $\mathbf{v}_\perp$ and **B** remains constant throughout the motion and is equal to $\pi/2$. $\mathbf{v}_\perp$ and **B** do not change in magnitude. Hence the force $m d\mathbf{v}_\perp/dt$ is constant in magnitude and is perpendicular to $\mathbf{v}_\perp$. Therefore *Eq.* (28) describes the circular motion. The radius of the circle is given by the relation

$$m v_\perp^2/R = q v_\perp B. \qquad ...(29)$$

The superposition of motions due to $v_\parallel$ and $v_\perp$ gives the helix.

The principle is used in many instruments. Let us discuss below few of them.

(*a*) Magnetic focussing : In general the ions emitted from the ion source are of different velocities, the ions of same velocity may also emit in different directions. The source may emit ions of different masses.

(*i*) To focus ions of different velocities, they are collimated by an entrace slit *S*. The magnetic field is applied ⊥ to the plane of a paper and particles are bent in a circular path. After a turn of 180° they are recorded on a photographic plate *P*. The particles of different velocities are focussed at different points on the plate. By measuring the distances one can calculate velocity, momentum and energy of different ions.

$$mv^2/R = qvB \quad \text{or} \quad v = qBR/m$$

$$\textit{momentum } mv = qBR \text{ and energy } \tfrac{1}{2}\,mv^2 = q^2B^2R^2/2m$$

This relation is used when one employs magnetic fields to determine particle momentum. The product *BR* is often called the *magnetic rigidity*.

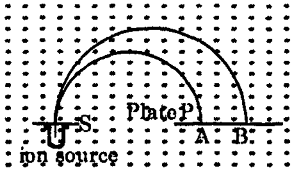


Fig. 11.4. Isotope separation.

This device is also used in isotope separation, the separation of various masses of the ions emitted from a source.

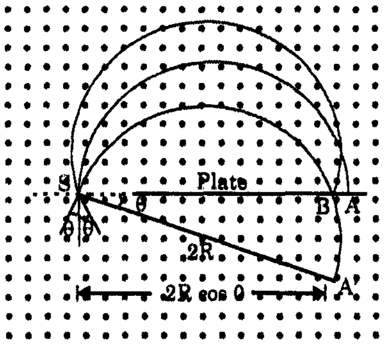


Fig. 11.5. Focussing of ions.

(*ii*) To focus ions of same velocity travelling in different directions, the ions are allowed to pass through a fine slit *S*. After a turn of 180° in a perpendicular magnetic field, they are focussed on a plate. All the particles leaving the slit normally will strike the plate at point *A* normally. The other particles making initial angle θ with the normal strike the plate at other point *B*. The width of the focussed image

$$\begin{aligned} AB &= SA - SB = 2R - 2R\cos\theta \\ &= 2R\,(1 - \cos\theta) = 2R\,[1 - 1 + \theta^2/2\,!] \\ &= R\theta^2. \end{aligned}$$

As θ is very small. The focussing power or the resolution

$$P = AB/SA = R\theta^2/2R = \theta^2/2. \qquad ...(30)$$

If the slit width is s, then

$$P = (s + R\theta^2)/2R = s/2R + \theta^2/2. \qquad ...(31)$$

This device is very useful in β-ray spectrometers.

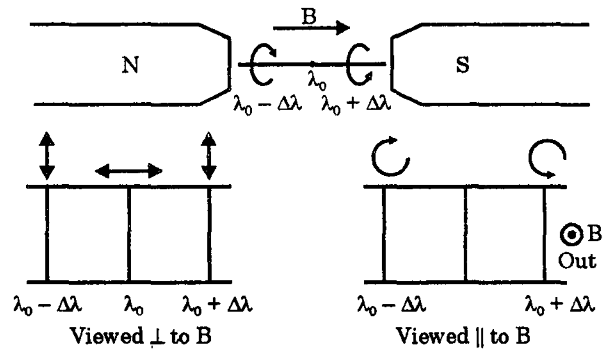

B
N
S
λ_0
$\lambda_0 - \Delta\lambda$
$\lambda_0 + \Delta\lambda$
B
Out
$\lambda_0 - \Delta\lambda$
λ_0
$\lambda_0 + \Delta\lambda$
$\lambda_0 - \Delta\lambda$
$\lambda_0 + \Delta\lambda$
Viewed ⊥ to B
Viewed || to B

Fig. 11.6. Normal Zeeman effect.

(b) Zeeman effect : In 1896, Zeeman discovered that spectral lines are split up into components when the source emitting the lines is placed in a very strong magnetic field. The simplest case is known as the *normal Zeeman Effect*. A single line of wavelength λ_0 in the absence of a magnetic field is split into three components with wavelengths $\lambda_0 - \Delta\lambda$, λ_0 and $\lambda_0 + \Delta\lambda$ when viewed at right angles to the magnetic field B. The shifted lines are plane polarized with the electric vector perpendicular to **B**, while the line of wavelength λ_0 is plane polarized with the electric vector parallel to **B**. Only the two components with wavelengths $\lambda_0 \pm \Delta\lambda$ are seen when viewed along B, which are circularly polarized.

Classically a charge performing simple harmonic motion radiates light. In the absence of a magnetic field, the electron of mass m and charge e is revolving round the nucleus in an orbit of radius r, such as the centripetal force mv^2/r $(= m\omega_0^2 r)$ is balanced by an elastic force proportional to the displacement, *i.e.*

$$m\omega_0^2 r = kr \quad \text{or} \quad \omega_0 = (k/m)^{1/2}$$

$$\therefore \qquad \text{frequency } \nu_0 = \frac{1}{2\pi}\sqrt{\left[\frac{k}{m}\right]}. \qquad ...(32)$$

In the presence of magnetic field **B**, the component of the motion parallel to **B** is unaffected and the angular velocity changes to ω due to the perpendicular component. In general any simple harmonic vibration can be regarded as the sum of two superposed circular motions with the same frequency but in opposite directions. Due to the additional force $e(\mathbf{v} \times \mathbf{B})$ due to the magnetic field **B**, the frequencies of these two circular motions are different. Viewed perpendicular to **B**, radiation from the circular motion would be plane polarized with the electric vector perpendicular to **B**. Now

$$m\omega^2 r = kr \pm evB = m\omega_0^2 r \pm er\omega B.$$

$$\text{or} \qquad \omega^2 - \omega_0^2 = (\omega + \omega_0)(\omega - \omega_0) = \pm e\omega B/m.$$

Since $\omega = 2\pi\nu$, and $\omega \approx \omega_0$, hence the modified frequency

$$\nu = \nu_0 \pm eB/4\pi m. \qquad ...(34)$$

As the electron is negatively charged, the slower rotation (longer λ) occurs in the anticlockwise direction as seen by an observer looking at the source through the hole in the pole piece. From his results Zeeman showed that the radiating particle was negative and e/m was found 1.6×10^{11} *C/kg*.

11.4 MOVING CHARGES IN NON UNIFORM MAGNETIC FIELD

Magnetic bottle : Consider a magnetic field which is strong at the ends a and b and is weak in the central region c. Such a non-uniform field is called *magnetic bottle*. A positive charged particle (or ion) at point d having axial velocity component, follows the spiral path,

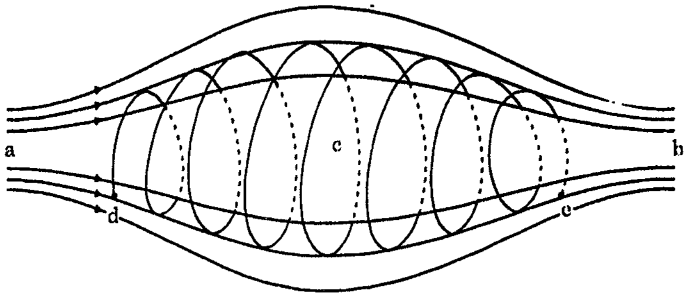


Fig. 11.7. Charged particle in a magnetic bottle.

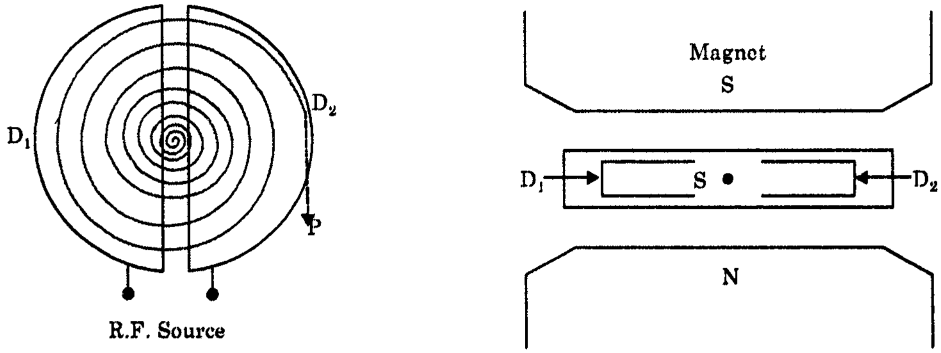
as shown in Fig. 11.7, the radial velocity component interacts with the magnetic field **B**. The lines of **B** are diverging in the region between *a* and *c*, thus produce an acceleration of the particle towards the right. When the particle passes into the region between *c* and *b*, where the lines of **B** are converging, it experiences deceleration, or the acceleration toward the left. Thus the particle is slowed down and reaches to the extreme point *e*. The particle then drifts to the left until it reaches the leftward extreme (at point *d*). Thus the particle spirals back and forth between the points *d* and *e*.

Such *magnetic confinement devices are used* in the study of high temperature ionized gases (*plasmas*) to prevent the ions from striking and losing energy to the walls of the system.

The *earth's radiation belts* consist of protons and electrons that are trapped in the earth's non-uniform magnetic field. The earth's dipole magnetic field is non uniform and resembles a magnetic bottle field. The protons are confined in the inner region, which is centred about 3000 km above the earth's surface. The electrons are in the outer region which is centred about 15,000 km above the earth's surface.

11.5 CYCLOTRON

In 1931 E. Lawrence and M. Stanley developed a machine named cyclotron, for the acceleration of charged particles, such as protons or deuterons. These particles (ions), starting from the central source are caused to move in circular orbits by magnetic field and are accelerated by the electric field.



(*a*) (*b*)

Fig. 11.8. Schematic diagram of cyclotron.

In its simplest form, it consists of two flat semi-circular hollow, metal chambers called dees because of their shape. These hollow chambers have their diametric edges parallel and slightly separated from each other. A radio-frequency alternating potential of the order of magacycles per second is applied between the dees D_1 and D_2, which act as electrodes. These dees are surrounded by a closed vessel containing gas like hydrogen, helium, deuterium at very low pressure. The vessel is placed between the poles *N* and *S* of a strong electromagnet which provides a magnetic field perpendicular to the plane of the dees. Ion source *S* is placed at the centre of the chamber (mid point of the gap). In the ion source a filament is heated and a small potential difference is applied between the filament and the chamber to increase the energy of emitted electrons. These electrons ionize some of gas atoms and thus produce positive ions. Several types of ion sources are used now a days.

Suppose that at any particular instant the alternating potential is in the direction which makes D_1 positive and D_2 negative. A positive ion of mass m, charge q and an initial velocity v starting from the source S will be attracted by the dee D_2. Due to magnetic field B it will move in a circular path of radius r inside the dee D_2, where

$$r = mv/Bq. \qquad ...(35)$$

In the interior of the dee, the speed of the ion remains constant. After it has traversed half a cycle the ion comes to the edge of D_2. If in the meantime, the potential difference between D_1 and D_2 has changed direction so that D_2 is now positive and D_1 negative, the positive ion will receive an additional acceleration, while going across the gap between the dees and then travel in a circular path of larger radius inside D_2 under the influence of magnetic field. After traversing a half cycle in D_2 it will reach the edge of D_1 and receive an additional acceleration between the gaps because in the meantime the direction of potential difference between the dees has again changed. The ion will continue travelling in a semi-circle of increasing radii, the direction of *pd* changes every time the ion goes from D_1 to D_2 and form D_2 to D_1. The time taken by the charged particle to traverse the semicircular path in the dee is given by

$$t = \pi r/v = \pi m/Bq. \qquad ...(36)$$

This relation indicates that time t is independent of the velocity of the particle and of the radius. For any given value of m/q, it is determined by the magnetic field intensity. By adjusting the magnetic field intensity the time can be made the same as that required to change the potentials. On the other hand the oscillator frequency can also be adjusted to the nature of a given ion and to the strength of the magnetic field. The frequency of the oscillations required to keep the ion in phase is called *cyclotron resonance frequency* and is given by the relation

$$f = 1/T = 1/2t = Bq/2\pi m. \qquad ...(37)$$

If the oscillation frequency is adjusted to keep the charged ion always in phase, each time the ion crosses the gap it receives an additional energy and at the same time it describes a flat spiral of increasing radius. Eventually, the ion reaches the periphery of the dee, where it can be brought out of the chamber by means of a deflecting plate charged to a high negative potential. This attractive force draws the ion out of its spiral path and thus can be used easily. If R is the radius of the dee, kinetic energy of the ion emerging from the cyclotron is thus given by

$$E = \tfrac{1}{2}m\,(BqR/m)^2 = B^2R^2q^2/2m = 2\pi^2R^2f^2\,m. \qquad ...(38)$$

This relation indicates that the maximum energy attained by the ion is limited by the radius R, magnetic field B or the frequency of the alternating potential f. It is independent of the alternating voltage. It can be explained by the fact that when the voltage is low the ion makes a large number of turns before reaching the periphery, but when the voltage is high the number of turns is small. The total energy remains same in both the cases provided B and R are unchanged.

The above relations are also true for relativistic velocities, provided $m = m_0/(1 - v^2/c^2)^{1/2}$. The quantity Br, often called rigidity of the beam of ions, is given as

$$Br = \frac{mv}{q} = \frac{Ev}{qc^2} = \frac{1}{qc}(E^2 - E_0^{\,2})^{1/2} \qquad ...(39)$$

$$(\text{As } E = E_0 + T = m_0c^2 + \tfrac{1}{2}\,mv^2)$$

Thus, we see that to keep the cyclotron resonance frequency constant, B would have to be increased with radius because of the relativistic increase of mass with velocity. For protons, the relativistic increase becomes important beyond 25 *MeV*. The magnetic field can be made to increase towards the periphery of the circular pole face by using a set of auxiliary coils (trim coils). The consequent defocussing tendency of the circulating beam (ions) is corrected by dividing the pole gap azimuthally into alternate regions of strong and weak fields. This is done by attaching sector shaped steel pieces over the pole faces of the magnet. The machines using such magnets are thus called sector focussed (*SF*) cyclotrons or Azimuthally Varying Field (AVF) cyclotrons.

Cyclotrons are usually described in terms of the diameter of the pole faces of the magnet. The first machine constructed by Lawrence and Livingston in 1932 had a magnet with pole faces 11 inches in diameter and produced 12 MeV protons. The first cyclotron (26" dia) in operation in India is at Physics Department, P.U. Chandigarh. The first Indian 88" dia cyclotron VEC is at Calcutta.

Table 11.1. Cyclotrons in India

Particulars	*Chandigarh Cyclotron*	*VEC Calcutta*
Type	Simple type transferred from Rochester University USA	AVF type Built by Indian Physicists and Engineers
Pole diameter	26"	88"
Pole gap	6.3"	7.7"
Maximum Magnetic field	14 k Gauss	21.5 k Gauss
Weight of the magnet	20 tones	262 tones
Frequency range of oscillator	10 – 20 MHz	5.5 – 16.5 MHz
Maximum Dee Voltage	40 kV	70 kV
Maximum power of oscillator	25 kW	300 kW
Ion source	Hooded arc type	PIG type
Vacuum	5×10^{-6} Torr	10^{-6} Torr
Particles and their energies	p(2 – 5 MeV) d(1 – 3 MeV), α (2 – 8 MeV)	*p*(6 – 60 MeV), d(12–65 MeV), α (25 – 130 MeV), heavy ions.

11.6 SYNCHRO-CYCLOTRON

The relativistic limitation on energy for fixed frequency cyclotron has restricted the useful size of the magnets. Theoretically the ions can be accelerated indefinitely if the applied frequency is varied to match exactly the ion revolution frequency.

In synchrocyclotron light positive ions (*p, d*, α, etc.) are accelerated to energies significant relative to the rest energy of the particle. The ions traverse in circular orbits with increasing radii as energy increases. The ions pass many times (two times in an one revolution) through the *rf*-field of a large *D*-shaped hollow electrode and experience an acceleration on each traversal of the accelerating gap. The orbit radius corresponding to a particular *K.E.* of the particle is given by

$$r = \frac{mv}{qB} = \frac{[T(T+2m_0c^2)]^{1/2}}{cqB}, \qquad ..(40)$$

The corresponding frequency for synchronism is given by

$$f = \frac{v}{2\pi r} = \frac{qB}{2\pi m} = \frac{qB}{2\pi m_0}\left(\frac{m_0c^2}{m_0c^2+T}\right) = \frac{f_0}{1+T/m_0c^2}, \qquad ...(41)$$

where $f_0 = qB/2\pi m_0$ = non relativistic cyclotron frequency.

Above relation shows that the fractional change in oscillator frequency is proportional to T/m_0c^2. Thus the electrons, because of the small rest mass cannot be readily accelerated in the synchrocyclotron. The change in frequency is ordinarily made with a rotating multibladed capacitor.

The ion source and *D*-shaped electrode are similar to that used for the standard cyclotron. The magnetic field is produced by a solid core magnet of large pole face area. It is uniform over the pole face and decreasing slightly with increasing radius to provide focussing. The radio frequency oscillator for giving power to dee depends for its frequency variation on the variable capacitor (resonant circuit). The magnetic deflector is used to produce an emergent beam.

The synchrocyclotron has been most valuable in providing information about the production and properties of mesons.

11.7 CHARGED PARTICLES IN PARALLEL MAGNETIC AND ELECTRIC FIELDS (Thomson Parabola Method)

Positive rays also called canal rays are produced in a discharge tube consisting of an anode (an aluminium disc) and a cathode (rod of *Al* in which is embedded a narrow copper tube of bore 0.1 mm). These rays are allowed to pass through the strong electric and magnetic fields acting parallel to each other and perpendicular to the ray direction. Let us calculate the effect of these fields. Consider a particle of charge *q* and mass *m* moving along the negative *z*-direction with an initial velocity *v*, with reference to the origin O at the centre of the screen, as shown in Fig. 11.9.

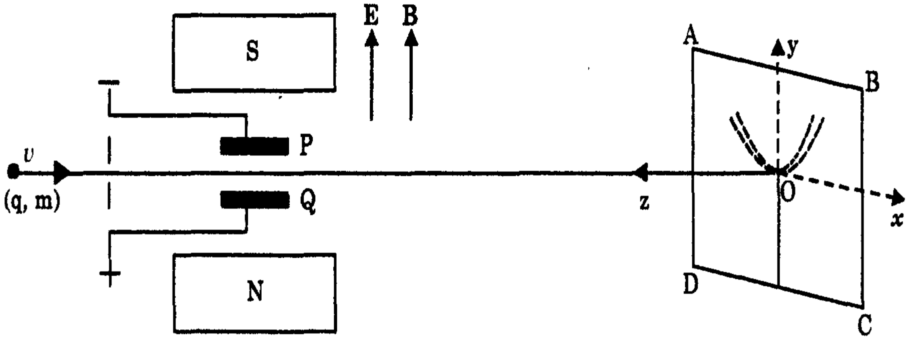


Fig. 11.9. Thomson parabola method for *q*/*m* of positive rays.

Let **E** and **B** fields be along the *y*-direction. Due to the magnetic field *B*, a force *q***v** × **B** will act on the charge particle along *x*-axis.

As : $\mathbf{B} = B\mathbf{j}, \mathbf{E} = E\mathbf{j}$ and $\mathbf{v} = -v\mathbf{k}$. here $\mathbf{F} = q(-v\mathbf{k}) \times (B\mathbf{j}) = qvB\mathbf{i}$.

Hence the acceleration produced will be given by the relation

$$a_x = qv\,B/m. \qquad \text{...(42)}$$

If the magnetic field exists across the length l_1, the time taken by the particle to traverse the magnetic field with velocity *v* is

$$t_1 = l_1/v.$$

Since the initial velicity has no component in the *x*-direction, hence the displacement of the particle in the *x*-direction at the end of the magnetic field

$$x_1 = \tfrac{1}{2} a_x t_1^2 = \tfrac{1}{2}\,(qvB/m)\,(l_1/v)^2. \qquad \text{...(43)}$$

If the screen is placed at a distance L_1 from the end point of magnetic field the further displacement of the charged particle along *x*-axis is

$$x_2 = L_1 \tan\theta = L_1 \frac{v_x}{v} = \frac{L_1 a_x t_1}{v} = \frac{qBL_1 l_1}{mv}.$$

$$\therefore \quad \text{Total displacement } x = x_1 + x_2 = \frac{qB}{mv}\left(\frac{l_1^2}{2} + L_1 l_1\right). \qquad ...(44)$$

As discussed earlier in article 11.2 the displacement along y-direction due to the electric field in y-direction on a charged particle moving along z-direction with a velocity v is given as

$$y = \frac{qE}{mv^2}\left(\frac{l_2^2}{2} + L_2 l_2\right), \qquad ...(45)$$

where l_2 is the length of electric field region and L_2 is the distance from the end of electric field to the screen.

Eliminating v from Eqs. (44) and (45), we get

$$x^2 = \frac{q}{m}\frac{B^2\left(\frac{1}{2}l_1^2 + L_1 l_1\right)^2}{E\left(\frac{1}{2}l_2^2 + L_2 l_2\right)}y = \frac{q}{m}Ky, \qquad ...(46)$$

where the constant K can be calculated from the knowledge of the E and B fields and the geometry of the apparatus.

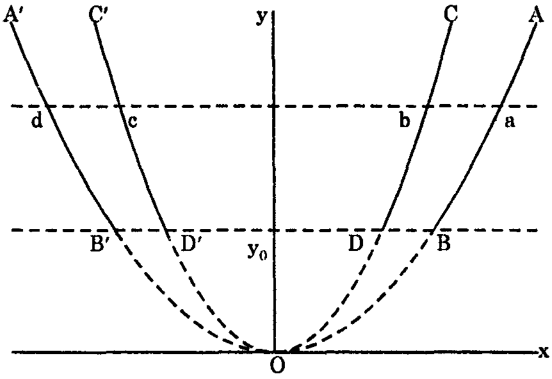


Fig. 11.10. Mass parabolas.

For the particles having same q/m ratio, equation (46) represents a parabola in x-y plane. By measuring ordinates of any point on a parabola the value of q/m can be determined. The particles of different q/m will trace the different parabolas on the screen, as shown in Fig. 11.10.

For the ions with the same charge but with mass m' greater than m, the parabola will be lower because the magnetic displacement is smaller for the heavier ions. If the mass m of ions giving the parabola AB is known accurately, then the mass m' corresponding to the parabola CD can be obtained in the following way. The magnetic field is reversed after half the time interval of the exposure of plate and a pair of curves $A'B'$ and $C'D'$ is obtained, which are the mirror images of AB and CD respectively. By drawing lines parallel to x-axis, we can find distances between the curves and their mirror images.

From Eq. (46), we have

$$\frac{q/m}{q/m'} = \frac{x_1^2/y}{x_2^2/y} = \frac{x_1^2}{x_2^2} = \frac{(2x_1)^2}{(2x_2)^2} = \frac{(ad)^2}{(bc)^2}$$

or

$$m' = m\,(ad/bc)^2. \qquad ...(47)$$

Thus the unknown mass m' is calculated.

It is clear from the Fig. 11.10 that all parabolic arcs terminate at a minimum distance y_0 from apex O, y_0 is called *cut off value*, which is independent of the charge and mass of the ion or charged particle.

The maximum initial velocity v_m of the particle due to the potential difference V is given by

$$\frac{1}{2}mv_m^2 = qV.$$

The corresponding minimum electrostatic deflection is given by Eq. (45).

$$y_0 = \frac{qE}{mv_m^2}\left(\frac{l_2^2}{2} + L_2 l_2\right) = \frac{E}{2V}\left(\frac{l_2^2}{2} + L_2 l_2\right) \qquad ...(48)$$

Thus we see that y_0 is independent of q/m and depends upon E and V.

11.8 CHARGED PARTICLES IN CROSSED ELECTRIC AND MAGNETIC FIELDS

Consider a particle of charge q and mass m, emitted at the origin with zero initial velocity into a region of uniform electric and magnetic fields. The field **E** is acting along x-axis and field **B** is along y-axis. The force $q\mathbf{E}$ due to electric field will act along x-axis. As the charged particle is placed in a magnetic field **B** it will experience a force $q\,(\mathbf{v} \times \mathbf{B})$ acting in the direction perpendicular to **v** and **B** both. Hence the particle will bent. The resultant force also known as Lorentz force is thus given by

$$\mathbf{F} = q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]. \qquad ...(49)$$

In our particular case $\mathbf{E} = E\,\mathbf{a}_x,\ \mathbf{v} = v_x\,\mathbf{a}_x + v_y\,\mathbf{a}_y + v_z\,\mathbf{a}_z$ and $\mathbf{B} = B\,\mathbf{a}_y$.

$$\therefore \quad \mathbf{F} = q[E\mathbf{a}_x + v_xB\,\mathbf{a}_z - v_zB\,\mathbf{a}_x] = q[E - v_z B]\,\mathbf{a}_x + qv_xB\,\mathbf{a}_z.$$

To find velocity components v_x and v_z, above relation can be written as

$F_x = m(dv_x/dt) = qE - qv_zB,\ F_y = m(dv_y/dt) = 0$ and $F_z = m(dv_z/dt) = qv_xB$.

On solving these equations, we get

$$m\frac{d^2v_x}{dt^2} = -\,qB\frac{dv_z}{dt} = -\frac{q^2B^2v_x}{m}$$

$$\text{or} \quad \frac{d^2v_x}{dt^2} + \frac{q^2B^2}{m^2}v_x = 0. \qquad ...(50)$$

This is the general equation of S.H.M. Its solution is

$$v_x = A\sin(\omega t + \phi), \text{ where } \omega = qB/m. \qquad ...(51)$$

As at $t = 0$ $v_x = 0$, hence $\phi = 0$.

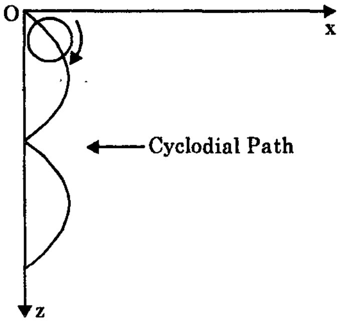


Fig. 11.11. Combined effect of electric and magnetic fields.

Again $dv_x/dt = A\omega\cos\omega t$. Hence at $t = 0$, $dv_x/dt = A\omega$, which should be same as qE/m.

$$\therefore \quad A\omega = qE/m \text{ or } A = qE/m\omega = E/B.$$

Hence Eq. (51) becomes $v_x = (E/B)\sin\omega t$.

Now substituting value of v_x in the equation for F_x, we get

$$m\frac{E}{B}\,\omega\cos\omega t = qE - qv_zB \text{ or } v_z = \frac{E}{B}\,[1 - \cos\omega t]. \qquad ...(52)$$

On integrating equations for v_x and v_z and knowing that at $t = 0, x = 0$ and $z = 0$, we get

$$x = \frac{E}{B\omega}[1 - \cos\omega t] \text{ and } z = \frac{E}{B\omega}[\omega t - \sin\omega t] \qquad ...(53)$$

These equations are the equation for a *cycloi*, which is defined as the path generated by the point on the circumference of a circle, rolling along a circle as shown in Fig. 11.11. In the present case the radius of the rolling circle is $E/B\omega$, the maximum displacement along x-direction is $2E/B\ \omega$. The x-displacement becomes zero at $t = 0,\ 2\pi/\omega,\ 4\pi/\omega$. The *z-displacement in time* $t = 2\pi/\omega$ is $2\pi E/B\omega$.

Let us now discuss few important applications of the combined electric and magnetic fields acting perpendicular to each other.

(*i*) Velocity selector : Since most of the electrons or ions emitted from the source are with a wide range of velocities, a velocity selector is often essential to obtain ions of same velocity. This device requires both electric and magnetic forces. A capacitor like arrangement AB provides a uniform electric field **E** in the plane of the paper. A uniform magnetic field **B** is provided ⊥ to the plane of the paper in upward direction. Collimated charged particles move along SC if the force due to **E** is balanced by the force due to **B** as shown in Fig. 11.12. Hence the particle will move with velocity v, given by $qE = qv\,B$ or $v = E/B$. Thus by adjusting E and B, particles of different velocities are obtained in the SC-direction.

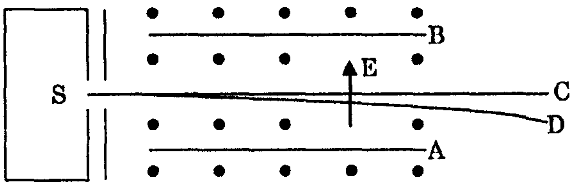


Fig. 11.12. Velocity selector.

***(ii)* Hall effect** : In 1879, E.H. Hall devised an experiment that gives the useful information regarding current carriers in conductors. Suppose we have a steady current i, hence current density **j**, flowing in a uniform conducting strip along the x-axis. If this strip is placed in a uniform magnetic field **B**, acting along the y-axis. The field **B** will exert a deflecting force **F** on the current carriers, may be electrons or holes (missing electrons), in the direction ⊥ to the plane containing vector **B** and current density vector **j**. If the current is carried by positive charge q moving with velocity **v** along **j**, then the force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$.

As clear from Fig. 11.13, this force is in upward direction whether the charge cariers are positive or negative. If the charge carriers are electrons, an excess negative charge accumulates at the upper edge of the slab, leaving an excess positive charge at its lower edge. Thus a transverse potential difference, known as *Hall emf* is produced. This name is after its discoverer. The sign of this emf will be opposite if the charge carriers are positive. Thus the sign of the charge carriers is determined by the sign of this Hall emf, which can be measured with a potentiometer. Experiment shows that in metals the charge carriers are electrons. However semiconductors exhibited a Hall emf opposite to that of the metals as if their charge carriers were positively charged. These charge carriers are now known as *holes*.

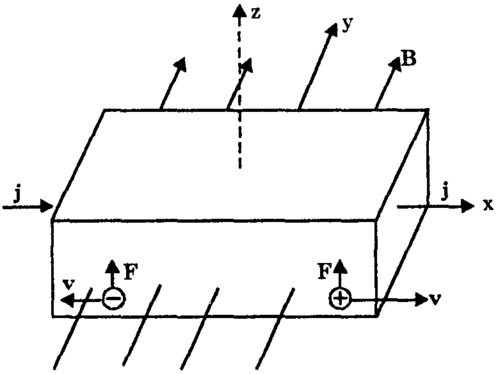


Fig. 11.13. Hall emf.

The displacement of charge carriers gives rise to a transverse field, known as *Hall electric field* $\mathbf{E}_H$, which acts inside the conductor, to oppose the side ways drift of the carriers. An equilibrium is reached in which the magnetic deflecting force on the charge carriers is just balanced by the electric force caused by the Hall electric field.

$$q\,(\mathbf{v} \times \mathbf{B}) + q\mathbf{E}_H = 0 \text{ or } \mathbf{E}_H = -\,\mathbf{v} \times \mathbf{B}. \qquad \text{...(54)}$$

The velocity of the deflecting charge carriers **v** can be measured by measuring $\mathbf{E}_H$ and **B**, where E_H = Hall emf (V_H)/distance (d) between two edges. The Hall emf is thus given by

$$V_H = \mathbf{E}_H.\mathbf{d} = vBd. \qquad \text{...(55)}$$

We know that the current density $\mathbf{j} = nq\mathbf{v}$, hence $E_H = jB/nq$

or density of charge carriers $\qquad n = jB/qE_H. \qquad$ ...(56)

The ratio of the Hall electric field to the product of current density j and magnetic induction B is known as the *Hall coefficient or simply Hall constant R.* Thus

$$R = E_H/jB = 1/nq. \qquad \text{...(57)}$$

The Hall constant R is negative if charge carriers are electrons (negative q) *and is positive if these are holes* (positive q). Observed and calculated values of the Hall constant are in good agreement for most of the metals. Its unit is m^3/coul.

The exact result for semiconductors is slightly different from Eq. (57) because the electrons in the semiconductor are not entirely free. The Hall coefficient is found to be

$$R = 3\pi/8\, nq = 1.18/nq. \qquad \ldots(58)$$

Since the current density is defined as $j = nq < v >$, hence we get

$$E_H = jBR = nq < v > B.(1/nq)$$
$$= < v > B = \frac{< v >}{\mathcal{E}} \mathcal{E}B = \mu\mathcal{E}B, \qquad \ldots(59)$$

where $\mathcal{E}$ is the emf responsible for current density j and μ is called the *mobility* of charge carriers. The mobility can be determined by measuring E_H when a known $\mathcal{E}$ and B are applied to the crystal. The mobility may also be measured by measuring the *Hall angle*, defined as

$$\tan\theta = E_H/\mathcal{E} = \mu B. \qquad \ldots(60)$$

The mobility μ of the charge carriers may also be determined by measuring the conductivity σ, defined as

$$\sigma = nq\mu = \mu/R, \quad \text{or} \quad \mu = \sigma R. \qquad \ldots(61)$$

Measurement of Hall coefficient gives us the number of current carriers per unit volume and the *relaxation time* (the average time between two collisions of an electron). Its sign tells us whether the current carriers are positive or negative. The mobility of the carriers can be measured by the conductivity of the material and the Hall coefficient. Hall effect may be used to measure magnetic fields by using *Hall probe*.

Hall probe : It is a simple instrument used to measure the flux density of a magnetic field. In it a wafer of semiconductor has two contacts on opposite sides which are connected to a high impedance voltmeter V. An electric current $i(< 1$ amp) is allowed to pass through the semiconductor in the normal direction and is measured with the ammeter A. Araldite encapsulation is done so that the wires may not be detached from the wafer. Combining relations

$$E_H = jB/nq, \; V_H = E_H d \text{ and } i = jA, \text{ we get}$$

$$V_H = \frac{jBd}{nq} = \frac{iBd}{nqA},$$

$$\therefore \quad B = \frac{V_H nqA}{id} = \frac{V_H}{i}\left(\frac{nqA}{d}\right) \qquad \ldots(62)$$

Here (nqA/d) is constant for a given semiconductor and can be determined easily. Thus the measurement of V_H and i, *i.e.*, the voltmeter and ammeter readings will give the value of B of magnetic field in which the semiconductor or Hall probe has been placed.

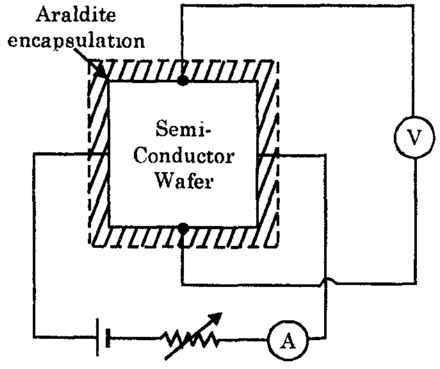


Fig. 11.14. Hole probe.

(*iii*) Determination of *e/m* of an electron (Thomson method) : J.J. Thomson in 1897, devised an experiment for the determination of *e/m* (specific charge) of the electron which he called cathode corpuscles by using electric and magnetic fields in mutually ⊥ directions. The electrons are allowed to pass in the direction ⊥ to **E** and **B**.

The discharge is maintained by the application of high *pd* between the cathode C and anode A of a discharge tube containing air at a very low pressure (~10^{-2} mm of *Hg*). The electrons emerging from cathode C are allowed to pass through slits A_1 and A_2 also kept at the potential of A and thus act as

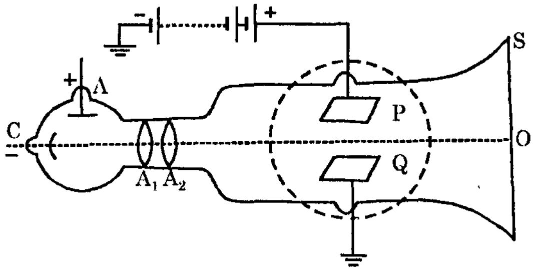


Fig. 11.15. Thomson's apparatus for determination of *e/m* for electron.

anodes. The beam then passes along the axis of the tube and produces a spot of light at O on the fluorescent screen S. The electric field **E** is applied between two horizontal plates P and Q. The magnetic field **B** is applied in the direction ⊥ to the plane of paper by passing the current through coils, in the region within the dotted circle. It is clear from Fleming's left-hand rule, that **F** due to **E** is in upward direction, while due to **B** in downward direction. Hence fields **E** and **B** can be adjusted so that the electrons suffer no deflection and strike at point O on the screen. In this case

$$eE = evB \text{ or } v = E/B. \qquad ...(63)$$

Thus for a given electron speed v, the condition for no deflection can be obtained by adjusting E or B. For this the position O of the spot on the fluorescent screen is first noted with E and B both equal to zero. A fixed electric field E is applied and the position of the spot on the screen is noted. Now a magnetic field is applied simultaneously and adjusted until the beam deflection is restored to zero, *i.e.* the spot regains to its original position O. The crossed electric and magnetic fields thus serve as a *velocity selector.*

The magnetic field is removed and the deflection OO' caused by the electrostatic field alone is measured. As discussed in section 11.2, the total y-displacement at a distance L from the ends of the plates (each of length l) of a charge particle entering the gap between the plates with velocity v is given by.

$$y = OO' = \frac{eEl}{mv^2}\left[\frac{l}{2} + L\right]. \qquad ...(64)$$

Substituting the value of v from Eq. (63) into Eq. (64) and solving we get

$$\frac{e}{m} = \frac{E}{B^2}\frac{y}{l(l/2 + L)}. \qquad ...(65)$$

If the geometry of the apparatus is known, one can calculate e/m from the measured displacement of the spot on the screen and the applied electric and magnetic fields.

Thomson found that e/m determined in this way was independent of the kind of gas in the tube and of the electrodes. Thomson's value for e/m was 1.7×10^{11} coul/kg, which is in excellent agreement with the modern value of 1.7890×10^{11} coul/kg.

***(iv)* Mass spectrograph :** Instruments are used to measure mass numbers, to determine the relative abundance of isotopes or to produce separated isotopes. Instruments in which we use a photographic plate and obtain the series of lines (called the mass spectrum) on it are known as *mass spectrographs.* Instruments in which an electric or a magnetic field is adjusted to make each part of the spectrum in turn to fall on a fixed detecting slit and measured electrically are called *mass spectrometers.*

Now a days mass spectrographs or mass spectrometers are not only used for the study of isotopes, but are useful in the analysis of complex mixtures such as petroleum, in establishing the structures of certain organic molecules and even in identification of new unstable compounds. Few of them are discussed here in the next articles.

11.9 ASTON MASS SPECTROGRAPH

The Thomson parabola method for analysing positive rays could not yield precise values of isotopes masses and abundances on account of the following reasons : (*i*) The intensity available for photography is very much reduced by collisions and spreading, (*ii*) The traces on the photographic plate are blurred with no definite edges, (*iii*) The influence of secondary rays make analysis difficult and (*iv*) the resolving power is very poor.

Aston's mass spectrograph is the improvement of the Thomson parabola method and has a greater sensitivity and high precision. In this arrangement the electric and magnetic fields are separated and are at right angles to each other. The photographic plate is slightly inclined to the

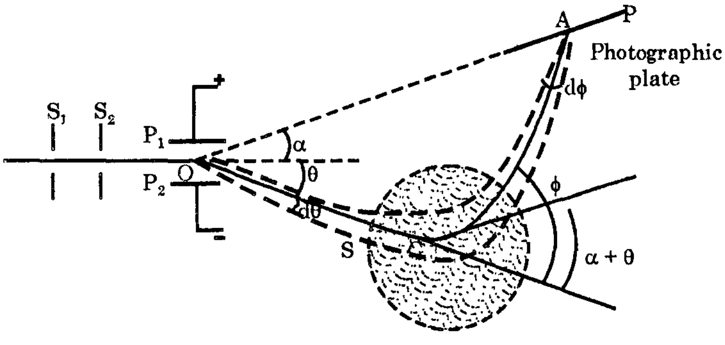


Fig. 11.16. Aston's mass spectrograph.

beam of positively charged particles. The positive rays from a discharge tube are allowed to pass through two very narrow parallel slits S_1 and S_2. A very fine beam thus obtained enters the space between the metal plates P_1 and P_2. The electric field between these plates causes the deflection of ions toward P_2. The particles of different velocities are deflected to the different amounts. Thus a narrow beam is broadened as it passes through the field. The linear diplacement s of the ion of charge q and mass m due to the electric field E is given by the relation

$$s = \frac{1}{2}\text{at}^2 = \frac{1}{2}\left(\frac{qE}{m}\right)\left(\frac{l}{v}\right)^2 = \frac{qEl^2}{2mv^2},$$

where l is the path length in the electric field. The angular displacement θ is thus given by

$$\theta = s/l = qEl/2mv^2, \qquad \text{...(66)}$$

The broadened beam limited by a relatively wide slit S enters the magnetic field, indicated by the dotted circle, perpendicular to the plane of paper ($\mathbf{B} \perp \mathbf{E}$). The magnetic field causes deflection, the more slowly moving ions being deflected more than the fast ones. Thus the beam can be focussed at some point A on the photographic plate P. The linear displacement s' of the ions due to magnetic field is given by

$$s' = \frac{1}{2}a't'^2 = \frac{1}{2}\left(\frac{qvB}{m}\right)\left(\frac{l'}{v}\right)^2 = \frac{qBl'^2}{2mv},$$

where l' is the path length of the ion in the magnetic field. Thus the angular displacement ϕ is given by

$$\phi = s'/l' = qBl'/2mv. \qquad \text{...(67)}$$

Let $d\theta$ and $d\phi$ be the dispersion angle due to electric field and the convergence angle due to magnetic field respectively, then from Eqns (66) and (67), we get

$$\frac{d\theta}{dv} = -\frac{2qEl}{2mv^3} = \frac{-2}{v}\theta \text{ or } \frac{dv}{v} = -\frac{1}{2}\frac{d\theta}{\theta} \qquad \text{...(68)}$$

and
$$\frac{d\phi}{dv} = -\frac{qBl'}{2mv^2} = -\frac{1}{v}\phi \text{ or } \frac{dv}{v} = -\frac{d\phi}{\phi}. \qquad \text{...(69)}$$

$$\therefore \quad d\theta/\theta = 2\,d\phi/\phi, \text{ or } d\theta/d\phi = 2\,\theta/\phi. \qquad \text{...(70)}$$

Let $OC = a$ and $CA = b$. The width of beam at $O = ad\theta$. If this beam has travelled a further distance b in the absence of magnetic field, the width of the beam would be $(a + b)\,d\theta$. This divergence is balanced by the convergence due to the magnetic field and the beam is focussed at point A. Hence we have

$$(a + b)\,d\theta = b\,d\phi \text{ or } d\theta/d\phi = b/(a + b). \qquad \text{...(71)}$$

From Eqns. (70) and (71), we get

$$\frac{2\theta}{\phi} = \frac{b}{a+b} \text{ or } \frac{b}{a} = \frac{2\theta}{\phi - 2\theta}. \qquad \text{...(72)}$$

This is the condition for focussing all the ions on the photographic plate. If $\phi = 2\theta$, $b = \infty$ and the photographic plate is parallel to emergent beam. In this case $\alpha = \theta$. $\phi < 2\theta$ is not permissible by relation (72). For $\phi = 4\theta$, $b = a$. If the instrument is properly designed, ions having the same value of q/m but slightly different velocities can be focussed at one point on a photographic plate. The ions of different values of q/m are focussed at different points on the plate. In practice these are not points, but lines. The unknown masses of the ions can be obtained by comparing the positions of the lines caused by the masses with the positions of the lines caused by standard substances whose masses are accurately known.

Aston was able to prove that neon has two isotopes (20 and 22). He also showed that chlorine has two isotopes (35 and 37). He could determine the masses with an accuracy of 1 part in 10,000.

11.10 DEMPSTER'S MASS SPECTROMETER

Dempster built an instrument which was basically simpler and most suited for making accurate mass measurements. It was called mass spectrometer because the ions were not recorded on a photographic plate, but measured electrically by an electrometer. A schematic diagram of the Dempster's mass spectrometer is shown in Fig. 11.17. It works on the principle of 180° magnetic focussing. A beam of positive ions of the element to be analysed is produced by bombarding salts with electrons or by heating salt on a platinum strip. Upon emerging from the source S, the ions are accelerated through a potential difference V applied between the plates P and Q each having narrow slits. The collimated beam enters a magnetic field B applied perpendicular to the plane of the paper and follows the semicircular path.

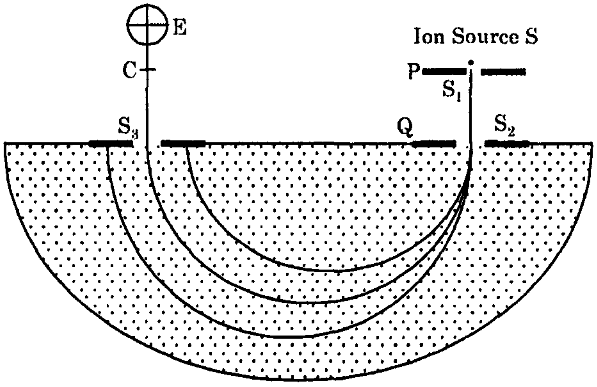


Fig. 11.17. Dempsters mass spectrometer.

If the charge on the positive ions is q, the K.E. of the ion of mass m will be same as the energy acquired in passing through a potential difference V, *i.e.* $\frac{1}{2}mv^2 = qV$.

If R be the radius of curvature of the ion path in a magnetic field $\mathbf{B}$, then $qvB = mv^2/R$.

Eliminating v from the preceding two equations, we get

$$q/m = 2V/B^2R^2. \qquad ...(73)$$

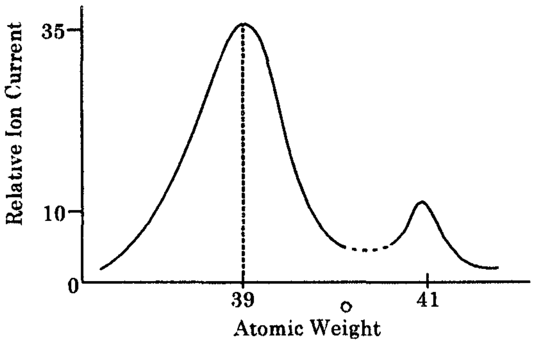


Fig. 11.18. Mass spectrum of potassium.

Since the radius of the circle must have a certain definite value in order that the ions enter the slit S_3 and detected by an electrometer E. The current recorded by the electrometer is proportional to the number of ions falling per unit time on the collector plate C. Thus ions with only one particular value of q/m will be detected for a given combination of V and B. The ions with different values of q/m are detected by varying the potential difference V.

Mass spectrum of potassium as observed by Dempster is shown in Fig. 11.18. Two peaks in the spectrum show the existence of two isotopes of potassium of atomic masses 39 and 41. The relative areas under these peaks are 18 : 1 respectively. It shows that ^{39}K is more abundant.

11.11 BAINBRIDGE'S MASS SPECTROGRAPH

It is an accurate method for determining the atomic masses. A schematic diagram of Bainbridge's mass spectrograph is shown in Fig. 11.19. In it a velocity selector is used to obtain ions of particular velocity v. The velocity selector allows a beam of a positive ions having the same velocity v to pass undeviated through crossed electric and magnetic fields E and B. In this case forces due to these fields are equal and opposite.

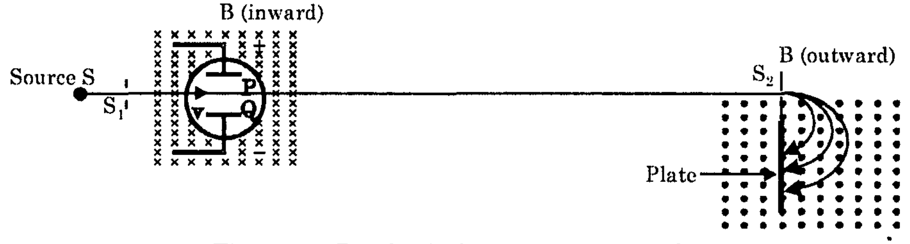


Fig. 11.19. Bainbridge's mass spectrograph.

$$\text{Force due to electric field } qE = \text{Force due to magnetic field } Bqv$$

or
$$v = E/B. \qquad \text{...(74)}$$

All the ions having the same velocity ($v = E/B$) enter the analysing chamber through the slit S_2. In this chamber another magnetic field B' is applied perpendicular to the plane of paper and in outward direction. Due to this field ions of different masses move in circles of different radii such that

$$\text{Magnetic force } qvB' = \text{centrifugal force } mv^2/r$$

or
$$\frac{q}{m} = \frac{v}{B'r} = \frac{E}{BB'r}. \qquad \text{...(75)}$$

Thus we see that the ions of different charges will follow different circular paths. Radius r of circular orbit is directly proportional to the mass m. The value of r is obtained from the position of the line on the photographic plate and hence q/m is calculated using the above relation.

Resolving power : Mass of the ion $m = qBB'r/E$

$$\therefore \qquad \frac{dm}{m} = \frac{dr}{r} \text{ or } \frac{m}{dm} = \frac{2r}{2dr}$$

But $2dr = W$ = Width of the slit S_2

$\therefore$ Resolving power = $m/dm = 2r/W$. ...(76)

It has symmetrical images in a linear mass scale, which Aston could not get. Its resolving power and precision is high in comparison to that of Dempstor's mass spectrometer.

11.12 ELECTRON OPTICS (ELECTRON-MICROSCOPE)

Lenses are used to converge or diverge the optical rays. An optical microscope is a device to magnify small objects. To magnify very minute objects electron beam is used instead of the light rays because (*a*) electrons have a wave nature, similar to the light rays but of much shorter wavelength, (*b*) electrons can be focused by electric and magnetic fields, very much like light rays which are focused by glass lenses.

For the useful magnification, the high resolution and clear perceptibility are the two important conditions. If the perceptibility decreases with increase in resolution., the image would get blurred and no useful purpose is served. The resolving power of a microscope depends on the wavelength of light and the numerical aperture. The smaller the wavelength of light used the greater will be the resolving power. The electrons accelerated across a potential difference of 60 kV have wavelength ~10^{-12} m, 10^5 times smaller than that of the visible light. Hence the resolving power of the electron microscope is 10^5 times greater than that of optical microscope.

The clear perceptibility under high resolution is secured by a electron focussing which are of following two types :

(1) **Electrostatic focussing** : An electron lens, consisting of succession of anodes at increasing potentials, is used to converge and to get focussed the electrons flying out in different directions. The electrons from the source 0 are accelerated successively in the regions and are focussed at some point I on the axis similar to the light rays that are focussed through a lens. The focal length of the electron lens can be changed at will by altering the potential of the cylinders. This can be shown by the Fig. 11.20.

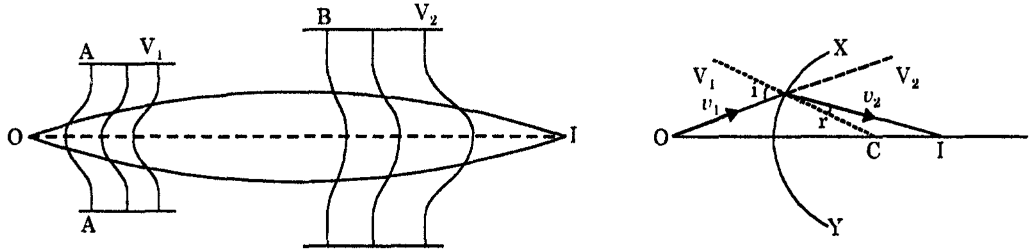


Fig. 11.20. Electrostatic focussing.

If v_1 and v_2 be the velocities of the electron in voltage regions V_1 and V_2 respectively. If c is the charge on the electron and m its mass, then we can write

$$\tfrac{1}{2}mv_1^2 = V_1e \text{ and } \tfrac{1}{2}mv_2^2 = V_2e,$$

$$\therefore \quad v_1/v_2 = (V_1/V_2)^{1/2}. \qquad \text{...(77)}$$

If the velocities are resolved in two components, parallel and perpendicular to the equipotential surface XY, then it is clear that there is no change in the components || to the surface XY, hence

$$v_1 \sin i = v_2 \sin r \quad \text{or} \quad v_2/v_1 = \sin i/\sin r = (V_2/V_1)^{1/2}, \qquad \text{...(78)}$$

Thus for $V_2 > V_1$, $i > r$, *hence the beam is deflected toward the normal, similar to refraction of light.*

(2) **Magnetic focussing** : Magnetic focussing is chiefly used in the electron microscope, when a very fast beam of electrons is required.

Consider an electron at point O inside a uniform magnetic field **B** moving with a velocity v and making an angle θ with the direction of **B**. Its component $v \cos\theta$ along **B** will not be affected and will proceed in a straight line, while the perpendicular component $v \sin\theta$ will describe a circle. Hence, the resultant motion is a helix. The circular motion is governed by the relation.

$$Bev \sin\theta = m\,(v \sin\theta)^2/R,$$

or the time taken by the electron to describe the circle

$$t = \frac{2\pi R}{v \sin\theta} = \frac{2\pi m}{Be}. \qquad \text{...(79)}$$

Hence, the pitch of the helix $L = v \cos\theta \times t$

$$= 2\pi\, vm \cos\theta/Be \qquad \text{...(80)}$$

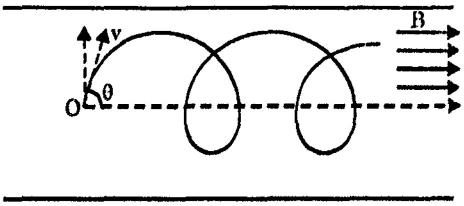


Fig. 11.21. Magnetic focussing.

It is clear from the expression for t that this time does not depend on θ, hence all the electrons diverting from point O in any direction are all focussed on the same point I.

The distance L depends upon **B**, hence the focal length of this magnetic lens can be altered by changing the value of **B**. The magnetic field represents the refractive medium which acts upon the electron beam as the glass does on a light beam.

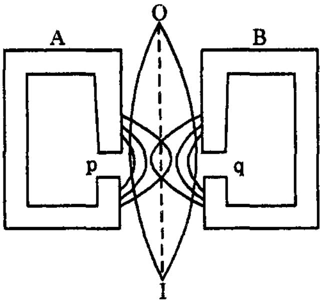


Fig. 11.22. Magnetic electron lens.

There are several types of magnetic electron lens. Gabor in 1927 suggested that the magnetic field could be concentrated within a shorter distance along the axis of the solenoid coil if the latter was encased in an iron shield which reduced stray magnetic fields considerably. For the large magnifications, minimum focal length should be used. The schematic diagram of magnetic electron lens is shown in Fig. 11.22. AB is the section of the coil of an electromagnet. The coil is surrounded by a shoft iron shield except for the gap pq. The magnetic field is strong across the gap between p and q and is symmetrical about the axis OI and thus focusses the electron beam at I.

Electron microscopes are very useful in many fields of modern research in industry, medicine and study of atomic structure. The difficulty that the electrons destroy the object can be removed by different methods, one is to soak the specimen in a solution of osmium salts.

EXERCISES

Example 1. *A proton, with initial velocity of 5 × 10⁶ m/sec, passes through an electric field (transverse) of 200 volts/cm. Calculate the transverse deflection in travelling a distance of 1 meter.*

Let the proton direction as x and that of the electric field as y.

Time taken to travel a distance L along the x-axis

$$t = L/v = 1/5 \times 10^6 = 2 \times 10^{-7} \text{ sec.}$$

Force acting on the proton due to electric field $= qE_y = mf_y$.

$\therefore$ Proton acceleration $f_y = qE_y/m$.

Hence the transverse deflection $y = \frac{1}{2} f_y t^2 = \frac{1}{2} qE_y t^2/m$

$$= \frac{1}{2} \times 1.6 \times 10^{-19} \times 2 \times 10^4 \times (2 \times 10^{-7})^2/1.6 \times 10^{-27}$$
$$= 0.04 \text{ meter.}$$

Example 2. *A stream of protons and deuterons in a vacuum chamber enters a uniform magnetic field. Both protons and deuterons have been subjected to the same accelerating potential, hence the kinetic energies of the particles are the same. If the magnetic field is perpendicular and the protons move in a circular path of radius 15 cm, find the radius of the path traversed by the deuterons.*

If m_1 and m_2 are the masses of proton and deuteron respectively and v_1 and v_2 are their velocities, then

$$K.E. = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_2 v_2^2. \qquad ...(i)$$

If r_1 and r_2 be the radii of the paths of protons and deuterons respectively in the magnetic field B, then

for protons $m_1 v_1^2/r_1 = Bqv_1$ or $v_1 = Bqr_1/m_1$

and for deuterons $m_2 v_2^2/r_2 = Bqv_2$ or $v_2 = Bqr_2/m_2$.

Substituting these values in the relation (*i*), we get

$$\frac{1}{2} m_1 (Bqr_1/m_1)^2 = \frac{1}{2} m_2 (Bqr_2/m_2)^2 \text{ or } r_1^2/m_1 = r_2^2/m_2.$$

As deuteron is an isotope of hydrogen and consists of a proton and a neutron, hence $m_2 \approx 2m_1$. Therefore we have

$$r_2 = r_1 (m_2/m_1)^{\frac{1}{2}} = 0.15\sqrt{2} = 0.212 \text{ meter}$$

Example 3. *A proton is shot with a speed of 1 × 10⁶ m/sec along a line at an angle of 30° to the x-axis, parallel to which a uniform field of 0.20 wb/m² exists. Describe the motion of the proton.*

One can split the velocity vector into two components, one along x-axis and other along y-axis. Since the field is in the x-direction, the force $F_x = qv_xB \sin 0 = 0$, and hence v_x is unaffected by the field. Thus the proton moves in the x-direction with a constant velocity $v_x = (1 \times 10^6) \cos 30° = 0.866 \times 10^6$ m/sec.

As **B** is $\perp$ to v_y the proton travels in a circle which is in y-z plane. Equating magnetic to centripetal force, we get, $mv_y^2/r = Bqv_y$.

or
$$r = \frac{mv_y}{Bq} = \frac{1.67 \times 10^{-27} \times 1 \times 10^6 \times \frac{1}{2}}{0.20 \times 1.6 \times 10^{-19}} = 2.6 \times 10^{-2}\ m$$

Hence time period
$$T = \frac{2\pi r}{v_y} = \frac{2 \times 3.14 \times 2.6 \times 10^{-2}}{1 \times 10^6 \times \frac{1}{2}} = 3.26 \times 10^{-7} \text{ sec.}$$

During this time the proton will have travelled a distance $v_x T$ parallel to the x-axis. The distance
$$v_xT = 0.866 \times 10^6 \times 3.26 \times 10^{-7} = 0.282 \text{ meter.} = 28.2 \text{ cm.}$$

Thus the proton will follow a helix like path with radius 2.6 cm and pitch 28.2 cm.

Example 4. *A deuteron travels in a circular path of radius 40 cm in a magnetic field of flux density 1.5 Wb / m². (a) Find the speed of the deuteron, (b) the time required for it to make one half a revolution (c) through what pd would the deuteron have to be accelerated to acquire this velocity?*

Due to magnetic field **B**, the particle of mass m and of charge q moves in a circular orbit of radius R. The motion is given by the relation
$$mv = BqR \quad \text{or} \quad v = BqR/m.$$

As the deuteron is the nucleus of the isotope of hydrogen, consisting of one neutron and one proton, hence its mass is $2u$ (= $2 \times 1.66 \times 10^{-27}$ kg). Now substituting numerical values, we get
$$v = 1.5 \times 1.6 \times 10^{-19} \times 0.4/2 \times 1.66 \times 10^{-27} = 2.89 \times 10^7 \text{ m/sec.}$$

Time required for one half revolution,
$$t = \pi R/v = 3.14 \times 0.4/2.89 \times 10^7 = 4.34 \times 10^{-8} \text{ sec.}$$

If potential difference required to accelerate is V, then $qV = \frac{1}{2}mv^2$

or
$$V = mv^2/2q = 2 \times 1.66 \times 10^{-27} \times (2.89 \times 10^7)^2/2 \times 1.6 \times 10^{-19}$$
$$= 8.55 \times 10^6 \text{ volts.}$$

Example 5. *Deuterons in a cyclotron describe a circle of radius 32.0 cm just before emerging from the D's. The frequency of the applied alternating voltage is 10 MHz. Find (a) the magnetic flux density and (b) the energy and speed of the deuterons upon emergence.*

Frequency of the applied emf = cyclotron frequency $f = Bq/2\pi m$.

$\therefore$ Magnetic flux density $B = 2\pi mf/q = 2 \times 3.14 \times 2 \times 1.66 \times 10^{-27} \times 10 \times 10^6/1.6 \times 10^{-19}$
$= 1.30\ wb/m^2$.

The maximum velocity $v_{\max} = BqR/m = 2\pi fR$
$= 2 \times 3.14 \times 10 \times 10^6 \times 0.32 = 2.01 \times 10^7$ m/sec.

Hence the energy $E_{\max} = \frac{1}{2}m\,(v_{\max})^2 = \frac{1}{2} \times 2 \times 1.66 \times 10^{-27} \times (2.01 \times 10^7)^2$
$= 4.22\ MeV$.

Example 6. *A particle of mass 1 × 10^{-26} kg and charge 1.6 × 10^{-19} C travelling with a velocity 1.28 × 10^6 m / sec in the positive x-direction enters a region in which a uniform electric field E and a uniform magnetic field of induction B are present such that $E_x = E_y = 0$, $E_z = -\,102.4$ kV / m and $B_x = B_z = 0$, $B_y = 8 \times 10^{-2}$ T. The particle enters this region at time t = 0. Determine the location of the particle at t = 5 × 10^{-6} sec. If the electric field is switched off at this instant, what will be the position of the particle at t = 7.45 × 10^{-6} sec.*

In the vector form $\mathbf{E} = -\,102.4\,\hat{z}$ (kV/m),
$$\mathbf{B} = 8 \times 10^{-2}\ \hat{y}\ (T) \text{ and } \mathbf{v} = 1.28 \times 10^6\ \hat{x} \text{ (m/sec).}$$

∴ The force on the charged particle due to the electric field E is

$$\mathbf{F}_e = q\mathbf{E} = -1.6 \times 10^{-19} \times 102.4 \times 10^3 \hat{z} = -1.638 \times 10^{-14} \hat{z} \text{ (N).}$$

The force due to magnetic field

$$\mathbf{F}_m = q(\mathbf{v} \times \mathbf{B}) = 1.6 \times 10^{-19} (1.28 \times 10^6 \hat{x} \times 8 \times 10^{-2} \hat{y}) = 1.638 \times 10^{-14} \hat{z} \text{ (N).}$$

Thus, we see that F_e and F_m are equal in magnitude but opposite in direction. The net force is zero, therefore the particle continues to travel along x-axis with the uniform velocity. The distance travelled by the particle is $D = v \times t = 1.28 \times 10^6 \times 5 \times 10^{-6} = 6.4\ m$.

When the electric field is switched off, the particle moves in a circular path due to magnetic field in the x-z plane in an anticlockwise direction if seen along the y-axis. The radius of the path is given by

$$R = \frac{mv}{qB} = \frac{1 \times 10^{-26} \times 1.28 \times 10^6}{1.6 \times 10^{-19} \times 8 \times 10^{-2}} = 1.0\ m.$$

The revolution time of the particle is

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{qB} = \frac{2 \times 3.14 \times 10^{-26}}{1.6 \times 10^{-19} \times 8 \times 10^{-2}} = 4.9 \times 10^{-6} \text{ sec.}$$

∴ Number of revolutions completed by the particle in time $\Delta t = t_2 - t_1 = 2.45 \times 10^{-6}$ sec. is

$$n = \frac{2.45 \times 10^{-6}}{4.9 \times 10^{-6}} = \frac{1}{2} \text{ revolution.}$$

Thus the distance travelled by the particle along z-direction = $2R$ = $2m$. Therefore the coordinates of the particle at $t = 7.45 \times 10^{-6}$ sec are ($6.4\ m$, 0, $2m$).

Example 7. *A potential difference of 600 V is applied across the plates of a parallel plate capacitor. The separation between the plates is 3 mm. Neglecting the edge effects. Find the magnetic field in the region between the plates, required to keep the electron of velocity 2×10^6 m/sec projected parallel to the plates, undeflected.*

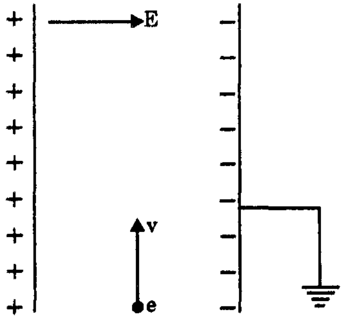


Fig. 11.23. Example 7.

The electric field between the plates normal to them is

$$E = V/d = 600/3 \times 10^{-3} = 2 \times 10^5\ Vm^{-1}.$$

It is directed from positive to negative plates. Velocity of the electron parallel to the plates = 2×10^6 m/sec. It is acting in upward direction.

For no deflection of electron when the magnetic field is applied, Lorentz force will be zero and the system will act as a velocity selector.

$$\therefore \qquad \boldsymbol{E} = -\boldsymbol{v} \times \boldsymbol{B}.$$

Thus $\boldsymbol{B}$ must be perpendicular to the plane of $\boldsymbol{E}$ and $\boldsymbol{v}$ (*i.e.,* plane of the paper) and into the plane.

$$B = \frac{E}{v} = \frac{2 \times 10^5}{2 \times 10^6} = 0.1 \text{ Wbm}^2.$$

Example 8. *In a synchrocyclotron producing 400 MeV protons, what must be the ratio of the oscillator frequency at the beginning of an accelerating cycle to that at the end? Calculate the velocity of such protons in terms of velocity of light.*

The frequency at the beginning of an accelerating cycle = non-relativistic cyclotron frequency = $f_0 = qB/2\pi m_0$,

The synchro cyclotron frequency is given by

$$f = \frac{qB}{2\pi m} = \frac{qB}{2\pi m_0}\left(\frac{m_0c^2}{m_0c^2 + T}\right) = \frac{f_0}{1 + T/m_0c^2}$$

$$\therefore \qquad \frac{f_0}{f} = 1 + \frac{T}{m_0c^2} = 1 + \frac{400 \text{ MeV}}{938.3 \text{ MeV}} = 1.426.$$

Since the rest mass energy of proton = m_0c^2 = 938.3 MeV. The relativistic relation for kinetic energy

$$T = (m - m_0)c^2 = m_0c^2\left(\frac{1}{\sqrt{1 - v^2/c^2}} - 1\right)$$

or $$\frac{f_0}{f} = 1 + \frac{T}{m_0c^2} = 1 + \frac{1}{\sqrt{1 - v^2/c^2}} - 1 = 1.426$$

or $$v^2/c^2 = 1 - 0.4917 = 0.5083 \text{ or } v/c = 0.71.$$

Thus the velocity of 400 MeV protons will be 0.71 times the velocity of light.

Example 9. *A mass spectrometer is being used to monitor air pollutants. It is difficult to separate molecules with nearly equal mass such as CO (28.0106u) and N_2 (28.0134u). How large a radius of curvature must a spectrometer have to separate these molecules by 0.20 mm on the film?*

Two molecules of masses m_1 and m_2 follow the radii r_1 and r_2 respectively in the spectrometer. The two lines on the film are separated by a distance Δx given by

$$\Delta x = 2(r_1 - r_2) = 2\Delta r.$$

In a spectrometer, we have $mv^2/r = qvB$ and $\Delta r/r = \Delta m/m$

or $$r = \frac{\Delta r}{\Delta m}m = \frac{\Delta x.m}{2\Delta m} = \frac{0.20 \times 10^{-3} \times 28.0120}{2 \times 0.0028} = 1 \text{ meter.}$$

Thus the spectrometer must have a radius of curvature of 1m to separate the given molecules by 0.2 mm on the film.

Example 10. *A doubly ionized helium atom of mass 6.7×10^{-27} kg is accelerated by a voltage of 2800 V. What will be its radius of curvature in a uniform magnetic field of 0.24T? Determine the period of revolution.*

A doubly ionized atom has charge q = 2 × proton charge = $2 \times 1.6 \times 10^{-19}$ *C.*

Its velocity v after being accelerated will be given by

$$\tfrac{1}{2}mv^2 = qV \text{ or } v = (2qV/m)^{1/2}$$

It will move in a circular orbit when a magnetic field B is applied normally. The radius R of the orbit is given by

$$mv^2/R = qvB \text{ or } R = mv/qB$$

$$\therefore \quad R = \frac{1}{B}\left(\frac{2mV}{q}\right)^{1/2} = \frac{1}{0.24} \times \left(\frac{2 \times 6.7 \times 10^{-27} \times 2800}{2 \times 1.6 \times 10^{-19}}\right)^{1/2} = 4.5 \text{ cm.}$$

The time of revolution

$$T = \frac{2\pi R}{v} = \frac{2\pi m}{qB} = \frac{2 \times 3.14 \times 6.7 \times 10^{-27}}{2 \times 1.6 \times 10^{-19} \times 0.24} = 5.48 \times 10^{-7} \text{ sec.}$$

Example 11. *In a Dempster mass spectrograph, calculate the distance between the point where the beam enters the magnetic field and the point where it strikes the photographic plate, if V = 2kV and B = 0.3 T, for (a) the singly ionized ^{12}C atoms and (b) singly and doubly ionized ^7Li atoms of mass 7.01601u.*

In the Dempster's mass spectrometer, the velocity gained by a charge q and mass m while passing through a potential difference of V is given by

$$\tfrac{1}{2}mv^2 = qV \quad \text{or} \quad v = (2q\,V/m)^{1/2}$$

When the charge particle passes in the magnetic field B, the radius R of the circular path is given by

$$qvB = mv^2/R \text{ or } R = mv/qB.$$

$$\therefore \quad R = (2\text{m } V/qB^2)^{1/2}.$$

The distance between the point where the beam enters the field and the point where it strikes the plate

$$D = 2R = 2\,(2mV/q)^{1/2}\,(1/B).$$

(*a*) For singly ionized ^{12}C atoms, $q = 1.6 \times 10^{-19}\ C$ and $m = 12u = 12 \times 1.66 \times 10^{-27}$ kg.

$$\therefore \qquad D = 2\left(\frac{2 \times 12 \times 1.66 \times 10^{-27} \times 2 \times 10^{3}}{1.6 \times 10^{-19}}\right)^{1/2} \times \frac{1}{0.3} = 14.86 \text{ cm}$$

(*b*) For the singly ionized ^{7}Li atoms, $q = 1.6 \times 10^{-19}\ C$ and $m = 7.0160 \times 1.66 \times 10^{-27}$ kg.

$$\therefore \qquad D = 2\left(\frac{2 \times 7.0160 \times 1.66 \times 10^{-27} \times 2 \times 10^{3}}{1.6 \times 10^{-19}}\right)^{1/2} \times \frac{1}{0.3} = 11.40 \text{ cm.}$$

For the doubly ionized ^{7}Li atoms $q = 2 \times 1.6 \times 10^{-19} C$

$$\therefore \qquad D = 11.40/\sqrt{2} = 8.06 \text{ cm.}$$

Example 12. *In a Bainbridge mass spectrograph the electric field between the plates of a velocity selector is 1000V/cm and the magnetic induction in both magnetic fields is 0.5 Wb/m*^{2}*. A stream of singly charged neon ions moves in a circular path of 8.74 cm radius in the magnetic field. Find the mass number of the neon isotope.*

Since the neon ions are singly charged, therefore charge q = proton charge = 1.6×10^{-19} coulomb.

Magnetic fields are equal, therefore $B = B' = 0.5\ Wb/m^2$.

Electric field $E = 1000\ V/cm = 10^5\ V/m$

Radius of the circular path = 8.74 cm = $8.74 \times 10^{-2}\ m$.

Substituting these values in the relation for q/m of an ion in Bainbridge mass spectrograph, we get

$$\frac{q}{m} = \frac{E}{BB'r} \quad \text{or} \quad m = \frac{qBB'r}{E}$$

$$\begin{aligned} \therefore \qquad m &= 1.6 \times 10^{-19} \times (0.5)^2 \times 8.74 \times 10^{-2}/10^5 \text{ kg.} \\ &= 1.6 \times 10^{-19} \times (0.5)^2 \times 8.74 \times 10^{-2}/1.66 \times 10^{-27} \times 10^5 u. \\ &= 21.06u. \end{aligned}$$

Since the mass number is an integer closest to the atomic mass in atomic mass scale, therefore the mass number of the neon isotope = 21

ORAL QUESTIONS

1. What type of motion do you expect, when the proton is placed in uniform electric field.
2. An electron is moving with velocity **v** in uniform **E** or **B** field. If the velocity remains unchanged then (*a*) $\mathbf{v} \perp \mathbf{E}$, (*b*) $\mathbf{v} \perp \mathbf{B}$, (*c*) $\mathbf{v} \parallel \mathbf{B}$.
3. What type of motion do you expect, when the proton moves (*a*) along **B**, (*b*) ⊥ to **B** or (*c*) along a line at an angle of θ° to the direction of **B**?
4. How do you know whether a moving electron in a certain region is being deflected by an electric field or by a magnetic field or by both?
5. The dees of a cyclotron are good conductors. Is there any effect of the alternating voltage applied on protons when they are inside the dee.
6. Why is not the cyclotron used for accelerating electrons?
7. What limits the maximum energy attainable for protons in an ordinary cyclotron? How is this overcome in a synchro cyclotron?
8. What would be the transverse Hall field, if the charge carriers are electrons and holes of equal density?
9. What would be the effect of a magnet if brought close to a television?
10. Can you set a resting electron into motion with a magnetic field? With an electric field?
11. Can we use the Dempster mass spectrometer for determination of the mass of light particles such as electrons and mesons?
12. On which principle does the electron microscope work?

PROBLEMS

1. An electron is shot at the speed of 10^6 m/sec between two parallel charged plates separated by a distance 0.5 cm and producing an electric field of 10^3 *N/C*. Where will the electron strike the upper plate if it starts from the one end of the negatively charged plate? (0.75 *cm*)
2. A small sphere of mass *m* and charge *q* hangs as a pendulum of length *l* in a region of uniform electric field *E* which is directed downward. Find the period of the pendulum. $2\pi\sqrt{l[g + qE/m}]$.
3. An electron initially travelling with velocity 3×10^7 m/sec enters in an electric field of 18 volts/cm. If the transverse deflection is 2mm in traversing a length of 10 cm. Find the value of *e/m*. [2×10^{11} *coul/kg.*]
4. A coil of wire is wound on an evacuated hollow glass tube with inner diameter 2.00 cm and produces a uniform magnetic field of $3 \times 10^{-2} T$ parallel to the axis of the tube. Protons with speed 5×10^5 m/sec are shot into the tube at a point on its axis. What is the maximum angle that a particle makes with the axis as it enters the tube if it is not to hit the wall as it spirals down the tube? (1.65°)
5. A bullet of mass 7 gm and charge $3.5 \times 10^{-9} C$ moves with a speed of 300 m/sec perpendicular to the earth's magnetic field of $5 \times 10^{-5} T$. Find the distance by which it is deflected from its path due to the earth's magnetic field after it has travelled 600*m*. ($1.5 \times 10^{-8} m$)
6. A proton, a deuteron and an α-particles have equal kinetic energies. Compare the radii of their paths when a normal magnetic field is applied. ($r_d = \sqrt{2}\, r_p = \sqrt{2}\, r_\alpha$)
7. A beam of protons with velocity 4×10^5 m/sec enters a uniform magnetic field of 0.3*T* at an angle of 60° to the magnetic field. Find the radius of the helical path taken by the proton beam. Determine the pitch of the helix. (1.205 *cm*, 4.372 *cm*)
8. The diameter of a cyclotron dee is 1.2m and the magnetic induction is 0.80*T*. The cyclotron is used to accelerate protons and α-particles. Calculate the K.E. and the speed of these accelerated particles. Determine the frequency of the alternating voltage that must be applied to the dees. (*p* : 11 *MeV*, 4.6×10^7 *m/sec*, 12.2 *MHz*; α : 11 *MeV*, 2.3×10^7 *m/sec*, 6.1 *MHz*)
9. A certain cyclotron can produce 10 MeV protons. Calculate the time of each revolution if the magnetic field is 10^4 *G*. If the accelerating voltage across the dee gap is 40kV, how many revolutions are necessary to accelerate the protons to the final energy? How long does the acceleration process take? (6.53×10^{-8} *sec*, 250, 1.63×10^{-5} *sec*)
10. Deuterons are accelerated in a fixed frequency cyclotron to a maximum dee orbit radius of 88 cm. The magnetic field is 1.4 Wb/m^2. Calculate the energy of the emerging deuteron beam and the frequency of the dee voltage. What change in magnetic flux density is necessary if doubly ionized helium ions are accelerated. [36.3 *MeV*, 10.67 *Mc/sec*. 1.39 *Wb/m*^{2}]
11. A relativistic charged particle of charge *q* moves in a plane perpendicular to a uniform magnetic field *B*. For an extreme relativistic particle ($K >> m_0c^2$), show that the radius of its circular path is *K/eBc*. Calculate the orbit radius for the cosmic ray proton with K = 10^{12} eV in the spiral arm of our galaxy where the average field is $10^{-10} T$. Calculate the cyclotron frequency for the particle. [222 A.U. where 1 astronomical unit = Earth–sun distance = 1.5×10^{11} m, 1.43×10^{-6} Hz]
12. A current of 0.10A flows through a semi conductor slab 1.0 mm thick, 6.0 mm wide and 1.5 cm long, which is placed so that its flat surface is perpendicular to the magnetic field of 1.2*T*. If the density of electrons in the semiconductor is $1.0 \times 10^{24}\ m^{-3}$, find the potential difference between the two sides of the slab. (0.75 *mV*)
13. A particle having a mass of 0.5 gm carries a charge of $2.5 \times 10^{-8} C$. The particle is given an initial horizontal velocity of 6×10^4 *m/sec*. Find the magnitude and direction of the minimum magnetic field that will keep the particle moving in a horizontal direction. [3.27 *Wb/m*^{2}, ⊥ to v]
14. An ion of mass *M* and charge *q* is accelerated by potential difference *V* and allowed to enter a field of magnetic induction **B**. In the field it moves in a semi-circle, striking photographic plate at a distance *x* from the slit. Show that the mass *M* is given by $M = B^2 qx/8V$.
15. Protons move in a circle of radius 8.10 cm in a 0.58 *T* magnetic field. What value of electric field could make their path straight? In what direction it will point? ($2.6 \times 10^6 V/m$)
16. In a mass spectrometer, germanium atoms have radii of curvature equal to 21.0, 21.9 and 22.8 cm. The largest radius corresponds to an atomic mass of 76*u*. What are the atomic masses of the other isotopes? (70, 73*u*)
17. In a Dempster mass spectrograph, the accelerating voltage is 20*kV*, and the Ytterbium group of lines is centered at 25 cm from the point of enterance in the magnetic field. Find the magnetic field required. Calculate the range Δ*x* for the isotopic lines from mass numbers 168 and 176.
18. In a Bainbridge mass spectrograph, the electric field between the plates is 2.18×10^4 *V/m* and $B = B' = 0.57T$. The source contains boron isotopes ^{10}B and ^{11}B. How far apart are the lines formed by the singly charged ions of each isotope on the photographic plate? (1.04 *mm*)

12

ELECTROMAGNETIC INDUCTION

12.1 INTRODUCTION

We have seen how charges at rest produce static electric fields and moving charges or electric currents produce static magnetic fields. In this chapter we will discuss the effects of time varying magnetic field.

The magnetic effect due to electric current was discovered by Oersted. Scientists then began to suspect the reciprocal effect, *i.e.*, a ***magnetic field can produce an electric current.*** Faraday performed several unsuccessful experiments. Finally in 1831, he discovered that an electric field could be produced by a time varying magnetic field, almost simultaneously and independently Joseph Henry (in USA) made the same discoveries, but his results were reported a few months later.

12.2 FARADAY'S EXPERIMENTS

In 1831 Michael Faraday in England performed a series of experiments and discovered that if a closed circuit moved across a magnetic field current flowed even though there were no battries present. *In one of his experiments,* he took one closed circuit consisting of a coil and a galvanometer. When a bar magnet was moved in the neighbourhood of this coil, following observations were obtained :

(*i*) If the magnet is moved towards the coil with its north pole facing the coil, the galvanometer deflects, the electric current is thus produced in the circuit.

(*ii*) If the magnet is moved away from the coil, the galvanometer again deflects but in opposite direction.

(*iii*) If the south pole end instead of the north pole end of the magnet is moved towards the coil the deflections are reversed.

(*iv*) The galvanometer also shows deflection if the magnet is kept stationary and the coil is moved.

(*v*) The deflection in galvanometer is found to be proportional to the relative speed of the magent and the coil.

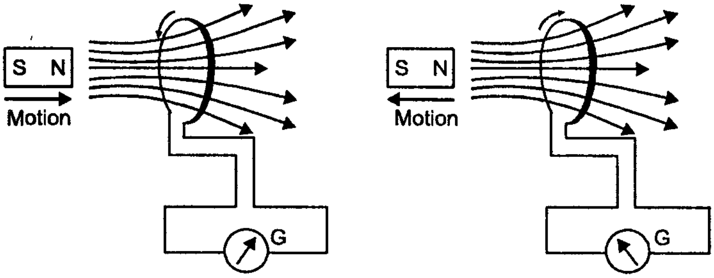


Fig. 12.1. Current in a coil due to relative motion of a magnet.

(*vi*) The galvanometer shows no deflection when the magnet is held stationary with respect to the coil.

In his another experiment, two coils were placed close together, but at rest with respect to each other. One coil was connected to a battery and the other to a galvanometer. When the battery circuit, known as *primary circuit,* was closed, there was a sudden deflection in the galvanometer circuit, known as *secondary circuit.* Following results were observed.

(*i*) When the primary circuit is closed or broken, the galvanometer needle deflects momentarily. The directions of deflection are opposite to each other.

(*ii*) The galvanometer also shows deflection if the current in the primary circuit is varied. *This deflection depends upon the rate of change of current and not the value of the current.*

(*iii*) The current in the secondary circuit does not exist when the current in the primary is constant, *i.e.*, either established or reduced.

(*iv*) When there is a relative motion between these coils, the deflection also exists.

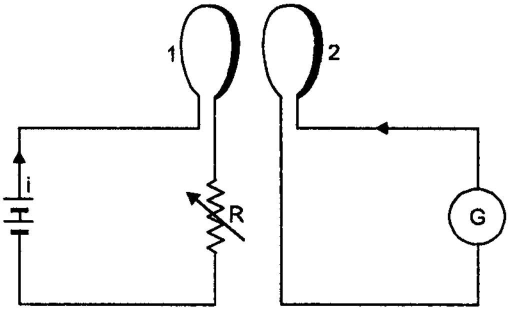


Fig. 12.2. Current in a closed loop due to current in neighbouring loop.

The deflection in galvanometer in above experiments is due to the current developed in the secondary circuit, this current is called an *induced current and the emf giving rise to such current is called an induced electromotive force.* This phenomenon was given the name *electromagnetic induction* by Faraday. *This is due to the change of magnetic flux through the closed circuit by (i) the relative motion of the circuit and the magnetic lines of force produced by either the magnet or the primary coil and (ii) starting and stopping a current in a primary coil.*

12.3 FARADAY'S LAWS OF INDUCTION

From his experimental results, Faraday gave two laws :

(*i*) *When the flux of magnetic induction through a circuit is changing an electromagnetic force is induced in the circuit.*

(*ii*) *This emf is equal to the negative rate of change of the magnetic flux, i.e.,*

$$\mathcal{E} = -d\Phi_B/dt. \qquad ..(1)$$

This equation is known as *Faraday's law of induction.* It is found to be independent of the way in which the flux is changed, the circuit may be moved or the value of **B** at various points inside the circuit may be changed. The flux through a circuit can also be changed by changing its shape by squeezing or stretching it.

The relation representing that the induced emf in a circuit is equal to the rate of change of magnetic flux linked with the circuit is known as *Neumann's law*. The minus sign is an indication of the direction of the induced emf, will be discussed in the next article as a Lenz's law.

If the coil through which magnetic flux is changing consists of N turns and the coil is tightly wound that each turn covers nearly the same area, therefore the flux through each turn will be same and the induced emf in such a device (of N turns such as toroids and solenoids) is given by

$$\mathcal{E} = -N(d\Phi_B/dt) = -d(N\Phi_B)/dt, \qquad ...(2)$$

where $N\Phi_B$ is the measure of flux linkage in the device.

Consider the simple case of an open surface S bounded by a closed loop C *and* located in a region in which there is a time varying magnetic field B ($\mathbf{r}$, t). The magnetic flux through $\mathbf{S}$ is given by the surface integral of $\mathbf{B}.d\mathbf{S}$ as

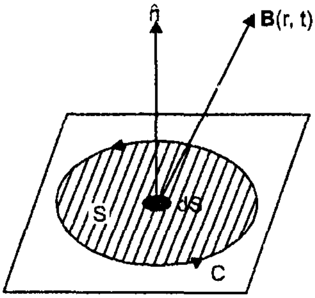


Fig. 12.3. Time varying field linked with the loop C.

$$\Phi_B = \int_S \mathbf{B}\cdot d\mathbf{S} = \int_S \mathbf{B}\cdot\hat{n}dS, \qquad ...(3)$$

where $\hat{n}$ is the unit normal vector perpendicular to the plane surface S.

If the electric field induced in space is $\mathbf{E}$, then the induced emf around closed loop C is given by the line integral

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{l} \qquad ...(4)$$

As the circuit C be bounded by an open surface S, therefore

$$\mathcal{E} = \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}. \qquad ...(5)$$

This is called the *integral form of Faraday's law of induction*. Thus the Faraday's law tells us that the line integral of the electric intensity around an fixed closed path is equal to the time rate of decrease of flux of magnetic induction through the surface enclosed by the curve.

For a fixed curve C and a fixed surface S, neither S nor C depends on t and as a result we can move d/dt inside the integral, *i.e.*,

$$\oint_C \mathbf{E}\cdot d\mathbf{l} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}. \qquad ...(6)$$

Here partial derivative is used inside the integral because $\mathbf{B}$ is a function of position and time. Using Stoke's theorem, the left hand side of Eq. (6) can be changed to surface integral, as

$$\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}.d\mathbf{S}.$$

This is true for any arbitrary surface S, we may thus equate the integrands and have

$$\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t. \qquad ...(7)$$

This is the *differential form of Faraday's law*. It relates the space derivatives of E at a particular point to the time rate of change of $\mathbf{B}$ at that point. It is thus one of the fundamental equations of electromagnetism.

Equation (7) describes the law of generation of an electric field at a certain point due to a variation of the magnetic induction at the same point. The field is thus called the induced field. Thus *electric field is created not only by electric charges but by a varying magnetic field as well.*

In a time varying magnetic field $\partial \mathbf{B}/\partial t \neq 0$ and hence curl $\mathbf{E} \neq 0$. This means that unlike the electrostatic field created by fixed charges, the induced electric field is not a potential field. The work done in this field over a closed path is not equal to zero. Hence this field cannot be represented as a gradient of a certain scalar function. The field is thus a *non conservative* field. $\boldsymbol{E}$ is conservative in the special case when $\boldsymbol{B}$ is constant in time. There is one restriction. The time variation of the field must be slow so that the change of magnetic field during a time interval equal to the size of the circuit divided by the speed of light is negligible, Thus we can neglect the effects associated with the finite propagation time of the field.

12.4 LENZ'S LAW

Until now we have said nothing about the negative sign in the Faraday's law equation. The negative sign tells us the direction of induced emf and is very important.

A German scientist H.F.E. Lenz without the knowledge of the work of Faraday and Henry duplicated many of their experiments nearly simultaneously and deduced a law in 1834 for the direction of induced current. This is thus known as *Lenz's law*. It states that *direction of the induced current is always such as to oppose the cause that produced it.* Thus if the flux increases, through a closed circuit, the induced emf tends to cause currents in such a direction as to decrease the flux. If we thrust one pole of a magnet into a coil, induced emf tends to cause currents in such a direction that the magnetic field produced by it tends to repel the pole.

In general the cause of the current may be the motion of the conductor in a magnetic field or the change of magnetic field. In the first case, the direction of the induced current in the moving conductor is such that the force acted upon the conductor is opposite to its motion. In the second case, the induced current sets up a magnetic field which with in the area bounded by the circuit is (*a*) in the direction opposite to the original field if this is increasing and (*b*) in the direction of the original field if it is decreasing. Thus we see that the *induced field does not oppose the magnetic field but opposes the change in it.*

Since this law refers to induced currents, hence we must have a closed circuit. If a conductor does not form a closed circuit, we can usually think in terms of what would happen if it was closed and in this way find the direction of the induced emf.

When a conductor moves in a magnetic field the direction of its motion along with the directions of the magnetic field and the emf is given by the Fleming's right hand rule. It states that *if the thumb and the first two fingers of the right hand are spread out in such a way that they are at right angles to one another. If the index (or fore) finger represents the direction of the magnetic field and the thumb the direction of motion of the conductor, then the central finger will give the direction of the induced emf and thus of the induced current.*

When a magnet is moved before the coil, either towards or away from it, a work is always required. Lenz's law is not the violation of the law of conservation of energy. Here energy is converted from one form into another. The mechanical energy produces induced current by electromagnetic induction. *Thus mechanical energy is converted into electrical energy.* Lenz's law may be considered as the *electrical analogue of Newton's law of action and reaction or as an electrical application of the general principle of Le Chatellier*, which states that *if any state of equilibrium is disturbed a reaction is induced in such a sense as to oppose the change.*

12.5 PROOF OF FARADAY'S LAW

Helmholtz and a little later Kelvin showed that Faraday's law was a direct consequence of the principle of the conservation of energy. Consider a straight conductor *PQ* which slides on a pair of horizontal rails separated by a distance *l* and are connected to a source of emf $\mathcal{E}_0$. If the magnetic field induction ***B*** is perpendicular to the plane of this circuit and in inward direction. Due to field ***B*** and the current *i in the* circuit the conductor *PQ* experiences a force ***F*** of magitude *iBl* to the right. Because of this it slides with a velocity *v* at a particular instant *t* and thus work is being done at the rate *Fv*.

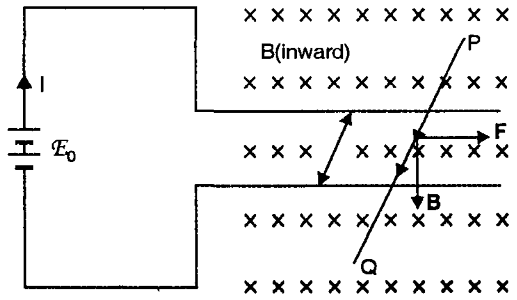


Fig. 12.4. Faraday's law from law of conservation of energy.

From the principle of conservation of energy, power supplied by the emf $\mathcal{E}_0$ is equal to the power loss in heat generation + power loss in doing work. Therefore

$$\mathcal{E}_0 i = i^2R + Fv = i^2R + iBlv$$

$$\text{or} \quad i = (\mathcal{E}_0 - Blv)/R, \qquad \text{...(8)}$$

Thus we see that the current is less than the original value, $\mathcal{E}_0/R$. The decrease in current is due to $-Blv$, the emf induced by the motion of the conductor in a magnetic field **B**. We have assumed here that the resistance of the closed circuit is so large or current *i* is small that it has a negligible effect on the value of **B**. The induced emf can be written as

$$\mathcal{E} = -Blv = -Bl\,(dx/dt) = -B(dA/dt) = -d\Phi/dt.$$

It is same as given by Faraday's law of electromagnetism. If **v** is arbitrarily oriented with respect to **l**, then the only component of **v** which is ⊥ to **l** contributes $\mathcal{E}$. Thus $\mathcal{E}$ is proportional to **l** × **v**. As $\mathcal{E}$ is a scalar quantity, hence

$$\mathcal{E} = \mathbf{B}\cdot(\mathbf{l} \times \mathbf{v}). \qquad \text{...(9)}$$

12.6 INDUCED ELECTRIC FIELD DUE TO SOLENOID

Consider a long solenoid of radius *R* and of *n* turns per unit length and carrying a varying current $i = i_0 \sin \omega t = i_0 \sin 2\pi ft$. We know that the magnetic induction inside the solenoid of length >> diameter is given by the relation $B = \mu_0 ni$. It is nearly zero outside the solenoid. To find the electric field at point *P* inside the solenoid we draw a circular path C_1 passing through *P* and of radius *r*. For path C_1 the integral form of Faraday's law (Eqn. 5) gives

$$\mathcal{E} = \oint_{C_1} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S} = \frac{d}{dt}(B\pi r^2),$$

where $B\pi r^2$ = flux linking area enclosing curve C_1, as *B* is assumed constant. From the circular symmetry *E* will be constant on the dotted circular path C_1. Thus we have

$$E(2\pi) = \pi r^2\,(dB/dt) = \pi\mu_0\, nr^2\,(di/dt),$$

$$\therefore \qquad \mathrm{E} = \pi\mu_0\, ni_0 f r \cos 2\pi ft. \qquad \text{...(10)}$$

Thus we see that *E* varies directly with *r* and sinusoidally with time. For the point *P′* outside the solenoid, *i.e.*, on the curve C_2, we have

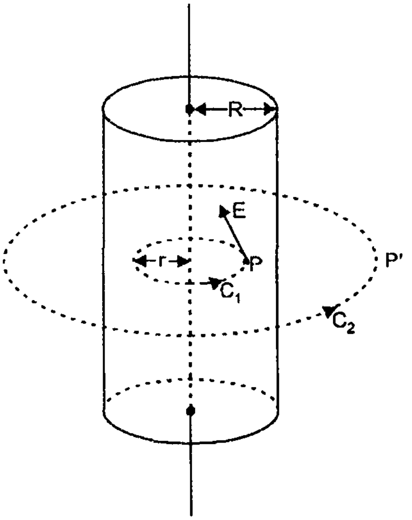


Fig. 12.5. Pulsating solenoid.

$$\mathcal{E} = \oint_{C_2} \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}.$$

Since the magnetic induction is $B = \mu_0 ni$ inside the solenoid and is zero out side, therefore $\mathbf{B}\,.d\mathbf{S} = B\,\pi R^2$ and

$$E\,(2\pi r) = \pi R^2 (dB/dt) = \pi\mu_0 n R^2 (di/dt)$$

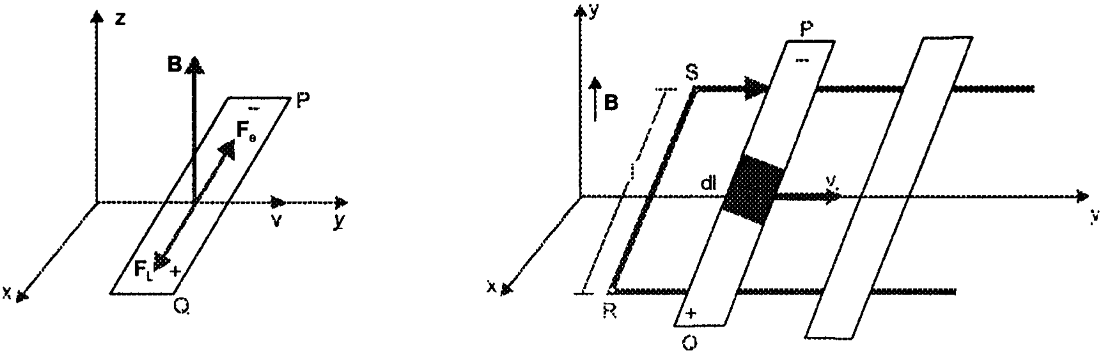
$$\therefore \qquad E = (1/r)\,\pi R^2 \mu_0 n i_0 f \cos 2\pi f t. \qquad \text{...(11)}$$

Thus we see that **E** varies inversely as r outside the solenoid.

12.7 MOTIONAL ELECTROMOTIVE FORCE

From Faraday's law of induction, we know that the electric field is produced due to the change of magnetic flux. Let us now examine how this law links electric and magnetic fields. The variation of magnetic flux with time may be caused in various ways : (*i*) Time varying loop area in a static B-field, (*ii*) stationary loop in a time varying magnetic field and (*iii*) time varying loop area in a time varying magnetic field.

(*a*) Moving Conducting Rod in a Constant Magnetic Field–Consider a conducting rod PQ of length l, parallel to x-axis, in a uniform magnetic field **B**, along z axis as shown in Fig. 12.6(*a*). The rod is moving along y-axis with velocity **v** and a charge q within it is acted on by a force **F**, known as *Lorentz force* and having value q (**v** × **B**).



(*a*) (*b*)

Fig. 12.6. A conducting rod moving through a constant magnetic field.

It is in the direction ⊥ to the plane containing **v** and **B**, hence along x-axis. Thus the free electrons in the rod will move along –*ve* x-axis, leaving the end Q positively charged. This accumulation of excess charges at the ends continues till it establishes an electrostatic field $\mathbf{E}_e$ which opposes the displacement of charges due to the Lorentz force. Electric field $\mathbf{E}_e$ (produced due to separated + *ve* and –*ve* charges in the rod) at every point inside the rod thus balances $\mathbf{E}_L$ the field due to Lorentz force **F**.

$$\therefore \qquad \mathbf{E}_e = -\mathbf{E}_L = -\mathbf{F}/q = -(\mathbf{v}\times\mathbf{B}). \qquad \text{...(12)}$$

This is the expression for the electric field produced due to the motion of a conductor in a uniform magnetic field.

From the observer moving with the rod, an electric field **E′** is observed every where in space having a constant value, given by

$$E' = v \times B. \qquad \text{...(13)}$$

(b) Conducting rod sliding along a stationary *U*-shaped conductor placed in a uniform magnetic field–Let us consider a conducting rod *PQ* sliding along a stationary *U*-shaped conductor placed in a uniform magnetic field **B**, as shown in Fig. 12.6(*b*). As the *U*-shaped conductor is stationary, hence the force acting on the free charge carriers of it is zero. But due to the motion of the conducting rod *PQ* in the magnetic field **B**, the opposite charges inside this rod are separated and are collected at the ends (+ *ve* at *Q* and – *ve* at *P*). As the circuit is closed through *RS*, the electric current will flow. This flow reduces the strength of the charges at the ends of the rod, but compensated immediately by the further displacement of free electrons caused by the magnetic force $\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$. Hence so long as the rod is in motion, a current flows through the closed circuit *PQRS*. Thus a moving rod in a perpendicular magnetic field produces some kind of emf. This emf is known as *motional induced emf*, and the current produced by it as *induced current*. The motional electric field $\mathbf{E}_m$ may be defined as

$$\mathbf{E}_m = \mathbf{F}_m/q = \mathbf{v} \times \mathbf{B}.$$

Consider a small element *dl* of a moving conducting rod as shown in Fig. 12.6(*b*). The motional electromotive force

$$\mathcal{E} = -\int \mathbf{E}_e \cdot d\mathbf{l} = \int \mathbf{E}_m \cdot d\mathbf{l} = \int (\mathbf{F}_m / q).d\mathbf{l} = \int (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \quad ...(14)$$

As $$(\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l} = \mathbf{B} \cdot (d\mathbf{l} \times \mathbf{v})$$

$\therefore$ $$\mathcal{E} = \int \mathbf{B} \cdot (d\mathbf{l} \times \mathbf{v}). \quad ...(15)$$

As $d\mathbf{l} \times \mathbf{v}$ is the area swept per unit time by the length *dl* and hence $\mathbf{B} \cdot (d\mathbf{l} \times \mathbf{v})$ is the flux of induction through this area. Therefore the *motional emf is equal to the flux of induction cut by the conductor per unit time*. If the length is *l* and the **l**, **v** and **B** are mutually perpendicator to each other, then

$$E = Blv. \quad ...(16)$$

The observer moving with the rod sees the *U*-shaped section of the loop moving with velocity $\mathbf{v}' = -\mathbf{v}$. The moving observer will find an emf $|Blv|$ due to the moving loop. The sense of current flow will be the same as was found above. Thus the two observers find the same current in the loop.

By applying Stoke's theorem to equation (14), we get

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{S}$$

or $$\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{v} \times \mathbf{B}). \quad ...(17)$$

(c) Rectangular Loop of Wire moving through a non-uniform Magnetic Field– Consider a rectangular loop of wire *abcd* moving with a velocity **v** in the y-direction through a non-uniform magnetic field **B**, which is along the z-direction, as shown in Fig. 12.7.

Let its position at some instant *t* be such that the magnetic induction is B_1 at the left side of the loop (*i.e.*, arm *ad*) and B_2 at the right side of the loop (*i.e.*, arm *bc*). Let **F** be the force acting on the charge *q* that moves along the loop. The force will be zero for the parallel *ab* and *cd* (as $d\mathbf{l} \parallel \mathbf{v}$). Thus the line integral of **F** around the whole loop is given by

$$\oint \mathbf{F}\cdot d\mathbf{l} = \int_{bc} \mathbf{F}\cdot d\mathbf{l} + \int_{da} \mathbf{F}\cdot d\mathbf{l} = -qvB_2w + qvB_1w$$

$$= qv\,(B_1 - B_2)\,w, \qquad \text{...(18)}$$

where w = width of the loop = length *bc or ad*. If the charge takes negligible time in moving around the loop, so that the position of loop has not changed appreciably, then Eq. (18) gives the work done by the force **F**. Thus the work done per unit charge

$$\frac{1}{q}\oint \mathbf{F}\cdot d\mathbf{l} = v(B_1 - B_2)w. \qquad \text{...(19)}$$

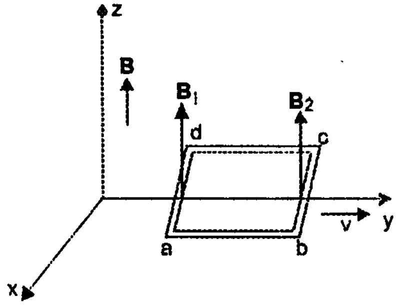


Fig. 12.7. Rectangular loop moving through a non-uniform field.

The left hand side is the emf $\mathcal{E}$ also known in this case as *motional electromotive force.*

The loop covers a distance $v\,dt$ in time dt. Thus the net change in flux through the loop in a time dt

$$d\Phi = B_2vw\,dt - B_1vwdt = -(B_1 - B_2)\,vwdt$$

or

$$d\Phi/dt = -(B_1 - B_2)vw. \qquad \text{...(20)}$$

From Eqs. (19) & (20), we get

$$\mathcal{E} = -d\Phi/dt.$$

Thus the Faraday's law is also valid for non-uniform magnetic field.

(*d*) Stationary Loop in Time Varying Magnetic Field–The induced emf due to a stationary loop in a time varying magnetic field **B** is often referred to as *transformer emf,* since it is due to transformer action. For a circuit with a single turn, we have the Faraday's law in terms of electric and magnetic fields as

$$\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\int_S \mathbf{B}\cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}.$$

By using Stoke's theorem, as discussed earlier, we get

$$\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S} \text{ or } \nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \qquad \text{...(21)}$$

(*e*) Moving Loop in Time Varying Magnetic Field–This is the general case in which a conducting loop is moving in a time varying magnetic field. In this case both motional emf and transformer emf are present and the total emf is thus given by

$$\mathcal{E} = \oint \mathbf{E}.d\mathbf{l} = \oint (\mathbf{v}\times\mathbf{B}).\,d\mathbf{l} - \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}.$$

By using Stoke's theorem, we get

$$\int_S (\nabla\times\mathbf{E})\cdot d\mathbf{S} = \int_S \nabla\times(\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S} - \int_S \frac{\partial \mathbf{B}}{\partial t}\cdot d\mathbf{S}$$

or

$$\nabla\times\mathbf{E} = \nabla\times(\mathbf{v}\times\mathbf{B}) - \partial\mathbf{B}/\partial t. \qquad \text{...(22)}$$

Few General Cases :

(i) Circuit moving in a magnetic field. Motion may be due to the translation of the circuit as a whole or the circuit may have a movable arm which slides back or forth relative to the remaining part of the circuit. In any case the periphery or its part moves in a magnetic field.

Let C_1 and C_2 be the locations of the circuit, moving with velocity v, at the time t and $t + dt$ respectively (Fig. 12.8). If the circuit is taken positive in counter clockwise direction, its area should point outwards. Thus due to the motion of the circuit C_1, the area *PQRS* (= dS) is added to the elementary length dl of this circuit in time dt. Thus

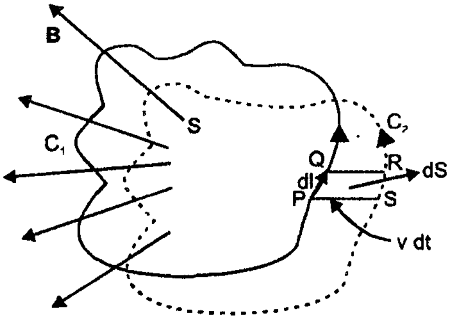


Fig. 12.8. Circuit moving in a magnetic field.

$$d\mathbf{S} = \mathbf{v}\,dt \times d\mathbf{l}$$

Change in flux in time dt, $d\Phi = \Phi(t + dt) - \Phi(t)$

$$= \Phi(t) + \int_{\Delta S} \mathbf{B}.d\mathbf{S} - \Phi(t) = \int_{\Delta S} \mathbf{B}.d\mathbf{S}\cdot \quad ...(23)$$

Hence the change in flux in time dt is the flux through the rim ΔS. Therefore the integral over the surface ΔS can be written as an integral around the path C, as

$$d\Phi = \int_{\Delta S} \mathbf{B}\cdot d\mathbf{S} = \int_C \mathbf{B}\cdot(\mathbf{v}\,dt \times d\mathbf{l}).$$

Since dt is a constant for integration, we can thus write

$$\frac{d\Phi}{dt} = \int_C \mathbf{B}\cdot(\mathbf{v}\times d\mathbf{l}) = -\int_C (\mathbf{v}\times\mathbf{B})\cdot d\mathbf{l}. \quad ...(24)$$

(from the rule for scalar triple product of vectors)

The integral in the right hand side is the line integral of force per unit charge around the circuit, which is known as the *e.m.f.* Thus

$$\mathcal{E} = -d\Phi/dt.$$

Thus we see that the relation for Faraday's law is valid for arbitrary shape and motion of the loop or circuit.

(ii) Circuit is stationary and the field source is moving : Consider two frames S and S', the latter be moving with a velocity **v** in the y-direction relative to the former frame, as shown in Fig. 12.9. The circuit will appear to be stationary in frame S' and to be moving with velocity **v** relative to the frame S. A stationary magnetic field source in frame S will appear to be moving with a velocity **v**′ (= – **v**) relative to the frame S'. The electric field in frame S' due to Lorentz force is given by

$$\mathbf{E}' = -(\mathbf{v}'\times\mathbf{B}') = \mathbf{v}\times\mathbf{B}'. \quad ...(25)$$

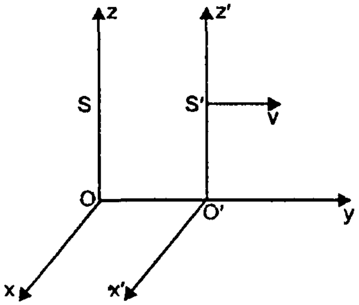


Fig. 12.9. Relative motion of the source.

This will produce an emf in frame S'

$$\mathcal{E} = \int \mathbf{E}'\cdot d\mathbf{l}' = \int -(\mathbf{v}'\times\mathbf{B}')\cdot d\mathbf{l}'$$
$$= -d\Phi'/dt. \quad ...(26)$$

This is the *same result as obtained for moving circuits.*

(iii) The circuit is moving relative to the observer and the magnetic induction B is changing at every point with the time (the most general case). Referring again to Fig. 12.8, if the circuit remains fixed in the position C_1, the emf is due to the change in B and is equal to the rate of decrease of flux of induction. In addition, we have a motional emf in the case of moving circuit equal to the rate of decrease of the flux due to the motion of the periphery. Thus the total emf

is equal to the total rate of decrease of flux of magnetic induction through the circuit due to the change in B and to the motion of the circuit, *i.e.*,

$$\mathcal{E} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_C (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{l}. \qquad ...(27)$$

In the first term the integral of $\partial \mathbf{B}/\partial t$ is taken over the original surface S. We have shown earlier in this article that the motional emf due to the motion of the circuit $\oint_C (\mathbf{v} \times \mathbf{B}).\, d\mathbf{l} = -\int_S \mathbf{B}.(\partial \mathbf{S}/\partial t)$. Hence C stands for the contour of the loop or circuit and S for the surface bounded by C. Thus Eq. (27) may be written as

$$\mathcal{E} = -\oint_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} - \int_S \mathbf{B} \cdot \frac{\partial \mathbf{S}}{\partial t} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S} = -\frac{d\Phi}{dt}. \qquad ...(28)$$

Thus the induced emf in the most general case is also equal to the negative of the total time derivative of Φ. This implies that Faraday's law is also valid when the loop of arbitrary shape moves in a time varying magnetic field.

If there are N closed circuits arranged in series with linked fluxes Φ_1, Φ_2, $\Phi_3, ...\Phi_N$ respectively, the total induced emf

$$\mathcal{E} = -\left[\frac{d\Phi_1}{dt} + \frac{d\Phi_2}{dt} + \ldots \frac{d\Phi_N}{dt}\right].$$

If the closed circuits occupy exactly the same position (the case of a coil of N turns closely wound), the $\Phi_1 = \Phi_2 = \ldots \Phi_N = \Phi$ (say). Thus we get

$$\mathcal{E} = -N(d\Phi/dt). \qquad ...(29)$$

12.8 VECTOR AND SCALAR POTENTIALS IN VARYING ELECTROMAGNETIC FIELD

Since the law of electromagnetic induction (Faraday's law) is not related to the laws of generation of magnetic field, the divergence of magnetic induction remains unchanged, *i.e.* div $\mathbf{B} = 0$. It follows that $\mathbf{B}$ can be written as $\nabla \times \mathbf{A}$. Thus Eq. (7) may be written as

$$\nabla \times \mathbf{E} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}. \qquad ...(30)$$

Here we have changed the order of differentiation, as space and time coordinates are independent to each other. Eq. (30) may be written as $\nabla \times (\mathbf{E} + \partial \mathbf{A}/\partial t) = 0$.

We know that the irrotational vector field can be written as the gradient of a scalar field.

Hence $$\mathbf{E} + \partial \mathbf{A}/\partial t = -\nabla\varphi \quad \text{or} \quad \mathbf{E} = -\nabla\varphi - \partial \mathbf{A}/\partial t, \qquad ...(31)$$

where φ is a scalar field. Thus for varying fields the electric field strength is expressed as the sum of two terms $\mathbf{E}^q$ and $\mathbf{E}^i$, where

$\mathbf{E}^q = -\nabla\varphi$ = electric field generated by electric charges

$\mathbf{E}^i = -\partial \mathbf{A}/\partial t$ = Electric field generated by induction in accordance with Faraday's law of electromagnetic induction.

The field $\mathbf{E}^q$ is nonsolenoidal and conservative and the field $\mathbf{E}^i$ is non conservative and solenoidal. Thus

$$\nabla \times \mathbf{E}^q = 0 \text{ and } \nabla \times \mathbf{E}^i = -\partial \mathbf{B}/\partial t. \qquad \text{...(32)}$$

Integrating the curl equation of $\mathbf{E}^i$ over an area **S** and substituting $\mathbf{B} = \nabla \times \mathbf{A}$, we get

$$\int_S \nabla \times \mathbf{E}^i \cdot d\mathbf{S} = \int_S -\frac{\partial}{\partial t}(\nabla \times \mathbf{A}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t}\int_S \nabla \times \mathbf{A} \cdot d\mathbf{S} \qquad \text{...(33)}$$

By using Stoke's Theorem, above equation reduces to

$$\oint_C \mathbf{E}^i \cdot d\mathbf{l} = -\frac{d}{dt}\oint_C \mathbf{A} \cdot d\mathbf{l}. \qquad \text{..(34)}$$

12.9 INDUCED CHARGE

Whenever the magnetic flux linked with a coil of N turns changes, an emf is induced in the coil. If the coil is a closed circuit, a current flows and the charge is carried by the electrons. If R be the resistance of the circuit of which the coil is a part, the induced current

$$i = \frac{\mathcal{E}}{R} = -\frac{N}{R}\frac{d\Phi}{dt}. \qquad \text{...(35)}$$

Hence δq, the quantity of electricity set in motion in time δt, is given by

$$\delta q = i\delta t = -\frac{N}{R}\frac{d\Phi}{dt}\delta t = -\frac{N}{R}\delta\Phi.$$

$$\therefore \qquad \text{Induced charge } q = -\frac{N}{R}\int_{\Phi_1}^{\Phi_2} d\Phi = \frac{N}{R}[\Phi_1 - \Phi_2]. \qquad \text{...(36)}$$

Thus, we see that the total quantity of induced electricity is independent of the rate at which the flux changes, but depends upon the resistance of the circuit and the total change in flux $N(\Phi_1 - \Phi_2)$.

12.10 ROTATING LOOP IN A STATIONARY MAGNETIC FIELD (A Simple Dynamo)

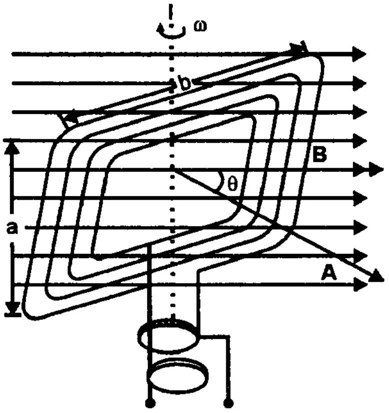


Fig. 12.10. Rotating loop in a stationary magnetic field.

The simple law of induction is the basis on which all dynamos or machines, for creating an emf are based. The principle of these instruments is the principle of rotation of a current carrying loop in a magnetic field.

Let us consider a rectangular loop of length a and width b rotating with uniform angular velocity ω about its axis in a uniform magnetic field **B**. If at a particular time t, θ is the angle which the normal to the plane of the loop makes with **B** which is ⊥ to the axis of rotation of the coil, then the magnetic flux through the coil of area A $(= ab)$ is

$$\Phi = \mathbf{B} \cdot \mathbf{A} = BA\cos\theta = BA\cos\omega t. \qquad \text{...(37)}$$

On account of the motion of the loop through the field, and emf induced in the coil

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int \mathbf{B} \cdot d\mathbf{S} = BA\,\omega\sin\omega t. \qquad \text{...(38)}$$

If number of turns in the loop is N, then the total induced emf, as each turn contributes the same flux is

$$\mathcal{E} = NBA\,\omega \sin \omega t.$$

Thus, the emf varies sinusoidally with the time. The maximum emf occurs, when $\sin \omega t = 1$, and

$$\mathcal{E} = NBA\,\omega = E_{\max}.$$

$$\therefore \qquad \mathcal{E} = E_{\max} \sin \omega t. \qquad \text{...(39)}$$

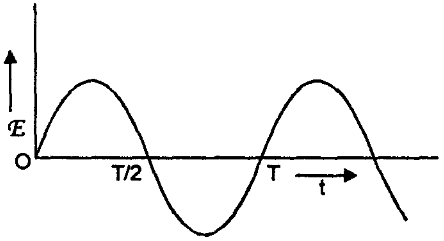


Fig. 12.11. Emf produced in a rotating coil.

From above relation it is clear that both flux and the induced emf reverse their senses relative to the circuit twice in each revolution. If the rotation is uniform, the one is maximum when the other is zero and vice versa. The emf is maximum, when θ = 90° or 270° and is zero when θ = 0° or 180°. The emf depends upon area and no. of turns and not on its shape. It depends not on the flux through the loop but on its rate of change. The rotating loop can be utilized as the source in an external circuit by making connections to *slip rings* which rotate with the loop. *Carbon brushes* against the rings are connected to the output terminals. The instantaneous terminal voltage on open circuit is the instantaneous emf, known as alternating emf. The variation of $\mathcal{E}$ with t is as shown is Fig. 12.11. The instantaneous values of I, $\mathcal{E}$, ... are generally represented by small letters *i.e.*, thus Eq. (39) becomes

$$e = e_{\max} \sin \omega t. \qquad \text{...(40)}$$

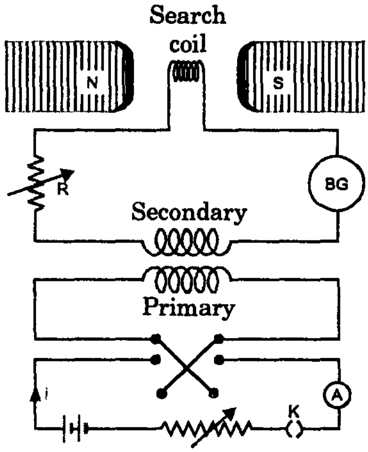


Fig. 12.12. Search coil experiment.

12.11 MEASUREMENT OF MAGNETIC FIELD

(a) Search Coil Method. The search coil is a closely wound small coil of about 50,100 or 200 turns of fine wire and about 1sq. cm in cross-sectional area. This coil in combination with the properly calibrated ballistic galvanometer or Grassot fluxmeter is used for the measurement of magnetic induction of several *tesla*.

The search coil is connected in series with a ballistic galvanometer through a rheostat and a secondary coil of a standard solenoid (which is used to calibrate ballistic galvanometer). The primary of the solenoid is connected through a reversing key with a battery, rheostat, and ammeter in series as shown in Fig. 12.12. First of all the primary circuit of the solenoid is kept off and the search coil is inserted in the magnetic induction **B**, to be measured. If the coil is placed with its plane ⊥ to a magnetic induction, the magnetic flux linked with the coil of turns N, each of area A is $\Phi = BAN$. If the coil is not normal, then $\Phi = N\mathbf{B}.\mathbf{A} = NBA \cos \phi$. If the coil is first inserted with its plane ⊥ to **B** and then removed out quickly, the flux through the coil thus decreases rapidly from BAN to zero. During the time that the flux is decreasing, an emf of short duration is induced in the coil and a kick is imparted to the ballistic galvanometer. The amount of charge passing through galvanometer

$$q = \int_0^t i\,dt = \int_0^t \frac{\mathcal{E}}{R}\,dt = \int_\Phi^0 \left(-\frac{1}{R}\right) d\Phi = \frac{\Phi}{R} = \frac{BAN}{R}.$$

If θ is the first throw of the galvanometer due to this charge, then

$$q = BAN/R = K\theta\left(1+\tfrac{1}{2}\lambda\right). \qquad ...(41)$$

where K is the ballistic constant and λ is the logarithmic decrement of ballistic galvanometer.

To find the constant K, R and λ, the galvanometer is cailibrated by means of a standard flux produced by a standard solenoid. A known current i ampere flowing through the primary coil of the solenoid is reversed by means of the reversing key and the corresponding throw of the galvanometer is observed. Magnetic field at the middle region of the solenoid of n_1 turns per unit length due to the current i through it is $B_S = \mu_0 n_1 i$. Hence the total flux linked with the secondary coil, wrapped in the middle of the solenoid, of n_2 turns is $\Phi = n_2 A' B_S = \mu_0 n_1 n_2 i A'$, where A' is the area of each turn of the secondary coil. When the current in the primary of solenoid is reversed, the magnetic flux linked with the secondary changes to $-\mu_0 n_1 n_2 i A'$. The net change in flux $2\mu_0 n_1 n_2 i A'$ thus produces a kick in the galvanometer. If θ' is the corresponding first throw, then

$$q = 2\mu_0 n_1 n_2 i A'/R = K\theta'\left(1+\tfrac{1}{2}\lambda\right). \qquad ...(42)$$

Thus from equations (41) and (42), we get

$$\frac{BNA}{2\mu_0 n_1 n_2 i A'} = \frac{\theta}{\theta'} \quad \text{or} \quad B = \frac{2\mu_0 n_1 n_2 i A'}{NA}\cdot\frac{\theta}{\theta'} \qquad ...(43)$$

In this way B may be calculated. Here the deflection θ is maximum for the particular orientation of the coil, when the coil was ⊥ to the field direction. Thus the magnitude and direction of an unknown field can both be found by this method. This method is used to measure strong magnetic fields, *e.g.* the field between pole pieces of powerful magnet.

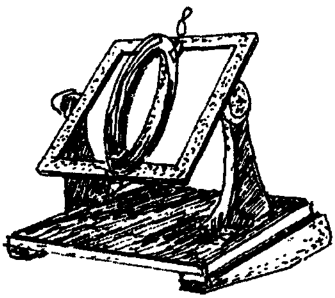


Fig. 12.13. Earth inductor.

(*b*) **Earth Inductor Method.** Earth inductor is an instrument used for the measurement of earth's magnetic field. It has a coil consisting of a large number of turns of insulated *Cu*-wire wound over a wooden or ebonite frame. The coil is fitted in a wooden frame and can be rotated quickly by means of a handle about an axis passing through the centre of the coil and lying in its plane. The rotation is confined to 180° exactly by means of stops provided in the frame. The frame itself can be rotated about an axis which is ⊥ to the axis of rotation of coil. Thus the coil can be adjusted vertical or horizontal as desired. The ends of the coil wire are connected to two binding screws on the frame.

It works on the principle of the rotation of a coil in the magnetic field. It is generally used for the determination of the horizontal component of earth magnetic field H (or B_H), vertical component V (or B_V) and angle of dip ϕ. To find B_H, the frame is kept vertical such that H acts along the normal of the plane of coil. The ends of the inductor coil are connected with the ballistic galvanometer, the coil is rotated through an angle 180° by the handle and the first throw in the galvanometer is noted.

If at an instant t the coil makes an angle θ with B_H, then the flux linked with the coil of N turns,

$$\Phi = N(\mathbf{A}\cdot\mathbf{B}_H) = NAB_H \cos\theta.$$

$$\therefore \quad \text{Induced current } i = \frac{\text{Induced emf}}{R} = -\frac{1}{R}\cdot\frac{d\Phi}{dt}.$$

Thus the charge given to the galvanometer coil

$$q = \int_0^t i\,dt = -\int_0^\pi \frac{NAB_H}{R}\,d(\cos\theta) = -\frac{NAB_H}{R}[\cos\theta]_0^\pi$$

$$= 2NAB_H/R.$$

If θ_H is the first throw in galvanometer, then we have

$$q = K\theta_H\left(1+\frac{\lambda}{2}\right) = \frac{2NAB_H}{R} \text{ or } B_H = \frac{K\theta_H R}{2NA}\left(1+\frac{\lambda}{2}\right) \qquad \text{...(44)}$$

To eliminate K, R and λ, the circuit is made as shown in Fig. 12.12, except that the search coil is replaced by the earth inductor and *N– and S*–poles are replaced by the earth's magnetic field. The experiment can be performed as given earlier in the search coil method. If θ be the first throw in the ballistic galvanometer, when the current i in the primary of the solenoid of n_i turns per unit length is reversed, then we have

$$2\mu_0 n_1 n_2 i A'/R = K\theta\,(1 + \lambda/2), \qquad \text{...(45)}$$

where n_2 is the number of turns of the secondary of the solenoid and A' the area of each turn.

Equations (44) and (45) give

$$B_H = \frac{\mu_0 n_1 . n_2 i A'}{NA}\cdot\frac{\theta_H}{\theta} \text{ weber/m}^2. \qquad \text{...(46)}$$

To determine vertical component of earth's magnetic field B_V, the coil of the earth inductor is laid horizontally with the axis of rotation along the magnetic meridian. Coil is now rotated through 180° and the first deflection in galvanometer coil is noted. If it is θ_v, then

$$B_V = K\theta_v R\left(1+\tfrac{1}{2}\lambda\right)/2NA. \qquad \text{...(47)}$$

Proceeding as above, the combination of equations (45) and (47), gives

$$B_V = \frac{\mu_0 n_1 n_2 i A'}{NA}\cdot\frac{\theta_V}{\theta} \text{ weber/m}^2. \qquad \text{...(48)}$$

The angle of dip ϕ, thus can be given by

$$\tan\phi = \frac{B_V}{B_H} = \frac{\theta_V}{\theta_H} \text{ or } \phi = \tan^{-1}\frac{\theta_V}{\theta_H}. \qquad \text{...(49)}$$

(*c*) **Grassot Fluxmeter.** This instrument is really a special type of ballistic galvanometer in which both mechanical damping and restoring torque are made negligibly small but the electromagnetic damping is relatively large. The coil of small cross section and of farily large open rectangular structure is suspended from a spring S by means of a single thread of silk and hangs with its parallel sides in the air gap between the pole pieces of permanent magnet *NS*. This silk fibre has negligible restoring torque and the flat spiral saves it from effects of shocks.

The current is led into the coil by spirals of very thin, annealed silver strips which induce very little mechanical damping of the motion. These leads are connected to the terminals T_1

T_2 to which a search coil may be connected. The search coil is small and usually mounted within a flat brass container with an insulated handle. With the help of this handle the search coil may be inserted in or withdrawn from the region of unknown magnetic field. The deflection of the coil of the fluxmeter is indicated by a pointer moving over a calibrated scale attached with the coil or by the lamp and scale arrangement.

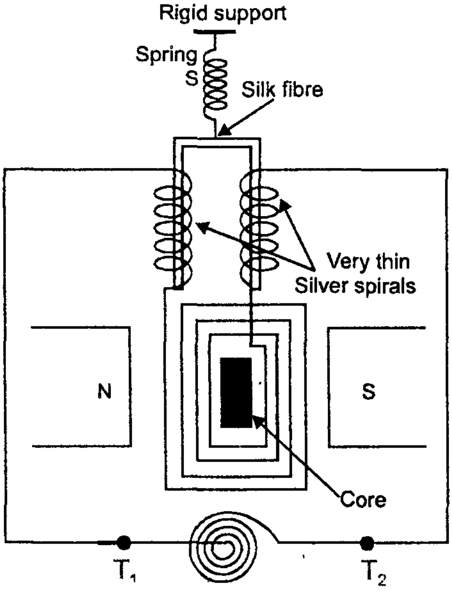


Fig. 12.14. Grassot fluxmeter.

When a search coil of turns N and of area of cross section of each turn A, is inserted in the unknown magnetic induction **B**, then the flux changes from 0 to $\Phi\,(= NAB)$. Due to this change in flux, an induced emf $\mathcal{E}$ is produced in the circuit. This induces the current

$$i = \mathcal{E}/R.$$

where R is the total resistance of the circuit.

Due to this current a deflecting couple will act upon the coil of fluxmeter and thus will impart an angular momentum. If I is the moment of inertia of the fluxmeter coil and ω its angular velocity, then

$$i\,n'B'A' = I(d\omega/dt)$$

$$\therefore \quad \mathcal{E} = iR = (IR/n'B'A')\,d\omega/dt. \qquad \text{...(50)}$$

Here n' = number of turns in the coil of fluxmeter, A' = area of its each turn and B' = magnetic field of permanent magnet.

Actually the induced emf which produces angular acceleration is the difference between the emf due to the rate of change of the flux in the search coil and the emf produced in the coil of fluxmeter due to its rotation, *i.e.*

$$\mathcal{E} = d\,\Phi/dt - n'A'B'\,\omega = d\Phi/dt - n'A'B'\,d\theta/dt. \qquad \text{...(51)}$$

Combining Eqs. (50) and (51), we get

$$\frac{IR}{n'B'A'}\frac{d\omega}{dt} = \frac{d\Phi}{dt} - n'A'B'\frac{d\theta}{dt}.$$

On integrating, we get

$$\int_0^t \frac{d\Phi}{dt}dt - n'A'B'\int_0^t \frac{d\theta}{dt}dt = \frac{IR}{n'B'A'}\int_0^t \frac{d\omega}{dt}dt$$

or

$$\int_0^\Phi d\Phi - n'A'B'\int_0^\theta d\theta = \frac{IR}{n'B'A'}\int_0^0 d\omega$$

Integration limit of ω on the right hand side is from 0 to 0, as the coil is at rest before and after the change in flux.

$$\therefore \quad \Phi - n'A'B'\theta = 0 \quad \text{or} \quad \Phi = n'A'B'\theta = k\theta. \qquad \text{...(52)}$$

This relation shows that the total change of flux Φ in a search coil is proportional to the deflection of coil of fluxmeter. Since $\Phi = NAB$, therefore Eq. (52) gives

$$B = \Phi/NA = (k/NA)\,\theta. \qquad \text{...(53)}$$

This instrument is portable but not so sensitive as a ballistic galvanometer. It has the *following advantages* over the ballistic galvanometer : (*i*) The deflection is independent of time and is same whether the time taken for the change of flux be a fraction of a second or as much as one or two minutes, (*ii*) The deflection is independent of moment of inertia of the coil hence large M.I. is not required, (*iii*) The coil is suspended by a silk fibre hence the mechanical couple is reduced to a smaller value. The induction of the search coil circuit is unimportant and may be quite large with negligible effect upon the deflection.

12.12 THE BETATRON.

The cyclotron principle is inapplicable for the acceleration of electrons to high energies because of the large relativistic increase of mass at low energies. The first important machine for producing high energy electron was the betatron. The first betatron was constructed by Kerst at the University of Illinois in 1940.

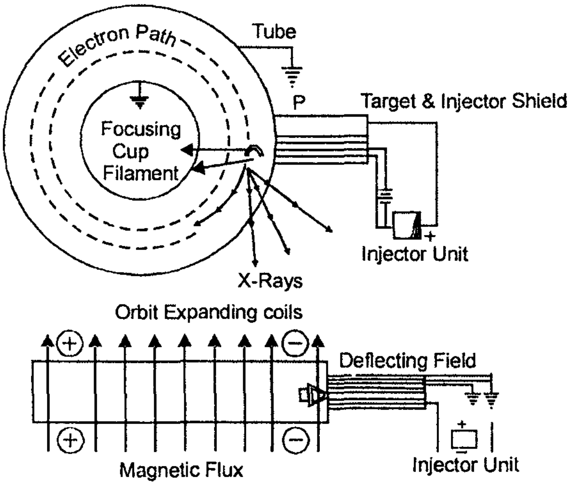


Fig. 12.15. Schematic diagram of Betatron.

The action of the betatron depends on the same fundamental principle as that of the transformer in which an alternating current applied to a primary coil induces an alternating current usually with higher or lower voltage in the secondary coil. In the betatron secondary coil is replaced by a doughnut shaped vacuum chamber. Electrons produced in the doughnut from a hot filament, are given a preliminary acceleration by the application of an electric field having potential difference of 20 *kV* to 70 *kV*. When an alternating magnetic field is applied parallel to the axis of the tube, two effects are produced : (1) an electromagnetic force is produced in the electron orbit by the changing magnetic flux that gives an additional energy to the electrons, (2) a radial force is produced by the action of magnetic field whose direction is perpendicular to the electron velocity which keeps the electron moving in a circular path. Instead of spiralling as in the cyclotron, conditions are arranged such that the increasing magnetic field keeps the electrons in a circular orbit of constant radius.

Using Faraday's law of induction, the work done on an electron of charge e in one revolution

$$W = e\mathcal{E} = ed\Phi dt.$$

If F is the tangential force acting on the electron, the work done W can also be expressed as $W = 2\pi RF$.

$$\therefore \qquad F = \frac{e}{2\pi R}\frac{d\Phi}{dt}. \qquad \text{...(54)}$$

For an electron moving in a circular orbit.

$$Bev = Mv^2/R \quad \text{or} \quad Mv = BeR.$$

This relation indicates that the magnetic field B at the orbit must increase as the electron velocity increases otherwise radius R will not be constant. If R is kept constant, then

$$\frac{d}{dt}(Mv) = eR\frac{dB}{dt}. \qquad ...(55)$$

From Newton's second law, the rate of change of momentum is equal to the force. Hence on combining equations (54) and (55), we get

$$\frac{d\Phi}{dt} = 2\pi R^2\frac{dB}{dt} = 2\frac{d}{dt}(\pi R^2 B). \qquad ...(56)$$

This relation shows that the rate of change of flux Φ within the orbit of radius R is always twice what it would have been if the magnetic field intensity was uniform throughout the orbit. This relation is known as the *betatron condition*. Since the induced emf is determined by the rate of change of flux, the iron core is laminated as in a transformer and alternating potential at 50 or 150 cycles is used to produce varying magnetic field. The electrons are injected at an instant when the magnetic field is just rising from its zero value in the first quarter cycle. The increasing magnetic field induces a potential within the doughnut which increases the energy of the electrons. When the field strength passes its maximum value and starts to decrease, the direction of the induced emf is changed and the electrons start to slow down. This effect is avoided by removing the electrons from their stable orbit by passing a pulse of current through an auxiliary coil when the field reaches its positive maximum. These high energy electrons leave the path tangentially to strike a target which then emits X-rays. The emission from the machine consists therefore of a succession of short pulses of X-raditions, each occurring at the peak of the ac. maximum. Only the first quarter of each a.c. cycle is usefully employed in acceleration.

The energy of electrons can be estimated from the time average of the energy given to the electrons and the total number of revolutions made by the electrons. If the flux variation is given by the relation $\Phi = \Phi_0 \sin \omega t$, the time during which acceleration takes place will be $\pi/2\omega$. Hence the energy gained by the electron per turn when the flux changes will be

$$\Delta E = e\,\mathcal{E} = ed\Phi/dt = e\omega\Phi_0 \cos \omega t.$$

Thus, during the acceleration period, its average value will be

$$E_{av} = \frac{\int_0^{\pi/2\omega} e\omega\Phi_0 \cos\omega t\, dt}{\int_0^{\pi/2\omega} dt} = \frac{e\omega\Phi_0(1/\omega)}{\pi/2\omega} = \frac{2}{\pi}e\omega\Phi_0.$$

For most of the time electrons travel with a velocity close to the velocity of light and therefore the total distance L travelled during the acceleration process will be $c\pi/2\omega$. If R is the radius of the orbit, the number of revolutions N will be given by

$$2\pi RN = L = c\pi/2\omega \quad \text{or} \quad N = c/4\omega R. \qquad ...(57)$$

Hence the total energy gained by electrons

$$E = E_{av} \times N = ec\Phi_0/2\pi R. \qquad ...(58)$$

The energy of electrons can also be estimated by the help of relativistic equation for energy

$$K.E = pc = BeRc. \qquad ...(59)$$

where B is the magnetic field at the acceleration. The energy obtained is therefore limited by the radius and the peak strength of the magnet of the orbit.

Most of the modern betatrons are operated from a 50 cycles/sec, *a.c.* source. If the length of the circular path within the doughnut of a betatron is supposed to be 3 metres and the energy gained by the electron in each revolution 400 *eV*, the electrons will then have a total energy of 167 *MeV*. In practice the energy of electrons can be varied from 10 *MeV* to 100 *MeV* by applying the orbit shifting magnetic field at different times during the quarter cycle in which the field is increasing.

At a certain limit the energy gain is balanced by the radiation loss (because of the centripetal force, the circulating electrons are expected to emit radiations). Schwinger (1946) suggested that the energy radiated per revolution was proportional to E^4/R. Thus the radiation loss is very serious at very high energies and this device cannot be used for acceleration of electrons to a very large energies.

12.13 INDUCTION CURRENT IN CONTINUOUS MATERIAL MEDIUM

The induction effect is observed not only if the conductor is in the form of a wire or a coil. It is also observed even if it is in the form of a sheet or a slab. The phenomenon of induction currents in continuous material medium due to varying magnetic field can be discussed with the differential form of the Faraday's law of electromagnetic induction as :

$$\text{curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t$$

$$\therefore \quad \text{curl curl}\,\mathbf{E} = -\,\text{curl}\,(\partial\mathbf{B}/\partial t). \qquad \text{...(60)}$$

Using the Ohm's law $\mathbf{j} = \sigma\mathbf{E}$, and assuming the conductivity σ of the medium as constant or the medium as homogeneous, we get

$$\text{curl curl}\,\mathbf{j} = -\sigma\,\text{curl}\left(\frac{\partial\mathbf{B}}{\partial t}\right) = -\sigma\frac{\partial}{\partial t}(\text{curl}\,\mathbf{B}) \qquad \text{...(61)}$$

Since $\text{curl curl}\,\mathbf{j} = \text{grad div}\,\mathbf{j} - \nabla^2\mathbf{j}$.

As there is no accumulation of charge inside the material, therefore

$$\text{div}\,\mathbf{j} = 0, \text{ thus we have curl curl}\,\mathbf{j} = -\nabla^2\mathbf{j}. \qquad \text{...(62)}$$

From Eqs. (61) and (62), we get

$$\nabla^2\mathbf{j} = \sigma\frac{\partial}{\partial t}(\text{curl}\,\mathbf{B}) = \mu_0\sigma\frac{\partial\mathbf{j}}{\partial t}. \qquad (\because \text{By Ampere's law curl}\,\mathbf{B} = \mu_0\mathbf{j}) \text{...(63)}$$

If A be the area of cross section of the conductor, then $j = i/A$. Thus Eq. (63) may be written as

$$\nabla^2 i = \mu_0\sigma(\partial i/\partial t). \qquad \text{...(64)}$$

This is the equation for the currents induced in the continuous material medium (or conducting medium) by the varying magnetic field. Such currents are known as *Focault's or induction currents.* The solution of Eq. (64) based upon imposed boundary conditions gives the nature of the induction current in any particular case.

EDDY CURRENT (FOCAULT CURRENT)

Focault in 1895 discovered that whenever a conductor is moving in a magnetic field or placed in a region of varying magnetic field, induced currents are set up by induction. These induced currents circulate throughout the volume of the metal. Because of their general circulatory nature these are referred to as *eddy currents*. The direction of circulation of these currents is governed by Lenz's law.

If a magnetic field is applied to the limited portion of a rotating disc in the direction ⊥ to the plane of the disc as shown in Fig. 12.16 and points into the paper. The element *ob* is moving across the field and has an emf induced in it. Elements *oa* and *oc* are not in the field, but, with all other elements which are outside the field, provide return conducting paths along with charges displaced along *ob* can return from *b* to *o*. A general eddy current circulation is thus set up in the disc as shown in Fig. 12.16 (*b*).

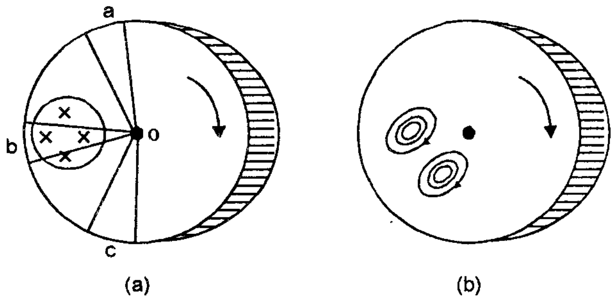


Fig. 12.16. Eddy currents.

It shows that **j** in the region of the field is almost unidirectional and is in a sense to produce a torque that oppose the rotation. Eddy currents dissipate mechanical energy through Joule heating. Thus in the absence of external source, the result of eddy currents is always to produce a braking action.

To find the direction of the eddy current, Arago performed a simple experiment. A *Cu*-disc was pivoted horizontally and a magnetic needle was suspended just above the disc, without touching it. When the disc was rotated rapidly, the needle was found rotating in the direction of the disc. If the direction of rotation of the disc was reversed, the needle was observed as rotating in reverse direction. By applying Fleming left-hand rule, it is clear that the motion of the needle is as due to the current sets up in the disc.

Eddy currents are formed in the interior of the iron cores of the rotating armatures of motors and dynamos and also in the cores of transformers. These eddy currents are very undesirable both because of the energy which they dissipate and because of the flux which they themselves set up. These currents are nearly, although not completely, eliminated by the use of a laminated core, instead of a single solid piece of iron, Fig. 12.17(*a*). Thus core consists of thin sheets of lamina, insulated from each other as shown by curved lines in Fig. 12.17(*b*). In this way the eddy currents are restricted to the individual laminations instead of flowing across the whole core. The resulting length of path is greatly increased with consequent increase in resistance. Hence the induced currents and their heating effects are minimized.

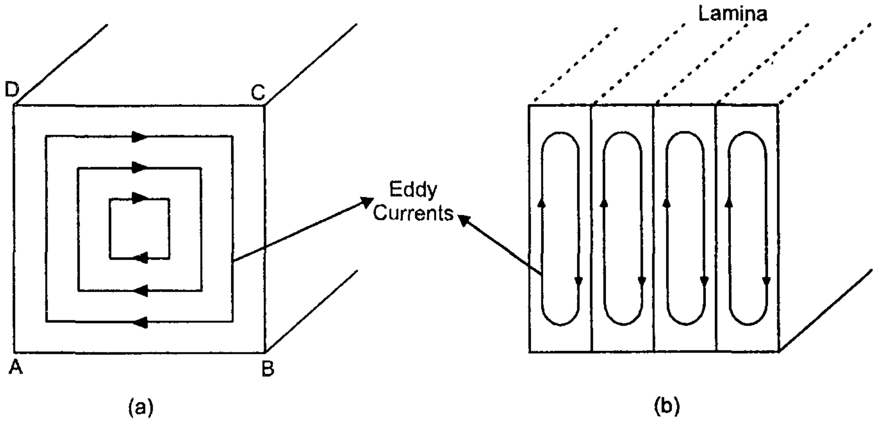


Fig. 12.17. Eddy currents in the iron cores.

As will be discussed in example 29, the circulating current in a block of a conductor in a changing field will be proportional to a^2 and thus the power dissipation will be proportionate to a^4. If we now subdivide the block into four sections, the current cannot flow from one section into another [similar to Fig. 12.17 (*b*)]. The power dissipation in each section will be proportional to $(a/2)^4$. The power dissipation in all four blocks will be proportional to $4\,(a/2)^4 = a^4/4$. Thus it is reduced to 1/4 if divided into 4 sections. In general, the power loss due to eddy currents is reduced to $1/N^{th}$ part if the block is subdivided into N parts.

Applications : *(i) Electromagnetic damping :* An important application of the eddy current is made in galvanometers to make them dead beat. For this the coil is wound on a light metal frame. When the coil and frame rotate in the field of a permanent magnet, the eddy currents are set up in the frame. These currents produce a force which opposes the motion of the frame. The motion of the frame is thus damped. This damping is known as electromagnetic damping. The oscillations of a moving ballistic galvanometer are stopped by short circuiting the coil, where the induced current is set up in the coil itself.

(ii) Induction furnace : Heating effect of eddy currents is used in induction furnaces, in which the substance to be heated is placed in a rapidly changing magnetic field. Heat produced by eddy current in the metal is no much that melting takes pace. This process is used in extracting the metal from ore. Now the heating effect of eddy current is used for the heating of tissues.

(iii) Induction motor : A metallic cylinder, when placed in a rotating magnetic field, rotates due to the eddy currents set up in it. The rotation is as given by Lenz's law. This is the principle of an induction motor.

(iv) Electric Brakes : When a strong magnetic field is suddenly applied to a rotating conducting drum the eddy currents set up in the drum exert a torque which stops the motion of the drum. This principle is used in stopping electric trains.

SKIN EFFECT

The current density $\mathbf{j}$ remains constant over any given section when the steady current passes through the uniform wire. But when the alternating voltage is applied between the ends of the wire, the distribution of current is no longer uniform, due to the emf induced in the wire itself owing to changing flux. Since the magnetic flux is always present inside and outside the current carrying conductor. A change in current produces a greater change in flux across the inner core than across the outer layer (as $B \propto 1/r^2$). Hence the induced emf due to this change of flux and therefore the effective resistance is greater at the centre than in the outer layers and the current is concentrated in the outer layers. When the frequency is very large, the eddy currents reduce the value of dB/dt inside the conductor so as to limit the current penetration into a conductor. Thus the current is almost entirely confined to the surface layer. Such a phenomenon is known as *skin effect.* As the central parts of thick wires would not carry any appreciable part of the current and would therefore be useless, hence the conductors required to carry high frequency ac are made up of a number of stands of fine wire, insulated from each other, in order to have a large surface for any given area of cross section.

The eddy or induction current in continuous material medium of conductivity σ placed in a varying magnetic field is given by the relation

$$\nabla^2 i = \mu\,\sigma\,(\partial i/\partial t). \qquad \text{...(65)}$$

In order to simplify the solution of this equation, we assume that the current flows through an infinite homogeneous conductor occupying the half space $z > 0$, along the y-axis. The plane $z = 0$ is the surface of the conductor. The currents circulate parallel to this $x - y$ plane. The amplitude of the currents is the function of z only. Thus Eq. (65) reduces to

$$\partial^2 i_y / \partial z^2 = \mu\sigma(\partial i_y / \partial t). \quad ...(66)$$

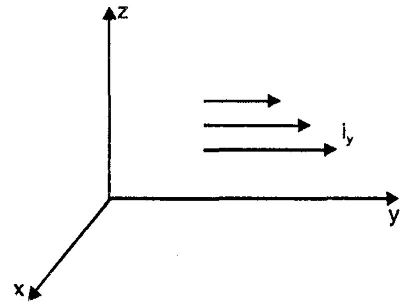


Fig. 12.18. Skin effect in an infinite conductor.

For the alternating current, the inducing magnetic flux varies in a simple periodic manner. Thus the induced current will also depend harmonically on time t, as

$$i_y(z, t) = i_o(z)e^{j\omega t}. \quad ...(67)$$

Substituting this value in Eq. (66), we get

$$d^2 i_o/dz^2 = j\omega\mu\sigma i_o. \quad ...(68)$$

Let the solution of this equation be of the form

$$i_o = Ae^{\alpha z}, \quad ...(69)$$

where α be a constant, which can be determined by solving Eq. (68) with the help of Eq (69), Thus we have

$$\alpha^2 = j\,\omega\,\mu\sigma \text{ or } \alpha = \pm\sqrt{j\omega\mu\sigma} = \pm(\omega\mu\sigma/2)^{1/2}(1+j) = \pm k(1+j)$$

$$[\because 2j = 1 + 2j - 1 = (1+j)^2]. \text{ Here } k = (\omega\mu\sigma/2)^{1/2}$$

$$\therefore \quad i_o = A_1 e^{k(1+j)z} + A_2\, e^{-k(1+j)z}. \quad ...(70)$$

The first term will increase indefinitely with z. Thus for the physically admissible situation A_1 must be zero and

$$i_0 = A_2 e^{-kz}.\, e^{-kjz} \qquad \therefore i_y(z, t) = A_2\, e^{-kz}\, e^{j(\omega t - kz)} \quad ...(71)$$

Thus we see that the induced current varies periodically. Its amplitude decreases exponentially with z. At a depth z below the surface, the damping is large for the high frequencies of the current. The current entering the conductor is damped to $1/e = 0.369$ of its value at the surface in a distance

$$\Delta = 1/k = (2/\omega\mu\sigma)^{1/2}. \quad ...(72)$$

The distance Δ is called the *skin depth* or the *penetration depth*. Thus, the skin depth may be very small for very high frequencies. For a good conductor like *Cu*, $\Delta = 0.93$cm for frequencies of 50 cps and 0.71×10^{-3} cm for 100 Mc/sec. It shows that the current flows only on the surface of the conductors in high frequency circuits.

Since the effective cross sectional area through which a current flows decrease with increasing frequency, *the resistance of a conductor increases with increasing frequency.* Because of the expulsion of flux from the interior of the conductor the *inductance of a conductor decreases with increasing frequency.* Owing to the skin effect, the joule's heat is released mainly from the surface layer of the conductor and allows the hardening of a surface layer. Thus it is easy to *harden the metals by high frequency currents.*

The phenomenon of skin effect is valid only under the condition that the skin depth is much larger than the mean free path of freely moving electrons. The situation radically changes at very low temperatures, since the conductivity increases rapidly and so does the mean free path, while the skin depth decreases. This phenomenon is called the *anomalous skin effect.*

12.14 MUTUAL INDUCTANCE

As illustrated in the article 12.2, a changing current through one coil can produce an induced emf in a neighbouring coil. It is due to the change in flux in the second coil. The first coil in which current passes is known as *primary coil* and the other coil, in which induced emf is set up is known as *secondary coil*. Let n_1 and n_2 be the number of turns in primary and secondary respectively. A current in coil 1 sets up a magnetic lines of force and some of these lines pass through the second coil 2. If flux Φ_2 is the result of the magnetic field arising from current i_1 in coil 1. Since $B \propto i_1$, hence we have total flux associated with n_2 turns of secondary

$$n_2\Phi_2 \propto i_1 \quad \text{or} \quad n_2\Phi_2 = Mi_1. \qquad \text{...(73)}$$

The *constant of proportionality M* depends upon the geometrical and coil construction factors as; the closeness of two coils, their orientations and sizes and the number of turns in primary. This is known as the *mutual inductance* or *coefficient of mutual inductance of the two coils.* Actually mutual inductance is the property of the two coils regarding the mutal action of the magnetic fields. It is true only if no ferromagnetic materials are present.

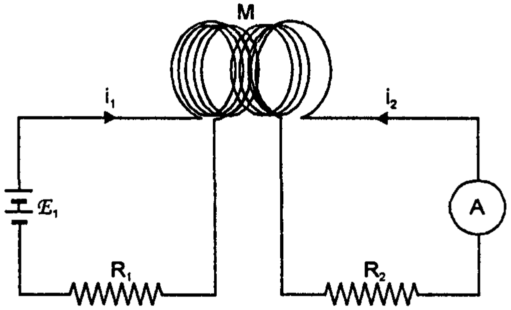


Fig. 12.19. Mutual inductance.

If $\mathcal{E}_2$ is the induced emf in the secondary due to variation of $n_2\Phi_2$, then

$$\mathcal{E}_2 = -\frac{d}{dt}[n_2\Phi_2] = -\frac{d}{dt}[Mi_1] = -M\frac{di_1}{dt},$$

$$\text{or} \qquad M = -\mathcal{E}_2/(di_1/dt). \qquad \text{...(74)}$$

Thus from relations (73) and (74) we can define M as :

(i) The mutual inductance of two coils is numerically equal to the magnetic flux linked with one coil due to the unit current in the other.

(ii) It can also be defined as the emf induced in one coil when the rate of decrease of current in the other coil is unity.

$$\text{The unit of } M = \frac{\textit{unit of } \mathcal{E}}{\textit{unit of } (di/dt)} = \frac{\textit{volt}}{\textit{amp./sec}} = \textit{volt sec/ampere.}$$

This unit is given the special name *henry*, in honour of Joseph Henry.

Mutual Inductance between two arbitrary circuits (Newmann's formula)

Consider two current carrying loops C_1 and C_2, as shown in Fig. 12.20. The current i_1 in the circuit C_1 produces a magnetic flux. The part of this flux which is associated with the surface of circuit C_2 is given by

$$\Phi_{21} = \int_{S_2} \mathbf{B}_1 \cdot d\mathbf{S}_2, \qquad \text{...(75)}$$

where S_2 is the integration surface stretched over the loop C_2. The magnetic induction $\mathbf{B}_1$ generated by the current i_1 is given by

$$\nabla \cdot \mathbf{B}_1 = 0.$$

If $\mathbf{A}_1$ be the vector potential describing the field $\mathbf{B}_1$, then $\mathbf{B}_1 = \nabla \times \mathbf{A}_1$.

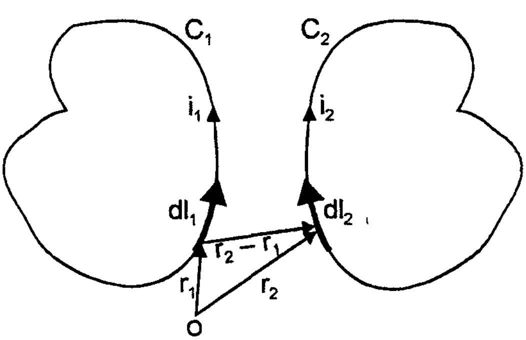


Fig. 12.20. Two current circuits.

$$\therefore \qquad \Phi_{21} = \int_{S_2} (\nabla \times \mathbf{A}_1).\, d\mathbf{S}_2 = \oint_{C_2} \mathbf{A}_1 \cdot d\mathbf{l}_2, \qquad \text{...(76)}$$

Here we have used the Stoke's theorem.

We may write the vector potential in terms of the parameters of the circuit by using equation

$$\mathbf{A}_1 = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{i_1 d\mathbf{l}_1}{r_{12}} = \frac{\mu_0}{4\pi} \oint_{C_1} \frac{i_1 d\mathbf{l}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}, \qquad \text{...(77)}$$

$$\therefore \qquad \Phi_{21} = \oint_{C_2} \frac{\mu_0 i_1}{4\pi} \oint_{C_1} \frac{d\mathbf{l}_1 . d\mathbf{l}_2}{|\mathbf{r}_2 - \mathbf{r}_1|} = \frac{\mu_0 i_1}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_1 . d\mathbf{l}_2}{|\mathbf{r}_2 - \mathbf{r}_1|}, \qquad \text{...(78)}$$

where the mutual inductance M_{21} of the loop C_2 with respect to the loop C_1 is given by

$$M_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\mathbf{l}_1 . d\mathbf{l}_2}{|\mathbf{r}_2 - \mathbf{r}_1|} \qquad \text{...(79)}$$

This is known as *Newmann's formula* for the mutual inductance. This formula shows that mutual inductance depends only on geometrical characteristics of the loops and their mutual arrangement.

Proceeding in the same manner, the flux linked with the circuit C_1 due to the current i_2 in the circuit C_2 is given by

$$\Phi_{12} = \frac{\mu_0 i_2}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_2 . d\mathbf{l}_1}{|\mathbf{r}_1 - \mathbf{r}_2|} = M_{12} i_2, \qquad \text{...(80)}$$

where

$$M_{12} = \frac{\mu_0}{4\pi} \oint_{C_1} \oint_{C_2} \frac{d\mathbf{l}_2 \cdot d\mathbf{l}_1}{|\mathbf{r}_1 - \mathbf{r}_2|}. \qquad \text{...(81)}$$

Since $d\mathbf{l}_1$ and $d\mathbf{l}_2$ are independent integration variables, the order of integration can be interchanged. Considering also that $d\mathbf{l}_1 \cdot d\mathbf{l}_2 = d\mathbf{l}_2 \cdot d\mathbf{l}_1$ and $|\mathbf{r}_1 - \mathbf{r}_2| = |\mathbf{r}_2 - \mathbf{r}_1|$, we conclude from Eqs. (79) and (81) as

$$M_{12} = M_{21}. \qquad \text{...(82)}$$

This relation is known as **reciprocity theorem** for mutual inductance of two current circuits.

It should be born in mind that the emf induced in each loop increases not only due to the change in magnetic flux of the field created by the currents in the other loops, but also due to the variation of magnetic flux of the field created by the current in this loop. For the system of two current loops, the flux linked with loop C_1 is given by

$$\Phi_1 = \Phi_{12} + \Phi_{11},$$

where Φ_{12} = flux through C_1 due to current i_2 around C_2

Φ_{11} = flux through C_1 due to current i_1 around C_1.

Similarly the total flux linked with loop C_2,

$$\Phi_2 = \Phi_{21} + \Phi_{22}.$$

Thus the emf's induced in the C_1 and C_2 loops are given by

$$\mathcal{E}_1 = -\frac{d\Phi_1}{dt} = -\frac{d\Phi_{12}}{dt} - \frac{d\Phi_{11}}{dt} \qquad \text{...(83)}$$

and $$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -\frac{d\Phi_{21}}{dt} - \frac{d\Phi_{22}}{dt}. \qquad ...(84)$$

The reciprocity theorem can also be proved as :

The induced emf produced in the second circuit C_2 due to the change in current i_1 in the primary circuit C_1 is given by

$$\mathcal{E}_{12} = -M_{21}\,(di_1/dt). \qquad ...(85)$$

where M_{21} represents the mutual inductance of coil 2 with respect to coil 1.

This induced emf will produce current i_2 in the secondary coil which will also produce induced emf in the primary coil. Thus

$$\mathcal{E}_{21} = -M_{12}\,(di_2/dt), \qquad ...(86)$$

where M_{12} is the mutual inductance of coil 1 with respect to coil 2.

The force experienced by an element $\delta\mathbf{l}_2$ of the secondary carrying current i_2 and assumed to be placed in a magnetic field $\mathbf{B}_1$ (due to current i_1 in primary) is given by

$$\delta\mathbf{F}_1 = i_2\delta\mathbf{l}_2 \times \mathbf{B}_1. \qquad ...(87)$$

∴ Amount of work done in displacing this element by a distance $\delta\mathbf{r}$

$$\delta W_2 = i_2(\delta\mathbf{l}_2 \times \mathbf{B}_1).\,\delta\mathbf{r} = i_2\,\delta\Phi_{21}, \qquad ...(88)$$

where $\delta\Phi_{21}$ is the flux with in the volume of the parallelopiped of sides of $\delta\mathbf{l}_2$, $\mathbf{B}_1$ and $\delta\mathbf{r}$.

This relation is same for the motion of the secondary coil. Thus δW_2 represents the work done in displacing a secondary coil by a distance δr and $\delta\Phi_{21}$, the additional flux linked with the secondary by virtue of the current in primary.

The total work done in bringing the secondary coil from far apart to a point where the flux linkage is Φ_{21}

$$W_2 = \int dW_2 = \int i_2 d\Phi_{21} = i_2\Phi_{21}. \qquad ...(89)$$

Similarly the work done in bringing the primary keeping the secondary fixed

$$W_1 = \int i_1 d\Phi_{12} = i_1\Phi_{12}, \qquad ...(90)$$

where Φ_{12} is the flux linked with the primary. Since the final states are equivalent regardless of the way, hence

$$W_1 = W_2 \quad \text{or} \quad i_1\Phi_{12} = i_2\Phi_{21}.$$

$$\therefore \quad i_1\,(M_{12}i_2) = i_2(M_{21}i_1) \quad \text{or} \quad M_{12} = M_{21}. \qquad ...(91)$$

Hence the mutual inductance of the coils remains same whether the current is flowing in coil 1 or in 2. This is known as *reciprocity theorem of mutual inductances.*

If $M_{12} = M_{21} = M$(say), then the amount of work done and thus the mutual potential energy of the coils is

$$W = W_1 = W_2 = Mi_1i_2. \qquad ...(92)$$

Measurement of Mutual Inductance : In this article we are discussing the *dc* method only. Few more methods (ac) are given in chapter (16). Here P and S are the two coils whose mutual inductance M is be determined. By short circuiting resistance r, a suitable current i is allowed to pass through the primary coil P. If the positions 1 and 2 of the commutator C are connected, then the induced current will flow of the circuit consisting secondary coil S.

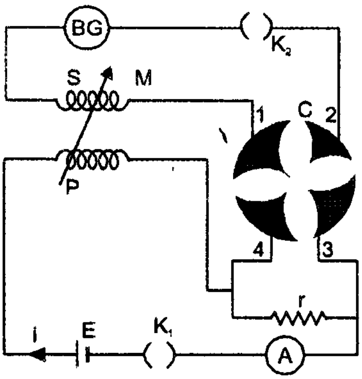


Fig. 12.21. Measurement of mutual inductance.

$$\text{Induced emf } \mathcal{E} = -M\,(di/dt).$$

$$\therefore \quad \text{Induced current} = \mathcal{E}/R = -(M/R)\,di/dt,$$

where R is the resistance in the secondary circuit.

Hence the total charge through the ballistic galvanometer

$$q = \int_0^t \left(-\frac{M}{R}\right)\frac{di}{dt}\,dt = -\frac{M}{R}\int_0^i di = -\frac{Mi}{R}.$$

The – ve sign may be omitted as it gives the direction of flow of charge.

If θ be the first throw in the ballistic galvanometer, then

$$q = \frac{Mi}{R} = K\frac{T}{2\pi}\theta\left(1+\frac{\lambda}{2}\right). \qquad \text{...(93)}$$

To eliminate K and λ, the contacts between 1 and 4 and that between 2 and 3 are made. In this way small resistance r (~ 0.2 Ω) is placed in the primary circuit and a steady current is now allowed to pass in the primary. The current will be same, as very small resistance r does not effect its value. The *pd* across resistance r thus sends a steady current ir/R though the galvanometer and a steady deflection θ' is observed. Then

$$ir/R = K\theta'. \qquad \text{...(94)}$$

From relations (93) and (94), we get

$$M = r\frac{T}{2\pi}\frac{\theta}{\theta'}\left(1+\frac{\lambda}{2}\right). \qquad \text{...(95)}$$

Thus mutual inductance M can be calculated.

12.15 SELF INDUCTANCE

Faraday induction is of equal importance in a single circuit. If a current flows in a single coil, the current causes a flux through the same coil. If the current changes, the flux also changes and emf is induced in the coil. The direction of the induced emf is such as to oppose the change of current in the circuit. It is given by Lenz's law, this is against the current when current is increasing and in the direction of current when the current is decreasing. The phenomenon (the production of induced emf in the circuit when the current through it changes) is known as *self induction* and the induced emf as self induced emf. It is sometimes referred to as a back emf as it is opposite to the emf of the battery.

Suppose a circuit has N turns of wire and a flux Φ passes through the each wire. The total flux linked with the circuit ($N\Phi$) will be proportional to the current i flowing through the circuit, *i.e.*,

$$N\Phi \propto i \quad \text{or} \quad N\Phi = Li, \qquad \text{...(96)}$$

where L is a constant, known as the *self inductance* of a circuit. It depends upon the size, shape and number of turns. It also depends on the magnetic properties such as permeability of the medium in which the circuit is immersed.

From the Faraday's law of induction, we have induced emf

$$\mathcal{E} = -d(N\Phi)/dt = -L\,di/dt. \qquad \text{...(97)}$$

When a coil is connected to a battery, the current will flow through the coil and a work will be done on the coil by the battery. If the current is changing with the rate di/dt, this results in an emf $L\ (di/dt)$. This opposes the current flow through the coil. Thus to keep the current same, power P is applied to the inductor.

$$\text{Power } P = \mathcal{E}i = iL\ (di/dt).$$

∴ Total work done by the battery on the coil is given by the relation

$$W = \int P dt = L \int_0^I i di = \tfrac{1}{2} LI^2. \qquad ...(98)$$

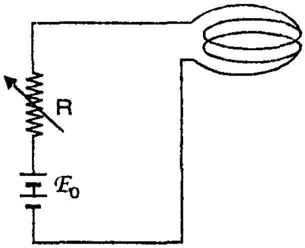


Fig. 12.22. Self inductance.

After the current has reached its final steady value, $di/dt = 0$, the power input is zero. The energy that has been supplied to the inductor (coil) is stored as potential energy. When the circuit is opened the magnetic field around the induction collapses and the energy is returned to the circuit. Relation (98) is similar to the expression for the mechanical kinetic energy $\left(\frac{1}{2}mv^2\right)$ and thus shows that *self inductance L* plays the same role in an electrical circuit as the mass or inertia does in mechanical motion. The relation (97) is similar to the expression for the external force ($F_{ext} = dv/dt$). The relation (96) is similar to the expression for the momentum ($p = mv$) in classical mechanics. Thus we find the following analog between the moving particle and the coil carrying a time varying current.

Moving particle	*Current carrying coil*
mass (m)	self inductance (L)
displacement (x)	charge (q)
velocity (v)	current (i)
momentum (mv)	induced emf ($\mathcal{E}$)
$K.E.\left(\frac{1}{2}mv^2\right)$	magnetic energy $\left(\frac{1}{2}Li^2\right)$

Self inductance L may be defined, from relation (96), (97) or (98) as :

(*i*) The flux linked with the circuit when unit current flows in it.

(*ii*) The induced emf produced in the circuit when the rate of change of current in it is unity.

(*iii*) The twice the work done in establishing the magnetic flux associated with the unit current in the circuit.

Its unit is henry as that of the mutual inductance.

Measurement of 'L' by Rayleigh's bridge Method : Lord Rayleigh gave a method for the measurement of self inductance of a coil using the principle of Wheatstone's bridge. The coil whose self-inductance is to be measured, with a very small resistance r in series, is connected in the unknown arm CD. This resistance may be short-circuited by a key K_3. A sensitive ballistic galvanometer along with the lamp and scale arrangement is placed in galvanometer arm BD.

Initially keeping the key K_1 closed, P is made equal to Q and the resistance R is adjusted to a value for which galvanometer coil does not deflect while pressing key K_2. In this case galvanometer current is zero and the inductance is not effective.

If galvanometer key K_2 is closed first and then the battery key K_1, galvanometer will observe the induced emf produced in CD arm due to the growth of the current through the coil. This extra emf ($\mathcal{E} = -L di/dt$) causes proportionate currents in every branch of the circuit. let the current in the galvanometer arm be kL (di/dt). It is for the time in which current in CD arm reaches to its steady value i_0 (say). Hence an impulse of charge q will pass through the galvanometer.

$$q = \int_0^t kL \frac{di}{dt} dt = \int_0^{i_0} kL \, di = kLi_0.$$

If θ is the first throw of the galvanometer coil due to this charge, then

$$q = kLi_0 = \frac{T}{2\pi} K \theta \left(1 + \frac{\lambda}{2}\right). \quad ...(99)$$

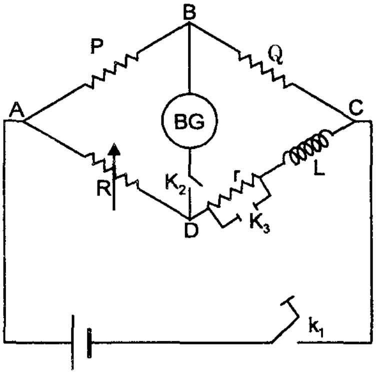


Fig. 12.23. Rayleigh's Bridge.

To eliminate k and i_0, a small resistance r is connected in series by taking out the plug from key K_3. This small resistance will not affect the steady current i_0 and the distribution of currents in the arms of the bridge and will introduce an additional emf ri_o in the arm CD. This will produce a steady current kri_o in the galvanometer arm. If the corresponding steady throw is θ', then

$$kri_o = K\theta'. \quad ...(100)$$

From relations (99) and (100), we get

$$L = r \frac{T}{2\pi} \frac{\theta}{\theta'} \left(1 + \frac{\lambda}{2}\right). \quad ...(101)$$

12.16 CALCULATION OF SELF-INDUCTANCE

(*a*) **A long Solenoid :** Let us consider a long solenoid of length l, total number of turns N, and of area of cross-section A. If a current i is passing through it, then the magnetic field inside the solenoid is

$$B = \mu_0(N/l)\, i \text{ and flux } \Phi = BA = \mu_0 NiA/l.$$

$\therefore$ Total magnetic flux linked with N turns $\mu_0 iA\ (N/l).\ N$

The induced emf around N turns

$$\mathcal{E} = -\frac{d}{dt}\left(\frac{\mu_0 iN^2A}{l}\right) = -\frac{\mu_o N^2 A}{l}\frac{di}{dt}.$$

Its comparison with equation $\mathcal{E} = -L\, di/dt$ gives

$$L = \mu_0 N^2A/l = \mu_o n^2 Al. \quad ...(102)$$

where $n (= N/l)$ is the number of turns per unit length.

Now if a second short coil (*known as secondary*) of m turns is wound on the solenoid, insulated from the first, the flux linked with the secondary due to the current i in the primary is

$$\Phi = BmA = \mu_o nmiA = Mi \text{ (By definition)}$$

or Mutual inductance $M = \mu_0 nmA$. ..(103)

If the solenoid is wound on the core of a material of constant permeability μ, then we have

$$L = \mu n^2 Al \text{ and } M = \mu nmA. \qquad ...(104)$$

These relations are valid only if the length of the solenoid is much greater then its diameter, and the secondary coil is placed near its centre. However, if the loop or secondary coil is placed at one end of the solenoid, where $B = \frac{1}{2}\mu_0 ni$, then the mutual inductance would be $\frac{1}{2}\mu_0 nm\, A$. Thus for two long solenoids of nearly same length and one surrounding the other, the mutual inductance will lie between the values $\mu_0\, nmA$ and $\frac{1}{2}\mu_0 nmA$. Similarly the self inductance of solencoid will be between the values $\mu_0 n^2 Al$ and $\frac{1}{2}\mu_0 n^2 Al$.

(*b*) **Solid cylindrical conductor :** Consider a long solid cylindrical conductor of radius a and carrying a current i, along its axis. Since the conductor is solid and of homogeneous material. The current flowing through it will be distributed uniformly over the cross section of area πa^2. The magnetic field at a point P inside the conductor at a distance r from the axis (*i.e.* $r \le a$) due to this current is given by Ampere's law as

$$B = \frac{\mu_0 ir}{2\pi a^2} = \frac{1}{2}\mu_0 jr.$$

The lines of B are circles about the z-axis, the axis of the cylinder. Fig. 12.24 shows the vertical cross section of the cylinder. We consider the magnetic flux through a strip of length l and width dr. The area of the strip is $l\ dr$. Thus the flux linkage corresponding to this element is

$$d\Phi = (\mu_0 i'\, r/2\pi a^2)\cdot l\, dr.$$

where l' is the fraction of the total current i through the area πr^2. Thus

$$i' = (\pi r^2/\pi a^2)i = (r^2/a^2)i.$$

$$\therefore \qquad d\Phi = \frac{(\mu_0 ir^3/2\pi a^4)\, ldr}{2\pi a^4}$$

The total flux linkage $\Phi = \int_0^a \frac{\mu_0 lir^3}{2\pi a^4}\, dr = \frac{\mu_0 li}{8\pi}$.

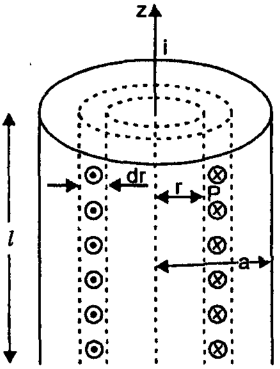


Fig. 12.24. Solid cylinder (vertical cross-section)

$\therefore$ Self inductance per unit length $= \Phi/il$

$$= \mu_0/8\pi. \qquad ...(105)$$

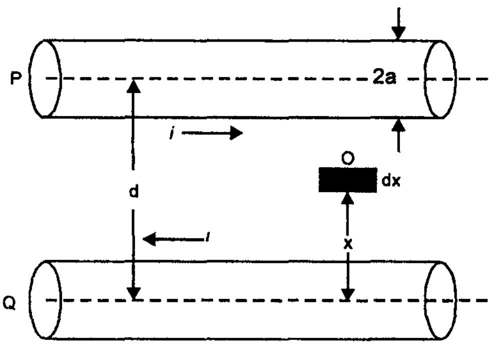


Fig. 12.25. Long parallel wires.

(*c*) **Long Parallel Wires ;** Let P and Q be two parallel wires placed at a distance d apart in air as shown in Fig. 12.25. If a be the radius of either wire and the currents in the wires are each equal to i and are in opposite directions. The magnetic field at any point O, at a distance x from one wire due to the current in the wires is

$$B = \frac{\mu_0}{2\pi}\left[\frac{i}{x} + \frac{i}{d-x}\right].$$

The flux through a small element of unit length and of width dx is

$$d\Phi = Bdx = \frac{\mu_0}{2\pi}\left[\frac{1}{x}+\frac{1}{d-x}\right] idx.$$

Hence the total flux passing through the entire area between two wires of unit length, is

$$\Phi = \frac{\mu_0 i}{2\pi}\int_a^{d-a}\left(\frac{1}{x}+\frac{1}{d-x}\right)dx = \frac{\mu_0 i}{2\pi}\left[\log_e\frac{d-a}{a}-\log_e\frac{a}{d-a}\right]$$

$$= \frac{\mu_0 i}{\pi}\log_e\frac{d-a}{a}.$$

$\therefore$ Self inductance per unit length $L = \dfrac{\Phi}{i} = \dfrac{\mu_0}{\pi}\log_e\dfrac{d-a}{a}$. ...(106)

(*d*) **Co-axial Cylinders :** An important method of transmitting radio frequency alternating currents over a distance is by means of a cable, a pair of coaxial cylinders of radii a and b ($> a$). The current in these cylinders is exactly equal in magnitude but in opposite direction. Due to this opposite current the field outside the cable is zero and thus the cable is non-radiating. The field is zero inside the inner cylinder, which is hollow. The magnetic field at a distance r from the axis and within the region between two cylinders is

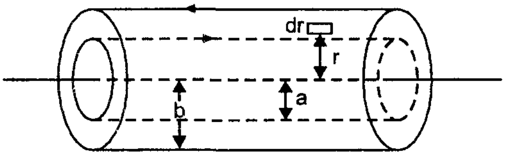


Fig. 12.26. Co-axial cylinders.

$$B = \mu_0 i/2\pi r.$$

Hence the flux through the area between r and $r + dr$ and of length unity is

$$d\Phi = (\mu_0 i/2\pi r)\, dr.$$

$\therefore$ Total flux between the unit length of the cylinders

$$\Phi = \int_a^b \frac{\mu_0}{2\pi}\frac{dr}{r} = \frac{\mu_0 i}{2\pi}\log_e\frac{b}{a}.$$

Hence the self inductance per unit length

$$L = \frac{\mu_0}{2\pi}\log_e\frac{b}{a}. \quad ...(107)$$

If the *inner cylinder is solid,* the above result is not valid. Since the current i is distributed uniformly over the cross-section of area πa^2. The field B is $\mu_0 ir/2\pi a^2$ inside the inner cylinder and is $\mu_0 i/2\pi r$ outside to it. The value of the magnetic flux between unit length of the cylinders

$$d\Phi' = \int_r^b Bdr = \frac{\mu_0 i}{2\pi a^2}\int_r^a rdr + \frac{\mu_0 i}{2\pi}\int_a^b \frac{dr}{r}$$

$$= \frac{\mu_0 i}{4\pi a^2}(a^2 - r^2) + \frac{\mu_0 i}{2\pi}\log_e\frac{b}{a}.$$

Since this flux does not link the entire current i, but the fraction of a current that is in an annular ring of area $2\pi r dr$. Hence the equivalent flux linkage

$$d\Phi = \frac{2\pi r dr}{\pi a^2} d\Phi' = \frac{\mu_0 i}{\pi a^2}\left[\frac{(a^2 - r^2)}{2a^2} rdr + \left(\log_e \frac{b}{a}\right) rdr\right]$$

$\therefore$ Self inductance per unit length $L = \frac{1}{i}\int_0^a d\Phi = \frac{\mu_0}{8\pi} + \frac{\mu_0}{2\pi}\log_e \frac{b}{a}$. ...(108)

Here we see that first term is the internal inductance of the inner solid cylinder and the second term is that of two hollow coaxial cylinders.

(*e*) Toroidal Coil of Rectangular Cross-section : Consider an endless toroidal coil of rectangular cross-section. A part of it is shown in Fig. 12.27.

Let a and b be the internal and external radii respectively. Let N be the number of turns distributed uniformly over its circumference. If a current i flows through this winding, it will produce lines of **B**, which are concentric circles. Applying Ampere's circuital law $\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0$ (current enclosed by the closed path) to a circular path of radius r, we get

$$B 2\pi r = \mu_0 Ni \text{ or } B = \mu_0 Ni/2\pi r. \quad ...(109)$$

To find the magnetic flux Φ for the cross section of the toroid, consider a rectangular strip of thickness dr and of height h (*the height of the toroid*) at a distance r from the centre. The magnetic flux linked with this strip

$$d\Phi = \mathbf{B} \cdot d\mathbf{A} = B\,(hdr) = (\mu_0 Nhi/2\pi r)\, dr.$$

$$\therefore \Phi = \int_a^b d\Phi = \frac{\mu_0 Nhi}{2\pi}\int_a^b \frac{dr}{r} = \frac{\mu_0 Nhi}{2\pi}\log_e\left(\frac{b}{a}\right).$$

Hence the self inductance of the rectangular toroid

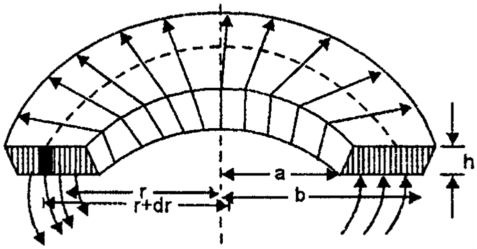


Fig. 12.27. Toroidal coil of rectangular cross section.

$$L = \frac{N\Phi}{i} = \frac{\mu_0 N^2 h}{2\pi}\log_e\left(\frac{b}{a}\right). \quad ...(110)$$

If the coil is wound over a core of constant permeability μ, then

$$L = (\mu N^2 h/2\pi)\log_e (b/a) \quad ..(111)$$

(*f*) Toroidal Coil of Circular Cross-section : Consider a toroidal coil of circular cross section, with N number of turns distributed uniformly over its circumference (Fig. 12.28). Applying Ampere's circuital law to the circular path of radius R, we get

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 Ni.$$

Since lines of B are concentric circles, hence $\mathbf{B} \cdot d\mathbf{l} = Bdl$.

$$\therefore \quad B\, 2\pi R = \mu_0 Ni \quad \text{or} \quad B = \mu_0 Ni/2\pi R,$$

where i is the current flowing in the toroid.

If A is the cross sectional area of the toroid or of each turn of the coil, then the total flux linked with toroid is

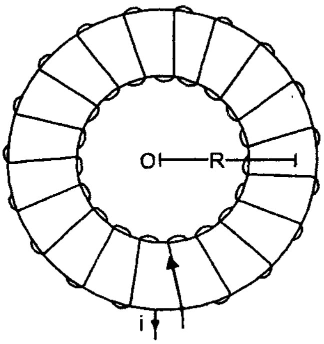


Fig. 12.28. Toroidal coil of circular cross section.

$$\Phi = BNA = \frac{\mu_0 Ni}{2\pi R}\,NA = \frac{\mu_0 N^2 Ai}{2\pi R}.$$

$\therefore$ Self inductance of the toroid of circular cross section

$$L = \mu_0 N^2 A/2\pi R. \quad \text{...(112)}$$

If the coil is wound over a core of constant permeability μ, then

$$L = \mu N^2 A/2\pi R. \quad \text{...(113)}$$

12.17 COEFFICIENT OF COUPLING OF THE TWO CIRCUITS

Let us now calculate mutual inductance between two circuits in terms of the self inductance of each circuit alone. This is the case of maximum flux linkage, when all the flux through one circuit links the other. For this two coils of equal area (but not necessarily of equal number of turns) should be placed next to each other. Calling the coils 1 and 2, we can write for the mutual inductance

$$M_{12} = N_1\Phi_{12}/i_2 \text{ and } M_{21} = N_2\Phi_{21}/i_1.$$

Similarly the self inductance of the coils is given by

$$L_1 = N_1\Phi_1/i_1 \text{ and } L_2 = N_2\Phi_2/i_2.$$

If all the flux of coil 2 links coil 1 (the case of maximum flux linkage), then

$$\Phi_{12}/i_2 = \Phi_2/i_2 \quad \text{or} \quad \Phi_{12} = \Phi_2.$$

or the flux Φ_{12} in coil 1 due to the current i_2 in coil 2 is the same as the flux Φ_2 in coil 2 due to the same current i_2. Similarly

$$\Phi_{21}/i_1 = \Phi_1/i_1 \quad \text{or} \quad \Phi_{21} = \Phi_1.$$

Since $M_{12} = M_{21} = M$(say), hence we have

$$M_{12}M_{21} = M^2 = N_1 N_2 \Phi_{12} \Phi_{21}/i_1 i_2 = N_1N_2\Phi_1\Phi_2/i_1i_2 = L_1L_2,$$

or

$$M_{max} = (L_1L_2)^{1/2}. \quad \text{...(114)}$$

This is the maximum possible value of M as the total flux associated with one coil is assumed to be linked with the other. In general only a fraction k_2 of Φ_2 passes through the coil 1. Hence $\Phi_{12} = k_2\Phi_2$.

Similarly only a fraction k_1 of Φ_1 is linked with coil 2, *i.e.*, $\Phi_{21} = k_1\Phi_1$. This gives

$$M_{12}M_{21} = M^2 = N_1N_2k_1k_2\Phi_1\Phi_2/i_1i_2 = k_1k_2L_1L_2$$

$$\therefore \quad M = (k_1k_2L_1L_2)^{1/2} \text{ or } M = k(L_1L_2)^{1/2}, \quad \text{...(115)}$$

where k is a constant, known as the coefficient of coupling of the coils (or loops). It purely depends on the geometry of the coils and their relative positions. It varies in between 0 and 1. In practice, however, the coefficient of coupling is inevitably less than unity since there will always be a certain amount of flux leakage.

12.18 COMBINATION OF INDUCTORS

An inductor is a circuit element, such as a tightly wound coil of wire that produces electro magnetic inductance. It is indicated by the symbol .

If n inductors are connected in series and are arranged so that there are no interactions through mutual inductance, the induced voltage across the system of inductors is given by

$$-L_1\frac{di}{dt}-L_2\frac{di}{dt}-L_3\frac{di}{dt}-\dots L_n\frac{di}{dt}=-L\frac{di}{dt}$$

or
$$L = L_1+L_2+L_3+\dots L_n=\Sigma L_n. \qquad \dots(116)$$

If these inductors are arranged in parallel, then we have

$$\mathcal{E}_1 = \mathcal{E}_2=\mathcal{E}_3\dots=\mathcal{E}_n=\mathcal{E}.$$

or
$$-L\frac{di_1}{dt} = -L_2\frac{di_2}{dt}=-L_3\frac{di_3}{dt}\dots=-L_n\frac{di_n}{dt}=-L\frac{di}{dt} \qquad \dots(117)$$

For the parallel arrangement, we also have

$$i = i_1+i_2+i_3+\dots\dots i_n$$

or
$$\frac{di}{dt} = \frac{di_1}{dt}+\frac{di_2}{dt}+\frac{di_3}{dt}+\dots\dots\frac{di_n}{dt}. \qquad \dots(118)$$

The combination of Eqns, (117) and (118) gives

$$\frac{\mathcal{E}}{L} = \mathcal{E}\left(\frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}+\dots\frac{1}{L_n}\right)$$

Hence the equivalent inductance of the parallel combination of n inductors is given by

$$\frac{1}{L} = \frac{1}{L_1}+\frac{1}{L_2}+\frac{1}{L_3}+\dots\frac{1}{L_n}. \qquad \dots(119)$$

The relations (116) and (119) are valid for pure inductors connected with no mutual inductance. Actually the inductor always has a certain resistance and the mutual inductance between the inductors cannot be ignored. The problem for n inductors will be difficult. For the simplicity, we consider two inductors in series, Fig. 12.29. The total voltage drop for the circuit

$$V = R_1i+L_1\frac{di}{dt}\pm M\frac{di}{dt}+L_2\frac{di}{dt}\pm M\frac{di}{dt}+R_2i = (R_1+R_2)i+(L_1+L_2\pm 2M)\frac{di}{dt}$$

Thus the circuit is equivalent to a resistance R_1+R_2 in series with an inductance $L_1+L_2\pm 2M$.

Total inductance $L = L_1+L_2+2M$, for the *positive coupling* where the currents traverse the loops in the same sense or the fluxes are in the same direction (fluxes enforce each other).

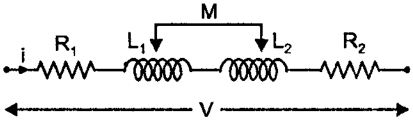


Fig. 12.29. Two inductor is series.

Total inductance $L = L_1+L_2-2M$, for the *negative coupling* where the currents traverse the loops in the opposite sense or the fluxes are in the same directions (fluxes opposite to each other).

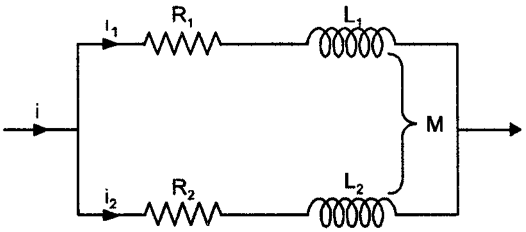


Fig. 12.30. Two inductors in parallel.

For the two inductors in parallel, Fig. 12.30, we get

$$V = L_1\frac{di_1}{dt}\pm M\frac{di_2}{dt}+R_1i_1 \qquad \dots(121)$$

and
$$V = L_2\frac{di_2}{dt}\pm M\frac{di_1}{dt}+R_2i_2. \qquad \dots(122)$$

Neglecting the resistance terms, multiplying Eq. (121) by L_2 and Eq. (122) by M and solving to eliminate di_2/dt, we get

$$V(L_2 \pm M) = (L_1 L_2 - M^2)(di_1/dt). \qquad ...(123)$$

Again neglecting the resistance terms multiplying Eq. (122) by L_2 and Eq. (121) by M and, solving to eliminate di_1/dt, we get

$$V(L_1 \pm M) = (L_1 L_2 - M^2)(di_2/dt). \qquad ...(124)$$

Since $i = i_1 + i_2$, hence $\dfrac{di}{dt} = \dfrac{di_1}{dt} + \dfrac{di_2}{dt}$. ...(125)

Thus the combination of Eqs. (123) to (125) gives

$$V = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}\frac{di}{dt}. \qquad ...(126)$$

Thus the effective inductance of two inductors in parallel

$$L = \frac{L_1 L_2 - M^2}{L_1 + L_2 \pm 2M}, \qquad ...(127)$$

where the sign of M depends on the way in which the inductors are connected.

12.19 ENERGY STORED IN A MAGNETIC FIELD

We know that the current $\mathbf{j}$ is related with the electric field $\mathbf{E}$ as $\mathbf{J} = \sigma\mathbf{E}$, where σ is the conductivity of the medium. This electric field intensity is the sum of two terms : $-\nabla V$ due to accumulation of charge and $-\partial\mathbf{A}/\partial t$ due to the changing magnetic field

$$\therefore \qquad \mathbf{j} = \sigma\mathbf{E} = \sigma(-\nabla V - \partial\mathbf{A}/\partial t).$$

Consider an element in the form of rectangular parallelopiped of length dl and area of cross section dS. If $\mathbf{j}$ is along dl, a charge jdS will go from left hand to right hand face in one second and thus will set a *pd* of $-\nabla V\cdot d\mathbf{l}$ across these faces. The power supplied by the source to this element of volume dv

$$dP = -\nabla V\cdot dljdS = -\nabla V\cdot\mathbf{j}\,dv = (\mathbf{E} + \partial\mathbf{A}/\partial t).\,\mathbf{j}dv.$$

$$\therefore \qquad P = \int_v\left[\mathbf{E}.\mathbf{j} + \frac{\partial\mathbf{A}}{\partial t}\cdot\mathbf{j}\right]dv = \int_v \frac{j^2}{\sigma}dv + \int\frac{\partial\mathbf{A}}{\partial t}\cdot\mathbf{j}dv. \qquad ...(128)$$

The first term is the *joule losses*. We are mainly concerned with the second term, the magnetic field energy. Let us write latter as P_m or dW_m/dt.

(*a*) In terms of B – We know that curl $\boldsymbol{B} = \mu_0\mathbf{j}$, hence

$$\frac{dW_m}{dt} = \frac{1}{\mu_0}\int_v\frac{\partial\mathbf{A}}{\partial t}\cdot(\nabla\times\mathbf{B})\,dv = \frac{1}{\mu_0}\int_v\left[\mathbf{B}\cdot\left(\nabla\times\frac{d\mathbf{A}}{\partial t}\right) - \nabla\cdot\left(\frac{\partial\mathbf{A}}{\partial t}\times\mathbf{B}\right)\right]dv$$

$$= \frac{1}{\mu_0}\int_v\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t}dv + \frac{1}{\mu_0}\int_S\left(\mathbf{B}\times\frac{\partial\mathbf{A}}{\partial t}\right)\cdot d\mathbf{S}. \qquad ...(129)$$

Here we have used relation $\nabla\cdot(\mathbf{A}\times\mathbf{B}) = (\nabla\times\mathbf{A}).\,\mathbf{B} - (\nabla\times\mathbf{B})\cdot\mathbf{A}$ and the divergence theorem for the conversion of volume integral into surface integral

If we choose S to be a spherical surface at infinity, as is shown for current loop, **A** falls off as $1/r^2$ and hence **B** as $1/r^3$ at large distances. Since the surface area $S \propto r^2$, hence the surface integral decreases as $1/r^2$ and thus vanishes. The rate of work done will be the rate of energy stored. Thus we have

$$\frac{dU_m}{dt} = \frac{dW_m}{dt} = \frac{1}{\mu_0}\int \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\,dv = \frac{1}{2\mu_0}\frac{d}{dt}\int B^2 dv.$$

Setting $U_m = 0$, when $\mathbf{B} = 0$, we get $U_m = \dfrac{1}{2\mu_0}\int B^2 dv.$ ...(130)

$\therefore$ Energy density $u = dU_m/dv = B^2/2\mu_0$. ...(131)

These relations show the similarity with the electrostatic energy in terms of **E**. As magnetic energy varies as B^2, hence the total energy due to several fields is not just the sum of energies calculated for separate fields.

(*b*) In terms of *j* and A : Using relation **B** = curl **A**, Eq. (130) becomes

$$U_m = \frac{1}{2\mu_0}\int_v \mathbf{B}\cdot(\nabla\times\mathbf{A})\,dv.$$

Using the vector identity and the divergence theorem as used before, we have

$$U_m = \frac{1}{2\mu_0}\int_v \mathbf{A}\cdot(\nabla\times\mathbf{B})\,dv + \frac{1}{2\mu_0}\int_S(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{S}$$

Here the surface integral vanishes, as $\nabla\times\mathbf{B} = \mu_0\mathbf{j}$, hence

$$U_m = \tfrac{1}{2}\int_v(\mathbf{j}\cdot\mathbf{A})\,dv, \qquad ...(132)$$

where v is any volume consisting all regions with $\mathbf{j} \neq 0$.

$\therefore$ Energy density $u = dU_m/dv = \frac{1}{2}(\mathbf{j}.\mathbf{A})$. ...(133)

(*c*) In terms of *i* and Φ : With filamentary currents one can replace **j** dv by $i\,d\mathbf{l}$, where $d\mathbf{l}$ is an element of the circuit carrying the current i, in Eq. (132). Thus we have

$$U_m = \tfrac{1}{2}\int i\,d\mathbf{l}\cdot\mathbf{A} = \tfrac{1}{2}i\oint\mathbf{A}\cdot d\mathbf{l} = \tfrac{1}{2}i\Phi. \qquad ...(134)$$

The directions of i and Φ are related by right hand screw rule.

(*d*) In terms of *i* and *L* : Since the magnetic flux can be replaced by the product of the self inductance L and the current i, hence

$$U_m = \tfrac{1}{2}i(Li) = \tfrac{1}{2}Li^2. \qquad ...(135)$$

12.20 ENERGY STORED BY COUPLED INDUCTORS

Consider two coils that are magnetically linked, coil 1 with n_1 turns and coil 2 with n_2 turns. Let the coils carry currents i_1 and i_2 respectively, due to externally applied sources of emfs $\mathcal{E}_{10}$ and $\mathcal{E}_{20}$. If i_1 is a function of time, it will induce $-L_1\,(di_1/dt)$ in coil 1 and $-M_{21}\,(di_1/dt)$ in coil 2. Similarly the time varying current i_2 in the coil 2 will induce $-L_2\,di_2/dt$ in coil 2 and $-M_{12}\,di_2/dt$ in coil 1. We assume the resistances of the coils to be negligible.

Thus at any instant of time t, the total induced emfs in the coils 1 and 2 are,

$$\mathcal{E}_1 = -L_1\frac{di_1}{dt} \pm M_{12}\frac{di_2}{dt}$$

and $$\mathcal{E}_2 = -L_2\frac{di_2}{dt} \pm M_{21}\frac{di_1}{dt}.$$

Thus the rate of work by the induced emf in these circuits or the power absorbed by each of the coupled inductors

$$P_1 = dW_1/dt = L_1 i_1\,(di_1/dt) \pm M_{12} i_1\,(di_2/dt)$$

and $$P_2 = dW_2/dt = L_2 i_2\,(di_2/dt) \pm M_{21}\,i_2(di_1/dt).$$

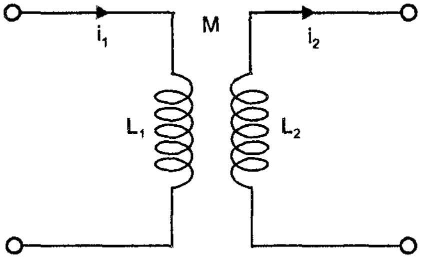


Fig. 12.31. Coupled inductors.

∴ Total power absorbed by both inductors

$$P = \frac{dW}{dt} = L_1 i_1\frac{di_1}{dt} + L_2 i_2\frac{di_2}{dt} \pm \left(M_{12} i_1\frac{di_2}{dt} + M_{21} i_2\frac{di_1}{dt}\right)$$

or $$\frac{dW}{dt} = L_1 i_1\frac{di_1}{dt} + L_2 i_2\frac{di_2}{dt} \pm M\frac{d}{dt}(i_1 i_2) \qquad ...(136)$$

$$\because \quad M_{12} = M_{21} = M \text{ and } i_1\frac{di_2}{dt} + i_2\frac{di_1}{dt} = \frac{d}{dt}(i_1 i_2).$$

If both the currents i_1 and i_2 start from zero at the same instant and reach values of I_1 and I_2 respectively. The total energy absorbed

$$U = W = L_1\int_0^{I_1} i_1\,di_1 + L_2\int_0^{I_2} i_2\,di_2 \pm M\int_0^{I_1,I_2} d(i_1 i_2)$$

$$= \tfrac{1}{2}L_1 I_1^2 + \tfrac{1}{2}L_2 I_2^2 \pm M I_1 I_2. \qquad ...(137)$$

This energy is stored as the magnetic field energy U_m. The first two terms represent self energies. The last term representing interaction energy, may be either positive or negative depending on the winding, but the total magnetic energy W_m must be positive (or zero) for any pair of currents.

If we denote I_1/I_2 by x, we may write

$$U_m = \tfrac{1}{2}I_2^{\,2}\,(L_1 x^2 + 2Mx + L_2) \geq 0.$$

On differentiating w.r.t. x and setting equal to zero, we get

$$x = -M/L_1. \qquad ...(138)$$

The second derivative w.r.t. x is positive, which shows that $W_m \geq 0$ for any x, W_m is minimum for $x = -M/L_1$. Thus we have

$$\frac{M^2}{L_1} - \frac{2M^2}{L_1} + L_2 \geq 0,$$

or $$L_1 L_2 \geq M^2. \qquad ...(139)$$

This is the result, we have stated earlier, but not proved.

The above relations may be generalized for n coupled loops. Again we assume the coils as of negligible resistances. At a given instant of time the induced emf in the i^{th} circuit is given by

$$\mathcal{E}_i = \sum_{j \neq i}^{n} M_{ij}\frac{di_j}{dt} - L_i\frac{di_i}{dt}, \qquad ...(140)$$

where M_{ij} is the mutual inductances between the i^{th} and j^{th} circuits and i_i and i_j are the currents in them respectively. The rate of work by the induced emf in the i^{th} circuit is

$$\frac{dW_i}{dt} = -\sum_{j \neq i}^{n} M_{ij}\, i_1 \frac{di_j}{dt} - L_i i_i \frac{di_i}{dt}$$

or $$dW_i = -\sum_{j \neq i}^{n} M_{ij}\, i_1\, di_j - L_i i_1 di_i . \qquad \text{...(141)}$$

The total differential work done in all circuits

$$dW = \Sigma dW_i = -\sum_{\substack{i,j=1 \\ i \neq j}}^{n} M_{ij}\, i_1 di_j - \sum_{i}^{n} L_i i_i di_i . \qquad \text{...(142)}$$

Let us write $M_{ij}\, i_i\, di_j + M_{ji}\, i_j\, di_i = M_{ij}\, d(i_i i_j)$, where $i < j$.

$$\therefore \quad dW = -\sum_{i,j}^{n} M_{ij} d\,(i_i i_j) - \sum_{i}^{n} L_i i_i di_i . \qquad (i < j)$$

On integrating from 0 to I_i and I_j, we get

$$W = -\sum_{ij}^{n} M_{ij}\, I_i I_j - \frac{1}{2}\sum_{i}^{n} L_i I_i^2 \qquad (i < j)$$

The total magnetic energy stored in the system

$$U_m (= W_m) = -W = \sum_{i,j}^{n} M_{ij} I_i I_j + \frac{1}{2}\sum_{i}^{n} L_i\, I_i^2 . \qquad (i < j) \qquad \text{...(143)}$$

We may express this energy in terms of the flux passing through the n – circuits. We may write Eq. (143) as

$$U_m (= W_m) = \frac{1}{2}\sum_{i \neq j}^{n} M_{ij} I_i I_j + \frac{1}{2}\sum_{i}^{n} L_i I_i^2 = \frac{1}{2}\sum_{i,j}^{n} M_{ij}\, I_i I_j , \qquad \text{...(144)}$$

where $M_{ij} = L_i$ for $i = j$. The total flux passing through the i^{th} circuit is given by

$$\Phi_i = \sum_{j=i}^{n} \Phi_{ij} = \sum_{j=1}^{n} M_{ij}\, I_j , \qquad \text{...(145)}$$

$$\therefore \quad U_m (= W_m) = \frac{1}{2}\sum_{i=1}^{n} \Phi_i\, I_i . \qquad \text{...(146)}$$

12.21 MAGNETIC FORCE BETWEEN TWO CIRCUITS

Consider two loops carrying currents i_1 in the same direction. The loops tend to move toward each other and are fixed in the space only if the magnetic forces are balanced by equal and opposite mechanical forces. We assume a small virtual displacement of one coil, without rotation. This displacement is very slow so the K.E. can be assumed as zero. For simplicity we neglect the Joule losses and thus have

Work done by sources = Increase in magnetic energy + Mechanical work.

The force between these coils can be found in two ways :

1. Currents are kept constant : Assumed that the loop 1 is allowed to move a distance *dr* toward loop 2. As the currents are kept constant, only the interaction energy changes and the self energies remain constant. Hence Eqn. (137) gives.

$$dW_m = I_1 I_2 dM = I_2\, d\Phi_{21} = I_1 d\Phi_{12},$$

where Φ_{12} is the flux originating in 2 and linking 1.

Since Φ_{12} and Φ_{21} increase in this case, hence dW_m is positive and the magnetic energy increases.

In loop 1, Φ_{12} increases and the induced emf is set in such a way as to oppose a magnetic field $\mathbf{B}_2$ (originating in 2 and linking 1). It will tend to oppose the current I_1. To keep the current constant, the source voltage must be increased at each instant by $d\Phi_{12}/dt$. The source in loop 1 therefore supplies an energy

$$dW_1 = I_1(d\Phi_{12}/dt)\, dt = I_1 d\Phi_{12} = I_1 I_2 dM.$$

Similarly the amount of energy supplied by the source in loop 2

$$dW_2 = I_1 I_2 dM.$$

$\therefore$ Total energy supplied by the sources $dW_s = 2I_1 I_2 dM.$ ...(147)

Since $I_1 I_2\, dM$ of energy is stored as a magnetic energy, hence the remainder $I_1 I_2\, dM$ would have gone into mechanical energy. If F_{12} is the force exerted by loop 2 on loop 1, then mechanical work done at constant current I_1 is given by

$$\mathbf{F}_{12} \cdot d\mathbf{r} = I_1 I_2 dM. \qquad \text{...(148)}$$

As it is +ve, hence to keep the scalar product + ve, $\mathbf{F}_{12}$ must point toward coil 2, like $d\mathbf{r}$. It is right as we have assumed the motion of coil 1 toward 2.

Let us write relation (148) in the form of an equation for the force between two current carrying loops as discussed in chapter 10. Using equation (79), we get

$$\boldsymbol{F}_{12} \cdot d\boldsymbol{r} = \frac{\mu_0}{4\pi} I_1 I_2 d\left[\oint_1 \oint_2 \frac{d\mathbf{l}_1 . d\mathbf{l}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right] = \frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 d\left(\frac{d\mathbf{l}_1 . d\mathbf{l}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right)$$

In the process of moving the coil 1 by a distance $\mathbf{dr}$, both $d\mathbf{l}_1$ and $d\mathbf{l}_2$ remain unchanged and thus differential is to be taken of $1/|\mathbf{r}_1 - \mathbf{r}_2|$.

$$\therefore \quad d\left(\frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r}_1 - \mathbf{r}_2|}\right) = -d\mathbf{l}_1 \cdot d\mathbf{l}_2 \frac{(\mathbf{r}_1 - \mathbf{r}_2) \cdot d\mathbf{r}}{|\mathbf{r}_1 - \mathbf{r}_2|^3}$$

Therefore we have

$$\mathbf{F}_{12} \cdot d\mathbf{r} = -\frac{\mu_0}{4\pi} I_1 I_2 \oint_1 \oint_2 \frac{d\mathbf{l}_1 . d\mathbf{l}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2) . d\mathbf{r}$$

Since $d\mathbf{r}$ on the right hand side is independent of $d\mathbf{l}_1$ and $d\mathbf{l}_2$, hence

$$\mathbf{F}_{12} = -\frac{\mu_0}{4\pi} I_1 I_1 \oint_1 \oint_2 \frac{d\mathbf{l}_1 \cdot d\mathbf{l}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} (\mathbf{r}_1 - \mathbf{r}_2). \qquad \text{...(149)}$$

It is same as obtained in example 10 of chapter 10. Relation (148) shows that at constant currents

Mechanical work done $\mathbf{F} \cdot d\mathbf{r}$ = Change in magnetic energy dU_m.

or $\mathbf{F} = \nabla U_m$ (constant current). ...(150)

2. **Fluxes are kept constant :** Another physical situation arises when the circuits are isolated from the external sources (batteries). In this case, the movement of one of the circuits

results in a change in the currents in them. According to Lenz's law, the amount of change in the induced current due to induced emf is such that the magnetic flux passing through the circuits remains the same. No energy is supplied by the sources. Joule losses are very negligible. Thus, the mechanical work is accompanied by a decrease in magnetic energy.

$$\mathbf{F}\cdot d\mathbf{r} = -dU_m$$

or

$$\mathbf{F} = -\nabla U_m \qquad \text{(constant flux)} \qquad ...(151)$$

Similarly, one can find the expressions for the magnetic torques acting on the circuits. If instead of a virtual translation, the circuit is allowed to have a rigid virtual rotation $d\theta$, then

$$\tau_\theta = \partial U_m/\partial\theta \qquad \text{(constant current)} \qquad ...(152)$$

represents the torque in the increasing θ direction in the case of constant currents and

$$\tau_\Phi = -\partial\, U_m/\partial\theta \qquad \text{(constant flux)} \qquad ...(153)$$

represents the torque in the increasing θ direction in case of constant fluxes.

12.22 DYNAMO

The principle of the dynamo has been discussed in article 12.10, but the actual form of the machine was not described. If the two ends of the rotating coil are connected to a pair of slip rings, the emf is enabled to produce an alternating current in the external circuit. To produce a current in one direction a device called *commutator* is used. It consists of a *split ring AB*, to each half of which is connected one end of the rotating coil. Two brass conductors *C* and *D*, called *brushes,* bear upon the split ring at opposite ends of a diameter. As the direction of the emf induced in the coil changes sign, each segment of the split ring moves on to the other brush and hence the emf always acts in one direction in the external circuit. As shown in Fig. 12.32, the emf and current in the external circuit are constant in direction but not in value. Thus we get pulsating direct current.

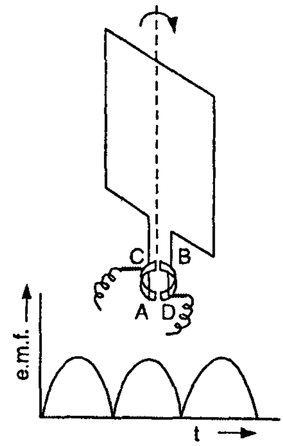


Fig. 12.32. Pulsating dc.

Continuous or direct current : Consider a pair of coils whose planes are at right angles to each other. As shown in Fig. 12.33 the ends *e* and *h are* joined to the commutator section *A*, and *f* and *g* are joined to section B. At the time of rotation of the coils, it will be seen that when the one coil is producing zero emf the other coil is producing its maximum emf. The resultant output emf is shown by thick curve (lower part of Fig. 12.33). It shows that the emf is not constant in value but varies between the limits *OA* and *OB*.

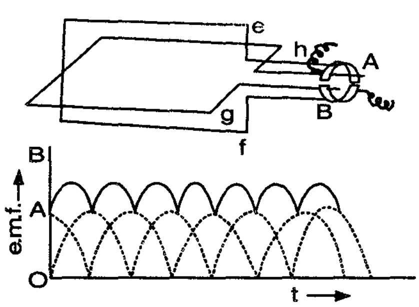


Fig. 12.33. Output of pair of coils.

By increasing the number of coils the emf and current curves may be made much smoother. Let us consider the dynamo with eight conductors, four coils. These conductors lie in slots *a b c d e f g h* in an iron core and are connected to the four commutator bars *ABCD* at the front of the armature, as shown by the continuous lines, and to each other at the back of the armature, as shown by the dottedlines. The conductors *b, c* and *d* are descending, hence the emf is directed from back to front, represented by the dot in a circle. The conductors *f*, *g* and *h* are ascending hence the emf is directed from front to back, represented by the cross in a circle. Current enters the armature by the brush *E* and passing through paths *EAadDgbCF* or *EAfcBheCF* exists to the external circuit through brush *F*.

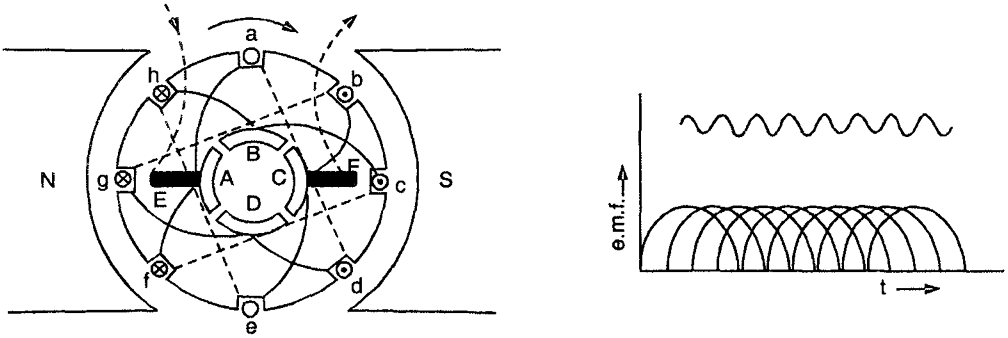


Fig. 12.34. Armature with 8-conductors (left), Output (right).

The way in which the mean voltage is increased and the ripple voltage is reduced by the superposition of a number of waves progressively different in phase as shown in Fig. 12.34 (right). By increasing the number of conductors, we can get nearly constant potential. The armature from its shape is called a *drum armature*.

Construction of a dc-dynamo : In a dynamo, the conductors in which the emf is induced are placed on the *armature*. The necessary motion is obtained by the rotation of the armature with in the magnetic field produced by electromagnet known as *field magnet*. The induced emf is an alternating emf. To obtain a direct or continuous emf, this emf is rectified by means of a *commutator* which rotates with the armature. The *shaft* mounted on bearings permits the necessary rotation of the armature. It also carries the commutator. The *brushes* are supported on a frame.

Armature : The armature consists of the *core* and the *winding*. The core obviously has to be magnetic in order to provide a path of low reluctance to the magnetic lines of force. To minimise the eddy currents in the core, the core is built up of thin iron sheets or discs insulated from one another by varnish, thin paper or oxide coating.

These sheets or discs are about 0.025″ thick. Upto about 3′ diameter they are cut out in complete rings, but for large armatures they are cut out in segments. A *ring armature,* useful for producing high potentials, but not for delivering a high current consists of sheets of shoft iron insulated from each other and packed together to form a ring. When the armature rotates in a magnetic field (say in a bipolar field, the portion of the armature coil near the north pole cuts magnetic lines of force in one direction and the rest part of the coil near the south pole cuts the lines on the opposite direction. Thus emf is produced.

The ring armature is now replaced by a *drum armature* which can produce heavy currents. In the drum armature a cylindrical drum made of shoft iron discs or stampings (transverse to the axis of rotation) is grooved or slotted. The insulated wire is wound in these grooves. To increase the magnetic flux, multipole magnet is used. The armature used in this case is known as *multipolar armature*.

The coils wound on the core are called *armature coils* or *windings* and the current through them as armature current. There are mainly two types of armature windings : (1) *Lap winding* and (2) *Wave winding*. The difference in these types consists mearly in the arrangement of the connections at the front of the armature. The length of the end connection at the back, is called the back pitch y_B and the length of the front connection is the front pitch y_F. These pitches are measured in terms of armature conductors. The two conductors which join two commutator bars constitute a *winding element*. The distance between the first and last conductors in an element is called the resultant pitch y. Therefore

$$y = y_B - y_F \qquad \text{for a lap winding.}$$

$$y = y_B + y_F \qquad \text{for a wave winding.}$$

Since the end connections of the consecutive conductors point alternately to right and left, hence the partial pitches y_B and y_F must be odd numbers and y is an even number. Since the poles are alternately north and south, the partial pitches y_B and y_F must be approximately equal to the *pole pitch* measured in armature conductors. For number of poles p and number of conductors n the pole pitch is n/p.

Field Magnets : Practically in all dynamos, the magnetic field is produced by the electromagnet consisting of a *circular yoke* with inwardly projecting *poles*. The polarity of these poles is alternately *north* and *south*. In small dynamos, cast iron is generally used for both yoke and poles. In large dynamos, yoke is fabricated of steel, because steel has roughly twice the permeability of cast iron. The poles may be cast steel, cast integral with the yoke, wrought iron or laminated.

The dynamos or d.c. generators are usually named according to the manner in which the exciting current is produced. If the current required to excite the magnet is drived from a separate source, the dynamo is said to be *separately excited*. If the current is taken from the dynamo itself, the dynamo is said to be *self excited*. The former is used in the large dynamos. In most dynamos the current for exciting the field magnets is drived from the machine itself, but the ways are different. The windings of the electromagnet and armature may be any of the following types :

(*a*) *Series wound dynamo :* If the field winding is such that the armature A and field coils are in series with the external circuit, this is called *series winding*. It is applicable for incandescent lamps, not for charging of accumulators. As the current flowing increases, the emf increases because the increasing current produces greater magnetic field and lines of force are cut by each conductor in every revolution.

(*b*) *Shunt wound dynamo :* If the field winding is such that the armature circuit and the field coils are in parallel, this is called *shunt winding*. It is particularly of much use for charging accumulators. In this case the emf of the machine drops as the current increases in the external circuit. As the whole emf of the armature is applied directly to the ends of the shunt coil, its resistance prevents an excessively great current flowing in it. The magnetisation therefore falls off and the e.m.f. thus decreases.

(*c*) *Compound wound dynamo :* As the emf rises with increasing current in series wound dynamo and the emf decreases in shunt wound dynamo, hence the combination of these two may give a constant emf for all external currents. Such an arrangement is called *compound winding*. The arrangements with their characteristic curves are shown in Fig. 12.35.

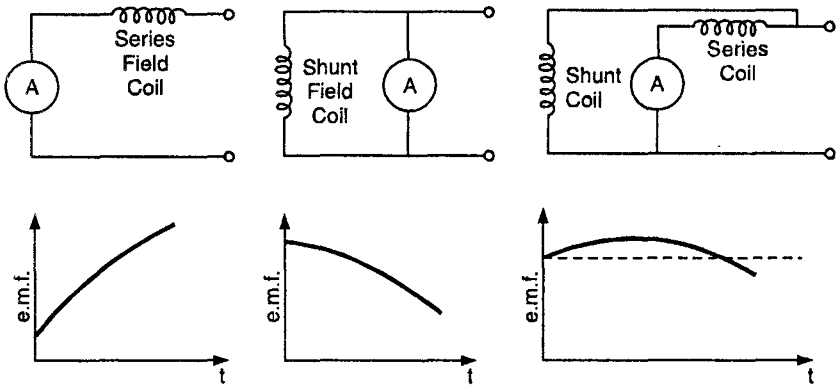


Fig. 12.35. Types of dynamos.

Commutator : The function of the commutator is to facilitate the collection of current from the armature. It is merely an extension of the armature windings and its function is to withstand the wear and tear of current collection. The commutator is a cylindrical structure built up of segments of high conductivity, hard drawn copper insulated from one another by mica.

Brushes : Two main types of brushes are generally used in dynamos : (*a*) The *metal brush* consists of thin sheet, wire or gauze of brass or copper, compressed into rectangular form. Its one edge bears against the commutator. (*b*) The *carbon brush* is made of compressed graphite. It rests normally on the commutator, hence does not require resetting if the direction of rotation is reversed.

As the commutator segment leaves the commutator brush, a back emf induced by the self inductance of the coil causes a sparking. This is reduced by using *carbon brushes,* as the area of the segment in contact with the brush diminishes, the increasing resistance makes the current change more smoothly through zero. The *second method* to reduce sparking is the alteration of the direction of the lines of magnetic induction produced by the current flowing through the armature. The main difficulty is the difficulty of adjusting the brushes to every variation in current. This problem is reduced by applying an external field from interpoles, which are connected with the armature and produce the field as required.

If the machine is with more than two poles, the number of brushes must be equal to the number of poles. If Φ be the total flux of magnetic induction from each pole, and p the number of poles, then the flux cut by each conductor per second is $\Phi\, pf$, where f is the frequency of revolution of the armature. If n is the number of conductors between successive brushes, then the emf between successive brushes is $\Phi\, pfn$.

Conditions for Self-excitation : The characteristics of the various types of dynamo show that there are certain definite conditions : (*i*) There must be some residual magnetism in the field magnet, (*ii*) This magnetism must be in the right direction with regard to the connections of the field winding to the armature, (*iii*) The resistance of the circuit must be less than the critical resistance in the *series dynamo,* (*iv*) The field is in series with the armature with respect to the path taken by the exciting current in the *shunt dynamo.* In this case, there is also a lower limit for the resistance of the external load below which the generator will fail to excite.

Efficiency of a generator : Efficiency of a generator (or dynamo) η = Output/(output + losses)

The losses are : (*i*) *Copper losses* (armature copper loss, field copper loss and loss due to brush contact resistance), (*ii*) *Iron losses* (hysteresis loss, eddy current loss) and (*iii*) *mechanical losses* (friction at bearings and commutator, winding of rotating armature). The magnetic (or iron loss) and mechanical losses are often grouped together and called *stray losses.*

12.23 DC MOTORS

The motor is a device to convert electrical energy into mechanical energy. Any direct current generator will run as a motor, if its field and armature are connected to a suitable electric supply. It is based on the principle that a force acts on a conductor carrying a current and placed in a magnetic field. We know that the toqure on a coil of N turns, carrying a current i placed in a magnetic field B and enclosing an area A is given by the relation

$$\tau = iNAB \sin \theta, \qquad ...(154)$$

where θ is the angle which the normal to the plane of the coil makes with the direction of magnetic field.

The ends of the coil are connected with each half of the split ring PQ and current is allowed to pass through the brushes B_1 and B_2, bear upon the split ring at opposite ends of a diameter. When the plane of the coil is vertical, *i.e.* $\theta = 0$, no toqure acts on the armature (coil). As the armature has sufficient inertia and the frictional forces are small, the angular momentum gained earlier carries it through the zero torque position. After the vertical position the current passes in opposite direction due to split ring, and hence a torque acts in the same direction. Thus we see that the *armature rotates in the same direction with two positions of no torque in each revolution.* In this way electrical energy is converted into mechanical energy.

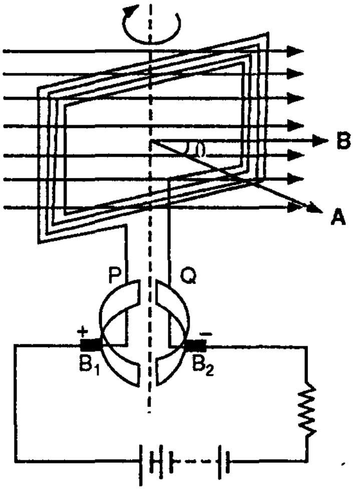


Fig. 12.36. Simple dc motor.

The motion of motor with a single coil and two poles is not continuous. To make the motor smooth running the single coil is replaced by several coils set at equal inclination to each other. A continuous drum winding may be preferred for this purpose. Refer to Fig. 12.34 dynamo with the eight armature conductors, let us apply some source of emf to the brushes. If current enters E and leaves F, the current in b, c and d will now be from back to front and in f, g and h from front to back. On applying left hand rule it will be seen that b, c, and d experience an upward force and f, g and h downward force. These forces will cause rotation in anti-clockwise direction, *i.e.* in the opposite direction to that in which it was previously driven by mechanical means. Thus we see that if *a* current be supplied by external means to a dynamo the machine will run as a motor, the rotation being in opposite direction when running as a motor to that in which it, as a dynamo, was driven.

Back e.m.f. in armature : When the armature is rotating, each conductor is cutting across magnetic lines of force and therefore emf is produced in it. This induced emf is proportional in value to the rate of change of number of lines of force, irrespective of the fact that the motion is due to this same magnetic field. To find its direction, apply the right hand law to the conductors b, c and d (of Fig. 12.34), it will be seen that emf due to cutting magnetic lines of force is from front to back and in f, g and h from back to front. Thus we get an emf in the opposite direction to the applied emf. It tends to reduce the current in the armature, hence is known as back emf.

In the beginning, the armature is at rest and there is no back emf. Hence the current depends upon the applied source and the resistance of the armature. As the armature picks up speed, the back e.m.f. increases and the current is consequently decreased. *This decrease is very much useful as, it prevents any injury in the insulation of the armature.*

If V be the potential applied at the brushes and E the induced back e.m.f., given by

$$\mathcal{E} = BAN\,(d\theta/dt) = BAN\,\omega, \qquad \text{...(155)}$$

then the current through an armature of resistance R

$$i = (V - \mathcal{E})/R.$$

Hence the torque acting over the coil

$$\tau = BANi = BAN\,(V - \mathcal{E})/R. \qquad \text{...(156)}$$

The mechanical energy produced per sec = The rate of doing work

$$= \tau d\theta/dt = BAN\,i\,\omega = \mathcal{E}i.$$

As $$V - \mathcal{E} = iR, \text{ or } Vi = \mathcal{E}\,i + i^2R, \qquad \text{...(157)}$$

This shows that the total power supplied to the armature is equal to the sum of the power converted into mechanical power and the power dissipated as heat due to the armature resistance.

$$\text{Electrical efficiency } \eta = \frac{\text{Mechanical energy obtained}}{\text{Electrical energy supplied}} = \frac{\mathcal{E}i}{Vi}$$

$$= \text{Back emf./Applied emf.} \qquad \text{...(158)}$$

η is maximum, when $\mathcal{E} = V$, but current becomes zero in this case and thus no work will be done. Work output is maximum when $\mathcal{E}i$ = maximum, or $\mathcal{E}\,(V - \mathcal{E})/R$ = maximum,

$$\therefore \quad \frac{d\mathcal{E}}{R}(V-\mathcal{E}) - \frac{\mathcal{E}}{R}d\mathcal{E} = 0, \quad \text{or} \quad \mathcal{E} = \frac{V}{2}. \qquad \text{...(159)}$$

Thus a motor develops the maximum mechanical power when the armature resistance or the armature current is such that the back emf is equal to one half of the applied emf. However this condition is not practicable, as the armature current will become too great to burn the armature coil and the efficiency of the motor will be below 50% due to the presence of other losses.

The speed of the motor adjusts itself to the load automatically, so that the electrical power required to drive the current through the armature is equal to the mechanical power required to drive the load (magnetic and mechanical losses). The speed of the motor is inversely proportional to the flux/pole.

Characteristics of DC Motors : Like the dynamos, the motors are also of three types depending upon the method of excitation. These are *series, shunt* and *compound wound motors.* In a *series wound motor*, the flux varies with the motor current. The speed is inversely proportional to the flux and thus the series motor is a variable speed motor. The speed being low on heavy load and dangerously high on light load. Thus the series motor is never run without some mechanical load. The torque is proportional to the square of current on light loads. When i is so great that the flux Φ becomes nearly constant, torque τ becomes propotional to i. Since the starting torque is very high, the series motor is used in cases where heavy masses have to be accelerated quickly, *e.g.*, cranes, lifts etc.

In a *shunt wound motor*, the exciting current is constant for a constant applied potential difference. The flux will have maximum value at no load and will decrease slightly as the load increases, but can be regarded as constant. Thus the speed decreases slightly with the increase in current and the shunt motor acts as a *constant speed machine*. The torque is proportional to the current sometimes, when starting a shunt motor, it is found that it takes a very large current and runs at a high speed in the wrong direction. This may be due to a break in the shunt winding. Shunt motors are suitable for driving light machine tools and for all purposes where constant speed is required, *e.g.*, fans, blowers, etc.

In a *compound wound motor*, the series field may be connected in the circuit either to help or oppose the shunt field, the motor is thus called *cumulatively compounded* or *differentially compounded* respectively. If the series field of the second type is adjusted so that the full load decrease in flux produced by it is just sufficient to make the full load speed equal to the speed on no load, then the speed will be constant approximately (first decreases slightly and then increases) for any load within this range. There are two disadvantages : (*i*) When the motor is being started, the shunt field will take some time to build up, the series field will be established first and the motor will move in opposite direction. When the shunt field is fully established, the total field will be too small to run the motor. (*ii*) If the motor becomes over loaded, the flux will decrease and the speed will increase and the motor will be seriously overloaded.

In the cumulatively compounded motor, the flux increases as the load increases. Thus the speed decreases, but not so rapidly as in the series motor. The mechanical characteristic is therefore between those of the series and shunt motors. The falling off in speed with the load makes the motor useful for driving heavy machine tools where sudden deep cuts may be taken. The motor also suitable for driving continuous running rolling mills where the motor is coupled to heavy fly wheel.

Motor Starter : As the current is too great to burn the coil in the very beginning, some high resistance is required in series with the armature A. This resistance may be cut out step by step until the full speed is attained and the back emf is sufficient to keep the current in the armature down to a safe value. Such a variable resistance is known as *starting resistance* and also as starter. One form of starting resistance for a shunt motor is shown in Fig. 12.37. When the motor is at rest the contact arm AO is pulled back by the spring S and the armature and field circuits are incomplete. When the arm is raised, it first makes whole resistance PQ in circuit. AO is

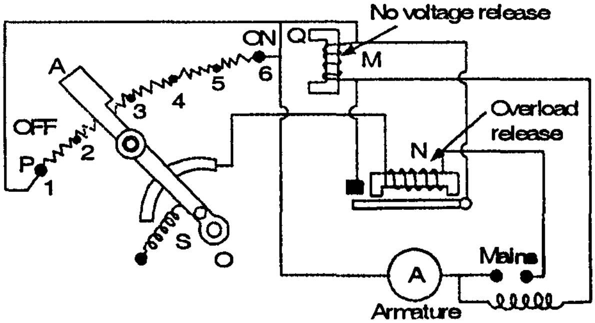


Fig. 12.37. Motor starter.

moved forward step by step as the speed of the motor increases. Finally all the resistance coils are cut out and the arm comes in contact with the poles of the electromagnet M, which is magnetised by the current also flowing in the field coils of the motor.

If the electric supply stops suddenly the electromagnet losses its magnetisation and the contact arm AO is pulled back by the spring and thus cuts off the motor from the mains. This avoids the burning out of the armature due to the large induced emf produced in opposite direction and is thus known as *no volt release.*

Another safety device which the starter contains is the another electromagnet N called the over load release. If the armature current becomes too great, the armature of this electromagnet is pulled up, thus short circuiting the electromagnet M. Hence the arm AO is released and goes back. In this way the supply current is automatically cut off, when it becomes too great. The starter is only used for large motors, not in small motors, as in latter case they gain speed rapidly.

The direction of rotation of motor can be reversed by reversing either the armature current or the field current, but not both as they will produce the rotation in the same direction as before.

Motor used as a brake : The modern electric trains and tram cars require powerful emergency brakes in addition to the mechanical brakes. These emergency brakes, also named as *magnetic brakes,* consist of an electro-magnet suspended by two strong springs just over the rails. When a current flows, the electromagnet is energiesed and the shoe is attracted and makes contact with the rails.

To apply the brake, the driver disconnects the motor from the mains and the resistance coil is placed in the circuit. Owing to the momentum of the tram or train, the armature is still driven and the motor acts as a dynamo. The current produced by this dynamo brings the train to rest. Thus we see that K.E. of the train is used to stop it.

12.24 WATTMETER

Wattmeter is an instrument to measure the power consumed in a circuit in watts. In direct current circuits power may be measured by an ammeter and voltmeter. The *dynamometer* type of instrument is also used for this purpose. In this a low resistance moving coil B is inserted in series with the main supply circuit, while the fixed coils A, A have high resistance and are in parallel with the load. The current through coils A, A is proportional to the voltage across the load, whilst the current through coil B is same as through the load. As the current I is passing through coil B suspended in the magnetic field due to coils A, A through which current proportional to V is flowing, hence the torque acting on the coil B is given by

$$\tau \propto VI.$$

This torque produces a deflection ϕ in the coil, given by

$$VI \propto \phi. \qquad \text{...(160)}$$

As VI is the power consumed. Hence the deflection of the coil B measures the power consumed. The scale over which the needle attached with the coil swings can thus be calibrated to read in watts.

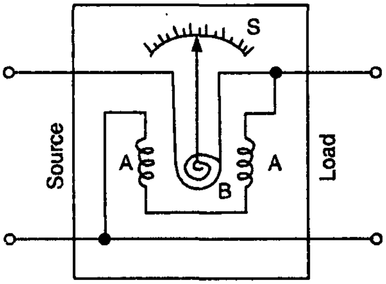


Fig. 12.38. D.C. Wattmeter.

12.25 TRANSIENT CURRENTS AND VOLTAGES

If a switch is used in a circuit to connect or disconnect the power source, the circuit is analysed at transition from one steady state condition prior to a switching operation to another steady state condition after the switching operation. This process is called *transient process*, or simply a *transient*. The transients are not caused solely by connection of the power source. *The transient processes take place when the emf sources and L, R, or C elements are connected or disconnected from a circuit.* During the transient process, the currents and voltages in the circuit are functions of time and are thus referred to as *transient currents and voltages*.

Let us consider transients arising in *RC, LR, LC* and *LCR* circuits.

(*a*) **RC–Series Circuit :** A capacitor C may be charged or discharged through a resistor R. **Charging** is done by connecting a battery E of emf $\mathcal{E}$, capacitor C and a resistor R in series by inserting the plug in the gap SA of a two way key as shown in Fig. 12.39. If i is the instantaneous value of current at an instant t through R, the potential difference across the capacitor will be $\mathcal{E} - Ri$. If q is the instantaneous charge on the capacitor at this instant t, then we have

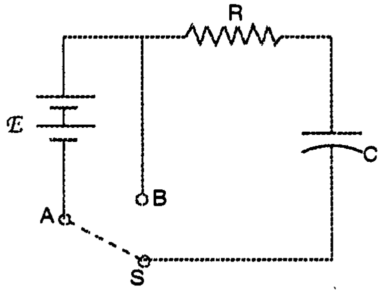


Fig. 12.39. R C-series circuit.

$$\mathcal{E} - Ri = q/C \quad \text{or} \quad \mathcal{E} - q/C = Ri.$$

Since $i = dq/dt$, hence

$$\mathcal{E} - \frac{q}{C} = R\frac{dq}{dt} \text{ or } \frac{dq}{\mathcal{E} - q/C} = \frac{dt}{R}.$$

On integrating, assuming $\mathcal{E}$ and C as constants, we get

$$-C \log_e (\mathcal{E} - q/C) = t/R + \text{const.}$$

Since the initial charge is zero (*i.e.* $q = 0$ at $t = 0$) hence const $= - C \log_e \mathcal{E}$ and therefore

$$\log_e (\mathcal{E} - q/C) = -t/RC + \log_e \mathcal{E}.$$

$$\therefore \quad \log_e \frac{\mathcal{E} - q/C}{\mathcal{E}} = -\frac{t}{RC} \text{ or } \frac{\mathcal{E} - q/C}{\mathcal{E}} = e^{-t/RC}$$

or

$$q = C\mathcal{E}(1 - e^{-t/RC}) = q_0 (1 - e^{-t/RC}) \qquad \text{...(161)}$$

where $q = C\mathcal{E}$ = maximum charge associated with the capacitor.

Thus the voltage across the capacitor during the transient process is given by

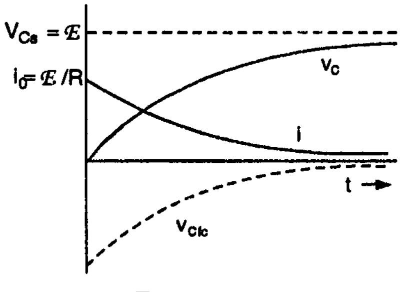


Fig. 12.40. Transient current and voltages curves for series RC–circuit.

$$v_C = \frac{q}{C} = \mathcal{E}(1 - e^{-t/RC}) \qquad \text{...(162)}$$

$$= \mathcal{E} - \mathcal{E}e^{-t/RC} = v_{Cs} + v_{Cfc} \qquad \text{...(163)}$$

The first term v_{Cs} is the steady state (forced) component of the transient voltage and v_{Cfc} is the force free component or free component. The current

$$i = \frac{dq}{dt} = \frac{C\mathcal{E}}{CR} e^{-t/RC} = \frac{\mathcal{E}}{R} e^{-t/RC}.$$

Since $\mathcal{E}/R = i_o$, the initial current, hence

$$i = i_0 e^{-t/RC}. \qquad \text{....(164)}$$

This is the free component i_{fc} of the charging current. The steady current $i_s = 0$.

The current and charge are therefore both exponential functions of time. At a time $t = RC$, the current has decreased to $1/e^{\text{th}}$ of its initial value. The product RC is known as the *time constant* or the *relaxation time* of the circuit. It is the time taken for the capacitor to acquire 63% of its maximum charge. It is represented by τ.

Eq. (151) and Fig. 12.40 show that the charge of a capacitor takes time to reach to its maximum value. It can be explained as :

When the circuit is closed, the initial current of $\mathcal{E}/R$ will set up in the circuit. Initially the capacitor experiences no potential difference as initial charge is zero. The flow of current through the resistor starts to charge the capacitor. Due to this charge, there will be a potential difference q/C across the capacitor. Since the sum of the potential differences (across the resistance and across the capacitor) must always equal $\mathcal{E}$, the applied emf, hence the potential difference across the resistor must decrease by the same amount q/C, which will reduce the charging current. Thus the charge of the capacitor builds up and the charging current decreases until the capacitor is fully charged. In time $t >> RC$ or practically equal to $4RC$, the capacitor is fully charged for all practical purposes.

If a capacitor of capacitance C having initial charge q_0 is allowed to **discharge** through a resistance R, connected in series, the charge does not vanish from the plates instantaneously but decays to zero exponentially. Let i be the instantaneous current in the circuit and q the charge on the capacitor at any instant t, the emf equation in this case will be

$$iR + q/C = 0 \quad \text{or} \quad R\,dq/dt + q/C = 0.$$

$$\frac{dq}{q} = -\frac{1}{RC}dt \quad \text{or} \quad \log_e q = -\frac{t}{RC} + \text{const.}$$

As $q = q_0$ at $t = 0$, hence const $= \log_e q_0$,

$$\therefore \quad q = q_0 e^{-t/RC}. \qquad \text{...(165)}$$

Current during discharge $i = \dfrac{dq}{dt} = -\dfrac{q_0}{RC}e^{-t/RC}$. ...(166)

Since $q_0 = C\mathcal{E}$ and $\mathcal{E}/R$ = Initial current i_0, hence

$$i = -i_0 e^{-t/RC}, \qquad \text{...(167)}$$

The current thus falls exponentially. The initial current is due to the charge on the capacitor. In the closed circuit it is in the direction opposite to that in the case of charging. Eqs (164) and (166) show that the charging or discharging is more rapid for smaller values of RC.

The discharge voltage may be obtained from Eq. (165) as

$$v_C = \mathcal{E}e^{-t/RC} \qquad \text{...(168)}$$

The variation of voltage and current with time is shown in Fig. 12.41. The steady voltage v_{Cs} and current i_s are zero. Only the free components exist.

$$v_{Cfc} = v_C = \mathcal{E}e^{-t/RC}$$

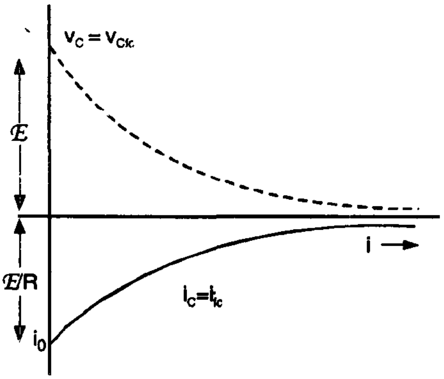


Fig. 12.41. Transient curves for RC series circuit (discharging).

and $$i_{fc} = i = -i_0 e^{-t/RC}. \qquad ...(169)$$

Let us now calculate the *energy* dissipated in the resistance during the charging or discharging of a capacitor. The energy dissipated in time δt is given by

$$\delta W = i^2 R\delta t = [(q_0/RC)\, e^{-t/RC}]^2\, R\delta t.$$

$$\therefore \qquad W = \frac{q_0^2}{RC^2}\int_0^\infty e^{-2t/RC}\, dt = \frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2} q_0 \mathcal{E}. \qquad ...(170)$$

This is independent of R. It shows that half of the total energy supplied by the battery is gained by the capacitor and rest half is dissipated in the resistor.

(*b*) **LR series circuit : (*i*) Growth of current :** Let us consider a circuit consisting of a battery, a coil of self inductance L, a resistor R and a key connected in series as shown in Fig. 12.42. If a plug is inserted in the gap SA of the two way key so that the battery is in circuit, the current at any instant t flowing in this circuit is given by

$$\mathcal{E} - L\,(di/dt) = Ri, \qquad ...(171)$$

where $L\,(di/dt)$ is the emf induced in the circuit at this instant t due to the rate of increase of current across the inductance L, and R is the total resistance in the circuit.

$$\frac{L\,di}{\mathcal{E} - Ri} = dt \quad \text{or} \quad L\int_0^i \frac{di}{\mathcal{E} - Ri} = \int_0^t dt$$

$$\therefore \qquad t = -\frac{L}{R}\log_e(\mathcal{E} - Ri) + \frac{L}{R}\log_e \mathcal{E} = -\frac{L}{R}\log_e \frac{\mathcal{E} - Ri}{\mathcal{E}}$$

$$\text{or } 1 - Ri/\mathcal{E} = e^{-Rt/L}.$$

$$\therefore \qquad i = \frac{\mathcal{E}}{R}(1 - e^{-Rt/L}) = i_0\,(1 - e^{-Rt/L}). \qquad ...(172)$$

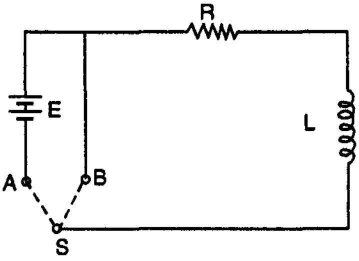


Fig. 12.42. *LR*–series circuit.

This is the relation for the current at any instant t. Here i_0 (= $\mathcal{E}/R$) is the steady value, the maximum value, of current when $t = \infty$. For $R = 0$, current will not be infinite but $(\mathcal{E}/L)\, t$, obtained by direct integration of Eq. (171).

Thus we see that the variation in the current when closing the *LR*-series circuit is similar to the variation in the voltage across the capacitor when closing the *RC*-series circuit.

The rate of growth of current $\dfrac{di}{dt} = \dfrac{R}{L} i_0\, e^{-Rt/L} = \dfrac{R}{L}(i_0 - i)$.

Thus we see that rate of growth of current decreases, as $i \to i_0$ and for any value of current i depends on R/L. The greater the value of L/R (the ratio known as *inductive time constant*), the smaller will be the rate of growth of current. The time constant L/R is denoted by τ and is expressed in seconds if L and R are in henrys and ohms respectively.

If $t = \tau$, then $i = i_0\,(1 - e^{-1}) = 0.63\, i_0$.

Hence time constant is the time taken for the current to rise to 0.63 times its maximum value.

As shown in Fig. 12.43, the rate of current rise during the transient period decreases gradually, therefore the current approaches asymptotically to a steady value i_0. Theoretically

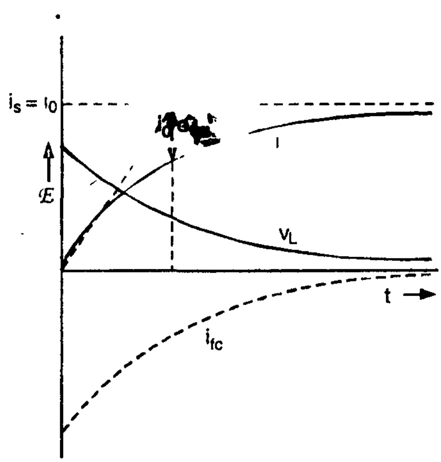


Fig. 12.43. Transient curves for *LR*–circuit (dc-source connected)

$i = i_0$ at $t = \infty$. In practice, the transient period is considered to be 4 τ, as at $t = 4\ \tau$ current *i* differs from the steady value i_0 by less then 2%.

Equation (172) shows that the transient current *i* may be written as

$$i = i_s + i_{fc}, \qquad ...(173)$$

where the steady state current $i_s = i_0$ and the free component

$$i_{fc} = -i_0 e^{-t/\tau}.$$

The potential difference across the inductor

$$v_L = L\frac{di}{dt} = \mathcal{E}e^{-Rt/L} = \mathcal{E}e^{-t/\tau} \qquad ...(174)$$

Under steady state conditions (*i.e.* at $t = \infty$), $v_{Ls} = 0$.

Thus the free component v_{Lfc} will be the inductor voltage during the transient period.

$$v_{Lfc} = v_L = \mathcal{E}e^{-t/\tau}. \qquad ...(175)$$

(*ii*) *Decay of current* : After the current attains steady value i_0, the battery is removed by connecting the common terminal *S* with *B*. The current will fall and the self inductance of the coil will oppose this fall of current. During this stage, if *i* current is flowing at any instant *t* in the circuit then

$$-L\frac{di}{dt} = Ri \text{ or } dt = -\frac{L}{R}\frac{di}{i},$$

On integration, we get

$$\int_{i_0}^{i}\frac{di}{i} = -\frac{R}{L}\int_0^t dt \quad \text{or} \quad \log_e\frac{i}{i_0} = -\frac{R}{L}t$$

$$\therefore \quad i = i_0 e^{-Rt/L}. \qquad ...(176)$$

Here we see that the greater the value of time constant τ, the more slowly is the decay of current. The current in a short circuited-coil decreases exponentially from i_0 to the steady state value $i_s = 0$. Thus the free component current i_{fc} = transient current $i = i_0 e^{-t/\tau}$.

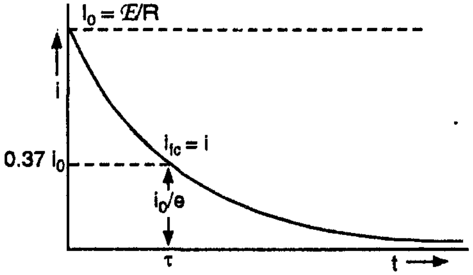


Fig. 12.44. Transient current for *LC*-circuit (source disconnected)

From above results we see that the effect of the inductance is to prevent the current from falling instantaneously to zero. If the battery is suddenly open circuited, a large induced emf $-L\ (di/dt)$may cause a spark across the point at which the circuit is broken. *With large inductances very large voltages may arise in this way which can damage the insulation, if the circuit is broken suddently.*

When the coil is switched on, the induced emf opposes the rise of current. The energy is supplied by the battery to do work against this emf. The total work done in bringing the current to its final value is stored as the magnetic field of the coil, which is taken back in to maintain the current when the current decays.

(c) LC–Circuit : Let us consider a circuit consisting of capacitor C, a inductance L and a battery E in series. The resistance of the circuit is assumed to be negligible. First of all the common terminal S of the two way key is connected with the battery terminal A, and the capacitor is charged. When the battery is removed from the circuit, the charge begins to flow from one plate to another via inductance L.

Let, at any instant t during discharge, q be the charge on the capacitor and i the current in the circuit. The *pd* across the capacitor is q/C and the induced emf set up in the inductance is $-L\ (di/dt)$. Hence the total effective emf in the circuit

$$= q/C - L\,di/dt = 0. \qquad \text{...(177)}$$

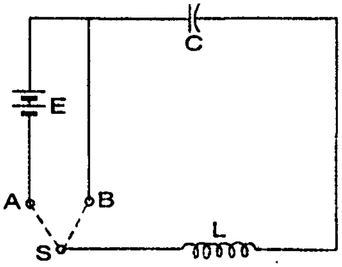


Fig. 12.45. *LC*-circuit.

As $i = -\,dq/dt$, the rate of decrease of charge, hence

$$\frac{q}{C} + L\frac{d^2q}{dt^2} = 0 \quad \text{or} \quad \frac{d^2q}{dt^2} + \frac{q}{LC} = 0$$

or $$d^2q/dt^2 + \omega^2 q = 0 \quad (\text{where } \omega^2 = 1/LC). \qquad \text{...(178)}$$

This is the linear differential equation of second order. Let its solution be

$$q = Ae^{\alpha t}, \qquad \text{...(179)}$$

where A and α are arbitrary constants, can be evaluated. Differentiating Eq. (179) with respect to t, we get

$$dq/dt = A\,\alpha\, e^{\alpha t} \quad \text{and} \quad d^2q/dt^2 = A\alpha^2 e^{\alpha t}.$$

Substituting these values in relation (178), we get

$$A\alpha^2 e^{\alpha t} + \omega^2 A e^{\alpha t} = 0 \quad \text{or} \quad A\,e^{\alpha t}(\alpha^2 + \omega^2) = 0.$$

As $A\,e^{\alpha t}$ cannot be zero, otherwise solution will be zero, hence

$$\alpha^2 + \omega^2 = 0 \quad \text{or} \quad \alpha = \pm j\omega \text{ where } j = \sqrt{(-1)}.$$

Thus we get two solutions of Eq. (178) corresponding to two values of α. Since the Eq. is linear, any linear combination of these two solutions will also be its solution. Hence the most general solution of equation (178) is

$$q = A_1 e^{j\omega t} + A_2 e^{-j\omega t}, \qquad \text{...(180)}$$

where A_1 an A_2 are arbitrary constants.

Using initial conditions : at $t = 0$, $q = q_0$ and $i = 0$, we get

$$q = q_0 = A_1 e^0 + A_2 e^0 = A_1 + A_2,$$

and $$i = dq/dt = 0 = j\omega\, A_1 e^0 - j\omega\, A_2 e^0 = j\omega\,(A_1 - A_2).$$

Solving these two, we get $A_1 = A_2 = q_0/2$

$$\therefore \quad q = \tfrac{1}{2} q_0 e^{j\omega t} + \tfrac{1}{2} q_0 e^{-j\omega t} = \tfrac{1}{2} q_0 (e^{j\omega t} + e^{-j\omega t}).$$

$$= q_0 \cos \omega t = q_0 \sin(\omega t + \pi/2). \qquad \text{...(181)}$$

This is the equation of the discharge of the capacitor through the inductance. As this equation represents a simple harmonic motion, hence the discharge of the capacitor is *oscillatory and simple harmonic*. The period time of the discharge is given by

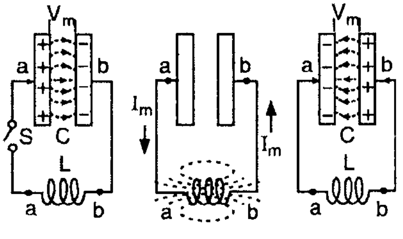
$$T = 2\pi/\omega = 2\pi\sqrt{LC},$$

and the frequency of oscillation is

$$f = \frac{1}{T} = \frac{1}{2\pi}\left(\frac{1}{LC}\right)^{1/2}. \qquad \text{...(182)}$$

It is obvious from Eq. (181) that the electrical oscillations are of constant amplitude q_0. It is the ideal case, as we have assumed zero resistance of the circuit. In actual practice, coil itself possesses a finite resistance, due to which the energy is dissipated gradually and the oscillations die away ultimately (will be discussed latter).

The production of oscillations can be explained by the Fig. 12.46. In the beginning when the capacitor is fully charged and the *pd* between the plates is V_m, the electric field is setup between the plates and the energy stored is $q_0^2/2C$. When the circuit is closed, the capacitor starts to discharge through the inductance L and causing current to flow. As the current rises from zero, it builds up a magnetic field around the inductance. At a later instant, represented by Fig. 12.46(*b*), the capacitor is completely discharged and pd between its plates is decreased to zero. The current becomes maximum I_m, the energy stored is $\frac{1}{2}LI_m^2$. At this time the whole electrical energy associated with the capacitor is completely transferred to the magnetic field energy linked with the inductance coil. The magnetic field now decreases, inducing an emf in the inductor in the same direction as the current. The current therefore persists, with diminishing magnitude, until the magnetic field is disappeared and the capacitor is again charged to the same potential. The charges so appeared on the capacitor are of opposite polarity. The process now repeats itself in the reversed direction. If there is no loss of energy, the above process continues and the process is known as *electrical oscillation.*



(*a*) (*b*) (*c*)

Fig. 12.46. Electrical oscillations in LC circuit.

At any instant the sum of the energy associated with the magnetic field of the inductor $\frac{1}{2}Li^2$ and that associated with the capacitor $q^2/2C$ is constant and is equal to the energy of the capacitor, when fully charged. *The electrical oscillations in an LC circuit can be very well compared with the oscillations of a mass spring system* (mass tied to a spring and vibrating without any friction) or of a *simple pendulum*. In the first case, Fig. 12.46(*a*), the capacitor is fully charged with one polarity and the current in the circuit is momentarily zero. Corresponding to this condition the mass is at rest and the spring at maximum compression; the bob of the pendulum is at rest and at maximum height. In the second case [Fig. 12.46(*b*)], the capacitor is discharged completely and the current in the circuit is maximum. The mechanical analog is that the spring is of its free length and the mass moving at maximum speed; the bob of the pendulum is at its equilibrium point and moving with maximum speed. Analogous to Fig. 12.46(*c*), when the current is zero and the capacitor is fully charged with opposite polarity, the mass is momentarily at rest and the spring at maximum extension; the bob of the pendulum is also at rest and at its maximum height at the opposite end of its swing.

(*d*) **LCR-Circuit :** Let us consider a circuit consisting of capacitor C, *a* coil of inductance L, a resistance R and a battery E of emf $\mathcal{E}$ in series, as shown in Fig. 12.47. When the switch

S is thrown over to A, the capacitor begins to charge. If at any instant t, i is the instantaneous value of current in the circuit and q is the charge on the capacitor, then we have effective emf in the circuit

$$\mathcal{E} - L\,di/dt - q/C = Ri.$$

As $i = dq/dt$ and $di/dt = d^2q/dt^2$, hence we have

$$\frac{d^2q}{dt^2} + \frac{R}{L}\frac{dq}{dt} + \frac{1}{LC}q = \frac{\mathcal{E}}{L}.$$

Putting $R/L = 2k$ and $1/LC = \omega^2$, we have

$$\frac{d^2q}{dt^2} + 2k\frac{dq}{dt} + \omega^2\left(q - \frac{\mathcal{E}}{L\omega^2}\right) = 0. \quad ...(183)$$

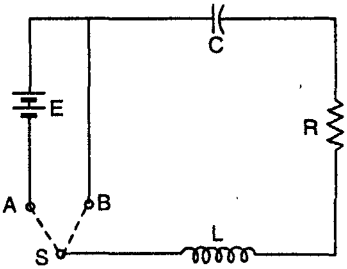


Fig. 12.47. *LCR*-series circuit.

Let us now replace quantity $q - \mathcal{E}/L\omega^2$ by a new variable x. Then we have $dq/dt = dx/dt$ and $d^2q/dt^2 = d^2x/dt^2$, hence Eq. (183) becomes

$$\frac{d^2x}{dt^2} + 2k\frac{dx}{dt} + \omega^2 x = 0. \quad ...(184)$$

Let its solution be $x = Ae^{\alpha t}$,

$$\therefore \quad dx/dt = A\alpha e^{\alpha t} \text{ and } d^2x/dt^2 = A\alpha^2 e^{\alpha t}.$$

Substituting these values in equation (184), we get

$$A\alpha^2 e^{\alpha t} + 2kA\alpha e^{\alpha t} + \omega^2 A e^{\alpha t} = 0 \quad \text{or} \quad Ae^{\alpha t}[\alpha^2 + 2k\alpha + \omega^2] = 0.$$

As $Ae^{\alpha t}$ cannot be zero otherwise solution will be zero, hence

$$\alpha^2 + 2k\alpha + \omega^2 = 0 \quad \text{or} \quad \alpha = -k \pm \sqrt{(k^2 - \omega^2)}.$$

Therefore the most general solution of Eq. (184) becomes

$$x = A_1 e^{[-k + \sqrt{(k^2-\omega^2)}]t} + A_2 e^{[-k - \sqrt{(k^2-\omega^2)}]t}.$$

As $\mathcal{E}/L\omega^2 = C\mathcal{E} = q_0$ = final steady charge on the capacitor,

$$\therefore \quad q = q_0 + A_1 e^{[-k + \sqrt{(k^2-\omega^2)}]t} + A_2 e^{[-k - \sqrt{(k^2-\omega^2)}]t}. \quad ...(185)$$

Using initial conditions : at $t = 0$, $q = 0$ and $i = dq/dt = 0$, we get

$$0 = q_0 + A_1 + A_2 \quad \text{or} \quad A_1 + A_2 = -q_0$$

and

$$0 = A_1[-k + \sqrt{(k^2-\omega^2)}] + A_2[-k - \sqrt{(k^2-\omega^2)}].$$

Combining these two equations we get

$$A_1 = -\frac{q_0}{2}\left[1 + \frac{k}{\sqrt{(k^2-\omega^2)}}\right] \text{ and } A_2 = -\frac{q_0}{2}\left[1 - \frac{k}{\sqrt{(k^2-\omega^2)}}\right]$$

$$\therefore q = q_0 - \tfrac{1}{2}q_0\left[1 + k/\sqrt{(k^2-\omega^2)}\right]e^{[-k+\sqrt{(k^2-\omega^2)}]t} - \tfrac{1}{2}q_0\left[1 - k/\sqrt{(k^2-\omega^2)}\right]e^{[-k-\sqrt{(k^2-\omega^2)}]t} \quad ...(186)$$

There are three possible cases :

(1) $k^2 > \omega^2$. In this case both the square roots are real and the charge gradually increases and acquires its steady value q_0 as shown by the lower curve in Fig. 12.48. The charge is known as *over-damped or dead beat.*

(2) $k^2 = \omega^2$. In this case the coefficients of the exponential terms of Eq. (185) becomes infinite. To simplify it let $(k^2 - \omega^2)^{1/2} = h$, where h is very small. Then Eq. (185) reduces to

$$\begin{aligned} q &= q_0 + A_1 e^{(-k+h)t} + A_2 e^{(-k-h)t} = q_0 + e^{-kt}\,[A_1(1 + ht + \ldots\ldots) + A_2(1 - ht + \ldots\ldots)] \\ &= q_0 + e^{-kt}(P + Qt), \end{aligned}$$

where $P = A_1 + A_2$ and $Q = h(A_1 - A_2)$. ...(187)

Applying initial conditions : *at* $t = 0$, $q = 0$ and $i = 0$, we have

$$0 = q_0 + P \qquad \text{or} \qquad P = -q_0$$

and $$0 = -kP + Q \qquad \text{or} \qquad Q = kP = -kq_0.$$

$$\therefore \qquad q = q_0 - q_0 e^{-kt}(1 + kt). \qquad \ldots(188)$$

Thus charge increases without any oscillations and called *critically damped.*

(3) $k^2 < \omega^2$. In this case $(k^2 - \omega^2)^{1/2}$ is imaginary. Let us put it as $j\beta$, where $\beta = (\omega^2 - k^2)^{1/2}$ Thus Eq. (186) reduces to

$$\begin{aligned} q &= q_0 - \frac{q_0}{2}\left(1 + \frac{k}{j\beta}\right)e^{(-k+j\beta)t} - \frac{q_0}{2}\left(1 - \frac{k}{j\beta}\right)e^{(-k-j\beta)t} \\ &= q_0 - \tfrac{1}{2} q_0 e^{-kt}\,[(e^{j\beta t} + e^{-j\beta t}) + (k/j\beta)(e^{j\beta t} - e^{-j\beta t})] \\ &= q_0 - q_0 e^{-kt}\,[\cos\beta t + (k/\beta)\sin\beta t]. \end{aligned}$$

Let us replace quantity k/β by $\tan\theta$, where θ is another constant.

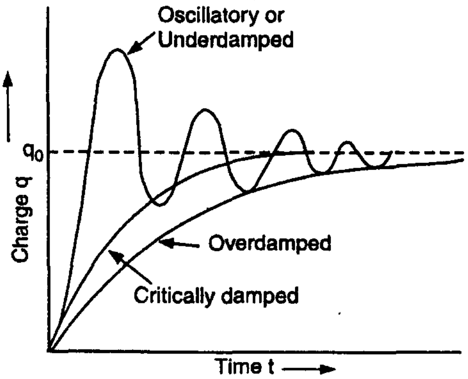


Fig. 12.48. Variation of charge q with time t in an LCR–circuit.

$$q = q_0\,[1 - (\omega/\beta)e^{-kt}\cos(\beta t - \theta)] \qquad \ldots(189)$$

This relation shows that the charge is *under damped* or *oscillatory,* as shown by the upper curve in Fig. 12.48. It fluctuates about the steady value q_0. The current at any instant t in the circuit can be obtained by differentiating Eq. (189) as

$$\begin{aligned} i &= \frac{dq}{dt} = q_0\,\frac{\omega}{\beta}\,[ke^{-kt}\cos(\beta t - \theta) + \beta e^{-kt}\sin(\beta t - \theta)] \\ &= q_0\,\frac{\omega^2}{\beta}\,e^{-kt}\,[\sin\theta\cos(\beta t - \theta) + \cos\theta\sin(\beta t - \theta)] \\ &= q_0\,\frac{\omega^2}{\beta}\,e^{-kt}\sin\beta t. \qquad \ldots(190) \end{aligned}$$

This is also the relation for *damped ocillations.*

Discharging : Consider a case when in the circuit (12.47), the capacitor is first charged and then the battery is removed from the circuit and the capacitor is allowed to discharge through the inductance and resistance. If the initial charge be q_0 and at any instant t during the discharge, let q be the charge left on the capacitor and i the current in the circuit. Thus we have

$$q/C - L\,di/dt = Ri. \qquad \ldots(191)$$

As the charge q is decreasing with time, hence $i = -dq/dt$.

Therefore, $L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{q}{C}=0$ or $\frac{d^2q}{dt^2}+\frac{R}{L}\frac{dq}{dt}+\frac{q}{LC}=0$

On assuming $R/L = 2k$ and $1/LC = \omega^2$, we get

$$\frac{d^2q}{dt^2}+2k\frac{dq}{dt}+\omega^2 q=0. \qquad ...(192)$$

As obtained earlier the most general solution is

$$q = A_1 e^{[-k+\sqrt{(k^2-\omega^2)}]t}+A_2 e^{[-k-\sqrt{[k^2-\omega^2)}]t} \qquad ...(193)$$

Applying the initial conditions : at $t = 0$, $q = q_0$ and $i = 0$, we get

$$q_0 = A_1 + A_2,$$

and

$$0 = A_1\,[-k+\sqrt{(k^2-\omega^2)}]+A_2\,[-k-\sqrt{(k^2-\omega^2)}]$$

$$\therefore \quad A_1 = \frac{q_0}{2}\left[1+\frac{k}{\sqrt{(k^2-\omega^2)}}\right] \text{ and } A_2=\frac{q_0}{2}\left[1-\frac{k}{\sqrt{(k^2-\omega^2)}}\right] \qquad ...(194)$$

Substituting these values in Eq. (193), we get

$$q=\tfrac{1}{2}q_0\,[1+k/\sqrt{(k^2-\omega^2)}]\,e^{[-k+\sqrt{(k^2-\omega^2)}]t}+\tfrac{1}{2}q_0[1-k/\sqrt{(k^2-\omega^2)}]\,e^{(-k-\sqrt{(k^2-\omega^2)})t} \qquad ...(195)$$

There are three possible cases :

(1) $k^2 > \omega^2$. $\sqrt{(k^2-\omega^2)}$ is thus real and positive. As t increases, the exponential terms decrease and become zero at $t = \infty$ (upper part of Fig. 12.49). This discharge is known as *over damped, non oscillatory or dead beat.*

(2) $k^2 = \omega^2$. As the coefficients of the exponential terms approach infinity, charge can be calculated as done earlier and we get

$$q = q_0\,(1+kt)\,e^{-kt}, \qquad ...(196)$$

which shows the non-oscillatory discharge. This discharge is known as *critically damped.*

(3) $k^2 < \omega^2$. In this case $(k^2-\omega^2)^{1/2}$ is imaginary. Let

$(k^2-\omega^2)^{1/2} = j\,(\omega^2-k^2)^{1/2} = j\beta$. Thus Eq. (195) reduces to

$$q=\frac{q_0}{2}\left(1+\frac{k}{j\beta}\right)e^{(-k+j\beta)t}+\frac{q_0}{2}\left(1-\frac{k}{j\beta}\right)e^{(-k-j\beta)t}$$

$$= q_0e^{-kt}\,[\cos\beta t+(k/\beta)\sin\beta t].$$

Putting $k/\beta = \tan\theta$ and solving, we get

$$q = q_0(\omega/\beta)\,e^{-kt}\cos(\beta t-\theta). \qquad ...(197)$$

This is the equation of the damped oscillations, hence the discharge is *damped oscillatory*. Thus the condition for oscillatory discharge is

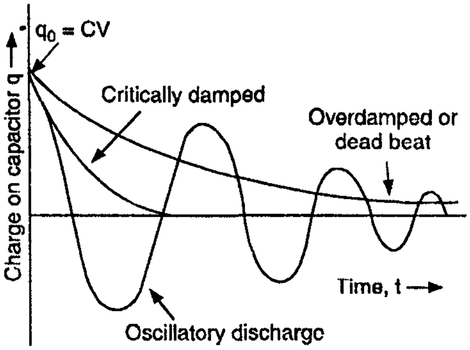


Fig. 12.49. Variation of q with time t in LCR-circuit (discharging).

$$k^2<\omega^2 \text{ or } \frac{R^2}{4L^2}<\frac{1}{LC} \quad \text{or} \quad R<2\left(\frac{L}{C}\right)^{1/2}.$$

The frequency of the discharge

$$f = \frac{1}{T}=\frac{\beta}{2\pi}=\frac{1}{2\pi}\left(\frac{1}{LC}-\frac{R^2}{4L^2}\right)^{1/2} \qquad ...(198)$$

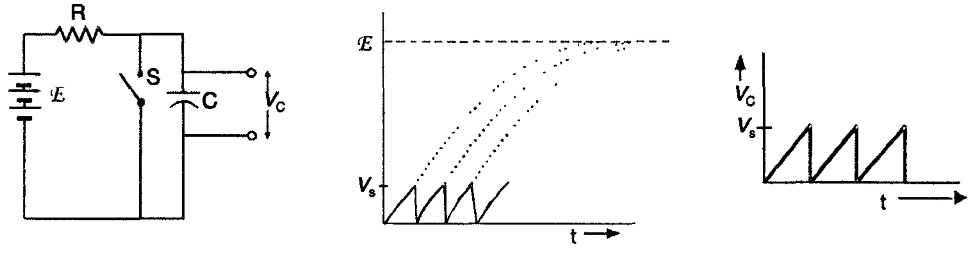
If R is very small

$$f = \frac{1}{2\pi}\left(\frac{1}{LC}\right)^{1/2}. \qquad ...(199)$$

With the proper electronic arrangement, energy can be fed into a R-L-C circuit at the same rate at which it is dissipated from resistor, (*i.e.*, radiation losses i^2R). In effect a negative resistance is inserted in the circuit so that its total resistance is zero and the circuit thus oscillates as the ideal L-C circuit and thus emitting electromagnetic waves continuously.

12.26. LINEAR TIME BASE CIRCUIT (SWEEP CIRCUIT)

Simple RC circuits have many important applications. The basic circuit for generating a *linear time base,* required in *CRO* and *TV* tubes is shown in Fig. 12.50. It is also called *sweep circuit.* It consists of steady voltage source E, a resistor R, capacitor C and a switch S. The switch has a property that it automatically closes when the potential difference between its terminals exceeds some preset value (v_s) and remains closed for a very small time interval Δt (1 to 10ps).



(*a*) (*b*) (*c*)

Fig. 12.50. Linear time base generating circuit and output.

Let $v_C = 0$ initially. It increases as time progresses. Since the switch is placed parallel to C, the potential difference between its terminals is also v_C. Thus the switch is initially open, v_C will increase with time exponentially with time constant $\tau = RC$. As soon as v_C reaches the preset value v_s ($<< \mathcal{E}$), the switch (which is an electronic circuit) closes allowing the capacitor voltage to drop drastically. The time constant for the decay of v_C is $\tau' = R'C$, where R' is the resistance of the switch which is extremely low ($\sim\mu\Omega$). If the switch stays closed for time Δt ($= n\tau'$). In this interval v_C decays from v_s to $(0.37)^n\, v_s$, which is effectively zero. The switch opens after an interval of Δt and v_C again rises exponentially. As soon as v_C reaches v_s, the switch closes once again. The process repeats and the output voltage v_C follows the sawtooth pattern [Fig. 12.50(*b*)]. Fig. 12.50(*c*) represents the sawtooth pattern on an expanded scale. The time dependence of v_C is nearly linear as it is a small part of the exponential curve. When $v_C = 0$, the current $i = E/R$ just before S closes, $v_C = v_s$ and the current $i = (E - v_s)/R$. Since $v_s << E$, the current in the circuit changes by nearly 1 percent during this time. At this stage charge on the capacitor $q = Cv_s$. Since the charging current is E/R, the time needed to accumulate a charge q on the capacitor is $T = q/i = Cv_sR/E$. This is of the order of *ms* and is very long compared with the time Δt. The time T is known as the *sweep time* for the circuit.

12.27 MEASUREMENT OF HIGH RESISTANCE BY LEAKAGE METHOD

If a capacitor of capacitance C and initial charge q_0 is connected across a resistance R, the charge leaks through the resistance. The charge remaining on the capacitor at any instant t is given by $q = q_0e^{-t/RC}$

or
$$R = -\frac{t}{C}\frac{1}{\log_e(q/q_0)} = \frac{t}{C}\frac{1}{\log_e(q_0/q)}. \qquad ...(200)$$

Resistance R can be found by first charging a known capacitor to a charge q_0 and then by allowing this charge to leak through the resistance R for a known time t. The ratio q/q_0 is determined experimentally by a ballistic galvanometer.

If the capacitor is given a charge q_0 and then immediately discharged through a ballistic galvanometer, then $q_0 \propto \theta_1$, where θ_1 is the first throw of the galvanometer.

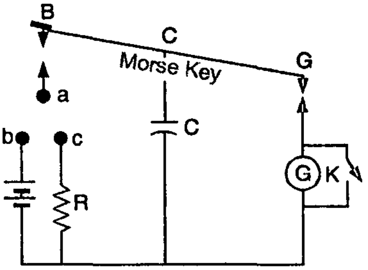


Fig. 12.51. Circuit diagram for leakage method.

Next the capacitor is given the same charge q_0 and then after being allowed the charge to leak through a resistance R for a time t, it is discharged through the same ballistic galvanometer. If ϕ_1 is the first throw of the galvanometer in this case we have $q \propto \phi_1$

$$\therefore \qquad q_0/q = \theta_1/\phi_1.$$

and
$$R = \frac{t}{C}\cdot\frac{1}{\log_e(\theta_1/\phi_1)} = \frac{t}{2.3026\,C\log_{10}(\theta_1/\phi_1)} \qquad ...(201)$$

For the purpose of charging and discharging of a capacitor Morse key is used. The electrical connections are made as shown in Fig. 12.51. By joining *a* and *b*, and pressing the Morse key the capacitor C is first charged to a constant *pd*. When the capacitor is fully charged, Morse key is released so that the whole of the charge will go to ballistic galvanometer. The first throw, θ_1 is thus noted.

Again the capacitor is charged to the same potential by pressing the Morse key, then keeping the knob B pressed, plug from gap *ab* is inserted in the gap *ac* and the charge is allowed to leak through the resistance R, say for 5 or 10 second. Now the Morse key is released and the first throw ϕ_1 is noted. Resistance R is now calculated by Eq. (200)

In practice, capacitor is slightly defective, has a self leakage resistance due to imperfection of its dielectric or of its insulated mounting. This self leakage resistance R_2 can be determined by allowing the charge to leak without the external resistance. The resistance determined in the first part of the experiment is really the actual resistance R_1 in parallel with R_2. Since $1/R = 1/R_1 + 1/R_2$, the value of R_1 can thus be determined.

12.28 COMPARISON OF CAPACITANCES, DE SAUTY'S DC METHOD

Two capacitors whose capacitances C_1 and C_2 are to be compared form the two adjacent arms *AB* and *AD* of a Wheatstone bridge. The remaining two arms BC and DC consist of non inductive variable high resistance R_1 and R_2 respectively.

A high resistance galvanometer *G* is inserted between *B* and *D* and a battery of high emf between *A* and *C* through a two way key or a Morse Key. The resistances R_1 and R_2 are so arranged that there is no deflection of the galvanometer on moving the connection of *a* to *b* or *c*. In both cases,at balance, the *pd* across the galvanometer will then be zero. The v_1 and v_2 the *pds* across the capacitors C_1 and C_2 respectively at any instant *t* can be given by equations.

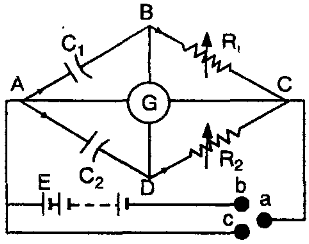


Fig. 12.52. De-Sauty's dc method for C_1/C_2.

$$v_1 = \frac{q_1}{C_1} = \frac{(q_0)_1}{C_1} e^{-t/R_1C_1} = v_0 e^{-t/R_1C_1}$$

and
$$v_2 = \frac{q_2}{C_2} = \frac{(q_0)_2}{C_2} e^{-t/R_2C_2} = v_0 e^{-t/R_2C_2}$$

At the time of balance, *B* and *D* are at the same potential at any instant *t*, hence

$$v_1 = v_2 \quad \text{or} \quad R_1C_1 = R_2C_2$$

or
$$C_1/C_2 = R_2/R_1 \qquad \text{...(201)}$$

The same condition for the balancing of the bridge would also result if the process of charging of the capacitors is considered.

EXERCISES

Example 1. *The magnetic flux through the coil perpendicular to its plane and directed into the paper is varying according to the relation*

$$\Phi = (4t^2 + 5t + 12) \text{ milliwebers.}$$

Calculate, (a) the magnitude of the emf induced in the loop when time t = 3 sec., and (b) what is the direction of the current through the resistance R ?

We know that the emf induced at any instant *t* is

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(4t^2 + 5t + 12) \times 10^{-3}$$

As $t = 3$ sec, $\quad \mathcal{E} = -10^{-3} \times (8t + 5]_{t=3} = -29$ millivolt.

As time *t* increases the flux through the loop increases. The –ve sign above shows that the direction of induced emf is such as to decrease the flux through the loop. It is possible when current passes along anticlockwise direction in the coil and hence from left to right through the resistor *R*.

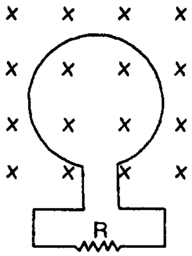


Fig. 12.53. Example 1.

Example 2. *A Cu-rod moves with a constant velocity of 200 cm/sec parallel to a long straight wire carrying a current of 10 amperes. Calculate the induced emf in the rod, if the ends of the rod from the wire are at distances 8 and 32 cms.*

Consider a small element of length *dx* of the rod at a distance *x* from the wire. The magnetic induction at this element due to the current *i* in wire is $B = \mu_0 i/2\pi x$.

As element dx moves a distance v per sec, hence the rate of change of flux will be $Bvdx$. According to the Faraday's law of induction,

Emf induced in this element $d\mathcal{E} = Bvdx$.

$$\therefore \quad \text{Total emf } \mathcal{E} = \int_a^b Bvdx = \int_a^b \frac{\mu_0}{2\pi} iv \frac{dx}{x} = \frac{\mu_0}{2\pi} iv \log_e \frac{b}{a}$$

$$= 2 \times 10^{-7} \times 10 \times 2 \times \log_e 4 = 5.55 \times 10^{-6} \text{ volts.}$$

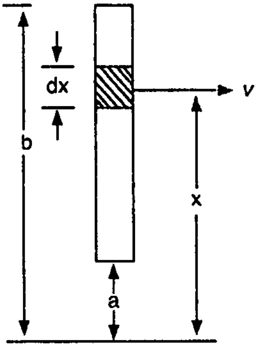


Fig. 12.54. Example 2.

Example 3. *Find the emf induced in a rectangular loop due to a current carrying long wire placed in the plane of the loop.*

Consider a long straight wire carrying current $I(t)$ in the plane of a rectangular loop with sides l and b. The emf induced around a rectangular loop C of area S can be found from the Faradays's law of induction.

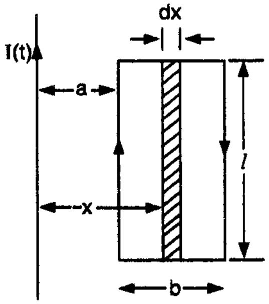


Fig. 12.55. Wire and a rectangular loop.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S}$$

To find the integral, consider a strip of thickness dx at a distance x from the wire. The magnetic field may be assumed to be constant at every point of this strip. Thus the magnetic field

$$B_x = \mu_0 I(t)/2\pi x$$

$$\therefore \quad \mathcal{E} = -\frac{d}{dt}\left[\int \frac{\mu_0 I(t)}{2\pi x} dx\, l\right] = -\frac{d}{dt}\left[\frac{\mu_0 l\, I(t)}{2\pi}\int_a^{a+b}\frac{dx}{x}\right]$$

$$= -\frac{\mu_0 l}{2\pi}\log_e\left(\frac{a+b}{a}\right).\frac{dI(t)}{dt}.$$

Example 4. *A metal disc of radius R rotates with an angular velocity ω, about an axis perpendicular to the plane of the disc and passing through its center. If a uniform magnetic field **B** is acting along this axis, find the value of emf induced between the rim and the center of the disc.*

Consider the radial segment of length dr at a distance r from the center of the disc, rotating with velocity ω. If at any instant t this element makes an angle θ with y-axis, then in time dt, this segment will rotate through an angle $d\theta$.

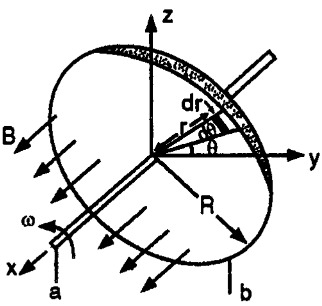


Fig. 12.56. Example 4.

$\therefore$ Area swept by this segment in time $dt = dA = rd\theta.\, dr$

and the flux through this area $d\Phi = \mathbf{B}\cdot d\mathbf{A} = BdA = Brdrd\theta$.

$\therefore$ Total change of flux in time dt due to the rotation of the disc

$$d\Phi' = \int_0^R Brdrd\theta = \tfrac{1}{2}BR^2 d\theta.$$

Hence the induced emf between the points $r = 0$ and $r = R$

$$\mathcal{E} = d\Phi'/dt = \tfrac{1}{2}BR^2\,(d\theta/dt) = \tfrac{1}{2}BR^2\omega.$$

If a wire is externally connected between the rim and the centre of the disc, current will flow in the wire due to this emf. This is the principle of an induction generator. Such a disc is called Faraday disc.

Example 5. *A stiff wire bent into a semicircle of radius R is rotated at a frequency f in a uniform magnetic field* **B**. *Calculate the amplitude and frequency of the induced voltage. If this circuit has negligible resistance and the internal resistance of the ammeter is 1000Ω, calculate the amplitude of the induced current.*

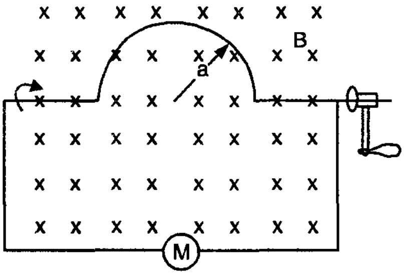


Fig. 12.57. Example 5.

Let the magnetic field **B** be perpendicular to the plane of paper. If at any instant t, the plane of the rotating semicircle makes an angle θ with the direction of **B**, then flux

$$\Phi = \mathbf{B}\cdot\mathbf{A} = B\left(\tfrac{1}{2}\pi a^2\right)\cos\theta$$

$$= \tfrac{1}{2}\pi a^2 B\cos\omega t = \tfrac{1}{2}\pi a^2 B\cos 2\pi ft.$$

As induced emf

$$\mathcal{E} = -d\Phi/dt = \pi^2a^2Bf\sin 2\pi ft.$$

∴ Amplitude of the induced emf $= \pi^2a^2Bf.$

As induced current $= \mathcal{E}/R = [\pi^2a^2Bf\sin 2\pi ft]/1000.$

∴ Amplitude of the induced current $= \pi^2a^2Bf \times 10^{-3}$ amp.

Example 6. *A Cu-rod of length L rotates at angular velocity ω in a uniform magnetic field* **B**. *Evaluate the emf developed between two ends of the rod.*

Consider a short portion of this rod of length dl at a distance l from center O. Its velocity $v = \omega l$. Since **v** is perpendicular to **B**, which is ⊥ to the plane of paper, hence a force **F** will act upon this element dl along its length and will produce an electric field **E** along it.

Electric field $\mathbf{E} = \mathbf{v}\times\mathbf{B},$

or $E = vB = \omega lB.$

This electric field will produce an emf

$$\mathcal{E} = \int \mathbf{E}\cdot d\mathbf{l} = \int E\,dl$$

$$= \omega B\int_0^L idl = \tfrac{1}{2}\omega BL^2.$$

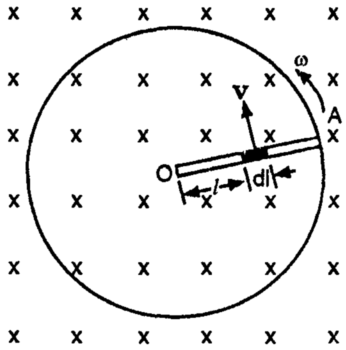


Fig. 12.58. Example 6.

Example 7. *A rectangular loop of wire is placed in a uniform magnetic field* **B** *acting normally to the plane of the loop. If we attempt to put it out of the field with velocity v, calculate the power required for this purpose.*

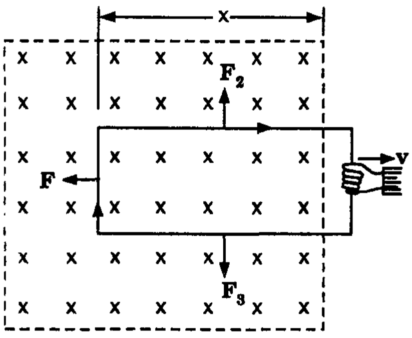


Fig. 12.59. Example 7.

If l is the width of the loop and a length x is with in the field region. The flux linked with the portion of the loop which is with in the field is

$$\Phi = \mathbf{B}\cdot\mathbf{A} = BA = Blx.$$

Due to the motion of the loop, the change in flux in time dt will be $d\Phi = Bldx$. Hence the induced emf.

$$\mathcal{E} = -d\Phi/dt = -Bldx/dt = Blv.$$

This induced emf sets up a current i in the loop. If R is the resistance of the loop, the current

$$i = \mathcal{E}/R = Blv/R.$$

From Lenz's law this current must be clockwise. It produces a force on each arm of the loop. Forces $\mathbf{F}_3$ and $\mathbf{F}_2$ are equal and opposite and having same line of action. Hence the net force is $\mathbf{F}_1$, where

$$\mathbf{F}_1 = i\boldsymbol{l} \times \boldsymbol{B} \text{ or } F_1 = ilB = B^2l^2v/R.$$

$\therefore$ Work done for the displacement dx against this force $= F_1 dx = (B^2l^2v/R)\, dx$

or the rate of doing work $= \dfrac{dW}{dt} = \dfrac{B^2l^2v}{R}\dfrac{dx}{dt} = \dfrac{B^2l^2v^2}{R}$.

It is the power required for the motion.

Example 8. *A metal rod OA, length a and mass m, can rotate about a horizontal axis O. It slides along a circular conducting ring of radius a in a uniform magnetic field B directed perpendicular to the plane of the ring. The axis O and the rim of the ring are connected to a source of emf E_0 to form a non-inductive closed circuit of resistance R. Find an expression for the emf of the source required to rotate the rod with constant angular velocity.*

As proved in the example 6, the induced emf in the rod $\mathcal{E} = \frac{1}{2}a^2\omega B$.

Since emf of the source is E_0, hence the net emf in the circuit

$$E = E_0 - \tfrac{1}{2}a^2\omega B.$$

$\therefore$ Current in the circuit $E/R = \left(E_0 - \frac{1}{2}a^2\omega B\right)/R$.

The magnetic force experienced by the rod $F_m = i\ a\ B$. It will be in the direction perpendicular to **B** and the length of the rod **a**, and acted at the center of the rod.

The gravitational force $F_g = mg$, acting in vertically downward direction through the mid point of the rod.

For the steady motion, the torque about O should be zero, hence

$$\frac{1}{R}\left(E_0 - \tfrac{1}{2}a^2\omega B\right) aB \times \frac{a}{2} = mg \times \frac{a}{2} \sin \omega t,$$

where ωt is the angle through which the rod rotates in time t sec, assuming the initial position of A at the y-axis.

$$\therefore \quad E_0 = \tfrac{1}{2}a^2\,\omega B + \frac{mg\,R\sin\omega t}{aB} = \frac{1}{2aB}[a^3\omega B^2 + 2mg\,R\sin\omega t].$$

Example 9. *A square wire of length l, mass m and resistance R slides down without friction over two parallel conducting rails of negligible resistance. The rails are connected to each other at the bottom by a resistanceless rails parallel to the wire, so that a closed loop is formed. The plane of the rails makes an angle θ with the horizontal and a uniform vertical field of magnetic induction **B** exists throughout the region.*

(a) Show that the wire acquires a steady state velocity of magnitude
$v = (mgR/B^2l^2)\tan\theta\sec\theta$.

(b) Show the validity of the principle of conservation of energy.

When the wire slides down, the area of the loop decreases and induced current is set up in this circuit. The induced emf is set up in this circuit. The induced emf

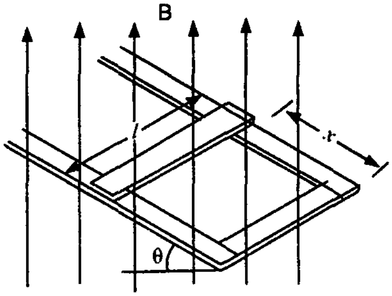


Fig. 12.60. Example 9.

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(\mathbf{B}\cdot\mathbf{A}) = \frac{d}{dt}(B.lx.\cos\theta)$$

$$= Bl\cos\theta\,(dx/dt).$$

∴ The induced current

$$i = \frac{\mathcal{E}}{R} = \frac{Bl\cos\theta}{R}\frac{dx}{dt} = \frac{Bl\cos\theta}{R}v.$$

The direction of current is according to the Fleming's right hand rule and is such as to increase the flux.

The force experienced by the square wire is

$$\mathbf{F} = i\mathbf{l}\times\mathbf{B} \text{ or } F = ilB.$$

The component of force along the rails = $F_H = Bil\cos\theta$.

At the steady state this force is balanced by the gravitational force along the rails, hence

$$B(Bvl\cos\theta/R)\,l\cos\theta = mg\sin\theta,$$

or

$$v = mgR\sin\theta/B^2l^2\cos^2\theta = (mg\,R/B^2l^2)\tan\theta\sec\theta.$$

For the second part of question, the rate of work done on the wire

$$P_1 = \mathbf{F}\cdot\mathbf{v} = Bilv\cos\theta = B^2l^2v^2\cos^2\theta/R.$$

The rate at which the joule heat is produced in the resistor

$$P_2 = i^2R = (B^2l^2\cos^2\theta v^2/R^2)R = B^2l^2v^2\cos^2\theta/R.$$

Hence, we get $P_1 = P_2$ or the principle of conservation of energy is valid for this problem.

Example 10. *A railway line 1.2 m wide runs along the magnetic meridian. The vertical component of the Earth's magnetic field is 0.5 oersted. Calculate the emf in volts that will exist between the rails when a train runs on the line at a speed of 60 km/hr.*

The induced emf $\mathcal{E}$ due to the rate of change of flux is given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(\mathbf{B}\cdot\mathbf{A}) = Blv\cos\theta.$$

In the present problem, rails are in magnetic meridian and magnetic field B is the vertical component of the earth's magnetic field, thus θ between **B** and **A** is zero and

$$\mathcal{E} = B_v lv.$$

Substituting the numerical values,

$$B_v = 0.5 \text{ oersted} = 0.5\times10^{-4} \text{ weber/meter}^2$$

$$\text{length } l = \text{distance between the rails} = 1.2 \text{ m}$$

andvelocity of the train $v = 60$ *km/hr* $= 60\times1000/60\times60 = 16.66$ m/sec,

∴ we get,

$$\mathcal{E} = 0.5\times10^{-4}\times1.2\times16.66 = 10^{-3} \text{ volt}.$$

Example 11. *A ballistic galvanometer of 2.2 × 10^{-9} ampere per cm current sensitivity and 10 sec periodic time is connected in a circuit of total resistance 500 ohms. Find the deflection, in cm, produced in the galvanometer if the magnetic flux linked with the circuit changes by 1000 emu.*

The charge q due to a change in magnetic flux, through the galvanometer is given as

$$q = \frac{\text{Change in magnetic flux } \Phi}{\text{Resistance } R} = \frac{1000 \text{ emu}}{500 \text{ ohm}}$$

$$= 1000 \times 10^{-8} \text{ weber}/500 \text{ ohm} = 2 \times 10^{-8} \text{ coulomb}.$$

The charge-sensitivity (charge per unit deflection, *i.e.*, q/θ) is related with the current sensitivity as

$$\text{Charge sensitivity } \frac{q}{\theta} = \frac{T}{2\pi} \times \text{current sensitivity } (S_i)$$

$$\therefore \quad q = \theta T S_i/2\pi \quad \text{or} \quad \theta = 2\pi q/TS_i.$$

Substituting numerical values, we get

$$\theta = 2 \times 3.14 \times 2 \times 10^{-8}/10 \times 2.2 \times 10^{-9} = 5.7 \text{ cm}.$$

Example 12. *Two infinite parallel wires separated by a distance a carry equal currents i in opposite directions. A square loop of wire of length a on a side lies in the plane of the wires at a distance a from one of the parallel wires. If the current i is increasing at the rate di/dt, find (a) the emf induced in the square loop. (b) Is the induced current clockwise or counter clock wise ?*

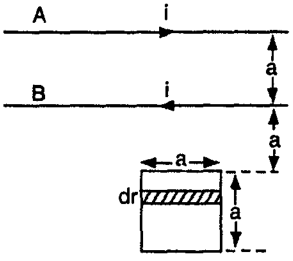


Fig. 12.61. Example 12.

The magentic field produced by an infinite straight current carrying wire A at a distance r from the wire is given by

$$B = \mu_0 i/2\pi r.$$

It is perpendicular to the wire directing into the page on the right side. The magnetic flux passing through the loop due to this field is

$$\Phi_1 = \int_{2a}^{3a} \frac{\mu_0 i}{2\pi r}(a \cdot dr) = \frac{\mu_0 ia}{2\pi} \log_e \frac{3a}{2a} = (\mu_0\, ia/2\pi) \log_e (3/2)$$

The flux due to the current in another wire B which is nearer the loop is given by

$$\Phi_2 = \int_{a}^{2a} \frac{\mu_0 i}{2\pi r}(a \cdot dr) = \frac{\mu_0 ia}{2\pi} \log_e \frac{2a}{a} = \frac{\mu_0 ia}{2\pi} \log_e 2.$$

It is perpendicular, but pointing out from the page. Hence the net flux

$$\Phi = \Phi_2 - \Phi_1 = \frac{\mu_0 ia}{2\pi}[\log_e 2 - \log_e \tfrac{3}{2}]$$

$$= (\mu_0 ia/2\pi) \log_e (4/3).$$

$$\therefore \text{ Induced emf in the square } \mathcal{E} = -\frac{d\Phi}{dt} = -\frac{\mu_0 a}{2\pi} \log_e \left(\frac{4}{3}\right) \frac{di}{dt}.$$

This induced emf or the induced current will oppose the change in magnetic flux. Therefore the magnetic field produced by the induced current will direct into the page. Thus by the right hand rule the induced current is clockwise.

Example 13. *A wire of infinite length carrying a current i is placed along + z-direction. A rectangular loop of wire of side a is connected to a voltmeter and moves with velocity v radially away from the wire. Calculate the voltmeter reading with direction.*

The magnetic field at a point of radial distance r due to an infinite wire

$$B = \mu_0 i/2\pi r.$$

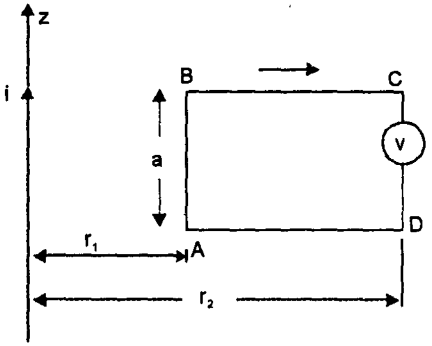


Fig. 12.62. Example 13.

Its direction is perpendicular to and pointing into the paper.

The induced emf in the rectangular loop is

$$\mathcal{E} = \oint (\mathbf{v} \times \mathbf{B}) . d\mathbf{l} = \oint v \frac{\mu_0 i}{2\pi r} \mathbf{a}_z . d\mathbf{l}$$

$$= \int_A^B \frac{\mu_0 v i}{2\pi r_1} dl + 0 + \int_C^D \frac{\mu_0 v i}{2\pi r_2} (-dl) + 0$$

$$= \frac{\mu_0 v i}{2\pi} a \left(\frac{1}{r_1} - \frac{1}{r_2} \right).$$

As $r_1 < r_2$, therefore $\mathcal{E}$ is positive, and terminal C is positive.

Example 14. *A fan blade of length 2a rotates with frequency f cycles per second perpendicular to a magnetic field* **B**. *Find the potential difference between the centre and end of the blade.*

Let us imagine the dotted circle along which blade slides. As the contact at point D slides down along v, the area A of the loop ODC increases, we thus have

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(BA) = -B\frac{dA}{dt}.$$

In one revolution (time = time period = $1/f$) blade sweeps out an area of πa^2.

$$\therefore \qquad \mathcal{E} = -B\frac{\pi a^2}{1/f} = -\pi B a^2 f.$$

This is the induced emf in the loop and is also the *pd* between O and D.

Same result can be obtained as :

$$V_D - V_O = -\int_0^D E_r dr = -\int_0^a vBdr = -\int_0^a r\omega B dr$$

$$= -\int_0^a 2\pi r f \, B dr = -\pi B a^2 f.$$

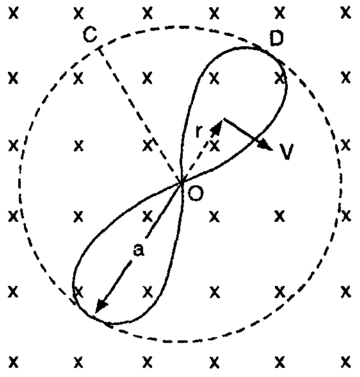


Fig. 12.63. Example 14.

It is very easy to understand that no potential difference exists between the two ends of the blade.

Example 15. *Find an expression for the induced emf produced between the ends of a coil of radius r having N turns and placed in a uniform magnetic field $B = B_0 \sin \omega t$ making an angle θ with the normal to the plane of the coil.*

The induced emf produced between the open ends of the coil of one turn is given by

$$\mathcal{E} = -\frac{\partial}{\partial t}\int \mathbf{B} \cdot d\mathbf{S} = -\frac{\partial}{\partial t}[(B_0 \sin \omega t) \pi r^2 \cos\theta)]$$

$$= -\pi r^2 \omega B_0 \cos\theta \cos\omega t.$$

Here the integral is evaluated by taking surface as within the periphery of the coil. For a coil of N turns, the net result is roughly equivalent to the notion that each turn is separately linked by the magnetic flux. In each turn the induced voltage is $\mathcal{E}$. These voltages are in series, hence the total emf across the ends of a coil of N turns is N times greater and given by

$$\mathcal{E}_{total} = N\mathcal{E} = -N\pi r^2\, \omega B_0 \cos\theta \cos\omega t.$$

Example 16. *In a certain betatron the maximum magnetic field was 4000 gauss, operating at 50 cycles/sec with a stable orbit diameter of 60 inches. Calculate the average energy gained per revolution and the final energy of the electrons.*

If B is the magnetic field at the end of the accelerating cycle at the orbit of radius R, the final energy

$$\begin{aligned} E &= BeRc = 4000 \times 10^{-4} \times 1.602 \times 10^{-19} \times 30 \times 2.54 \times 10^{-2} \times 3 \times 10^{8} \\ &= 91.4\, MeV. \end{aligned}$$

Total number of revolutions

$$N = \frac{c}{4\omega R} = \frac{3 \times 10^{8}}{4 \times 2 \times 3.14 \times 50 \times 30 \times 2.54 \times 10^{-2}} = 3.13 \times 10^{5}.$$

$\therefore$ Average energy gained per revolution

$$= 91.4 \times 10^{6}/3.13 \times 10^{5} = 291.7\, eV.$$

Example 17. *You are equipped with current sources and a machine shop for constructing simple linear electric components such as coils, capacitors, resistors and inductors. You have instruments to measure mechanical forces but not electrical meters. Devise an experiment to measure the ampere using the given equipment and your knowledge of the basic equations of electricity and magnetism.*

Two identical circular coils are made and arranged coaxially under a right pan of a balance to construct the Ampere's current balance. The mutual inductance between the coils is $M_{12}(z)$. The same current from a current source is allowed to pass in the coils. The force F_{12} between the coils can be measured with standard weights on the other pan. The magnetic energy stored in the coils is

$$W_{12} = M_{12}\, i_1 i_2 = M_{12} i^2 (\text{same current } i_1 = i_2 = i)$$

$$\therefore \text{ Interacting force } F_{12} = \frac{\partial W_{12}}{\partial z} = i^2\, \frac{\partial M_{12}(z)}{\partial z}.$$

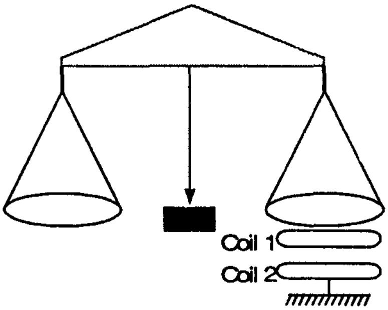


Fig. 12.64. Example 17.

Using the value of the force measured with the Ampere's balance and calculating $\partial M_{12}(z)/\partial z$, i can be measured, which is in amperes in *MKSA* unit of system.

Example 18. *Calculate the self inductance of the thin wire of circular cross section and bent into a curve whose radius of curvature is very large compared with the radius of cross section of the wire.*

The magnetic field at any point P on a surface rimmed by the wire may be calculated on the assumption that the current is concentrated along the axis of the wire. Hence if M is the mutual inductance between the circuits C and C_1 (the line of centres of the cross-section and the inner boundary of the wire), the flux through C_1 when unit current passes through the cross-section of the wire is approximately M.

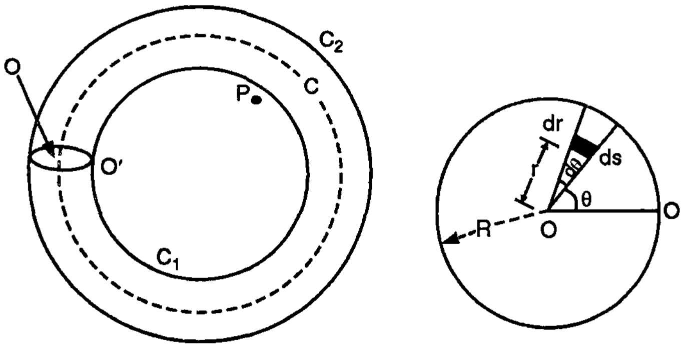


Fig. 12.65. Example 18.

The additional flux through C_1 and the filaments through an element ds of a cross sectional area is $l \int_r^R B dr$, where l is the length of the wire and R the radius of its cross section. Due to the unit current, distributed uniformly over its cross section, the magnetic field is given by

$$B = \mu_0 r/2\pi R^2.$$

$\therefore$ Total flux $\Phi =$

$$M + l\int_r^R \frac{\mu_0 r}{2\pi R^2}\, dr = M + \frac{\mu_0}{4\pi}\, l\, \frac{R^2 - r^2}{R^2}.$$

The self inductance of the wire $L = \iint \Phi di = \iint \Phi (ds/\pi R^2)$

$$= \iint \left(M + \frac{\mu_0}{4\pi} l - \frac{\mu_0}{4\pi} l \frac{r^2}{R^2} \right) \frac{ds}{\pi R^2} = M + \frac{\mu_0}{4\pi} l - \frac{\mu_0 l}{4\pi^2 R^4} \iint r^2 ds.$$

As $$\iint r^2 ds = \int_0^R \int_0^{2\pi} r^2 .dr.r d\theta = \int_0^R 2\pi r^3\, dr = \frac{\pi R^4}{2},$$

$$\therefore \qquad L = M + \frac{\mu_0}{4\pi} l - \frac{\mu_0 l}{4\pi^2 R^4} \frac{\pi R^4}{2} = M + \frac{\mu_0}{8\pi} l.$$

Example 19. *Calculate the mutual inductance between two plane coaxial coils placed at a distance d apart.*

Let us consider two coils P and Q of radii a and b and total number of turns n_1 and n_2 respectively. If the centres are at a distance d apart, where $d >> a, b$. The magnetic field at the centre of Q due to a current i in coil P is

$$B = \mu_0 a^2 n_1 i/2d^3.$$

As $d >> a, b$, hence the field can be assumed uniform therefore,

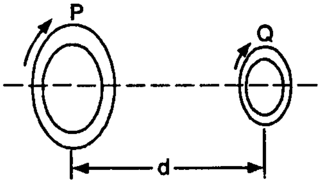


Fig. 12.66. Example 19.

Total flux through coil $Q = n_2 B.\ \pi b^2 = \mu_0 \pi a^2 b^2 n_1 n_2 i/2d^3$

Thus the mutual inductance $M = \Phi/i = \mu_0 \pi a^2 b^2 n_1 n_2/2d^3$.

Example 20. *A straight solenoid has 50 turns per cm in the primary and 200 turns in the secondary. The area of cross-section of the solenoid is 4 cm². Calculate the mutual inductance.*

The magnetic field at any point inside the straight solenoid of primary with n_1 turns per unit length carrying a current i_1 is given by the relation

$$B = \mu_0 n_1 i_1.$$

The magnetic flux through the secondary of n_2 turns each of area A is given as

$$\Phi = \mu_0 n_1 i_1 n_2 A.$$

If M is the mutual inductance of the solenoid, then

$$\Phi = M i_1$$

$$\therefore \qquad M = \mu_0 n_1 n_2 A.$$

In the present problem, $\mu = 4\pi \times 10^{-7}$ henry/*m*. $n_1 = 50$ turns/cm = 5000 turns/*m*, $n_2 = 200$ turns, $A = 4\text{ cm}^2 = 4 \times 10^{-4}\text{ m}^2$.

$$\therefore \quad M = 4\pi \times 10^{-7} \times 5000 \times 200 \times 4 \times 10^{-4} = 5.024 \times 10^{-4}\text{ henry.}$$

Example 21. *The total inductance of the two series connected coils equal 1.52 mH. On reversing the current in one of the coils, the total inductance is reduced to 0.88 mH. Find the mutual inductance of the coils.*

Total inductance of the coils in series is given by

$$L = L_1 + L_2 \pm 2M.$$

At an aiding connection of the coils, we have

$$L_{\text{aid}} = L_1 + L_2 + 2M$$

and at an oppositing connection

$$L_{opp} = L_1 + L_2 - 2M.$$

$$\therefore \quad L_{aid} - L_{opp} = 4M.$$

or

$$M = \frac{L_{aid} - L_{opp}}{4} = \frac{1.52 - 0.88}{4} = 0.16\ mH.$$

Example 22. *A coil of N turns with radius a, resistance R and self inductance L rotates in a uniform magnetic field B about a diameter perpendicular to the field. (a) Find the expression for current in the coil in terms of θ at a constant angular velcoity ω, where θ(t) = ωt is the angle between the plane of the coil and B-direction. (b) Find the external torque required to maintain this uniform rotation.*

The emf induced in the coil is given by

$$\mathcal{E} = -\frac{d}{dt}\int_S \mathbf{B}\cdot d\mathbf{S} = -\frac{d}{dt}\int B\sin\omega t\,dS$$

$$= -\frac{d}{dt}[\pi a^2 NB\sin\omega t]$$

$$= = -\pi a^2\omega NB\cos\omega t.$$

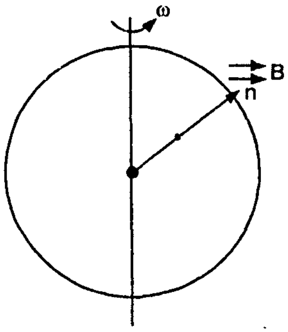


Fig. 12.67. Example 22.

The current in the circuit is given by

$$L\,(dI/dt) + IR = \mathcal{E}$$

Let $I = I_0 e^{i\omega t}$, we get $dI/dt = I_0\, i\omega e^{i\omega t} = i\omega I$.

$$\therefore \quad I_0 = -\frac{\pi a^2\omega NB}{i\omega L + R} = \frac{\pi a^2\omega NB}{(\omega^2L^2 + R^2)^{1/2}}\,e^{-i(\pi/2+\phi)}$$

where $\phi = \tan^{-1}(\omega L/R)$.

$$\therefore \quad I(t) = \frac{\pi a^2\omega NB}{(\omega^2L^2 + R^2)^{1/2}}\cos(\omega t - \phi - \pi/2)$$

$$= \frac{\pi a^2\omega NB}{(\omega^2L^2 + R^2)^{1/2}}\sin(\omega t - \phi)$$

(*b*) The magnetic dipole moment of the coil $\mathbf{m} = I\pi a^2 N\hat{n}$,

where $\hat{n}$ is a unit vector normal to the coil. At any time *t* the external torque on the coil

$$\tau = \mathbf{m} \times \mathbf{B}$$

with magnitude
$$\tau = |\,\mathbf{m} \times \mathbf{B}\,| = I\,\pi a^2 NB \sin(\pi/2 - \theta)$$
$$= \frac{(\pi a^2\ NB)^2 \omega}{(R^2 + \omega^2 L^2)^{1/2}} \cos\omega t \sin(\omega t - \phi).$$

Example 23. *The transformer windings have an inductance $L_1 = 6.0$ H, $L_2 = 0.05$ H and a coefficient of coupling $k = 0.9$. Find the emf induced in both windings when the primary current increases at the rate of 800 amp. / sec.*

The emf induced in the primary due to its self inductance

$$\mathcal{E}_1 = -L_1 \frac{di}{dt} = 6.0 \times 800 = 4800 \text{ volts.}$$

To find the emf acting in the secondary, first we calculate the mutual inductance of the windings, as

$$M = k\sqrt{L_1 L_2} = 0.9\sqrt{6 \times 0.05} = 0.49\, H.$$

Therefore the emf induced in the secondary

$$\mathcal{E}_2 = -M \frac{di}{dt} = 0.49 \times 800 = 392 \text{ volts.}$$

Example 24. *A piece of Cu-foil is bent into the shape as illustrated in Fig. 12.68. Assume $R = 2$ cm, $l = 10$ cm, $a = 2$cm, $d = 0.4$ cm. Estimate the lowest resonant frequency of this structure and the inductance measured between points A and B, when the inductance is measured at a frequency much lower than the resonant frequency.*

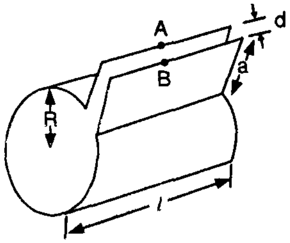


Fig. 12.68. Example 24.

Consider the current in the Cu-foil. As $d << R$, the currents in two sides of the cylinder may be considered in the same phase. Accordingly the maximum wavelength is $2\pi R$. Along the axial direction, the current densities are zero at both ends of the cylinder so that the maximum half wavelength is l or maximum wavelength is $2l$. Since $2l >> 2\pi R$, the maximum wavelength $2l = 20$ cm or the lowest resonant frequency is

$$f_o = \frac{c}{2l} = \frac{3 \times 10^8}{0.20} = 1.5 \times 10^9 \text{ Hz.}$$

For the frequencies much lower then f_o, the current is uniformly distributed over the cylindrical surface and varying slowly with time. Ignoring edge effects, the magnetic induction inside the structure is

$$B = \mu_0 i = \mu_0 I/l.$$

The magnetic flux crossing a cross-section of the structure

$$\Phi = B.S. = \frac{\mu_0 I}{l}[\pi R^2 + ad]$$

$$\therefore \quad \text{Self inductance } L = \frac{\Phi}{I} = \frac{\mu_0}{l}[\pi R^2 + ad] = \frac{4\pi \times 10^{-7}}{0.1}[\pi \times 0.02^2 + 0.02 \times 0.04)$$
$$= 1.68 \times 10^{-8} \text{ H.}$$

Example 25. *Calculate the force between two coaxial solenoids carrying constant currents, one of which extends a part of it within the other.*

To neglect end effects, let us assume that the solenoids are long and thin. If the diameters are assumed nearly equal, the currents flow in the same direction, n_1 and n_2 are the numbers of turns per meter on the two solenoids, l is the length of the 1 in the 2 and S their cross sectional area, then

The mutual inductance $M = (n_1 l)\Phi_{21}/i_2 = (n_2 l)\Phi_{12}/i_1$.

As $\Phi_{21} = (\mu_0 n_2 i_2)\,S$ and $\Phi_{12} = (\mu_0 n_1 i_1)S$,

hence $M = \mu_0 n_1 n_2 Sl$ and $\partial M/\partial l = \mu_0 n_1 n_2 S$.

$\therefore$ $F = i_1 i_2\,(\partial M/\partial l) = \mu_0 n_1 n_2 i_1 i_2 S$.

The force is axial and is attractive, since M increases with l.

Example 26. *What will happen to the above problem, if the fluxes are kept constant instead of currents.*

We know that the magnetic energy is given by

$$\frac{dW_m}{dt} = \frac{1}{\mu_0}\int \mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\,d\tau$$

or
$$dW_m = (1/\mu_0)\,[(B_1(dB_1)\tau_1 + B_2(dB_2)\tau_2 + B_1\,(dB_2)\,Sl + B_2\,(dB_1)\,Sl + B_1B_2Sdl]$$
$$= (1/\mu_0)\,[(B_1l_1S + B_2Sl)\,dB_1 + (B_2l_2S + B_1Sl)\,dB_2 + B_1B_2\,Sdl] \quad ...(i)$$

Since the total flux linkage for solenoid 1 is a constant, *i.e.*,

$$B_1n_1\,S + B_2n_1\,(l/l_1)\,S = \text{const.}$$

or
$$dB_1 + dB_2\,(l/l_1) + B_2\,(1/l_1)\,dl = 0. \quad ...(ii)$$

Similarly for the second solenoid we have

$$dB_2 + dB_1\,(l/l_2) + B_1\,(1/l_2)\,dl = 0. \quad ...(iii)$$

Solving these Eqs. we get

$$dB_1 = (lB_1 - l_2B_2)\,(l_1l_2 - l^2)^{-1}\,dl \quad ...(iv)$$

and
$$dB_2 = (lB_2 - l_1B_1)\,(l_1l_2 - l^2)^{-1}\,dl. \quad ...(v)$$

Substituting the values of dB_1 and dB_2 in Eq. (*i*) we get

$$dW_m = (1/\mu_0)B_1B_2Sdl$$

and the axial force $F = \partial W_m/\partial l = B_1B_2S/\mu_0$.

Example 27. *How much energy is being stored per unit length in a coaxial cable whose inner and outer conducting surfaces have radii R_1 and R_2 respectively and which carry a current I ? Where is the energy density highest ?*

We know that the self inductance per unit length of a co-axial cable carrying a current I is given by the relation

$$\frac{L}{l} = \frac{\mu_0}{2\pi}\log_e\frac{R_2}{R_1},$$

where R_1 and R_2 are respectively the inner and outer radii of the cable.

Hence energy stored per unit length

$$\frac{U}{l} = \frac{\frac{1}{2}LI^2}{l} = \frac{\mu_0 I^2}{4\pi}\log_e\frac{R_2}{R_1}.$$

Since $B = \mu_0 I/2\pi r$, the field is greatest near the surface of the inner conducting surface, so the energy density will be greatest there and given by

$$u = B^2/2\mu_0 = \mu_0 I^2/8\pi^2 R_1^{\,2}.$$

Example 28. *A current i is passing in opposite directions in the inner and outer coaxial cylinders. Inner is solid of radius a and the outer hollow with inner and outer radii b and c respectively. Calculate the magnetic energy per unit length in the different regions.*

In the region of the inner cylinder, let us apply circuital law to a path of radius ρ. Thus we get

$$2\pi\rho B = \mu_0 i\,(\pi\rho^2/\pi a^2) \text{ or } B = \mu_0 i\rho/2\pi a^2,$$

∴ Magnetic energy per unit length inside the inner cylinder

$$W_1 = \frac{1}{2\mu_0}\int_0^a \left(\frac{\mu_0 i\rho}{2\pi a^2}\right)^2 2\pi\rho\, d\rho = \frac{\mu_0 i^2}{16\pi}.$$

In the region between two cylinders $B = \mu_0 i/2\pi\rho$

and $$W_2 = \frac{1}{2\mu_0}\int_a^b \left(\frac{\mu_0 i}{2\pi\rho}\right)^2 . \, 2\pi\rho d\rho = \frac{\mu_0 i^2}{4\pi}\log_e\frac{b}{a}.$$

In the region of the outer cylinder, the current with in circular path of radius ρ is that in the center conductor i minus the current lying in the outer conductor between the radii b and ρ.

∴ $$B = \frac{\mu_0}{2\pi\rho}\left[i - i\left(\frac{\rho^2 - b^2}{c^2 - b^2}\right)\right] = \frac{\mu_0 i}{2\pi\rho}\,\frac{c^2 - \rho^2}{c^2 - b^2}$$

and $$W_2 = \frac{1}{2\mu_0}\int_b^c \left(\frac{\mu_0 i}{2\pi\rho}\right)^2 \left(\frac{c^2 - \rho^2}{c^2 - b^2}\right)^2 2\pi\rho d\rho$$

$$= \frac{\mu_0 i^2}{4\pi}\left[\frac{c^4}{(c^2 - b^2)^2}\log_e\frac{c}{b} - \frac{3c^2 - b^2}{4(c^2 - b^2)}\right]$$

In the fourth region, region outside the cylinders, $B = 0$ and $W_4 = 0$, the self inductance per unit length

$$L = \frac{2(W_1 + W_2 + W_3)}{i^2} = \mu_0\left\{\frac{1}{8\pi} + \frac{1}{2\pi}\log_e\frac{b}{a} + \frac{1}{2\pi}\left[\frac{c^4}{(c^2 - b^2)^2}\log_e\frac{c}{b} - \frac{3c^2 - b^2}{4(c^2 - b^2)}\right]\right\}$$

The second term is most important and is the inductance associated in annular region between the conductors.

Example 29. *Explain the reason of designing transformers with laminated insulated slabs of metals instead of a solid metal. Prove it mathematically.*

Whenever a conductor is placed in the varying magnetic field, induced currents are set up. These result in heating and energy loss. The losses for laminated slabs are smaller than the single solid metal. It is the reason of designing transformers with laminated cores.

Let us consider a block of sides $a \times a$ and thickness b. We shall find the eddy current losses in this case and will compared with the losses in a block divided into four insulated parts of sides $\frac{1}{2}a \times \frac{1}{2}a$. To solve this problem, consider the block broken into a set of thin square paths. If $2x$ be the length of the side, then the total length $l = 8x$. The area enclosed is $(2x)^2$.

If **B** is directed into the face of the block and dB/dt $(= k)$ is uniform over the block. The induced emf $\mathcal{E}$ around the path = $(dB/dt)\,(2x)^2$. The average induced electric field **E** along the path $= \dfrac{\mathcal{E}}{l} = \dfrac{4kx^2}{8x} = \dfrac{1}{2}kx.$

∴ Power dissipation due to this current

$$dP = Vi = Eli = EldAj = \sigma E^2 d\tau$$

or

$$P = \int \sigma E^2 \, d\tau = \sigma \int_0^a \left(\tfrac{1}{2}kx\right)^2 t8x\,dx$$

$$= \sigma b k^2 a^4/2.$$

If the block was divided into quarters, the power dissipation

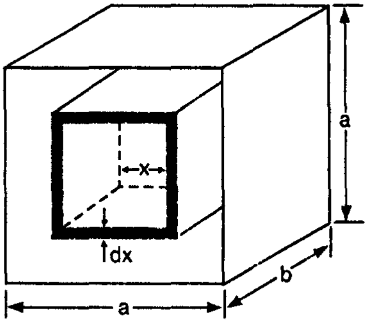


Fig. 12.69. Example 29.

$$P = 4\frac{\sigma b k^2}{2}\left(\frac{a}{2}\right)^4 = \frac{1}{4}\frac{\sigma b k^2}{2}a^4.$$

It is one fourth the energy dissipation in the solid block. Hence the transformers with the laminated cores are better than the solid blocks under eddy current effects.

Example 30. *A rectangular coil of dimensions a and b and resistance R moves with constant velocity v into a magnetic field B, as shown in Fig. 12.70. Find the force on the coil.*

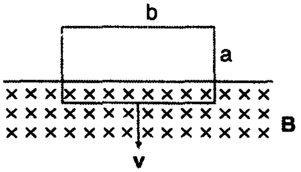


Fig. 12.70. Example 30.

As the coil moves into the magnetic field, an emf is induced in the coil

$$\mathcal{E} = \int (\mathbf{v} \times \mathbf{B}) . d\mathbf{l} = -Bvb$$

The induced current produced $I = \mathcal{E}/R = -Bvb/R$.

The minus sign indicates that the current flows counterclockwise.

The force on the coil $F = \int I \, d\mathbf{l} \times \mathbf{B} = I\,b\,B = -vb^2B^2/R.$

The minus sign shows that the direction of force is opposite to v, *i.e.* The force opposes the motion.

Example 31. *A rectangular loop of wire with dimensions b and a is released at t = 0 from rest just above a region in which the magnetic field is B, as shown in Fig. 12.71.The loop has resistance R, self inductance L and mass m. Consider the loop during the time that it has its upper edge in the zero field region. Find the current and velocity of the loop as a function of time assuming (a) L negligible and (b) R negligible.*

During the time the upper edge is in the zero field, the emf $\mathcal{E} = Bbv$ will be induced in the lower side of the loop, due to its downward motion.

This causes a current to flow in the loop, where $I = \mathcal{E}/R$.

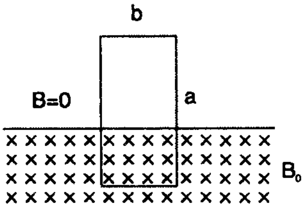


Fig. 12.71. Example 31.

Because of this current a force IbB acts on this lower side which will oppose the motion. Thus the equation of motion is

$$F = mg - BIb = m\,dv/dt. \qquad ...(i)$$

Due to the presence of inductance, the current is given by

$$\mathcal{E} - L\,dI/dt = IR. \qquad ...(ii)$$

(a) When L is negligible equation (i) can be solved by using $I = E/R = Bbv/R$.

$$m\frac{dv}{dt} = mg - \frac{B^2b^2v}{R}.$$

On solving, we get $\quad v(t) = \dfrac{Rmg}{B^2b^2}\left[1 - \exp\left(-\dfrac{B^2b^2}{mR}t\right)\right]$

$$\therefore \qquad \text{current } I(t) = \frac{Bbv(t)}{R} = \frac{mg}{Bb}\left[1 - \exp\left(-\frac{B^2b^2}{mR}t\right)\right]$$

(b) When $R = 0$, we have $L\,dI/dt = Bbv$ and $m dv/dt = mg - BbI$.

These give $\quad d^2v/dt^2 + \omega^2 v = 0,$

where $\quad \omega^2 = B^2b^2/mL.$

Thus the general solution of this differential equation is

$$v = C_1 \cos \omega t + C_2 \sin \omega t.$$

At $t = 0$, $v = 0$ and $I = 0$, therefore $C_1 = 0$, and $C_2 = g/w$

$$\therefore \qquad v = (g/w) \sin \omega t$$

and $\quad I = (mg/Bb)(1 - \cos \omega t).$

Example 32. *In a circuit, a resistor of 10 mega ohms, a capacitor of 0.2 μF and a battery of emf 20 volts are connected in series. Calculate the (a) rate of growth of charge, (B) energy stored by the capacitor, (c) heat appeared in the resistor and (d) the energy delivered by the battery after 2 seconds.*

(a) Growth of charge in the circuit is given by the relation

$$q = q_0(1 - e^{-t/CR})$$

$\therefore$ Rate of growth of charge $\dfrac{dq}{dt} = \dfrac{q_0}{CR}e^{-t/CR}.$

But the maximum value $q_0 = \mathcal{E}C$ coulomb.

$$\therefore \qquad \frac{dq}{dt} = \frac{\mathcal{E}C}{CR}e^{-t/CR} = \frac{\mathcal{E}}{R}e^{-t/CR}$$

$$= \frac{20}{10 \times 10^6}\exp.[-2/0.2 \times 10^{-6} \times 10 \times 10^6] = 7.4 \times 10^{-7}\ C/\text{sec}.$$

(b) Energy stored by the capacitor at any instant t

$$= \frac{q^2}{2C} = \frac{q_0^{\,2}}{2C}(1 - e^{-t/CR})^2$$

∴ Rate at which the energy is being stored by the capacitor

$$\frac{dU}{dt} = \frac{q_0^2}{2C}\left[2(1-e^{-t/RC})\frac{1}{RC}\right] = \frac{q_0^2}{RC^2}e^{-t/RC}(1-e^{-t/RC}) = \frac{\mathcal{E}^2}{R}e^{-t/RC}(1-e^{-t/RC})$$

$$= \frac{20^2}{10\times10^6}e^{-1}(1-e^{-1}) = 9.3\times10^{-6} \text{ watt.}$$

(*c*) Heat energy dH produced in the resistor of resistance R when a current I passes through it for time dt is I^2Rdt.

∴ Rate of heat produced $dH/dt = I^2R = (dq/dt)^2 R = 5.5 \times 10^{-6}$ watt.

(*d*) Energy delivered by the battery $dU = \mathcal{E}Idt$.

$$\therefore \quad dU/dt = \mathcal{E}I = \mathcal{E}(dq/dt) = 14.8 \times 10^{-6} \text{ watt.}$$

Example 33. *A resistor R and a 2μF capacitor in series are connected at a 200 volts direct supply. Across the capacitor is a neon bulb that strikes at 120 volts. Calculate the value of R to make the bulb strike 5 sec after the switch has been closed.*

In the case of a charging of a capacitor connected with a resistance R in series, the charge at any instant t is given by

$$q = q_0(1-e^{-t/CR}).$$

As $q = CV$, hence the potential difference at this instant t will be

$$V = V_0(1-e^{-t/CR}).$$

In our problem, $V = 120$ volts, $V_0 = 200$ volts, $t = 5$ sec and $C = 2 \times 10^{-6}\, F$.

$$\therefore \quad 120 = 200\,[1 - \exp.(-5/2\times10^{-6}R)$$

$$e^{2.5\times10^6/R} = 2.5 \text{ or } 2.5\times10^6/R = \log_e 2.5$$

$$\therefore \quad R = \frac{2.5\times10^6}{2.3026\times0.4} = 2.73\times10^6\Omega = 2.73 \text{ megohms.}$$

Example 34. *A coil of resistance 20Ω and inductance 0.5 henry is switched to direct current 200 volt supply. Calculate the rate of increase of current (i) at the instant of closing the switch and (ii) at t = LR sec after the switch is closed. Also find the value of the steady current in the circuit.*

During the growth in RL-circuit, the current at any instant is given by

$$i = i_0(1-e^{-Rt/L})$$

$$\therefore \quad \frac{di}{dt} = i_0\frac{R}{L}e^{-Rt/L} = \frac{\mathcal{E}}{L}e^{-Rt/L}.$$

Hence at $t = 0$, $\frac{di}{dt} = \frac{\mathcal{E}}{L} = \frac{200}{0.5} = 400$ amp/sec.

at $t = \frac{L}{R}, \frac{di}{dt} = \frac{\mathcal{E}}{L}\cdot\frac{1}{e} = \frac{200}{0.5\times2.718} = 147.2$ amp/sec.

The steady value of current in the circuit

$$i_0 = \mathcal{E}/R = 200/20 = 10 \text{ amp.}$$

Example 35. *A 5 henry inductor is placed in series with a 10 Ω resistor, an emf of 5 volts being suddenly applied to the combination. Using these values, prove the principle of conservation of energy, for time equal to the time constant.*

At any instant t, current in LR-circuit is given as

$$i = i_0\,(1 - e^{-Rt/L}) = \frac{\mathcal{E}}{L}(1 - e^{-Rt/L}).$$

At the time t = time constant L/R, we get

$$i = \frac{\mathcal{E}}{R}\left(1 - \frac{1}{e}\right) = \frac{5}{10}\left(1 - \frac{1}{2.178}\right) = 0.316 \text{ amp.}$$

∴ The rate at which the energy is delivered by the battery is

$$P_B = \mathcal{E}i = 5 \times 0.316 = 1.58 \text{ watts.}$$

At this time some energy will appear as joule heat in the resistor, given by

$$P_J = i^2R = (0.316)^2 \times 10 = 0.998 \text{ watt.}$$

Part of the energy is stored in magnetic field, linked with the inductor. It is given by the relation

$$P_m = Li\,(di/dt).$$

As
$$\frac{di}{dt} = \frac{\mathcal{E}}{L}e^{-Rt/L} = \frac{\mathcal{E}}{L}\cdot\frac{1}{e},$$

Therefore
$$P_m = Li.\frac{\mathcal{E}}{L}.\frac{1}{e} = \frac{\mathcal{E}i}{e} = \frac{5 \times 0.316}{2.718} = 0.581 \text{ watt.}$$

$$\therefore \qquad P_J + P_m = 0.998 + 0.581 = P_B.$$

It is same as required by the principle of conservation of energy.

Example 36. *If a battery, of emf 100 volts, is connected in series with an inductance of 10 mH, a capacitor of 0.05 μF and a resistance of 100 Ω, find (i) the frequency of the oscillatory current, (ii) the logarithmic decrement and (iii) the final capacitor charge.*

$$\frac{R^2}{4L^2} = \frac{100^2}{4 \times (10^{-2})^2} = 2.5 \times 10^7$$

and
$$\frac{1}{LC} = \frac{1}{10^{-2} \times 0.05 \times 10^{-6}} = 2 \times 10^9,$$

As in this problem $R^2/4L^2 < 1/LC$, hence the charging of the capacitor is oscillatory. The frequency of oscillation

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC} - \frac{R^2}{4L^2}\right)} = \frac{1}{2 \times 3.14}\sqrt{(2 \times 10^9 - 2.5 \times 10^7)}$$

$$= 7076.6 \text{ cycles/sec.}$$

The logarithmic decrement $= \dfrac{RT}{2L} = \dfrac{100}{2 \times 10^{-2} \times 7076.6} = 0.707.$

The final capacitor charge $q_0 = \mathcal{E}C = 100 \times 0.05 \times 10^{-6} = 5\mu C.$

Example 37. *In any leakage experiment a capacitor of capacitance 1μF was first charged to a suitable potential and discharged at once, the throw was 10 cm. The capacity was again charged and was short-circuited with a resistor R for 20 sec and the throw was 7.5 cm when was discharged through a ballistic galvanometer. Calculate the resistance of the resistor.*

In the first part of the experiment, the throw d_0 was due to a full charge q_0, thus $q_0 = \mathcal{E}C = K\theta = Kd_0$.

If the charge q_0 reduces to q due to the leakage through resistance R, then

$$q = q_0 e^{-t/RC} = Kd \quad \text{or} \quad Kd_0 e^{-t/RC} = Kd$$

or

$$R = \frac{t}{C \log_e (d_0/d)} = \frac{t}{2.3026 C \log_{10}(d_0/d)} = \frac{20}{2.3026 \times 10^{-6} \times \log_{10}(10/7.5)}$$

$$= 68.5 \times 10^6 \text{ ohm} = 68.5 \text{ megohm.}$$

ORAL QUESTIONS

1. State Lenz's law.
2. Write Maxwell's equation for electromagnetic induction.
3. What is meant by magnetic coupling and mutual flux linkage of closed circuits.
4. How is the coefficient of coupling of two magnetically coupled coils found from their inductance and mutual inductance ?
5. What is the difference between an aiding and opposing connections of coils?
6. Define emf of self and mutual inductance.
7. Give examples of the utilization and harmfulness of eddy currents.
8. What is meant by uniform magnetic circuit ?
9. A rectangular loop carrying a current is situated near a long straight wire carrying a steady current *i* such that the wire is parallel to one of the sides of the loop and is in the plane of the loop. Then the current loop will (*a*) remain stationary, (*b*) move away from the wire, (*c*) move towards the wire or (*d*) rotate about the straight wire.
10. For a circuit arrangement shown in Fig. 12. 72, The variable resistance is adjusted such that the bulb just glows. If the circuit is switched off (*a*) the bulb stops glowing, (*b*) momentarily bulb becomes very bright and stops glowing, (*c*) bulb becomes dim and then stop glowing.
11. Out of various combinations of *R*, *L* and *C*, which one has dimension of time : (*a*) *RC*, (*b*) *LC*/*R* (*c*) *R*/*L*.

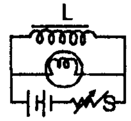


Fig. 12.72.

12. A Cu-ring lies on a table with a hole through its center. If a bar magnet with its south pole in the lower side is dropped vertically through the hole, describe the current in the ring.
13. A current carrying solenoid is moved toward a conducting loop as shown in Fig. 12.73. As viewed from the solenoid, what will be the direction of the induced current in the loop ?
14. Describe the motion of a bar magnet which is dropped lengthwise (vertically down) in a very long *Cu*-pipe.
15. A loop is placed vertical in the gap between pole pieces of an electromagnet. What is the direction of induced current in it if pulled out vertically?

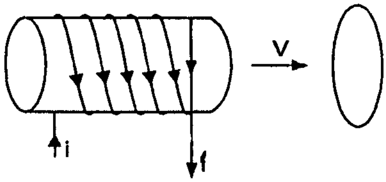


Fig. 12.73.

16. What is the advantage of placing two electric wires carrying ac close together ?

17. The two ends of the coil of a ballistic galvanometer are generally connected together to stop swinging at once. Explain the reason.
18. Why is the induced emf much larger when the switch is opened in a circuit consisting inductance and battery than when it is suddenly closed.
19. A *Cu*-ring and a wooden ring of the same dimensions are placed in the same region of varying magnetic field. Compare the induced emf's.
20. It is possible to wind a coil of wire on a straight wooden rod in such a way that the coil has essentially zero self inductance. How ?
21. How would you shape a given wire of fixed length to obtain the greatest self inductance ? The least?
22. A small coil is placed inside a long solenoid. How does the mutual inductance of the system change with the orientation of the coil ?
23. How would you arrange two flat circular coils so that their mutual inductance was (*a*) maximum, (*b*) minimum.
24. Why do you use, first quarter of a cycle of alternating magnetic field in the betatron?
25. Explain the force experienced while pulling in or out the metallic sheet placed in a magnetic field.
26. Explain how the dynamo can be used as a motor ?
27. How the motor is used as a brake in tram-cars ?
28. Is the energy density greatest near the ends of a solenoid or near its center ?
29. In a *L-R* circuit, can the induced emf across the coil of self inductance *L* ever be larger than the battery emf?
30. How a solenoid whose ends are connected together can by itself oscillate like an *LC* circuit?
31. In the analog between *LCR* oscillations and damped harmonic motion, discuss physically how *L, R* and $1/C$ act like *m, b* and *k* respectively.

PROBLEMS

1. The magnetic field inside a certain long solenoid is given by $B = 0.20 \sin (300\,t)$ tesla. A coil of 100 turns of 2.0 cm^2 area is placed inside the solenoid. Find the induced emf in it when its axis makes angles 30° and 90° with the solenoid axis. [$-$ 1.04 cos 300*t*, 0]
2. How many volts are generated in wire 12cm long which cuts directly across a flux whose intensity is 1.4 tesla, if it moves at a speed of 2 meters/sec ? [0.336 volts]
3. What is the maximum electromotive force induced in a coil of 4000 turns and average radius 12cm rotating at 30 revolutions per sec in the earth's magnetic field of intensity 0.5×10^{-4} web/m^2. [0.27 weber]
4. A circular loop with a radius *R* consists of *N* turns of wire and is placed in a magnetic field directed perpendicular to the plane of the loop. The magnetic field at a distance *r* from the center of the loop in its own plane is $B = B_0\,(1 - r/2R) \cos \omega t$. Determine the induced emf in the loop by *B* alone. [$(2\pi/3)\omega R^2 N B_0 \sin \omega t$]
5. A metal sphere of radius *a* moves with speed *v* parallel to a long wire that carries a constant current *I*. If the sphere remains at a distance *a* from the wire, show that the induced emf between the extreme points (one closest and other farthest) on the sphere is $(\mu_0\, Iv/2\pi) \log_e [(d + a)/(d - a)]$.
6. A horizontal metallic rod rotates at a frequency ν about a vertical axis passing at a distance equal to $1/k$ of its length from one of its ends. The length of the rod is *l*. Find the potential difference between the ends of the rod if it rotates in a uniform magnetic field *B*.[$\pi \nu l^2 B(k - 2)/k$]
7. A closed coil of 50 turns of 300 cm^2 area and of resistance 40Ω is held at right angles to a uniform field of 2×10^{-2} tesla. Find the charge induced in the coil, when turned through an angle 30° about the axis ⊥ to the field. (10^{-5} *coulomb*)]

8. A uniform cylindrical coil in a vacuum has $r_1 = 1\text{m}$, $l_1 = 1m$ and 100 turns. Coaxial and at the centre of this coil is a smaller coil of $r_2 = 10$ cm, $l_2 = 10$ cm and 10 turns. Calculate the mutual inductance of the two coils. (39.5 μH)

9. A closely wound search coil has an area of 4 cm^2, 160 turns and resistance of 50Ω. It is connected to a ballistic galvanometer whose resistance is 30Ω. When the coil rotates quickly from a position parallel to a uniform magnetic field to one perpendicular, the galvanometer indicates a charge of 40 μ*C*. What is the magnetic flux density? (0.05 *T*)

10. A uniform magnetic field *B* fills a cylindrical volume of radius *R*. A metal rod of length *L* is placed as shown in Fig. 12.74. If *B* is changing with time, show that the emf produced between the ends of the rod is given by $\mathcal{E} = \frac{1}{2}(dB/dt)\, L\, (R^2 - L^2/4)^{1/2}$.

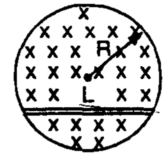


Fig. 12.74.

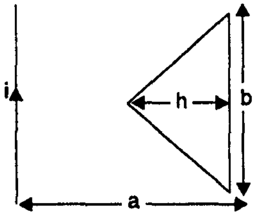


Fig. 12.75.

11. A very long straight conductor and an isoscales triangular conductor lie in a plane and are separated from each other as shown in Fig. 12.75. If $a = 20$ cm, $b = 20$ cm and $h = 10$ cm. Find the coefficient of mutual inductance. (1.2×10^{-8} H)

12. If a rectangular loop of area 400 cm^2, $R = 2\Omega$ is rotating with angular velocity $\omega = 10$ rad sec^{-1}, (*a*) about *x*-axis, (*b*) about *z*-axis, or (*c*) about an edge parallel to the *y*-axis. What is the maximum induced emf in each case. If the magnetic field $B = 0.5$ Wb/cm^2 is along *x*-axis. [0.02*V*, 0.2*V*]

13. A iron ring of radius 10 cm and of cross sectional area 12 cm^2 is evenly wound with 1200 turns of insulated wire. There is an air gap in the ring of length 1 mm. The permeability of the iron is 700 and is assumed independent of the field. Ignoring the hysteresis losses, calculate (*a*) the magnetic field in the gap when a current of 1 amp passes through the coil, and (*b*) the self inductance of the coil with this core. [$B = \mu_0 ni/[d + (2\pi R - d)/\mu_r]$, $L = BAN/i$] (0.79T, 1.14 H)

14. Two parallel conductors of infinite length and separated by a distance 2*a* carry a current *I* in opposite directions. A circular conducting ring of radius *a* in the plane of parallel wires lies between the two straight conductors and is insulated from them. Find the coefficient of mutual inductance between the circular conductor and the straight conductors. ($2\,\mu_0\, a$)

15. The flux density at all points within a circular portion equals to 0.5 Wb/m^2 and is decreasing at the rate of 0.1 Wb/m^2-sec. If a ring of radius 10cm and resistance 2Ω is placed in this field in such a way that the flux is ⊥ to the plane of the ring. Calculate the (*a*) magnitude and direction of the induced electric field at any point of the ring, (*b*) induced emf and (*c*) current in the ring. [Clockwise 0.005*V*/*mV*, 3.14*m*, 1.57*mA*]

16. Two conducting loops face each other a distance *d* apart. An observer sights along their common axis. If a clockwise current is suddenly established in the larger loop, find the direction of the induced current and of the force acting on the smaller loop.

17. Find the induced electric field *E* both outside and inside the solenoid of *n* loops per unit length and carrying a varying current $i = i_0 \sin 2\pi ft$. [$(\mu_0 \pi a^2 n i_0 f \cos 2\pi ft)/r$, $\mu_0 \pi n i_0 f r \cos 2\pi ft$)]

18. When a copper disc of radius 10cm makes 20 rotations per sec with its plane perpendicular to a uniform magnetic field, an induced emf 3.14 milli volts is generated between the center and the edge of the disc. Calculate the intensity of the magnetic field. [5×10^{-3} *weber*/*m*]

19. A circular coil of radius 7 cm and number of turns 100 is rotated about a diameter perpendicular to a magnetic field of strength 100 gauss with 3600 rotations per minute. Find the maximum emf developed. (1 volt)

20. In a betatron the electron completes 2×10^5 rounds around the loop before ejected to produce *X*-rays. If during this time, rate of change of flux is 400 *V*, find the energy and speed of ejected electron. [80 *MeV*]

21. A circular disc of metal with conductivity σ, thickness t, rotates about its axis with angular velocity ω between the pole faces of magnet which produces a uniform magnetic field over a small square area of size a^2 at the average distance r from the axis. If B is ⊥ to the disc, calculate the torque acting on the disc. [$\frac{1}{3}B^2a^2r^2\,\omega\sigma t$]

22. Calculate the mutual inductance of the coplanar concentric rings consisting of n_1 and n_2 turns respectively. The radius of the inner ring is r_1 and that of outer ring is r_2. [$\mu_0\pi r_1{}^2\, n_2 f/2r_2$]

23. A toroidal coil has N turns, a mean radius b and a circular cross section of radius a. Find the self inductance per unit length of the coil. [$\mu_0 N^2 (b - \sqrt{b^2 - a^2})$]

24. A square coil with 80 turns is wound on 18cm × 18cm frame. It rotates about an axis through the diagonal of the square in a uniform magnetic field of $0.8T$. If **B** makes an angle of 45° with the axis of rotation, what must be the angular velocity of the coil to produce a maximum instantaneous induced emf of $14V$? [3.04 π radians/sec]

25. Two coils having 50 and 60 turns form respectively, the primary and secondary of transformer. In the primary coil, direct current is passed which reaches a maximum value of 5 amp in 0.2 sec. A magnetic flux 3×10^{-2} weber is produced in the core passing through both the coils. Calculate the mutual inductance and the emf induced in the secondary coil.

26. A long solenoid is 1.0 m long and 3.0 cm in mean diameter. It has five layers of winding of 850 turns each and carries a current of 5 amp. What is the magnetic induction B at its center? What is the magnetic flux for a cross section of the solenoid through its center. [1.88×10^{-5} web]

27. A circular loop of wire 10cm in diameter carries a current of 100A. Calculate the energy density at the center of the loop. (0.157 J/m^3)

28. An inductor L is connected with a battery of emf $\mathcal{E}$ and a resistance R in series. Show that the current after a time t is $x(1 - x)\ \mathcal{E}/R$ and that if contact be then broken the current after a further time t is $x(1 - x)\ \mathcal{E}/R$, where $L \log_e x + Rt = 0$.

29. If $C = 1\ \mu F$ and $R = 1M\Omega$ in a CR circuit, find the time required by v_c to attain 50% of its maximum value. What will happen if the switch in the circuit also has a small resistance of 10Ω?

30. The time constant of certain inductive coil was found to be 2.5×10^{-3} sec. With a resistance of 80 ohms added in series, a new time constant of 0.5×10^{-3} sec was obtained. Find the inductance and resistance of the coil. [5.0×10^{-2} henry; 20 ohms]

31. A coil of resistance R and inductance L is connected to a battery of emf $\mathcal{E}$ volts. In what time the current rises to 99% of its final value ? When will the pd, across inductance equal to that across resistance ? [4.6 L/R, 0.69 L/R]

32. If a capacitor of 20 μF be charged on a pd of $15kV$ and then suddenly short-circuited by a coil of negligible resistance and of impedance $2mH$, calculate the frequency and maximum amplitude of the resultant current oscillations. [759 cycles/sec. 1500 amp.]

33. The capacitor of an oscillator circuit of low resistance is enclosed in a container. When the container is evacuated, frequency of the circuit is 100 kilo-cycles and when it is filled with CO_2 the frequency changes by 50 cycles. Calculate the dielectric constant of the gas. [1.001]

34. A condenser of capacity 0.1 μF and resistance 10 megohms is charged to a certain potential and then insulated. Find the time the potential will take to fall to half its original value. [0.693 sec]

35. A $12V$ battery is connected in series with a 30Ω resistor and a 220 mH inductor. Calculate (*i*) the time constant, (*ii*) time in which the current reaches to half of its maximum value. Also find the rate of energy being delivered by the battery and being stored in the inductor at this instant. [7.3 ms, 5ms, 2.4W, 1.2W]

36. A potential difference of 1 volt (dc) is applied to a coil of resistance 1 ohm and self inductance 1 henry. What will the current be in the circuit after 1/19 second ?

37. A condenser of capacity 0.25 μF if charged upto a certain voltage and then discharged through a circuit of inductance 0.1 henry and resistance 100 ohms. Show that the discharge is oscillatory. Also calculate the frequency of the discharge. [1004c/sec]

38. In an oscillating circuit L = 0.20 henry, $C = 1.2nF$. Find the maximum value of R so that the circuit may oscillate. [2.58×10^4 *ohms*]

39. A LCR circuit is critically damped with L = 0.2 henry and R = 100 ohms. What is the value of C ? [80 μF]

40. Examine if the discharge of a condenser of capacity $2\mu F$ in a circuit containing an inductance $L = 0.15H$ and resistance $R = 15\ \Omega$ shall be oscillatory.

13 MAGNETIC PROPERTIES OF MATTER

13.1 MAGNETIC INDUCTION & MAGNETIC MATERIALS

We know that electric lines of force change when a dielectric is placed between the parallel plates of a capacitor, this was explained by the simple atomic theory leading to the idea of electric dipoles aligned in the dielectric. Let us now see the effect of placing a piece of iron in a magnetic field. For this an experimental arrangement, known as *Rowland Ring,* is used. This consists of a toroidal winding with magnetic material, say iron, inside it in the form of a ring. This is equivalent to a toroidal solenoid supported on a core of magnetic material. Secondary few turns are connected to a ballistic galanometer G, as shown in Fig. 13.1(a).

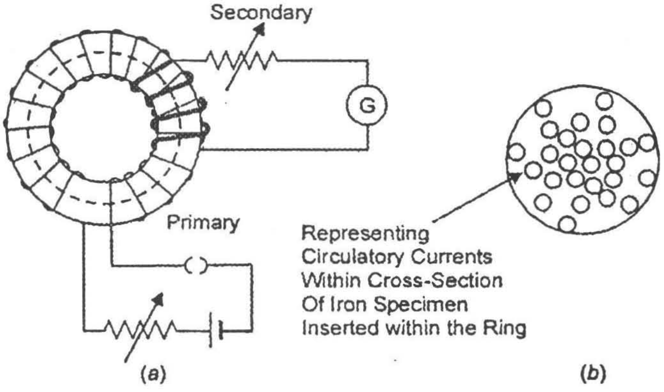

Secondary
G
Primary
Representing Circulatory Currents Within Cross-Section Of Iron Specimen Inserted within the Ring
(a)
(b)

Fig. 13.1 Circular currents in a specimen.

With the space inside the toroid empty, the change in flux is observed with the help of ballistic galvanometer by turning off the current suddenly. The experiment is repeated with matter filling the entire toroid, the deflection is found for the same current. Experiments show that the magnetic induction **B** is modified by the presence of matter in the toroid. This experiment can be repeated with the core of different materials. Few substances such as O_2, air, Pt, Al, Cr, Mn, etc., show a very small increase in flux in the interior of the toroidal solenoid and are said to be *paramagnetic*. Few other substances, such as H_2, H_2O, *Bi*, *Cu*, *Zn*, *Sb*, *Au*, etc., show a very small decrease in flux, and are said to be *diamagnetic*. There are other substances like *Fe*, *Co*, etc., through which the flux changes to a larger value and are known as *ferromagnetic*.

A circular cross section of the Rowland ring, when the external field is applied, exhibits magnetic effects. This effect can be represented by series of minute current loops, where all current directions are the same [Fig. 13.1 (*b*)]. The resultant current is equivalent to the sum of these mesh currents and equal to the current round the boundary of the ring and is known as the *surface current*.

13.2 THE MAGNETIC DIPOLE CONCEPT OF ATOM

According to the atomic theory, atom consists of electron rotating along closed orbital paths round the nucleus. The nucleus is a central heavy core consisting of neutrons and protons. Let us consider a simple model of one electron of charge e moving with speed v in a circular path of radius r round a nucleus with a proton charge. At any instant, the electron and the proton would appear as an electric dipole of dipole moment p_E, but the average value of p_E over a complete round is zero. Hence *net electric field due to an atom at a certain distance is zero.*

Magnetic Moment of an Electron : The electron rotating about a nucleus with velocity v in a circular path of radius r is equivalent to a current

$i = e \times$ frequency $\nu = e(\omega/2\pi) = ev/2\pi r$.

We have discussed in article 10.12(*B*) that the current carrying loop is equivalent to a magnetic dipole. Hence the magnetic dipole moment of circulating electron is

$$p_{m', l} = m_l = \text{circulating current} \times \text{area of loop} = (ev/2\pi r)\,\pi r^2 = \tfrac{1}{2} evr. \qquad \text{...(1)}$$

This is known as the orbital magnetic moment of an electron. In ordinary lump of matter there must be as many electrons going round one way as the other and m_l can be assumed as distributed evenly over all directions in space. Hence the *net orbital magnetic moment will be zero in the absence of an external magnetic field.*

Let us now see the effect of an external magnetic field of flux density **B**. If the electron of mass m_e is rotating along anticlockwise direction and the magnetic field **B** is applied ⊥ to the plane of the orbit in downward direction. When the field **B** changes, the flux through the circular orbit will change and an induced emf $\mathcal{E}$ will thus set up, such that

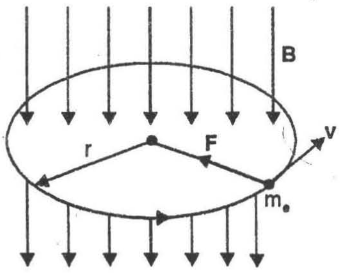


Fig. 13.2. Circulating electron in a magnetic field.

$$\mathcal{E} = -d\Phi/dt = -\pi r^2\, dB/dt.$$

But $$\mathcal{E} = -\oint \mathbf{E}\cdot d\mathbf{l} = -2\pi r E,$$

Hence $$E = \tfrac{1}{2} r\,(dB/dt).$$

The electric field will accelerate the electron, hence

$$m_e \frac{dv}{dt} = eE = \frac{er}{2}\frac{dB}{dt} \text{ or } dv = \frac{er}{2m_e} dB.$$

As e, r and m_e are fixed, the net change in v in the process of bringing the field upto the final value $\mathbf{B}_0$ is

$$\Delta v = erB_0/2m_e.$$

$$\therefore \quad \Delta\omega = \Delta v/r = eB_0/2m_e. \qquad ...(2)$$

Thus we see that the electron in a magnetic field acquires an additional angular velocity, called the *Larmor angular velocity* and is written as ω_L. It is characterized by the frequency, know as *Larmor frequency.*

ω_L coincides with **B** in direction. Since the electron charge is negative, therefore,

$$\omega_L = -(e/2m_e)\,\mathbf{B}_0. \qquad ...(3)$$

This additional angular velocity represents the additional rotation of the atom. The total frequency of electron rotation is equal to the sum of its frequency ω_0 of rotation in the atom and the frequency ω_L of rotation of the atom, where the centripetal force due to electrostatic attraction

$$m\omega_0^2 r = \frac{1}{4\pi\varepsilon_0}\frac{-Ze^2}{r^2}. \qquad ...(4)$$

The increase in velocity means the increase in the upward magnetic moment $\mathbf{m}_l$. The negatively charged electrons are thus decelerated and the magnetic moment is decreased. Therefore from Eq. (1), the change in magnetic moment

$$\Delta\mathbf{m}_l = -\frac{er}{2}\left[\frac{er\,\mathbf{B}_0}{2m_e}\right] = -\frac{e^2r^2}{4m_e}\mathbf{B}_0, \qquad ...(5)$$

The revolving electron also possesses angular momentum ***L***. Its magnitude is given by

$$L = m_e v r.$$

The magnetic moment of the orbital electron is thus given by

$$m_l = \tfrac{1}{2}evr = \tfrac{1}{2}e(L/m_e) = eL/2m_e \ amp.\,m^2. \qquad ...(6)$$

The ratio $m_l/L = e/2m_e$ is called the *orbital gyro-magnetic ratio* or the orbital magneto-mechanical ratio for the electron.

From the quantum idea, the orbital angular momentum of the electron is expressed as $L = nh/2\pi$, where n is known as *quantum number.*

$$\therefore \quad m_l = (e/2m_e)nh/2\pi = neh/4\pi m_e = nM_B, \qquad ...(7)$$

where $M_B = eh/4\pi m_e =$ *Bohr magneton.*

Above relation shows that the orbital magnetic moment of the electron is an integer multiple of Bohr magnetron.

Orbital angular momentum **L**, a vector of magnitude $m_e vr$ is in the upward direction ⊥ to the plane containing vectors **r** and **v**.

As the electron is in anticlockwise motion, hence the current will be in clockwise direction. Thus the magnetic moment $\boldsymbol{m}_l$ will be in downward direction and is given by

$$\boldsymbol{m}_l = -(e/2m_e)\boldsymbol{L} \ . \qquad ...(8)$$

Thus we see that the angular momentum **L** and the magnetic moment $\boldsymbol{m}_l$ have opposite directions.

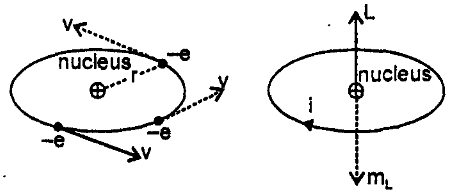


Fig. 13.3. **L** and $\mathbf{m}_l$ due to circulating electron.

An atom with an electron rotating in it can be visualized as a *gyroscope having* a magnetic moment or the *atom precesses in a magnetic field like a gyroscope.* The motion is called the *Larmor precession.*

Spin Magnetic Moment of an Electron : The electrons not only have orbital motion, but rotate about their own axes, thus possess angular momentum which is independent of orbital motion. This is called *intrinsic spin* of the electron. The magnetic moment associated with this intrinsic angular momentum (or spin angular momentum) is referred to as the *spin magnetic moment.* As the electron has spin $\frac{1}{2}$ or spin angular momentum $\frac{1}{2}(h/2\pi)$, hence the spin magnetic moment of electron is $eh/4\pi m_e$. It shows that

$$\mathbf{m}_s = -(e/m_e)\mathbf{S}. \qquad ...(9)$$

It has purely quantum mechanical explanation and thus can not be discussed here.

The magnitude of the spin magnetic moment ($\mathbf{m}_s$) is always the same, however it alignes with the applied magnetic field of flux density **B**. This orientation of $\mathbf{m}_s$ along **B** is due to the torque $\tau = \mathbf{m}_s \times \mathbf{B}$ experienced by $\mathbf{m}_s$ when placed in **B**.

In any atom, several electrons are usually present. Each has angular momentum and magnetic moment corresponding to both orbital and spin motions. As discussed earlier, the direction of magnetic moment is opposite to that of angular momentum. The ratio of these two is $e/2m_e$ for the orbital motion and e/m_e for the spin motion. Therefore for the atom as a whole, we may write

$$\mathbf{m} = -g(e/2m_e)\,\mathbf{J}, \qquad ...(10)$$

where 'g' (gyromagnetic ratio) is a factor which is characteristic of the state of the atom. It is called the '*Lande g-factor*'. It is unity for pure orbital motion and two for a pure spin motion and lies some where in between these two values for the combined motion, depending on the proportion and the manner in which the orbital motion of electrons and their spins contribute to the total magnetic moment. Here **J** is the total angular momentum of the atom. The above relation shows that the reorientation of magnetic moments takes place simultaneously with the reorientation of the corresponding angular momenta.

Magnetic Moment of a Proton : In the nucleus, there are neutrons and protons. Many nuclei possess magnetic moments, typically less than atomic moments by a factor of about 10^3 due to the inverse dependence of **m** on mass). The proton, like an electron, may move around in some kind of orbit and has an intrinsic spin. The proton is positively charged, the magnetic moment **m** is therefore parallel to the angular momentum **J**. In analogy with Eq. (10), the magnetic moment of a proton

$$\mathbf{m} = g(e/2m_p)\,\mathbf{J}, \qquad ...(11)$$

where m_p is the mass of the proton and g the nuclear g-factor. By analogy with the Bohr magneton M_B for atomic electrons we define a nuclear magneton M_n as

$$M_n = eh/4\pi m_p = M_B\,(m_e/m_p). \qquad ...(12)$$

Thus we see that the nuclear magneton is 1836 times smaller than the Bohr magneton.

If the spinning proton behaved like a uniformly charged classic sphere, then $g_s = 1$ and its magnetic dipole moment should be one half nuclear magneton. If the proton behaved like a particle following Dirac's relativistic quantum mechanics, then $g_s = 2$ and its magnetic dipole moment should be one nuclear magneton. Actually the spin magnetic dipole moment of the free proton is 2.763 nuclear magnetons.

Magnetic Moment of a Neutron : Neutron should not have magnetic dipole moment on account of its zero charge. Surprisingly the neutron is associated with a fairly large magnetic dipole moment (– 1.913 nuclear magnetons.) Actually the neutron has inner constitution such that its spin angular momentum $s = \frac{1}{2}$. The sign of spin magnetic moment for neutron is negative and therefore stimulates the rotation of negative charge in the spin direction.

An Atom in a Magnetic Field : We know that the electrons, protons and neutrons possess magnetic moments. Naturally most of the atoms may have net magnetic moments. If such an atom is placed in a magnetic field, its energy depends on the orientation of its magnetic moment relative to the magnetic field (*the external field*). According to quantum mechanics, the magnetic moment can point only in certain permitted directions. The restriction is called *space quantization. Classical* theory has no explanation for this quantization. Since the magnetic moment has a discrete set of possible orientations, the energy of the atomic level splits into a finite set of energies. In the absence of a magnetic field, all directions in space are equivalent and all the states have the same energy.

In an external magnetic field **B**, the interaction energy of the atom associated with each permitted orientation is given by

$$U = -\mathbf{m}\cdot\mathbf{B}. \qquad \text{...(13)}$$

Since the contribution of neutron and proton to the magnetic moment is negligible in comparison to the magnetic moment of the electrons. The latter may be written as

$$\mathbf{m} = -\frac{e}{2m_e}(g_s\mathbf{S} + g_l\mathbf{L}), \qquad \text{...(14)}$$

where **S** is the total spin and **L** the total orbital angular momenta of the electrons.

Since $g_s = 2$ and $g_l = 1$, therefore we have

$$U = \frac{e}{2m_e}[2\mathbf{S}\cdot\mathbf{B} + \mathbf{L}\cdot\mathbf{B}] \qquad \text{...(15)}$$

The energy shift of a state depends on whether or not **S·B** and **L·B** are constants of the motion. If they are not, the energy shift must be deduced as an average value of U. Two limiting cases are of special interest :

(a) The strong field limit in which the interaction energy of the atom with the external field greatly exceeds the *internal spin-orbit interaction* which is proportional to **S·L**.

(b) The weak field limit in which the interaction with the external field is much less than the spin-orbit interaction.

Let us choose the z-axis to lie along the direction of the external field **B**. If **B** is sufficiently great, then Eq. (15) may be written as

$$U = \frac{eB}{2m_e}(2S_z + L_z). \qquad \text{...(16)}$$

This interaction will change the pattern of spectral lines. Its influence is called the *Paschen-Back* effect.

In a weak field, the interaction energy is not sufficient to break down the L–S coupling and the total angular momentum **J** couples with the field **B**. We thus have

$$\mathbf{m} = -\frac{e}{2m_e}g_j\,\mathbf{J}. \qquad \text{...(17)}$$

and $$U = -m_z B = g_j \frac{e}{2m_e} J_z B. \qquad ...(18)$$

The influence of this weak interaction is called *Zeeman effect.*

Since J_z is a constant of the motion, given by $J_z = n_j h/2\pi$.

For a given j, the energy splits a level into $2j + 1$ equally spaced levels. Since the extreme values of J_z are $jh/2\pi$ and $-jh/2\pi$. Therefore the energy gap between these levels is

$$(\Delta U)_{max} = 2g_j \frac{e}{2m_e}\left(\frac{j\,h}{2\pi}\right)B, \qquad ...(19)$$

where the factor g_j may be written as

$$g_j = \frac{3J^2 + S^2 - L^2}{2J^2}. \qquad ...(20)$$

In the quantum mechanical form, *i.e.*, in terms of the quantum numbers j, s and l, we get

$$g_j = \frac{3j\,(j+1) + s(s+1) - l(l+1)}{2j\,(j+1)} \qquad ...(21)$$

13.3 MAGNETIZATION OF MATTER

A material body is consisting of large number of atoms and thus large number of electrons. Each electron produces orbital and spin magnetic moments and can be assumed as a dipole. The nucleus also contributes to the total magnetic moment, although it is about 10^{-3} that of electronic moments. In the absence of any external magnetic field, the dipoles of individual atoms are randomly orientated due to thermal agitation and the magnetic moments thus cancel. In some materials the cancellation is complete and net magnetic moment is zero.

When we apply an external magnetic field of the flux density **B**, two processes may occur :

(1) All atoms which have non zero magnetic moment are aligned with the applied field due to the torque τ. The increase in magnetic moment is along **B**.

(2) If the atom has a zero magnetic moment, the applied magnetic field distorts the electron orbit and thus induces magnetic moment. The induced magnetic moment is given by Eq. (5).

In materials, referred to as *diamagnetic,* such as *Cu, Bi, Zn, Ag, Pb* and *Hg*, the individual atoms do not have an intrinsic magnetic moment. When an external magnetic field is applied the second process occurs. The induced magnetic moment is thus set-up in the direction opposite to **B**. In this case the magnetic flux density in the interior of the body will be less than that of the external field **B**.

In materials, referred to as *paramagnetic,* such as, *Al, Pt, Cr, Mn*...etc., the constituent atoms have intrinsic magnetic moments, and the cancellation of magnetic moment vectors is not complete. When an external magnetic moment is applied, both of the above processes occur and the resultant magnetic moment is always in the direction of magnetic field **B** as the first effect predominates over the second.

Thus we see that the material body can be assumed as a volume distribution of magnetic dipoles, if it is acted on by an external magnetic field. If m is the average magnetic dipole moment per atom or molecule and there are N such effective dipoles per unit volume, the magnetization is defined as $\mathbf{M} = N\mathbf{m}$. It is also known as *magnetic dipole polarization per unit volume.* If the dipoles are not aligned parallel to each other, then **M** is the net magnetic moment per unit volume. It is measured in amperes meter and corresponds to the polarization **P** in the dielectric.

If the magnetic moment of i^{th} atom is $\mathbf{m}_i$, we may define the magnetization vector **M** as

$$\mathbf{M} = \underset{\Delta v \to 0}{\text{Limit}} \frac{1}{\Delta v} \Sigma \mathbf{m}_i, \qquad \text{...(22)}$$

where Δv the volume element of the material is made very small from the macroscopic point of view, but it contains a statistically large number of atoms. The quantity M thus becomes a vector point function. In the unmagnetized matter, M is zero as a result of random orientation of $\mathbf{m}_i$. The action of applied magnetic field changes the orientation of $\mathbf{m}_i$. The degree of magnetization depends on this applied field as well as on the material. From this point of view, all known materials fall into three groups (dia-, para-and ferro-magnetic).

In a vacuum the magnetic induction in the solenoid is given by

$$B = \mu_0(Ni/l) = \mu_0 j^s,$$

where j^s is the solenoidal current density, the number of amperes circulating around the solenoid per meter of its length. Its unit is *amp/m*. It is different from the current density j, defined as amp/m^2.

Rowland ring experiments show that **B** is modified by the presence of matter in the toroid. Thus **B** is not only due to solenoidal current density j^s_{free} in the toroidal coil, but also due to magnetization of the matter. The magnetic effects in magnetic materials are due to the atomic magnetic dipoles in the materials. These dipoles result from the orbital magnetic moments of the electrons spinning like a top and from the magnetic moments of the atomic nuclei. The rotating (spinning) and circulating electrons thus constitute a current known as *atomic current*. It is without a charge transport.

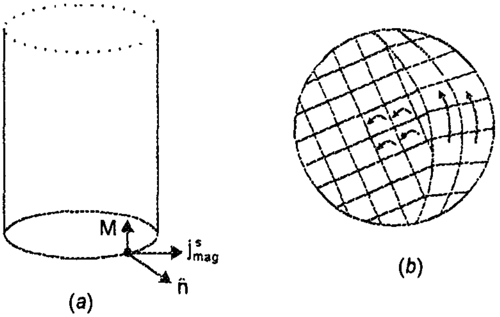


Fig. 13.4. Circulating currents and magetization.

On the other hand, the external current which corresponds to charge transport is called *free current*. The atomic current is defined in terms of an *effective solenoidal current density* j^s_{mag} that describes the magnetic effect of the matter.

In order to show the connection between magnetization **M** and the effective solenoidal current density j^s_{mag}, let us consider a cylindrical rod of meter length which has a *uniform magnetization* parallel to its axis. We know that such a uniform magnetization means a uniform density of circulatory currents everywhere inside the material. Fig. 13.4 (*a*).

A small slice of this cylinder, as shown in Fig. 13.4(*b*), may be considered to be made up of large number of cells of very small area. As the atomic current is passing through each cell, hence the current due to one cell is equal and opposite to that of the adjoining cell on every surface and the effects of all the current loops will cancel inside the slice, leaving only the circulating currents around the outer surface of the slice. Thus we have net circulating current on the surface of the rod. Due to this reason a uniformly magnetized rod is said to be equivalent to a long solenoid carrying an electric current, known as *Amperian current*.

For a cylinder of length l and area of cross section S, the magnetic moment round the periphery will be equal to the sum of the magnetic moments of all the cells, *i.e.*,

$$\begin{aligned} m &= \text{Circulating current} \times \text{area of the loop} \\ &= l \times \text{effective solenoidal current density } j^s_{mag} \times S \\ &= \text{Magnetization } M \times \text{volume of the cylinder } (lS). \\ \therefore \quad l j^s_{mag} S &= MlS \text{ or } j^s_{mag} = M. \qquad \text{...(23)} \end{aligned}$$

Thus we see that the surface current flowing per unit length at the surface of the magnetized volume is numerically equal to the magnetization M. It is also called *surface magnetization current density* or *linear current density*. In vector form $\mathbf{j}_{m,s} = \mathbf{M} \times \hat{n}$, ...(24)

where $\hat{n}$ is the out drawn normal to the surface.

Non-Uniform Magnetization : Let is now consider the most general case in which the magnetization is not uniform but varies from point to point in the material. We may assume magnetization **M** to be a vector point function. We know that in a uniform magnetization, the currents in the various loops tend to cancel each other out and the net effective current in the interior of the material is zero. In a non-uniform magnetization, the cancellation will not be complete and there will be a net current in the volume of the material.

To derive the relationship between the magnetization **M** and the *effective volume current density* $\mathbf{j}_{m.v}$ due to magnetization (or magnetization current density), consider two small volume elements, each of volume $\Delta x\ \Delta y\ \Delta z$, located next to each other in a piece of magnetic material. Fig. 13.5(a) shows two volume elements along y-direction (*i.e.*, in $x-y$ plane). If M_z be component of **M** along z-axis on element 1, then it will be $M_z + \Delta M_z$ on the element 2, which is at a distance Δy from the first element. Therefore we have

$$\Delta M_z = (\partial M_z / \partial y) \Delta y.$$

Let I_1 and I_2 be the circulating currents in these elements, as shown in Fig. 13.5 (a). Thus we have

$$M_z \Delta x\ \Delta y\ \Delta z = I_1\ \Delta x\ \Delta y \text{ and } (M_z + \Delta M_z)\ \Delta x\ \Delta y\ \Delta z = I_2\ \Delta x\ \Delta y.$$

Thus the net current on the common boundary of two elements due to circulating currents I_1 and I_2 is given by

$$I_2 - I_1 = \Delta M_z\ \Delta z = (\partial M_z / \partial y) \Delta y \Delta z.$$

It is along the x – direction. Thus the x-component of current density

$$j_{x,1} = I_x / \Delta y\ \Delta z = \partial M_z / \partial y. \qquad \text{...(25)}$$

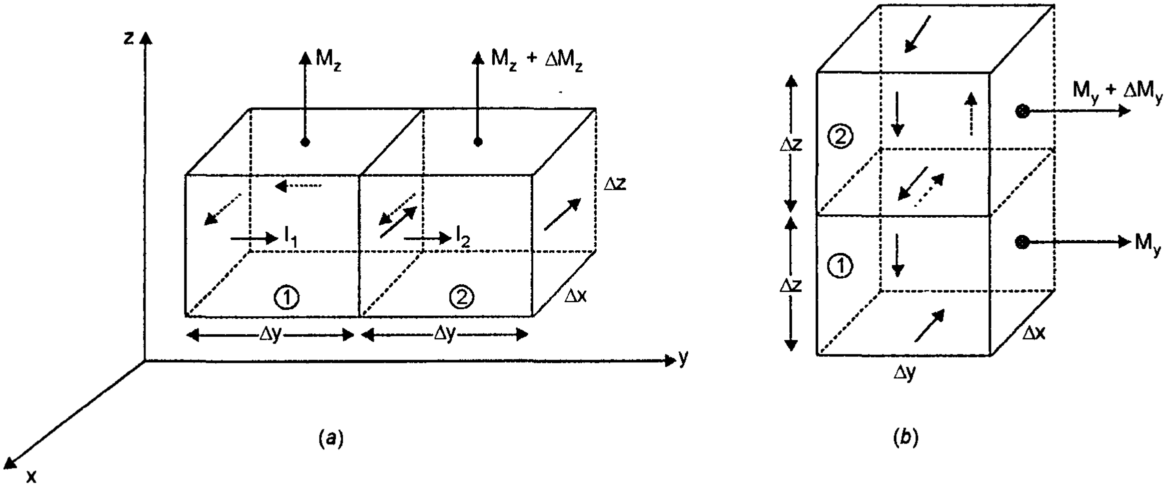


Fig. 13.5(a). Volume elements of magnetized material.

This current density has one more term due to variation of M_y, the y – component of magnetization **M** with z. For this we consider two volume elements along z – axis, *i.e.*, one on top of the other, as shown in Fig. 13.5(*b*).

If M_y be the component of **M** along y – direction on element 1 and $M_y + \Delta M_y$ on the element 2, which is at a distance Δz from the first element. Therefore

$$\Delta M_y = (\partial M_y/\partial z)\Delta z.$$

The net current in the common boundary of two elements due to circulating current I_1' and I_2', which are in the $x - z$ plane, is given by

$$I_1' - I_2' = -\Delta M_y\,\Delta y = -(\partial M_y/\partial z)\,\Delta y\,\Delta z.$$

It is along x – direction, thus the x – component of current density

$$j_{x,2} = -\partial M_y/\partial z. \qquad \text{...(26)}$$

Since the only surfaces of the elements which contribute to the current density in the x – direction are the x – y and $x - z$ surfaces. Therefore the net current density in the x – direction is given by

$$j_{m,x} = j_{x,1} + j_{x,2} = (\partial M_z/\partial y - \partial M_y/\partial z) = (\nabla \times \mathbf{M})_x.$$

Similar expression may be obtained for y – and z-components of the current density $\mathbf{j}_m$, *i.e.*,

$$j_{m,y} = (\nabla \times \mathbf{M})_y \text{ and } j_{m,z} = (\nabla \times \mathbf{M})_z,$$

$$\therefore \quad j_{m,x}\,i + j_{m,y}\,\mathbf{j} + j_{m,z}\,\mathbf{k} = (\nabla \times \mathbf{M})_x\,i + (\nabla \times \mathbf{M})_y\,\mathbf{j} + (\nabla \times \mathbf{M})_z\mathbf{k}$$

$$\text{or} \quad \mathbf{j}_{m,v} = \nabla \times \mathbf{M}. \qquad \text{...(27)}$$

This shows the net volume density of molecular current due to incomplete concellation of adjacent current loops in the interior of the magnetized body. Thus one can calculate **B** by replacing the magnetized material by its equivalent currents, $\mathbf{j}_{m,s}$ and $\mathbf{j}_{m,v}$.

13.4 AMPERE'S MODEL FOR THE EQUIVALENT CURRENTS

We know from article 10.15 that the vector potential **A** due to a current loop of magnetic dipole moment **m** is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\frac{\mathbf{m}\times\mathbf{r}}{r^3} = -\frac{\mu_0}{4\pi}\mathbf{m}\times\text{grad}\left(\frac{1}{r}\right). \qquad \text{...(28)}$$

Here r, the distance of the point from the centre of the loop, is large compared to the size of the loop. The contribution of the vector potential from the magnetic polarization of a material body will be obtained by integrating the contribution from each volume element of the material characterized by magnetic dipole element Mdv'. Hence

$$\mathbf{A}(x, y, z) = -\frac{\mu_0}{4\pi}\int_{v'}\mathbf{M}(x', y', z')\times\text{grad}'\left(\frac{1}{r}\right)dv', \qquad \text{...(29)}$$

where v' is the volume of the material body. Using vector identities for scalar function φ and vector function **F** as

$$\text{curl}\,(\varphi\mathbf{F}) = \varphi\,\text{curl}\,\mathbf{F} + \text{grad}\,\varphi\times\mathbf{F} \qquad \text{...(30)}$$

$$\text{and} \quad \int_v \text{curl}\,\mathbf{F}\,dv = \oint_S \hat{n}\times\mathbf{F}\,dS, \qquad \text{...(31)}$$

the integral (29) may be written as

15. Which physical properties are important to note when choosing the material for electromagnets.

$$\mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{v'}\left[\frac{1}{r}\nabla'\times\mathbf{M}\right]dv' + \int_{S'}\left(\frac{\mathbf{M}}{r}\times\hat{n}\right)dS'\right]. \qquad ...(32)$$

where S' is the surface bounding the volume v'.

If we compare this equation with the expression for **A** due to distribution of currents, we get

$\nabla' \times \mathbf{M}$ = Equivalent volume magnetization current density $\mathbf{j}_{m,\,v}$

and $\mathbf{M} \times \hat{n}$ = Equivalent surface magnetization current density $\mathbf{j}_{m,\,s}$.

$$\therefore \qquad \mathbf{A}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{v'}\frac{\mathbf{j}_{m,v}}{r}dv' + \int_{S'}\frac{\mathbf{j}_{m,s}}{r}dS'\right]. \qquad ...(33)$$

If the magnetization **M** is constant throughout the body, $\nabla' \times \mathbf{M}$ is zero and the first term is zero. The second term is zero when **M** is perpendicular to surface, *i.e.*,

$$\mathbf{M} \times \hat{n} = 0.$$

The equation (33) gives us an expression for the vector potential **A** due to an element of magnetized matter in terms of the equivalent current densities $\mathbf{J}_{m,\,v}$ and $\mathbf{J}_{m,\,s}$. We are actually interested in **B**, not in **A**. To calculate **B**, we take the curl of **A**, as

$$\mathbf{B} = \text{curl}\,\mathbf{A} = \nabla\times\mathbf{A}$$

$$\therefore \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{v'}\nabla\times\left(\frac{\mathbf{j}_{m,v}}{r}\right)dv' + \int_{S'}\nabla\times\left(\frac{\mathbf{j}_{m,s}}{r}\right)dS'\right] \qquad ...(34)$$

Using the vector identity (30), we get

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\left[\int_{v'}\frac{1}{r}(\nabla\times\mathbf{j}_{m,v})dv' + \int_{v'}\nabla\left(\frac{1}{r}\right)\times\mathbf{j}_{m,v}\,dv'\right] + \int_{S'}\frac{1}{r}(\nabla\times\mathbf{j}_{m,s})dS' + \int_{S'}\nabla\left(\frac{1}{r}\right)\times\mathbf{j}_{m,s}\,dS'\right]$$

Since the curl will operate with respect to unprimed coordinates (x, y, z) while the current densities $\mathbf{j}_{m,\,v}$ and $\mathbf{j}_{m,\,s}$ depend on primed coordinates (x', y', z'). Thus the first and third terms will not contribute to **B**.

$$\therefore \qquad \mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left[\int_{v'}\nabla\left(\frac{1}{r}\right)\times\mathbf{j}_{m,v}\,dv' + \int_{S'}\nabla\left(\frac{1}{r}\right)\times\mathbf{j}_{m,s}dS'\right] \qquad ...(35)$$

Equations (33) and (35) show that the vector potential **A** and the magnetic induction vector **B** due to the effect of a piece of magnetized matter can be computed with the help of surface current density $\mathbf{j}_{m,\,s}$ and a volume current density $\mathbf{j}_{m,\,v}$. These surface and volume currents are also known as *bound currents*. These are associated with molecular or atomic magnetic moments including the intrinsic magnetic moments of particles with spin. Let us represent the bound or equivalent magnetization current by $\mathbf{j}_m$.

In the presence of magnetized matter, the magnetic field is not only due to the conventional or true or free currents which can be measured in laboratory directly, but the bound currents also contribute effectively. The bound currents are equivalent surface currents only for the matter having uniform magnetization.

13.5 MAGNETIC FIELD INTENSITY H

When dealing with dielectric substances in an electric field, the electric displacement **D** is introduced in order to eliminate the effect of electric polarization **P**. A similar procedure is used for the magnetic substances.

We know that for steady currents and non-magnetic materials

$$\text{curl}\,\mathbf{B} = \mu_0\mathbf{j}. \qquad \ldots(36)$$

Rowland ring experiment shows that the magnetic induction **B** is modified by the presence of matter in the toroid (article 13.1). When the space inside the toroid is filled with a magnetic material, it gets magnetized due to the current in the toroid. The magnetic field **B** will be greater in the presence of magnetic material than without the material (or in vacuum). The large value of **B** is due to magnetization or magnetic polarization of the material. This increased value of **B** can be obtained even in the absence of the material in the toroid (or core) by increasing the current in the toroid (*known as free current*). Thus the magnetization of the magnetic material is equivalent in its effect on **B** to such an increase in current. This current is known as bound current (equivalent surface and volume magnetization currents) and is associated with molecular or atomic magnetic moments. Thus the total magnetic field **B** is not only due to the free current density $\mathbf{j}_f$ but also due to the bound or equivalent magnetization current density $\mathbf{j}_m$. Hence relation (36) must be replaced by

$$\text{curl}\,\mathbf{B} = \mu_0(\mathbf{j}_f + \mathbf{j}_m). \qquad \ldots(37)$$

Since the equivalent volume magnetization current density $\mathbf{j}_m$ is equal to the curl of the magnetic polarization **M** of the material, hence

$$\text{curl}\,\mathbf{B} = \mu_0(\mathbf{j}_f + \text{curl}\,\mathbf{M})$$

or

$$\text{curl}\,(\mathbf{B}/\mu_0 - \mathbf{M}) = \mathbf{j}_f. \qquad \ldots(38)$$

The vector $(\mathbf{B}/\mu_0 - \mathbf{M})$ is replaced by a new vector quantity **H** called *the magnetic field intensity* or *magnetic field strength.* Thus we have

$$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}. \qquad \ldots(39)$$

This vector eliminates the neccessity of dealing directly with the polarization as its curl directly gives the free current density. Thus we see that the magnetic field intensity **H** is related to the free current in the way **B** is related to the total current $(\mathbf{j}_{free} + \mathbf{j}_{bound})$.

We know that the divergence of a magnetic field **B** in a free space is always zero. It is also true if the magnetized matter is present. This is because **B** = curl **A** and the divergence of a curl **B** is always zero. It represents that under no conditions are there sources or sinks for lines of **B**.

As div **B** is always zero, hence Eq. (39) shows that **H** and **M** need not be separately divergenceless, but their sum must be. Therefore, we have

$$\text{div}\,\mathbf{H} = -\,\text{div}\,\mathbf{M}. \qquad \ldots(40)$$

It shows that div $\mathbf{H} \neq 0$ for **M** having non vanishing divergence. As Eq. (38) can be written as curl $\mathbf{H} = \mathbf{j}_f$, hence on integrating over a surface S we have

$$\int (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int \mathbf{j}_f \cdot d\mathbf{S}.$$

Using Stoke's theorem on the left hand side, we get

$$\int_C \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{j}_f \cdot d\mathbf{S} = i_f, \qquad \ldots(41)$$

where C is the curve bounding the surface S and i_f is the current of free charges or free current linking the curve C. The left hand term is called the *magnetomotance*, in analogus with the electromotive force in electric fields. The above equation is the more general form of *Ampere's circuital law*, as it can be used to calculate **H** even in the presence of magnetic material. The relation div **B** = 0 shows that the average macroscopic field inside the matter is **B** not **H**, or **B** is the *fundamental magnetic field vector*. Eq. (41) shows that the unit of H is ampere/meter.

In any region of space where there is no current carrying conductor, the free current i_f is zero. Thus Eq. (41) becomes

$$\oint \mathbf{H}\cdot d\mathbf{l} = 0. \qquad \text{...(42)}$$

H at some point inside the magnetized material is the vector sum of $\mathbf{B}/\mu_0$ and $-\mathbf{M}$. The lines of **H** are as it should be for if magnetic poles really were the source of magnetization, instead of electric current. We therefore conclude our results as :

(*i*) $\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$ in magnetic materials, while $\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}$ in dielectric. Thus **H** is the vector sum of **B** and **M**, while **D** is that of **E** and **P**.

(*ii*) The line integral of **H** is the free or true current and does not include the magnetizing current. Similarly the surface integral of **D** gives free charges only and not the polarization charges.

(*iii*) **H** can be directly calculated by Ampere's circuital law (Eq. 40), while **D** can be directly calculated by Gauss's theorem $\int_S \mathbf{D}.d\mathbf{S} = q$.

Thus we see that **H** plays the same role in magnetic materials as does the vector **D** in dielectrics. From this point of view, it should be more correct to call **H**, *the magnetic induction vector*. But this name belongs historically to **B**, although **B** plays a role in the magnetic matierals analogus to that of **E** in dielectrics. It follows that the analogus of permittivity ε in the electric field theory is $1/\mu$ in the magnetic field theory. It is impossible now to change the terminology, but we must remember the true significance of the magnetic field vectors **H** and **B**.

13.6 MAGNETIC SUSCEPTIBILITY AND PERMEABILITY

We know that both diamagnetic and paramagnetic substances develop a magnetic moment depending on the applied field. Leaving aside the saturation value of **M** at very low temperatures in fairly strong magnetic field the relation between **M** and **H** is linear. The ratio of magnetization **M** to magnetic field intensity **H** is a constant in general and is called the *magnetic susceptibility*. It is a measure of how susceptible the material is to polarization by an applied magnetic field. It is the characteristic of the material and is denoted by χ_m. It is known as *volume susceptibility*.

Thus, $$\mathbf{M} = \chi_m \mathbf{H}. \qquad \text{...(43)}$$

If the magnetic moment of unit mass of a material is $\mathbf{M}_{mass}$, then

$$\mathbf{M}_{mass} = \chi_{m,\,\text{mass}}\,\mathbf{H} \quad \text{or} \quad \chi_{m,\,mass} = \chi_m/\rho. \qquad \text{...(44)}$$

If the magnetic moment of one molecule is $\mathbf{M}_{\text{mol}}$ then

$$\mathbf{M}_{mol} = \chi_{m,\,mol}\,\mathbf{H} \quad \text{or} \quad \chi_{m,\,mol} = \chi_m A/\rho. \qquad \text{...(45)}$$

where ρ and A are the density and molecular weight of the material respectively. Here $\chi_{m,\,mass}$, and $\chi_{m,\,mol}$ are known as mass and molar susceptibilities, respectively.

χ_m is positive for paramagnetic and negative for diamagnetic materials. For these both cases it is less than one. Magnetic susceptibility of a paramagnetic material decreases with increasing temperature as $\chi_m = C/T$, where the constant C is known as *Curie constant.* χ_m is independent of temperature for diamagnetic substances.

Table 13.1. *The magnetic susceptibilities of paramagnetic and diamagnetic materials at approx, room temperature in SI unit*

Paramagnetic Materials	$\chi_{m,\,mass}$ 10^{-8} (m^3/kg)	Diamagnetic Materials	$\chi_{m,\,mass}$ 10^{-8} (m^3/kg)
Aluminium (Al)	0.82	Bismuth (Bi)	–1.7
Iron Auminium alloy	38.2	Cadmium (Cd)	–0.23
Calcium (Ca)	1.4	Copper (Cu)	–0.11
Chromium (Cr)	4.5	Diamond	–0.62
Cuprous oxide (CuO)	1.5	Germanium (Ge)	–0.15
Ferric oxide (Fe_2O_3)	26.00	Gold (Au)	–0.19
Magnesium (Mg)	0.69	Helium (He)	–0.59
Manganese (Mn)	1	Lead (Pb)	–0.18
Platinum (Pt)	1.65	Silver (Ag)	–0.25
Tantalum (Ta)	1.1	Zinc (Zn)	–0.20

Combining Eqs. (39) and (43), we get

$$\mathbf{B} = \mu_0 (\mathbf{H} + \chi_m \mathbf{H}) = \mu_0 (1 + \chi_m)\,\mathbf{H} = \mu \mathbf{H}, \qquad \text{...(46)}$$

where $\mu = \mu_0 (1 + \chi_m)$ is called the *magnetic permeability* of the material. The dimensionless quantity $\mu/\mu_0 = k_m$ is called the *relative permeability* of the material. k_m is unity for vacuum. Thus the permeability of vacuum is μ_0. For diamagnetic materials k_m is slightly less than unity and for paramagnetic materials it is slightly greater than unity. Thus for the former **M** and **H** are antiparallel and for the latter these are parallel. The permeability is a measure of the degree of penetration of magnetic field through the substance.

Table 13.2. Relative permeability (k_m or μ_r) of some materials

Diamagnetic	μ_r	Paramagnetic	μ_r	Ferromagnetic	μ_r
Bismuth	0.999833	Oxygen (STP)	0.999998	Cobalt	250
Mercury	0.999968	Air	1.00000037	Nickel	600
Silver	0.999974	Aluminium	1.000022	Soft iron	5000
Lead	0.999983	Tungsten	1.0008	Silicon-iron	7000
Copper	0.999990	Platinum	1.00032		
Water	0.999991	Manganese	1.0012		
Hydrogen (STP)	1.0				

13.7 BOUNDARY CONDITIONS FOR B AND H

The quantities **B** and **H** undergo abrupt changes at the boundary between magnetic materials with different values of μ. These are characterized by the boundary conditions. The procedure for deriving the boundary conditions is precisely the same as in the case of an electrostatic field (*i.e.*, for **E** and **D**).

(*i*) *Continuity of the normal component of* **B** : To find the boundary condition for the normal component of vector **B**, let us assume an interface between two magnetic media with magnetic permeabilities μ_1 and μ_2. Consider a small pillbox shaped Gaussian surface (short cylindrical imaginary volume) of small (negligibly) height, whose top and bottom faces are parallel and on either side of the interface.

Gauss's law for magnetism, similar to the law in electrostatics, tells us that the algebraic sum of the number of magnetic flux lines coming out of the surface of a volume is always zero, *i.e.*,

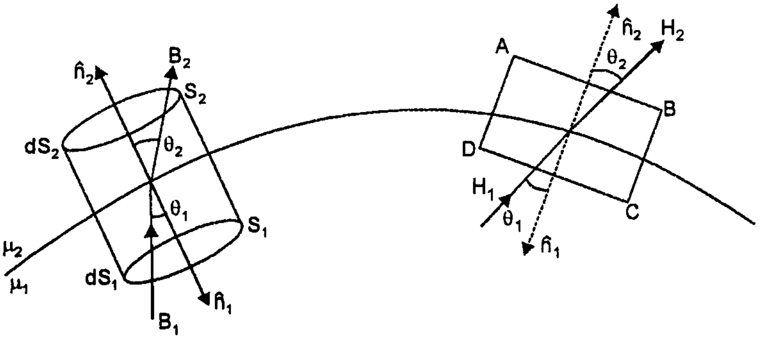


Fig. 13.6. Lines of **B** and **H** at the boundary.

$$\text{div } \mathbf{B} = 0, \text{ or } \Phi_{\mathbf{B}} = \int_S \mathbf{B}_2 \cdot d\mathbf{S} = 0$$

Thus the net flux through the pill box will also be zero.

$$\Phi = \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 + \int_C \mathbf{B} \cdot d\mathbf{S} = 0.$$

The height of the pill box is assumed to be so small that the magnetic flux through the curved surface is negligible.

$$\therefore \quad \int_{S_1} \mathbf{B}_1 \cdot d\mathbf{S}_1 + \int_{S_2} \mathbf{B}_2 \cdot d\mathbf{S}_2 = 0.$$

$$\text{As} \quad d\mathbf{S}_1 = \hat{n}_1\, dS,\ d\mathbf{S}_2 = \hat{n}_2\, dS \text{ and } \hat{n}_1 = -\hat{n}_2,$$

$$\therefore \quad \mathbf{B}_1 \cdot \hat{n}_1\, dS + \mathbf{B}_2 \cdot \hat{n}_2\, dS = 0 \text{ or } -\mathbf{B}_1 \cdot \hat{n}_2 + \mathbf{B}_2 \cdot \hat{n}_2 = 0$$

$$\therefore \quad B_1 \cos\theta_1 = B_2 \cos\theta_2 \quad \text{or} \quad B_{1n} = B_{2n}, \qquad \text{...(47)}$$

i.e., the normal component of **B** *is continuous across any interface.*

The corresponding boundary condition on the normal component of **H** may be obtained by using relation **B** = μ**H**.

$$\therefore \quad \mu_1 H_{1n} = \mu_2 H_{2n}. \qquad \text{...(48)}$$

(*ii*) *Continuity of the tangential component of* **H** : To find the boundary conditions on the tangential components of **H**, let us assume a closed rectangular path *ABCD* with its long sides *AB* and *CD* parallel to the interface and close to it. If no true current is enclosed by this loop, then according to Ampere's circuital law

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_A^B \mathbf{H}_2 \cdot d\mathbf{l}_2 + \int_B^C \mathbf{H} \cdot d\mathbf{l} + \int_C^D \mathbf{H}_1 \cdot d\mathbf{l}_1 + \int_D^A \mathbf{H} \cdot d\mathbf{l} = 0. \qquad \text{...(49)}$$

The path lengths *AB* and *CD* are equal and may be assumed as Δl. As the sides *BC* and *DA* are negligibly small, hence their contributions will be zero. Further assuming $\mathbf{H}_1$ and $\mathbf{H}_2$ to be constant for the length Δl. Thus Eq. (49) becomes

$$\mathbf{H}_2 \cdot d\mathbf{l}_2 + \mathbf{H}_1 \cdot d\mathbf{l}_1 = 0 \qquad \text{or} \qquad \mathbf{H}_2 \cdot \Delta \mathbf{l} - \mathbf{H}_1 \cdot \Delta \mathbf{l} = 0$$

$$\therefore \qquad H_2 \sin\theta_2 = H_1 \sin\theta_1 \qquad \text{or} \qquad H_{2t} = H_{1t}. \qquad \text{...(50)}$$

The corresponding boundary conditions for the tangential components of **B** are

$$B_1 \sin\theta_1/\mu_1 = B_2 \sin\theta_2/\mu_2 \qquad \text{or} \qquad B_{1t}/\mu_1 = B_{2t}/\mu_2. \qquad \text{...(51)}$$

Equations (47) and (50) are identical with those for **D** and **E** found for dielectric boundaries except that –(*a*) the normal component of **B** is strictly continuous across the boundary while that of **D** is continuous only if there is no surface charge, (*b*) the tangential component of **E** is strictly continuous, while that of **H** is continuous across the boundary only if there is no true surface current. *These differences confirm the existence of true electric charge and the absence of true magnetic charge.*

On combining equations (47) and (51), we get

$$\therefore \qquad \tan\theta_2 = (\mu_2/\mu_1)\tan\theta_1. \qquad \text{...(52)}$$

Thus the lines of **B** or of **H** are farthest away from the normal to the interface in the medium of larger permeability. *It is similar to the refraction of electric lines of force.*

13.8 MAGNETIC SPHERE IN A UNIFORM MAGNETIC FIELD

Consider a sphere of radius *a* made of a linear magnetic material of permeability μ_1 and embedded in a medium of permeability μ_2. If a uniform magnetic field $\mathbf{B}_0$ is initially present in the medium, it will be modified by the presence of a sphere. The problem is same as the case of a dielectric sphere placed in a uniform electric field, discussed in chapter 6.

Since there are no external currents and the materials are linear, the magnetic scalar potential satisfies Laplace's equation. Choosing the origin of the coordinate system at the centre of the sphere and the magnetic field pointing along the *z*– direction, the potentials are still symmetric about the ϕ – direction and thus will depend on *r* and θ only. The potential functions inside and outside the sphere can be written as a linear combination of the zonal harmonics, as

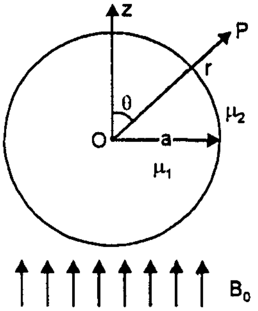


Fig. 13.7. A magnetic sphere in a uniform magnetic field.

$$\varphi_1(r,\theta) = A_1 r\cos\theta + A_2 r^{-2}\cos\theta \qquad r < a \qquad \text{...(53)}$$

and

$$\varphi_2(r,\theta) = C_1 r\cos\theta + C_2 r^{-2}\cos\theta \qquad r > a \qquad \text{...(54)}$$

Here 1/*r* term has been dropped out because it implies the existence of magnetic monopoles.

Since the potential φ is finite at the origin, therefore $A_2 = 0$. At a very large distance, $\varphi_2 = -(B_o/\mu_2)\, r\cos\theta$, therefore $C_1 = -B_o/\mu_2$.

Thus Eqs. (53) and (54) become

$$\varphi_1(r,\theta) = A_1 r\cos\theta \qquad \text{...(55)}$$

$$\varphi_2(r,\theta) = -(B_o/\mu_2)\, r\cos\theta + C_2 r^{-2}\cos\theta. \qquad \text{...(56)}$$

These relations required two more equations relating A_1 and C_2, which can be deduced from the boundry conditions on the surface of the sphere. (*i.e.* r = a). $H_{1t} = H_{2t}$ and $B_{1n} = B_{2n}$.

The continuity of the tangential components of H on the surface is equivalent to the continuity of the potentials on the surface. Thus

$$A_1 a = -(B_o/\mu_2)\, a + C_2\, a^{-2}. \qquad \text{...(57)}$$

For the continuity of the normal component of **B**, we require the relations :

$$B_{1n} = \mu_1 H_{1n} = -\mu_1\, \partial\varphi_1/\partial r = -\mu_1 A_1 \cos\theta$$

and

$$B_{2n} = \mu_2 H_{2n} = -\mu_2\, \partial\varphi_2/\partial r = B_o \cos\theta + 2\mu_2 C_2 r^{-3} \cos\theta,$$

$$\therefore \quad -\mu_1 A_1 = B_o + 2\mu_2 C_2 a^{-3}. \qquad \text{...(58)}$$

Solving Eqs. (57) and (58), we get

$$A_1 = -\frac{3B_o}{\mu_1 + 2\mu_2} \text{ and } C_2 = \frac{B_o a^3 (\mu_1/\mu_2 - 1)}{\mu_1 + 2\mu_2}.$$

Substituting the values of A_1 and C_2 in Eqs. (55) and (56) and using the relations $\mathbf{B} = -\mu\, \nabla\varphi$, we get the magnetic fields inside and outside the sphere as

$$\mathbf{B}_1 = \frac{3\mu_1 B_0 \mathbf{k}}{\mu_1 + 2\mu_2} \qquad \text{...(59)}$$

and

$$\mathbf{B}_2 = B_0\, \mathbf{r} + \frac{(\mu_1/\mu_2 - 1)}{(\mu_1/\mu_2 + 2)} B_0 \left(\frac{a}{r}\right)^3 (2\hat{r}\cos\theta + \hat{\theta}\sin\theta). \qquad \text{...(60)}$$

Relation (59) gives the magnetic field inside the sphere which is the sum of the external magnetic field $\mathbf{B}_0$ and the magnetic field created by the magnetisation of the sphere. The latter is called *demagnetizing* field H_{dem}, given by

$$H_{dem} = 2\frac{(1 - \mu_2/\mu_1)}{\mu_1 + 2\mu_2} B_0. \qquad \text{...(61)}$$

13.9 THE FIELD OF A PERMANENT MAGNET

Let us discuss the materials with permanent magnetic polarization. The permanent magnets can be made from alloys and compounds of the ferromagnetic substances. In the case of permanent magnets the alignment of dipoles results from internal forces within the matter. **M**, **H** and **B** all have non-vanishing values inside the magnet even though there is no true current. Let us take following three important cases only.

(*a*) **Bar Magnet :** To find **B** and **H** fields, we know that the self magnetization produces both inside and outside the matter, the use is made of equation $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$. From Gauss's law for **B**, $\int_S \mathbf{B}\cdot d\mathbf{S} = 0$

$$\text{or } \mu_0 \int_S \mathbf{H}.d\mathbf{S} + \mu_0 \int_S \mathbf{M}\cdot d\mathbf{S} = 0,$$

$$\therefore \int_S \mathbf{H}\cdot d\mathbf{S} = -\int_S \mathbf{M}\cdot d\mathbf{S}. \qquad \text{...(62)}$$

If A is the area of cross section of the magnet and **M** its magnetization then for the uniform magnetization, we have

$$-\int_S \mathbf{M}\cdot d\mathbf{S} = MA = \int_S \mathbf{H}\cdot d\mathbf{S}. \qquad \text{...(63)}$$

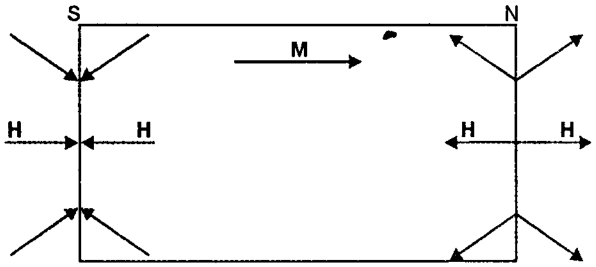


Fig. 13.8. A bar magnet.

The negative sign results from the fact that the lines of **M** are directed inward at the gaussian surface. Thus the **H** – field originates where the magnetization **M** ends and **H** field ends where the **M** originates.

As there are no real currents at the surface, hence $\oint \mathbf{H}\cdot d\mathbf{S} = 0$. Therefore the lines of **H** are uniformly distributed. This leads directly to the result that the effects of **H** originating from a small end element fall off as $1/r^2$. By analogy with the electrostatic case (*i.e.*, $-\int \mathbf{P}_S\cdot d\mathbf{S} = q_p$, the polarization charge), we can define the effects of the end discontinuity by

$$\int_S \mathbf{M}\cdot d\mathbf{S} = q_m\ amp/m. \qquad ...(64)$$

where q_m is called the *magnetic charge* or the magnetic pole strength. Gauss's law then becomes

$$\int_S \mathbf{H}\cdot d\mathbf{S} = MA = q_m\ amp/m. \qquad ...(65)$$

If σ_m be the surface density of magnetic charge, then $\mathbf{M}.\hat{n} = \sigma_m$. Thus the discontinuity in **M** is $\mathbf{M}.\hat{n} = M = \sigma_m$ on the right surface of the magnet and $\mathbf{M}.\hat{n} = -M = -\sigma_m$ on the left surface. σ_m is also called the *surface density of magnetic pole strength.* Its unit is A/m.

Similar to the polarization volume density ρ_p in the polarized dielectric, we have for the magnetized material the volume density of magnetic charge ρ_m, given by

$$\nabla\cdot\mathbf{H} = -\nabla\cdot\mathbf{M} = \rho_m. \qquad ...(66)$$

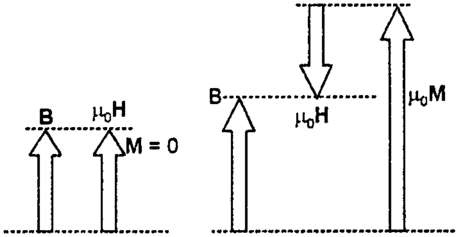
It is also called the *magnetic pole density*. Its unit is A/m^2.

The end of the magnet from which lines of **H** emerge is called the *north pole* and the end on which lines of **H** converge is called the *south pole*. The values of **B** and **H** at points outside and inside the magnet are as shown in Fig. 13.9(*a*) and (*b*) respectively. Thus we see that **M** and **H** are in opposite directions inside the magnet. At a point far away from the end of the magnet, so that the size of the end region is small compared with the distance r, the pole can be taken as a point. Now applying Gauss's law for **H** over a Gaussian sphere of radius r, we get.

$$\int \mathbf{H}\cdot d\mathbf{S} = q_m \quad \text{or} \quad 4\pi r^2 H = q_m$$

$$\text{or} \qquad H = \frac{q_m}{4\pi r^2} \qquad ..(67)$$

$$\text{and} \qquad B = \mu_0 H = \frac{\mu_0}{4\pi}\frac{q_m}{r^2}. \qquad ..(68)$$



(*a*) (*b*)

Fig. 13.9. Lines of **B** and **H** (*a*) outside and (*b*) inside the magnet.

The magnetic field at any point outside the bar magnet can be calculated by using the above equation for both the ends. The relation for no real current

$\oint \mathbf{H}.d\mathbf{l} = 0$, shows that **H** is *conservative* or *irrotational*. Thus curl **H** = 0. Since curl grad $\varphi = 0$, therefore similar to the case of electrostatic, we may define **H** as a negative gradient of a scalar potential φ_m. Thus we have

$$\mathbf{H} = -\nabla\varphi_m = \frac{q_m}{4\pi r^2} = \nabla\left(-\frac{q_m}{4\pi r}\right)$$

$$\text{or} \qquad \varphi_m = q_m/4\pi r. \qquad ...(69)$$

Thus the magnetic potential φ_m at any point outside the bar magnet can be calculated by using the above equation for a point magnetic charge q_m.

(*b*) **Cylindrical or disc shaped Magnet :** Let us consider a cylindrical magnet of length L and radius a, magnetized uniformly with a magnetization $\mathbf{M}_0$ in the positive z-direction. In the interior the volume density of polarization charge $\mathbf{j}_m = \nabla \times \mathbf{M}_0$. Since $\mathbf{M}_0$ is constant hence $\nabla \times \mathbf{M}_0 = 0$ and all the interior currents effectively cancel. Consequently, the entire field outside the cylinder is created by the surface current flowing in circles. Since $\mathbf{M}_0$ is parallel to the sides, hence there will be no surface charges on the sides. We have an equivalent magnetic charges at the ends of the cylindrical magnet. An equivalent magnetic charge.

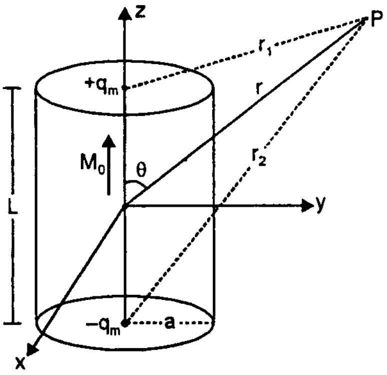


Fig. 13.10. Cylindrical Magnet.

$$q_m = -\int \mathbf{M}_0 \cdot d\mathbf{S} = \pi a^2 M_0. \qquad \text{...(70)}$$

Thus the charges on the upper and lower surfaces are $\pi a^2 M_0$ and $-\pi a^2 M_0$ respectively.

For the greater distances ($r >> a$) one can assume the charges at the centres of each side face. The magnetic scalar potential due to these point magnetic charges is thus given by

$$\varphi_m = \frac{1}{4\pi} q_m \left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{\pi a^2 M_0}{4\pi}\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad \text{...(71)}$$

Since $H = B/\mu_0 = -\nabla\varphi_m$, hence

$$H = -\frac{\pi a^2 M_0}{4\pi} \nabla\left(\frac{1}{r_1} - \frac{1}{r_2}\right). \qquad \text{...(72)}$$

For $r >> L$, we my write Eqs. (71) and (72) as

$$\varphi_m = \frac{\pi a^2 M_0}{4\pi} \frac{L\cos\theta}{r^2} = \frac{m\cos\theta}{4\pi r^2} \qquad \text{...(73)}$$

and

$$H = -\frac{m}{4\pi} \nabla\left(\frac{\cos\theta}{r^2}\right). \qquad \text{...(74)}$$

where m ($= \pi a^2 L M_0$) is the *total effective dipole moment of the bar magnet.*

(*c*) **Uniformly Magnetized Sphere :** Consider uniformly magnetized sphere of radius a that has a magnetization $\mathbf{M}$ along the z-direction. Since $\mathbf{M}$ ($= M_0\, \mathbf{k}$) is constant, then $\nabla.\mathbf{M} = 0$ and thus the scalar magnetic potential φ satisfies Laplace's equation. Taking the origin of our coordinate system at the centre of the sphere, we may express the potential as a sum of zonal harmonics ($\mathbf{M}$ along z-axis makes the potential dependent on r and θ). Since $\mathbf{M}.\hat{n} = M_0 \cos\theta$, we expect the angular dependence of φ to involve only $\cos\theta$, thus we have

$$\varphi_1 = A_1 r\cos\theta + A_2 r^{-2}\cos\theta, \qquad r < a \qquad \text{...(75)}$$

$$\varphi_2 = C_1 r\cos\theta + C_2 r^{-2}\cos\theta \qquad r > a \qquad \text{...(76)}$$

Since, φ_1 should be finite at $r = 0$, therefore $A_2 = 0$. At a very large distance φ_2 should vanish, therefore $C_1 = 0$. Using the boundary conditions at the surface of the sphere (*i.e.*, $r = a$).

(*i*) $H_{1t} = H_{2t}$ gives $\varphi_1 = \varphi_2$ at the surface,

$$\therefore \quad A_1\, a \cos\theta = C_2\, a^{-2}\cos\theta \text{ or } A_1 = C_2\, a^{-3}. \qquad ...(77)$$

(*ii*) $B_{1n} = B_{2n}$ or $\mu_0\,(\mathbf{H}_1 + \mathbf{M}).\hat{n} = \mu_0\mathbf{H}_2.\hat{n}$ gives

$$-\partial\varphi_1/\partial r + M_0\cos\theta = -\partial\varphi_2/\partial r \text{ at the surface,}$$

$$\therefore \quad -A_1\cos\theta + M_0\cos\theta = 2C_2\, a^{-3}\cos\theta$$

or

$$A_1 = M_0 - 2C_2\, a^{-3}. \qquad ..(78)$$

Combination of Eqs. (77) and (78) gives

$$A_1 = M_0/3 \text{ and } C_2 = M_0 a^3/3.$$

$$\therefore \quad \varphi_1 = \tfrac{1}{3}M_0\, r\cos\theta \qquad ..(79)$$

and

$$\varphi_2 = \tfrac{1}{3}M_0\,(a^3/r^2)\cos\theta. \qquad ..(80)$$

Thus we see that the internal potential $\varphi_1\left(=\tfrac{1}{3}M_0\, z\right)$ is linear with z and corresponds to a uniform magnetic field

$$\mathbf{H}_1 = -\tfrac{1}{3}M_0\,\hat{z}. \qquad ..(81)$$

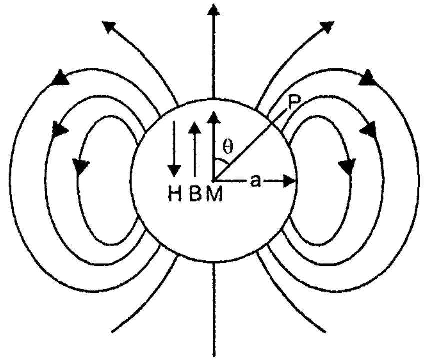


Fig. 13.11. Uniformly magnetized sphere.

It shows that $\mathbf{H}_1$ points along the $-z$ direction, *i.e.*, it opposes the magnetization $M_0\,\hat{z}$ and thus results in a *demagnetization effect*. Due to this reason $\mathbf{H}_1$ is called *demagnetizing field*.

The external potential φ_2 produced by the sphere may be written as

$$\varphi_2 = m\cos\theta/4\pi r^2, \qquad ...(82)$$

where $m = (4\pi/3)\, M_0\, a^3$ = volume of the sphere × magnetization M
= magnetic dipole moment.

The magnetic field corresponding to φ_2, outside the sphere

$$\mathbf{B}_2 = \mu_0\,\mathbf{H}_2 = -\mu_0\,\nabla\varphi_2$$

$$= \frac{\mu_0 M_0\, a^3}{3r^3}(2\hat{r}\,\cos\theta + \hat{\theta}\sin\theta) = \frac{\mu_0\, m}{4\pi r^3}(2\hat{r}\cos\theta + \hat{\theta}\sin\theta) \qquad ...(83)$$

$$= \mathbf{B}_r + \mathbf{B}_\theta, \qquad ...(84)$$

where

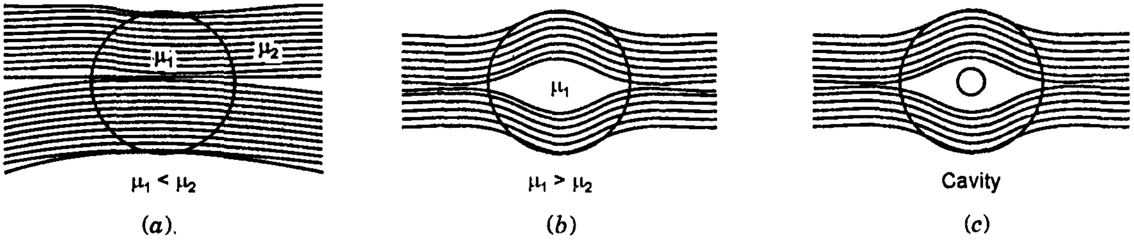
$$B_r = \frac{\mu_0}{4\pi}\frac{2m\cos\theta}{r^3} \text{ and } B_\theta = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}. \qquad ...(85)$$

It shows that the magnetic field outside the magnetized sphere will be same as due to a magnet of dipole moment m placed at the centre of the sphere along the direction of its magnetization.

13.10 MAGNETIC SHIELDING

It has been shown earlier [Eq. (61)] that the magnetic field is different inside the magnetic sphere if placed in a uniform magnetic field. For $\mu_1 < \mu_2$, the field inside is greater than that outside the sphere. (Fig. 13.12(*a*)).

For $\mu_1 > \mu_2$, the magnetic field inside the sphere is weakened, Fig. 13.12(*b*). It shows that the sphere screens its interior from the external magnetic field. It can be proved that the magnetic field cavity surrounded by the shell made of a magnetic material with a sufficiently high permeability μ_1 placed in a magnetic field is zero. The magnetic lines are concentrated mainly in the shell without penetrating into the cavity. Fig. 13.12(*c*). Thus the *shell made of magnetic material with a high μ operates as a screen which does not allow the magnetic field to penetrate into the space bounded by the shell,* or shields *the cavity from the external magnetic field.*



(*a*). (*b*) (*c*)

Fig. 13.12. Magnetic screening or shielding.

13.11 MAGNET AND EQUIVALENT SOLENOID

Consider a coil of N-turns, area **A** and carrying a current i amperes placed in a magnetic field of flux density **B**. If θ is the angle between **B** and **A**, then the torque acting on the coil is given by

$$\tau = Ni\,\mathbf{A} \times \mathbf{B} \quad \text{or} \quad \tau = NiAB \sin\theta. \qquad ...(86)$$

Let us compare this with the expression for the torque on an electric dipole of dipole moment $\mathbf{p}_E$ placed in an electric field **E**, *i.e.*, $\tau = p_E E \sin\theta$.

This comparison suggests that NiA can be taken as the *magnetic dipole moment* p_m (or m) also named as *electromagnetic moment.* Thus Eq. (86) becomes

$$\tau = p_m B \sin\theta = m\, B \sin\theta. \qquad ...(87)$$

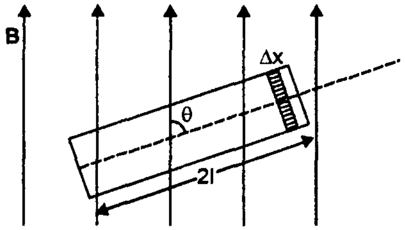


Fig. 13.13. Solenoid in a magnetic field.

The unit of m is ampere meter^2. It may therefore be defined as the maximum torque which can act on the body when it is immersed in a magnetic field of uniform flux density of unit strength, provided the torque is wholly due to the interaction between the body and the **B**.

A solenoid when immersed in a uniform magnetic field experiences a torque.

Consider a solenoid of length $2l$ and of n turns per unit length (Fig. 13.13). It may be divided up into a large number of sections, each of length Δx. The torque acting on the solenoid due to the interaction with **B** is

$$\begin{aligned} \tau &= \Sigma(n\Delta x)\, iAB \sin\theta = iAB \sin\theta\, \Sigma\,(n\Delta x) \\ &= NiAB \sin\theta. \end{aligned}$$

On comparison, we get the dipole moment of the solenoid as NiA.

We know that the magnetic field at a distance d (along the axis) from the centre of a solenoid of length $2l$, radius r and of total turns N is given as

$$B = \frac{\mu_0}{2}\frac{N}{2l}i(\cos\theta_2 - \cos\theta_1) = \frac{\mu_0 Ni}{4l}\left[\frac{d-l}{[r^2+(d-l)^2]^{1/2}} - \frac{d+l}{[r^2+(d+l)^2]^{1/2}}\right]$$

$$= \frac{\mu_0 Ni}{4l}\left\{\left(1-\frac{r^2}{2(d-l)^2}\right)-\left(1-\frac{r^2}{2(d+l)^2}\right)\right\}$$

$$= \frac{\mu_0 Ni}{4l}\frac{2r^2ld}{(d^2-l^2)^2} = \frac{\mu_0}{4\pi}\frac{2md}{(d^2-l^2)^2}, \qquad \text{...(88)}$$

where $m = \pi r^2 Ni$, the magnetic moment of the equivalent dipole. If the solenoid is short compared with the distance d, then we have

$$B = \frac{\mu_0}{4\pi}\frac{2m}{d^3}. \qquad \text{...(89)}$$

At any point on the perpendicular bisector of the solenoid, we may get

$$B = \frac{\mu_0}{4\pi}\frac{m}{d^3}. \qquad \text{...(90)}$$

Relations (89) and (90) show that m is a vector quantity and its direction is the same as the direction of **B** it generates.

To obtain the expression for the force on a magnetic pole, we modify the torque equation by replacing m by $q_m\,(2l)$, to obtain

$$\tau = q_m\,(2l)\,B\sin\theta.$$

This is exactly analogous to the case of an electric dipole in an electric field, where $F = qE$. By analogy, the force on a magnetic pole placed in a magnetic field **B** is given by the relation.

$$F = q_m B. \qquad \text{...(91)}$$

For a north pole, q_m is positive and the force is parallel to **B**, and for a south pole the force is antiparallel to **B**.

Thus the force *between the magnetic pole of pole strengths (or magnetic charges) q_m and q_m' separated by a distance r in a vacuum is given by

---

*In few standard books the reader may find the force equation as

$$F = \frac{1}{4\pi\mu_0}\frac{q_m q'_m}{r^2}, \qquad \text{...}(i)$$

similar to the Coulomb's law in the electrostatics, *i.e.*, $F = q_1q_2/4\pi\varepsilon_0 r^2$, where ε_0 has been replaced by μ_0 and the electrostatic charges by magnetic charges.

The above relation will be obtained, if we define the magnetic dipole moment of the current loop by

$$m = \mu_0 iAN. \qquad \text{...}(ii)$$

In this way the unit of m will be *web. m.*

The torque Eq. (87) will be $\tau = (1/\mu_0)\,mB\sin\theta = mH\sin\theta,$ ...(*iii*)

and the force Eq. (91) will be $F = (1/\mu_0)\,q_m B = q_m H.$ ...(*iv*)

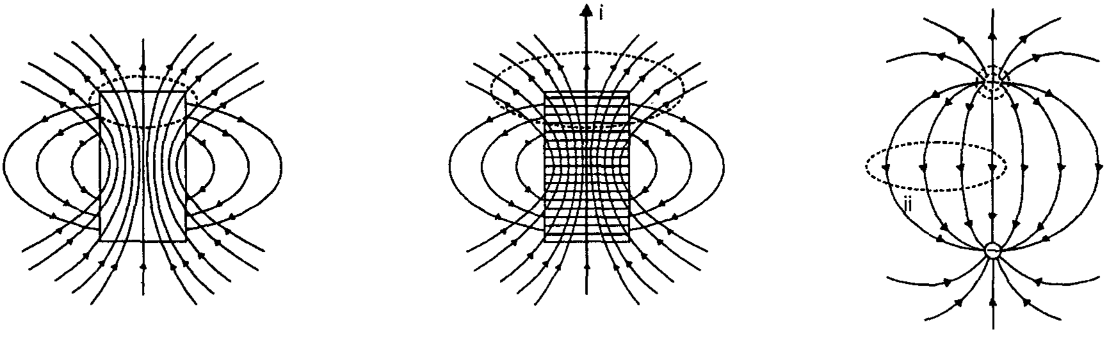
There is only one law of magnetostatics, and that is Biot and Savart's law. The Coulomb's law of force between magetic charges (or poles) cannot be considered a fundamental law as magnetic monopoles do not exist. On account of this reason, the author does not agree with the relations (*i*) to (*iv*).

$$F = \frac{\mu_0}{4\pi}\frac{q_m q'_m}{r^2}, \qquad ...(92)$$

where μ_0 is the permeability of vacuum, Eq. (92) is known as *Coulomb's law for magnetostatistics.*

Let us now consider two following simple cases :

(*a*) Bar Magnet : To compare the solenoid with the bar magnet, let us draw magnetic lines of induction for a bar magnet and for a short solenoid [Fig. 13.14(*a*) and (*b*)]. Lines of **E** for an electric dipole are also shown in Fig. 13.14(*c*). The comparison shows that the magnetic field **B** for a bar magnet is of same type as that due to a small solenoid. At large enough distances the fields in these two cases vary like those for a dipole. Hence the flux density on the axis and on the perpendicular bisector of a bar magnet may also be obtained by using equations (89) and (90) respectively. In this case $2l$ is slightly shorter than the actual length of the magnet. This similarity in the lines of forces show that the torque acting on a bar magnet placed in a magnetic field **B** may be given by Eq. (87). Hence a freely suspended magnet comes at rest when the magnet of magnetic moment (p_m) is parallel of the field **B**.



(*a*) (*b*) (*c*)

Fig. 13.14. Bar magnet and a solenoid.

(*b*) Magnetized sphere : We know that the magnetized body is equivalent to a solenoid and its effect can be simulated by the amperian surface current density $\mathbf{j}^s_m$, where

$$j^s_m = \text{Magnetization } M. \qquad ...(93)$$

Fig. 13.15 shows that a few of the turns of wire wound around the surface in which current i is producing magnetic field is equivalent to the field produced by M. The magnetic field at the centre of the sphere due to one turn at distance x from the centre

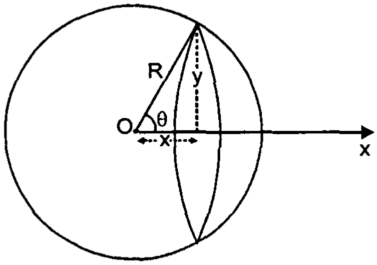


Fig. 13.15. Magnetized sphere.

$$dB = \frac{\mu_0 i}{2}\frac{y^2}{(x^2+y^2)^{3/2}}.$$

If dx is the thickness of one turn, then the current $i = dx$ × current density j. As $x = R\cos\theta$, $dx = -R\sin\theta\, d\theta$, and $y = R\sin\theta$, hence

$$dB = \frac{\mu_0 j}{2}\frac{R^3 \sin^3\theta d\theta}{R^3} = \frac{\mu_0 j}{2}\sin^3\theta d\theta.$$

or

$$B = \frac{\mu_0 j}{2}\int_0^{\pi}\sin^3\theta\, d\theta = \frac{2}{3}\mu_0 j. \qquad ...(94)$$

Since j is to simulate $\mathbf{j}_m^s$ and thus equal to M. Hence

$$B = \frac{2}{3}\mu_0 M = \frac{\mu_0}{4\pi}\frac{2m}{R^3}, \qquad ...(95)$$

where $m = \frac{4}{3}\pi R^3 M$, the total magnetic moment of the sphere.

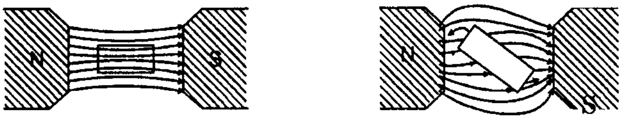
The Eq. (95) is same as due to a magnetic dipole. Hence the field at any point outside the sphere can be obtained by simple relations for electric or magnetic dipoles.

$$B_r = \frac{\mu_0}{4\pi}\frac{2m\cos\theta}{r^3} \text{ and } B_\theta = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^3}. \qquad ...(96)$$

13.12 PROPERTIES OF MAGNETIC MATERIALS

As discussed earlier, all substances, whether solid, liquid or gaseous may be classified into three categories, in terms of their susceptibility: (*i*) *paramagnetic*, (*ii*) *diamagnetic* and (*iii*) *ferromagnetic*.

Paramagnetic substances : Examples of such substances are platinum, aluminium, chromium, manganese, $CuSO_4$, solutions of salts of iron and nickel, oxygen and crown glass. They have the following properties :



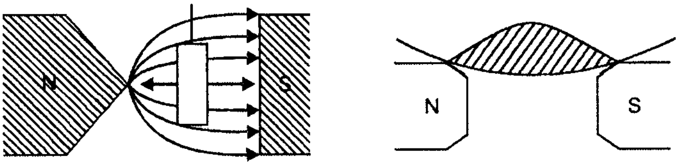
(*a*) (*b*)

Fig. 13.16. Paramagnetic substance in a magnetic field.

(1) The substances, when placed in a magnetic field, acquire a feeble magnetization in the same sense as the applied field. Thus the magnetic induction inside the substance is slightly greater than outside to it, as shown in Fig. 13.16(*a*).

(2) In a uniform magnetic field, these substances rotate until their longest axes are parallel to the field, Fig. 13.16(*b*).

(3) These substances are attracted towards regions of stronger magnetic field when placed in a non-uniform magnetic field. Fig. 13.17(*a*) shows a strong electromagnet in which one of the pole pieces is sharply pointed while the other is flat. Magnetic field is much stronger near the pointed pole than near the flat pole and the field is as diverging from pointed pole. If a small piece of test material is suspended in this region, a slight force can be observed by the slight displacement of the material, in the direction of arrow.



(*a*) (*b*)

Fig. 13.17. Paramagnetic substances in a non-uniform magnetic field.

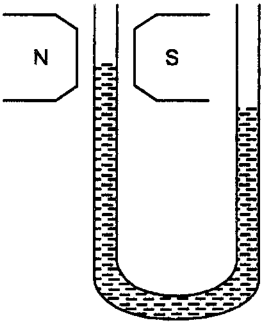


Fig. 13.18. Rise of paramagnetic liquid.

If a paramagnetic liquid is placed in a watch glass resting on the poles of a powerful electromagnet, the liquid is found to move up (or rise up) at point of greatest magnetic field, Fig. 13.17(*b*).

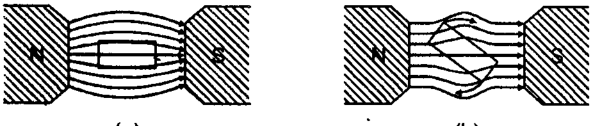
(4) If a paramagnetic liquid is filled in a narrow U-tube and one limb is placed in between the pole pieces of an electromagnet such that the level of the liquid is in line with the field, then the liquid will rise in the limb as the field is switched on (Fig. 13.18).

Similarly when a paramagnetic gas is allowed to ascend between the poles of an electromagnet, the gas spreads along the field.

(5) The relative permeability (μ_r or k_m) is slighty greater than one. k_m = 1.000021 for aluminium, 1.00036 for platinum and 1.00047 for chromium. The most paramagnetic element is manganese for which k_m = 1.0013.

(6) At a given temperature the susceptibility χ does not change with the magnetising field. However it varies inversely as the absolute temperature. As temperature increases, χ decreases. At some high temperature, χ becomes negative and the substance becomes diamagnetic.

Diamagnetic substances : Examples of such substances are bismuth, antimony, gold, quartz, water, alcohol, hydrogen. They have following properties.



(a) (b)

Fig. 13.19. A diamagnetic substance in a magnetic field.

(1) These substances, when placed in a magnetic field, acquire feeble magnetization in a direction opposite to that of the applied field. Thus the lines of induction inside the substance is smaller than that outside to it, Fig. 13.19(*a*).

(2) In a uniform magnetic field, these substances rotate until their longest axes are normal to the field, Fig. 13.19(*b*).

(3) In a non-uniform field, these substances move from stronger to weaker parts of the field. It is represented by the Fig. 13.20. If a diamagnetic liquid is placed in a watch glass resting on the pole of a powerful electromagnet, the liquid is found to accumulate on the sides, where the field is weaker.

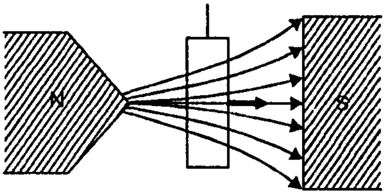


Fig. 13.20. A diamagnetic substance in a non-uniform field.

(4) If a diamagnetic liquid is filled in a narrow *U*-tube and one limb is placed in between the pole pieces of an electromagnet, the level of the liquid depresses when field is switched on.

Similarly, when a diamagnetic gas is allowed to ascend between the poles of an electromagnet, the gas spreads across the field.

(5) The relative permeability μ_r or k_m is slightly less than 1.

(6) The susceptibility χ of such substances is always negative. It is constant and does not vary with field or the temperature. There are few exceptions, *e.g.*, bismuth at low temperatures.

Ferromagnetic substances : Examples of such substances are iron, nickle, steel, cobalt and their alloys. These substances resemble to a higher degree with paramagnetic substances as regard to their behaviour. They have following additional properties :

(1) These substances are strongly magnetised by even a weak magnetic field. The relative permeability is very large and is of the order of hundreds and thousands. The susceptibility is positive and very large.

(2) The intensity of magnetization **M** is proportional to the magnetizing field intensity **H** for its smaller values, increases rapidly for larger values and attains a constant value for very large values of **H**, Fig. 13.21.

(3) Susceptibility χ thus remains constant for very small values of **H**, increases for larger values of **H** and then decreases for very large values of **H**.

(4) The magnetic induction **B** varies with **H** in a similar manner as **M** does, except that **B** does not attain a constant value for very large values of **H**.

(5) Permeability μ also varies as χ except at very high magnetic fields where μ decreases slowly in comparison to χ.

(6) The susceptibility χ decreases steadily with the rise of temperature. Above a certain temperature, known as Curie temperature, the ferromagnetic substances become paramagnetic. It is 1000°C for iron, 770°C for steel, 360°C for nickel and 1150°C for cobalt.

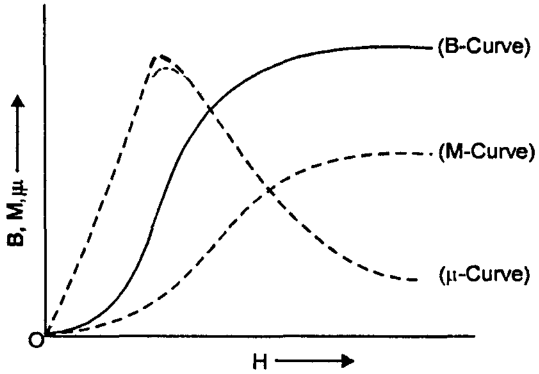


Fig. 13.21. Variation of *B*, *M* and μ with *H*.

13.13 DIAMAGNETISM

The origin of diamagnetism can be described using the simple model of the electronic motion around the nucleus. As discused in article 13.2, an electron of charge e rotating around the nucleus in a circular orbit of radius r and frequency ω is equivalent to a current loop with a current equal to $-e\omega/2\pi$. Hence the magnetic moment m of the atom is πr^2 $(-e\omega/2\pi)$ in the direction normal to the plane of the orbit.

Langevin's Theory : According to the classical theory of Langevin, in diamagnetic substances, the electrons associated with different atoms rotate in orbits oriented so that the net magnetic dipole moment is zero. When a magnetic induction B, whose direction is normal to the plane of the orbit, is applied on the atom, the net magnetic dipole moment is not zero, but is negative for these substances. Hence the magnetic susceptibility of diamagnetic substances is negative. Obviously diamagnetism is inherent in any substance. It may be explained as under :

The magnetic moments of the two oppositely circulating electrons in an atom are equal but in opposite directions and the net effect for the two circulating electrons is zero, Fig. 13.22(*c*).

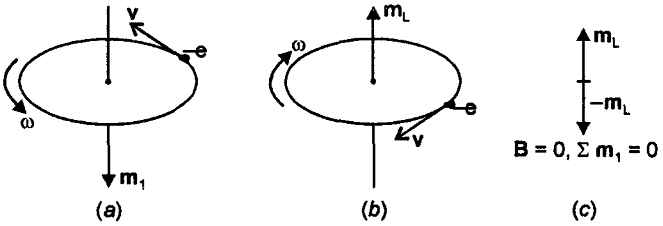


Fig. 13.22. Two electrons circulating in opposite directions.

If a magnetic field **B** whose direction is normal to the plane of the orbit, is now switched on the atom, an induced emf is produced on the electron. This emf results in an electric field, which acts tangentially to the direction of the electron motion. Thus the electron is accelerated. The increase in velocity $\Delta \mathbf{v}$ produces a change in angular velocity $\Delta\omega$, known as *Larmor frequency* ω_L. Consequently, an additional magnetic moment is produced, which is directed against the magnetic induction **B**. It is given by the relation

$$\Delta \mathbf{m} = -e^2 r^2 \mathbf{B}/4\, m_e. \qquad ...(94)$$

Thus for an atom with two electrons, rotating in opposite directions, the magnetic field **B** changes the angular velocity and hence the magnetic moment.

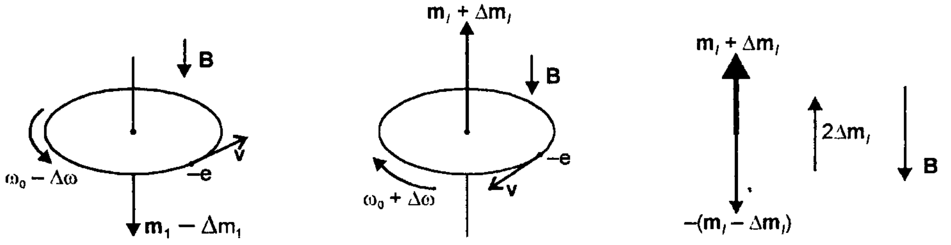

B
$\omega_0 - \Delta\omega$
$-e$
v
$\mathbf{m}_1 - \Delta \mathbf{m}_1$
(a)
$\mathbf{m}_l + \Delta \mathbf{m}_l$
B
$-e$
$\omega_0 + \Delta\omega$
v
(b)
$\mathbf{m}_l + \Delta \mathbf{m}_l$
$2\Delta \mathbf{m}_l$
B
$-(\mathbf{m}_l - \Delta \mathbf{m}_l)$
(c)

Fig. 13.23. Effect of magnetic field on the circulating electrons.

The net magnetic moment of two electrons rotating in opposite directions in a plane perpendicular to the magnetic field is 2 $\Delta \mathbf{m}_l$ in the direction opposite to the direction of **B**, Fig. 13.23(*c*).

Since the electrons are randomly oriented, the net induced magnetic moment per molecule is thus given by

$$\mathbf{m}' = \sum \Delta \mathbf{m} = -\frac{e^2 \mathbf{B}}{4m_e} \sum_{i=1}^{n} r_i^2. \qquad ...(95)$$

Here n is the number of electrons per molecule.

For a substance containing N-molecules per unit volume, the total magnetic moment per unit volume or the induced magnetization

$$\mathbf{M} = N\mathbf{m}' = -\frac{e^2 \mathbf{B}_0 N}{4m_e} \sum_{i=1}^{n} r_1^{\,2} = -\frac{e^2 \mu_0 \mathbf{H}_0 N}{4m_e} \sum_{i=1}^{n} r_i^2 . \qquad ...(96)$$

For diamagnetic materials, **H** differs very little from $\mathbf{H}_0$ hence the susceptibility of the diamagnetic substance is given by

$$\chi = \frac{M}{H_0} = -\frac{\mu_0 e^2 N}{4m_e} \sum_{i=1}^{n} r_i^2. \qquad ...(97)$$

In the derivation of this result, B_0 is assumed to be perpendicular to the planes in which electrons rotate. When the orbit is inclined, so that a normal to the i^{th} orbit makes an angle θ_i with the field B_0, then the above relations will be written as

$$\mathbf{M} = -\frac{e^2 \mu_0 \mathbf{H}_0 N}{4m_e} \sum_{i=1}^{n} r_i^2 \cos^2\theta_i, \qquad ...(98)$$

and

$$\chi = \frac{\mu_0 e^2 N}{4m_e} \sum_{i=1}^{n} r_i^2 \cos^2\theta_i\, . \qquad ...(99)$$

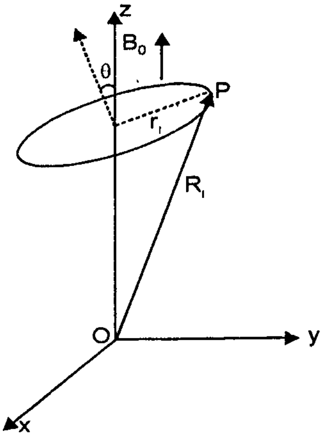


Fig. 13.24. Inclined orbit.

From Fig. 13.24, we have

$$R_i^2 = x_i^2 + y_i^2 + z_i^2,$$

where R_i is the distance between the i^{th} electron and the nucleus. For the random orientation of atoms in space, we have

$$<x_i^2> = <y_i^2> = <z_i^2> = <R_i^2>/3.$$

$$\therefore \quad r_i^2 \cos^2 \theta_i = x_i^2 + y_i^2$$

and

$$<r_i^2 \cos^2 \theta_i> = <x_i^2 + y_i^2> = \tfrac{2}{3} < R_i^2 >$$

$$<r_i^2 \cos^2 \theta_i> = \frac{1}{n}\sum_{i=1}^{n} r_i^2 \cos^2 \theta_i$$

$$= \tfrac{2}{3} < R_i^2 > = \tfrac{2}{3} < R^2 >,$$

where $<R^2>$ is the mean square distance between the electrons and the nucleus in an atom. Thus Eqs. (98) and (99) may be written as

$$\mathbf{M} = -\frac{e^2 \mu_0 \mathbf{H}_0 Nn}{6m_e} < R^2 >$$

and

$$\chi = -\frac{\mu_0 e^2 Nn}{6m_e} < R^2 >. \qquad ...(100)$$

Here we took into account that $\mu = \mu_0$, as the permeability of diamagnetics differs from the permeability of vacuum only insignificantly. The diamagnetic susceptibility for solids and liquids is of the order of 10^{-5}, while it is considerably lower (~ 10^{-8}) for gases due to smaller atomic concentration.

Relation (100) is known as Langevin's formula for diamagnetic susceptibility. It shows that χ is independent of temperature. This is due to the fact that the Larmor motion of electrons stabilizes very quickly. Consequently, *thermal motion as well as atomic collisions are unable to bring* the atom out of Larmor precession for any appreciable periods of time. Extremely small variation of χ with temperature may occur due to changing of lattice constants with temperature.

13.14 PARAMAGNETISM

The paramagnetic material has a positive but small magnetic susceptibility. Its molecules have a permanent magnetic dipole moments. The dipoles are free to be oriented under the influence of an external field. The dipole moment associated with each molecule is the vector sum of *orbital* and *spin* moments. The spin magnetic moment of each electron is first linked with the orbital magnetic moment of the same electron, thus forming the total magnetic moment of the electron. The magnetic moment of an atom is determined only by the electrons which occupy partially filled shells. The electron spins and orbital angular momenta in an outer shell tend to orient themselves in opposite directions so as to compensate each other to the maximum possible extent. Thus, *the magnetic moment of a free atom is due to the uncompensated spins of the outer electrons.* In a molecule, the atoms are bound by a chemical bond or ionic bond, which rearranges the outer electron shells, N_2 and *NaCl* molecules have total magnetic moment and are thus not paramagnetics. The oxygen gas is *paramagnetic*, as the spins of collective electrons are not compensated. NO and NO_2 gases are also paramagnetics, as the total number of electrons in these gases is odd and hence the spins of one electron is left uncompensated. Solids are generally diamagnetics, as they are composed of ions with closed shells. The transition elements are paramagnetics, whose electron shells are partially filled.

Langevin's Theory : A classical treatment of paramagnetism is very similar to that of the alignment of polar molecules placed in an electric field, discussed in chapter 7.

Langevin gave a theory for paramagnetic susceptibility. This theory provides a fairly accurate description for gases only, where the interaction between the molecules is negligibly small. The atoms or molecules of the magnetic material thus act as permanent magnetic dipoles. On account of the small interaction, these dipoles are randomly oriented and the net magnetic moment of all the molecules vanishes ($\Sigma m_i = 0$). The magnetic field on the other hand, exerts torques on the individual dipoles and tend to align them in the direction of the field. The alignment of dipoles is resisted by the collisions between the molecules due to thermal motion of the molecules. As a result, an equilibrium is reached. The degree of alignment for a particular sample of magnetic substance, thus depends both on the magnetic field B and on the temperature T.

Consider a molecular gas containing N molecules or atoms per unit volume at a temperature T. Each atom or molecule is assumed to possess a permanent magnetic moment **m**. If all the dipoles are along the applied magnetic field, the magnetization **M** will be N**m**. Actually the dipoles are pointing in all directions. The magnetic potential energy of the dipole, which is inclined at an angle θ with the applied field **B**, is given by

$$U = -\mathbf{m}\cdot\mathbf{B} = -mB\cos\theta. \qquad \text{...(101)}$$

According to the Boltzmann distribution law, the relative probability of finding a dipole making an angle θ with the **B** – direction is $e^{-U/kT}$, where k is the Boltzmann constant. Thus the number of dipoles dN, out of total number N, inclined between angles θ and θ + $d\theta$ with the magnetic field **B** will be given by

$$\begin{aligned} dN &\propto e^{-U/kT}\,d\Omega \\ &= C\,e^{-U/kT}\,d\Omega, \end{aligned} \qquad \text{...(102)}$$

where C is a constant of proportionality and $d\Omega$ is the solid angle subtended by the cones of semi-vertical angles θ and θ + $d\theta$ and given by

$$d\Omega = 2\pi\sin\theta\,d\theta$$

$$\begin{aligned} \therefore \quad dN &= C\,e^{-U/kT}\cdot 2\pi\sin\theta\,d\theta = C\cdot 2\pi e^{mB\cos\theta/kT}\sin\theta\,d\theta. \\ &= 2\pi C\,e^{\alpha\cos\theta}\sin\theta\,d\theta, \end{aligned} \qquad \text{...(103)}$$

where $\alpha = mB/kT$.

Integration of Eq. (103) will give the total number of dipoles per unit volume N. Thus

$$N = \int_0^{\pi} 2\pi C e^{\alpha\cos\theta}\sin\theta\,d\theta = \frac{4\pi C\sinh\alpha}{\alpha}$$

or

$$C = N\alpha/4\pi\sinh\alpha. \qquad \text{...(104)}$$

$$\therefore \quad dN = \frac{N\alpha}{2\sinh\alpha}e^{\alpha\cos\theta}\sin\theta\,d\theta. \qquad \text{...(105)}$$

The component of each dipole moment parallel to **B** is $m\cos\theta$, hence the total magnetic moment per unit volume

$$M = \int_0^\pi m\cos\theta\, dN = \frac{mN\alpha}{2\sinh\alpha}\int_0^\pi e^{\alpha\cos\theta}\sin\theta\cos\theta\, d\theta$$

$$= \frac{mN\alpha}{2\sinh\alpha}\left[\frac{e^{\alpha\cos\theta}}{\alpha^2} - \frac{\cos\theta\, e^{\alpha\cos\theta}}{\alpha}\right]_0^\pi$$

$$= Nm\,(\coth\alpha - 1/\alpha). \qquad ...(106)$$

Since Nm is the saturation value of the magnetic moment, say M_0, hence

$$M = M_0\,(\coth\alpha - 1/\alpha) = M_0 L(\alpha). \qquad ...(107)$$

This bracket term ($\coth\alpha - 1/\alpha$) is known is *Langevin function, $L(\alpha)$*. It is plotted as a function of α, in Fig. 13.25. This curve indicates that for large values of α the function tends to unity, *i.e.*, $M \to M_0$. It is the case when all the atomic dipoles are parallel to **B**. For small values of α, the curve is linear, as for smaller values of α the Langevin function

$$L(\alpha) = \coth\alpha - 1/\alpha = \alpha/3 = mB/3kT.$$

For most cases of paramagnetics, $\alpha << 1$ at ordinary temperature even for **B** as large as 1 tesla, therefore **M** can be approximated by the lower values of α, as

$$\mathbf{M} = \frac{M_0 m\mathbf{B}}{3kT} = \frac{Nm^2\mathbf{B}}{3kT}. \qquad ...108)$$

For a paramagnetic substances, **M** is very small positive value, hence $\mathbf{B} = \mu_0(\mathbf{H} + \mathbf{M}) \approx \mu_0\mathbf{H}$. Therefore, we have

$$\mathbf{M} = Nm^2\mu_0\mathbf{H}/3kT. \qquad ...(109)$$

For small values of α and thus for higher values of T

$$\text{Susceptibility} \quad \chi = \frac{M}{H} = \frac{M_0(\coth\alpha - 1/a)}{H} = \frac{m^2N\mu_0 H}{3kTH}$$

$$= \frac{\mu_0 M_0^2}{3NkT} = \frac{\text{const}}{T}. \qquad ...(110)$$

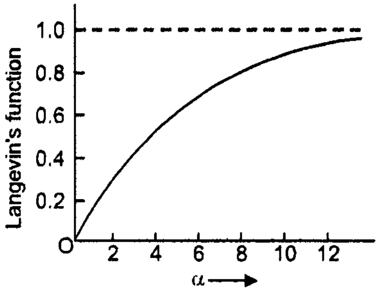


Fig. 13.25. Variation of Langevin function.

Here the constant is called the *Curie constant* and this relation is known as *Curie law*. Curie law is well verified experimentally and is very reasonable as at higher temperatures the thermal agitation opposes the dipole alignments along the **B**-direction. It breaks down at low temperatures. The temperature below which this law ceases can reach as low as 1.3°*K*, for some paramagnetic substances. At a very low temperature $\alpha >> 1$ and $M = Nm$, which is independent of magnetic field **B**. Thus M takes its saturation value M_0.

Langevin's theory applies strictly only to gases. In a gas or liquid the atoms are making collision with one another continuously, while in a solid the atoms are undergoing thermal oscillations. Under these conditions the various magnetic dipoles can interchange magnetic energy with the thermal energy of their environment. The thermal energy of the system tries to produce a completely random orientation of the magnetic dipoles, but the orientations along or near the applied field **B** have the lower magnetic energy and are thus favoured. For solids NkT in equation (110) is the total kinetic energy of the atoms. It may be regarded as a measure of the resistivity against the turning of the atoms (molecules) in the direction of **B**. The mutual action of the solid molecules affects the orientation of the molecules in the same way as their kinetic energy. Thus the effect of taking this action into account is equivalent to an increase of K.E. by a term Δ. Hence in the case of solids

$$\chi = \mu_0 M_0^2/3(NkT + \Delta), \qquad ...(111)$$

where the term Δ is independent of the temperature. Here we have not taken into account the contribution of diamagnetic susceptibility which is about 1% of the total value. Langevin theory can not explain the complicated dependence of susceptibility upon the compression, expansion, cooling of the gases and concentration of the solutions of the paramagnetic substances.

Diamagnetism and Paramagnetism : Since the diamagnetism involves induced magnetic moments, that are independent of the orientations of atoms or molecules. Hence thermal vibrations do not effect diamagnetic susceptibility. the diamagnetic susceptibility is negative and approximately independent of temperature. The diamagnetism is present in all types of matter, but its effect is masked generally by stronger effects of paramagnetism or ferromagnetism that is also present simultaneously in the material. Diamagnetism is prominent in materials which consist atoms with closed electron shells, as in these paramagnetic effects cancel out.

Some metals are diamagnetic, while others are paramagnetic. In the metals, χ is due to the ion cores and the conduction electrons. There is also a paramagnetic contribution from the electron spins. In some cases this paramagnetic contribution is more than required to cancel the diamagnetic term, hence *it is difficult to predict whether a given metal will be diamagnetic or paramagnetic. As paramagnetic susceptibilities decrease with temperature, hence all materials become diamagnetic at high enough temperatures.*

13.14 FERROMAGNETISM

In iron like elements and their alloys, known as *ferromagnetic substances,* the magnetization is not usually proportional to the applied magnetic field, as in the previously studied materials. The susceptibility is several thousand for these materials. Such a high magnetization may be due to magnetic moments of electron spin. In ferromagnetic substances the effective field acting on each spin magnetic dipole is the vector sum of the applied field plus a *strong interaction* field arising from all the neighbouring dipoles. This interaction field is so strong that thermal vibrations even at room temperature can not destroy the alignment. This interaction, known as *spin-spin or exchange interaction,* cannot be explained in terms of classical mechanics. The quantum mechanical effect resulting from the Pauli exclusion principle is responsible for such an interaction.

The quantum mechanical exchange interaction energy W_{ex} of a pair of atomic electrons is expressed in terms of their spins as

$$W_{ex} = -2I_{ex}\,\mathbf{S}_i \cdot \mathbf{S}_j, \qquad ...(112)$$

where $\mathbf{S}_i$ and $\mathbf{S}_j$ are the spins of the i^{th} and j^{th} interacting atoms respectively. The term I_{ex} is the exchange interaction integral. It depends on the distance between the interacting atoms. Since the atom's spin is associated with a magnetic moment and is given by the relation

$$\mathbf{m} = g(e\,\hbar/2m_e)\mathbf{S} = \beta\,\mathbf{S}.$$

Thus Eq. (112) may be written as

$$W_{ex} = -2I_{ex}\,\mathbf{m}_i\cdot\mathbf{m}_j/\beta^2 = -\mathbf{m}_j\cdot\mathbf{B}_{ex}, \qquad \text{...(113)}$$

where $\mathbf{B}_{ex} = 2I_{ex}\,\mathbf{m}_i/\beta^2$ is the magnetic induction of the exchange field produced by the i^{th} atom at the site of j^{th} atom.

Molecular Field Theory : Weiss devised a molecular field method, in which the attention is focused on a particular central atom. Since the spin-spin interaction falls rapidly with distance, only a few neighbouring atoms interact appreciably with this atom. The effect of these atoms on the j^{th} atom is the magnetic field produced by n neighbouring atoms, *i.e.*,

$$H' = 2I_{ex}\left(\sum_{i=1}^{n} m_i\right)/\mu_0\beta^2. \qquad \text{...(114)}$$

Let us assume that the atoms are identical, each atom having magnetic moment m. If N be the number of atoms per unit volume, then

$$H' = 2I_{ex}Nm/\mu_0\beta^2 = \gamma M,$$

where M is the magnetization and γ is known as *exchange interaction constant.* Thus the molecular field H at the site of j^{th} atom will be given by

$$\mathbf{H} = \mathbf{H}_0 + \mathbf{H}' = \mathbf{H}_0 + \gamma\,\mathbf{M}$$

or

$$H = H_0 + (2I_{ex}/\mu_0\beta^2)\,Nm. \qquad \text{...(115)}$$

Heisenberg explained the origin of the large value of γ. He showed that the large magnetic field is produced by the spin magnetic moments and by the electrostatic forces. Weiss-Heisenberg explained the way in which the magnetization of ferromagnet changes with temperature. This theory depicts ferromagnetism as the limiting case of paramagnetism in an extremely large magnetic field. The magnetization may thus be written in terms of Langevin's function, as

$$M = Nm(\coth\alpha - 1/\alpha), \qquad \text{...(106)}$$

where

$$\alpha = \frac{\mu_0 mH}{kT} = \frac{\mu_0 m(H_0+\gamma M)}{kT} = \frac{\mu_0 m[H_0+(2I_{ex}/\mu_0\beta^2)M]}{kT}$$

or

$$M = \frac{kT\beta^2}{2I_{ex}m}\left[\alpha - \frac{\mu_0 mH_0}{kT}\right] \qquad \text{...(116)}$$

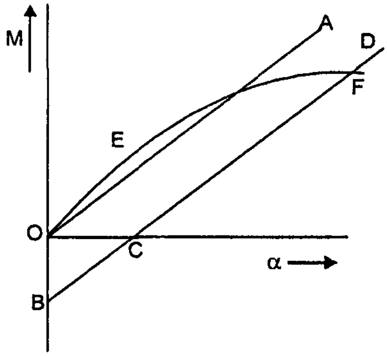


Fig. 13.26. Spontaneous Magnetization.

Spontaneous Magnetization : The spontaneous magnetization is the magnetization that exists even in the absence of the external magnetic field H_0. The materials that exhibit this property are called ferromagnetic materials. The magnetization for $H_0 = 0$ for a given temperature can be obtained by simultaneous solution of Eqs. (106) and (116), which is easily done by a graphical procedure (Fig. 13.26). For $H_0 = 0$, the Eq. (116) will represent a straight line OA, with slope $kT\beta^2/2I_{ex}m$. The graph OEF will represent Eq. (106). If the initial slope of the curve at $\alpha = 0$ is larger than the slope of the line OA, we get M that is consistent with both equations. For $\alpha \ll 1$, Eq. (106) gives

$$M \approx Nm\,\alpha/3. \qquad \text{...(117)}$$

As the temperature is increased, the linear curve becomes steeper, but the slope of Eq. (106) remains unchanged. Thus the interaction point shifts to the left and the spontaneous magnetization is decreased. The limiting case where the slopes at the origin are equal, the temperature is called *Curie temperature* T_C. Above this temperature spontaneous magnetization vanishes and the ferromagnetic material behaves as ordinary paramagnetic material. Thus we have

$$\frac{kT_C\beta^2}{2I_{ex}m} = \frac{Nm}{3} \text{ or } T_C = \frac{2m^2 I_{ex} N}{3k\beta^2}. \qquad \text{...(118)}$$

As T decreases, the spontaneous magnetization increases, until it reaches its maximum value $M_s = Nm$.

The exchange interaction constant $\gamma = \dfrac{3kT_C}{m^2 N\mu_0} = \dfrac{3kT_C\mu_0 N}{\mu_0{}^2 M_s{}^2}$.

For iron $\mu_0 M_s = 2.15\ T$, $T_C = 1043°\ K$. Thus $\gamma = 995$, which is much larger than the classical prediction of 1/3.

Curie-Weiss Law : Above the Curie temperature, ferromagnetic materials do not exhibit spontaneous magnetization. They exhibit paramagnetic properties. T_C, α can be taken small, thus the Eq. (106) becomes (117). The combination of Eqs. (116) and (117) gives

$$\frac{kT\beta^2}{2I_{ex}m}\left[\alpha - \frac{\mu_0 m H_0}{kT}\right] = Nm\frac{\alpha}{3}. \qquad \text{...(119)}$$

On combining with Eq. (118), we may write Eq. (119) in terms of Curie temperature as

$$\frac{T}{T_C}\left(\alpha - \frac{\mu_0 m H_0}{kT}\right) = \alpha \text{ or } \alpha = \frac{\mu_0 m H_0}{k(T - T_C)}. \qquad \text{...(120)}$$

For small values of α, the magnetization M is given by Eq. (117). The

$$M = Nm\frac{\alpha}{3} = \frac{\mu_0 m^2 N H_0}{3k(T - T_C)}.$$

$$\therefore \text{ Susceptibility } \chi = \frac{M}{H_0} = \frac{\mu_0 m^2 N}{3k(T - T_C)} = \frac{C}{T - T_C}. \qquad \text{...(121)}$$

This is the Curie-Weiss law. C is called Curie constant. For $T > T_C$, the ferromagnetic material behaves as the paramagnetic material. For $T < T_C$, this law is not applicable and a finite value of magnetization exists even when the applied field $H_0 = 0$. This magnetization is called the spontaneous magnetization.

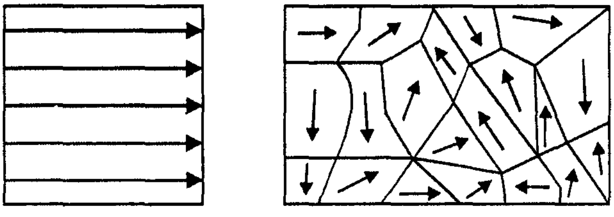

(a) (b)

Fig. 13.27. Ferromagnetic Domains.

Ferromagnetic Domains : A domain is a region composed of number of elementary magnets, each pointing with its axis in the same direction even in the absence of any external field, Fig. 13.27(*a*).

The size of the domain is now known to vary from about 10^{-6} cm^3 to 10^{-2} cm^3. In an unmagnetized ferromagnetic specimen the domains are oriented at random, so that their resultant magnetic moment is zero, as shown in Fig. 13.27(*b*).

Fig. 13.28(*a*) shows a ferromagnetic crystal as being made up of small cubes in each of which the domain vector is || to one of the directions of preferred magnetization. This picture is wrong as it violates the fundamental theorem of magnetism. The normal *component of magnetic field is to be conserved.* Fig. 13.28(*b*) is favourable as **B** is continuous across every domain. Using this concept the domain structure of a single crystal is illustrated in Fig. 13.28(*c*).

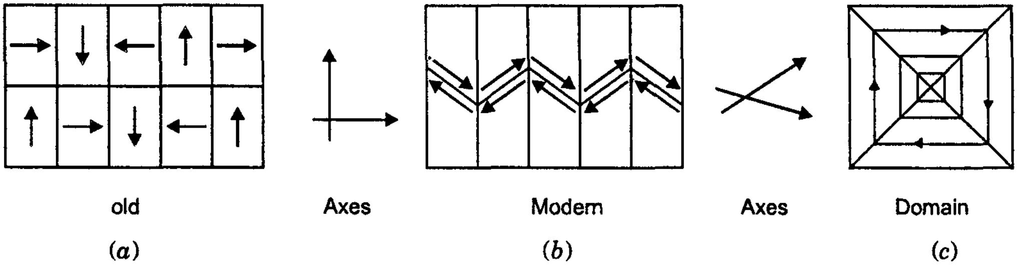

old
Axes
Modern
Axes
Domain
(a)
(b)
(c)

Fig. 13.28. Domain structures.

When the specimen is placed in a magnetic field the resultant magnetization may increase in two different ways : by an increase in the volume of the domains which are favourably oriented with respect to the field at the expense of unfavourably oriented domains, Fig. 13.29(*a*) or by the rotation of the domain magnetization toward the field direction Fig. 13.29(*b*). In weak fields the magnetization usually changes by the former method and in strong fields by the latter method. A magnetization curve for a ferromagnetic material is shown in Fig. 13.29(*c*). It shows the regions in which each process is dominant. In small fields the change is reversible, *i.e.* boundaries return to their original positions when the field is removed. If the magnetic field is further increased, irreversible boundary displacements predominate and the magnetization increases more rapidly. When the applied field is such that the knee of the curve is reached, the domain vectors will be so oriented that they are all parallel to the direction of those domains, whose volumes increases in the early stages. The further increase of the applied

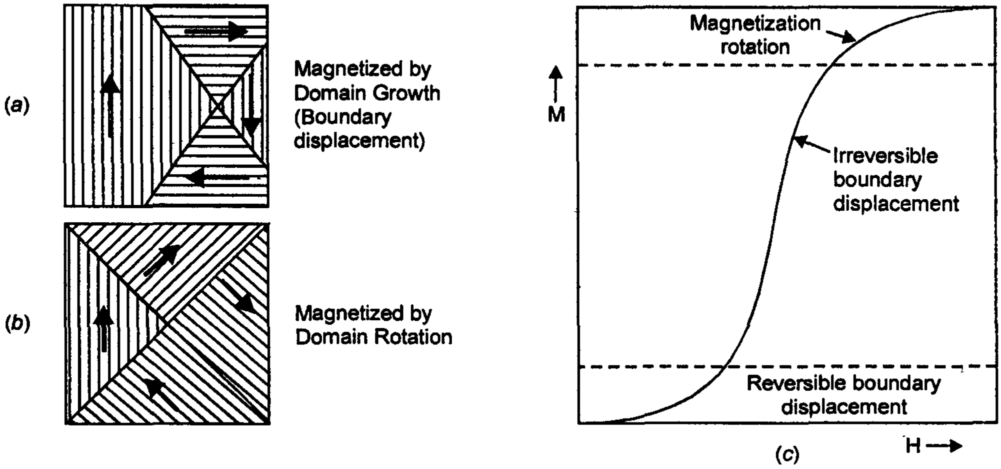

(a)
Magnetized by Domain Growth (Boundary displacement)
(b)
Magnetized by Domain Rotation
M
Magnetization rotation
Irreversible boundary displacement
Reversible boundary displacement
(c)
H

Fig. 13.29. Magnetization in a magnetic field.

field increases the magnetization by the rotation of the domain vectors in the direction of the applied field.

13.15 HYSTERESIS

The real distinguishing characteristic of a ferromagnetic material is not that it can be strongly magnetized but that the intensity of magnetization **M** is not directly proportional to the magnetizing field **H**. If a gradually increasing magnetic field **H** is applied to an unmagnetized piece of iron, its magnetization increases non-linearly, until it reaches a maximum. If **M** is plotted against **H**, a curve like *OA* is obtained, this curve is known as the *magnetization curve.* At this stage, all the dipoles are aligned and **M** has reached to maximum or saturated value. It is often more convenient to plot **B** than **M**, the difference is often small except for the factor μ_0, as $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ and **M** is usually much greater than **H.** If a magnetic field **H** is now reduced, the **M** (or **B**) does not return along magnetization curve but follows that AB. At **H** = 0, **M** (or **B**) does not come to its zero value, but its value is still near the saturated value. The value of **M** at this point (*i.e. OB*) is known as *remanence, remanent magnetization or retentivity.*

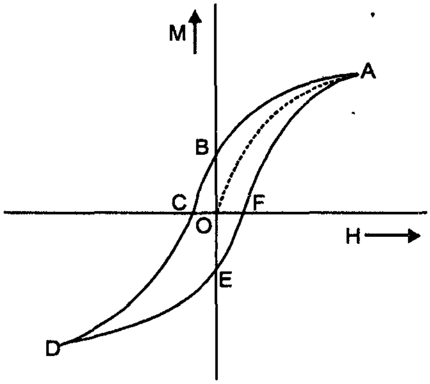


Fig. 13.30. Hysteresis loop.

The value of **B** at this point is known as the residual induction. The retentivity gives the state of permanent magnetization of the specimen.

On applying a reverse field the value of **M** falls and finally becomes zero. The abscissa *OC* represents the reversed magnetic field needed to demagnetize the specimen. This is known as *coercivity* (or coercive field) of the material. If the graph is in between **B** and **H**, then it must be noted that the numerical value of **H** to make **B** = 0 is not the coercivity, as $\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})$ hence **B** = 0 for **M** = –**H** and the magnetization **M** is left positive.

If the reverse field is further increased, a reverse magnetization is set up which quickly reaches the saturation value. This is shown as *CD.* If **H** is now taken back from its negative saturation value to its original positive saturation value, a similar curve *DEFA* will be traced. The whole graph *ABCDEFA* thus forms a closed loop, usually known as *hysteresis loop.* The whole process described above and the property of the iron characterized by it are called hysteresis. *Hysteresis is the name for the lag of magnetization behind the magnetizing field.*

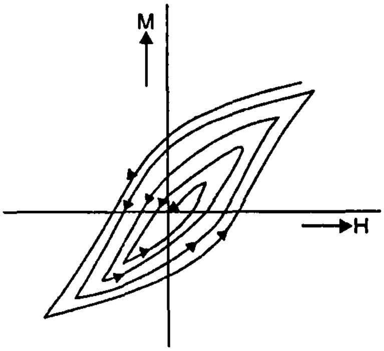


Fig. 13.31. Demagnetization.

Demagnetization : It is clear from the hysteresis loop that the intensity of magnetization **M** does not reduce to zero on removing the magnetizing field **H. M** is zero when the magnetizing field **H** is equal to the coercive force. At these points the magnetic induction is not zero, and the specimen is not demagnetized. To demagnetize a substance, it is subjected to several cycles of magnetization each time with decreasing magnetizing field and finally the field is reduced to zero. In this way the size of the hysteresis curve goes on decreasing and the area finally reduces to zero. Demagnetization is best obtained by placing the specimen in an alternating field of continuously diminishing amplitude. It is also obtained by heating, *ferromagnetic materials becoming practically nonmagnetic at sufficiently high temperatures.*

Ballistic method for B – H Curve : The $B - H$curve of a specimen can be obtained by a ballistic galvanometer method. In this method the specimen is taken in the form of a ring, known as anchor ring.

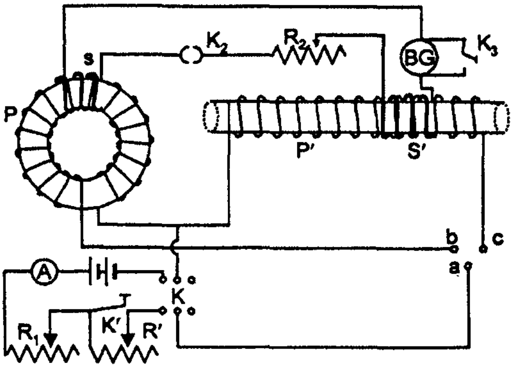


Fig. 13.32. Experimental arrangement for B-H curve.

A magnetizing coil is wound closely round the specimen ring, forming the primary P of an endless solenoid. The coil is connected through a reversing key K and two way key K_1 to a circuit containing a battery, an ammeter A, a rheostat R_1 and a resistance R'. The R' can be removed from the circuit by pressing the key K'. Over a small part of primary of the solenoid, a secondary coil S of few turns is wounded which is connected in series with ballistic galvanometer (*B.G.*), variable resistance R_2, a key K_2 and the secondary coil S' of the standard solenoid having primary P'. The two way key K_1 may connect either, the primary of the ring solenoid, or primary of the standard solenoid. To damp the motion of ballistic galvanometer, a tapping key K_3 is connected across it. The whole arrangement is shown in Fig. 13.32.

If the primary of the ring solenoid consists N_1 turns and the mean circumference of the ring is l, then the magnetic field produced in the ring due to the current i in the primary is given by

$$H = N_1 i/l. \qquad ...(122)$$

This magnetizing field produces a magnetic flux which passes through the secondary of the ring solenoid. If B is the flux density in the ring material, N the number of turns in the secondary and A the area of cross section of the ring then the total flux linked with the secondary

$$\Phi = NBA.$$

This is the change of flux in the secondary, it sets up an induced emf in the secondary circuit. If R is the total resistance of the secondary circuit, then the charge passing through the ballistic galvanometer

$$q = NBA/R. \qquad ...(123)$$

If θ is the first throw of the ballistic galvanometer coil, then

$$q = NBA/R = K\theta\,(1 + \lambda/2). \qquad ...(123)$$

To eliminate K and λ, the known current i' is passed through the primary of the standard solenoid by closing the key K_1 to the right. If n is the number of turns per unit length in the primary, then the flux density in the standard solenoid will be $\mu_0\, n\, i'$. The corresponding flux linked with the secondary of N' turns will thus be $\mu_0\, n\, i'\,(N' A')$, where A' is the area of cross-section of the secondary of the standard solenoid. This change in the magnetic flux sends charge through the ballistic galvanometer which deflects the coil. It θ' is the first throw in the galvanometer coil, then

$$\text{Charge} = \mu_0\, n\, N'\, i'\, A'/R = K\theta'\,(1 + \lambda/2). \qquad ...(124)$$

Dividing Eq. (123) by Eq. (124), we get

$$\frac{NBA}{\mu_0 nN'i'A'}=\frac{\theta}{\theta'},\ \text{or}\ B=\frac{\mu_0 nN'i'A'}{NA}\frac{\theta}{\theta'}. \qquad ...(125)$$

This is the expression for magnetic induction in the specimen corresponding to the magnetic intensity H, given by Eq. (122).

For the full cycle of magnetization, the following procedure is to be followed : The key K_1 is first closed to the left by inserting a plug between *ab* gap and the resistances R_1 and R' are decreased until on closing the commutator K the galvanometer gives full scale deflection from the zero. The current corresponding to this deflection is measured by an ammeter and is used as maximum current in the main experiment.

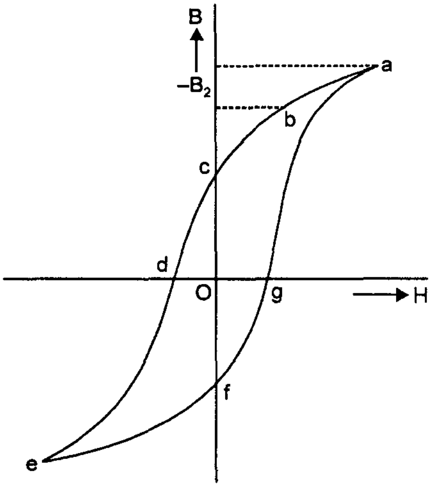


Fig. 13.33.

The residual magnetism in the specimen is reduced to zero as : The galvanometer circuit is first broken and the resistances R_1 and R' are reduced to the minimum. The current passing through the primary of the ring solenoid is then reversed many times by means of the commutator K, the resistances R_1 and R' are increased gradually simultaneously until the current becomes very small.

The galvanometer is again put in the circuit by closing key K_2. The key K' is closed and resistance R_1 is given a value corresponding to the maximum current. The galvanometer first throw is thus noted. The values of B and H, *i.e.*, B_1 and H_1 are thus calculated. The corresponding point on the B-H curve is a. The galvanometer circuit is again broken and the specimen is again demagnetized by reversing rapidly the commutator K as described before. The ballistic galvanometer is again put in the circuit. Now R' is given a small value and K' is opened. This decreases the current and thus the magnetizing field H. This decrease in H produces a decrease in magnetic induction B. Corresponding to this decrease B_1-B_2, a throw θ is observed. The value of magnetising field H_2 is calculated by noting ammeter reading. The corresponding point on the graph ($B - H$ curve) is denoted by the point b. This process is repeated by gradually increasing R' until current and hence H becomes zero. The graph corresponding to these readings is ac (Fig. 13.33). As after each measurement the specimen is returned to the state a by the reversal of maximum current, hence *point a works as the reference point.*

The *cde* of the graph is plotted by repeating the process in many steps with the reverse current. The point *e* is corresponding to the zero value of R'. The part *efga* can be drawn by symmetry or by repeating the experiment using *e* as the reference point and leaving the commutator on the left, if it was on the right previously.

CRO – Method for B – H curve : Ballistic or anchor ring method is quite tedious and laborious. A simple method for $B - H$ curve is the use of *CRO*, on the screen of which the complete hysteresis loop can be displayed.

The experimental arrangement is shown in Fig. 13.34. In this arrangement the specimen is taken in the form of ring with primary and secondary windings P and S. A supply is given to

P from an oscillator controlled by yoke. The current signal of the primary is applied across the *X*-plates of *CRO*. The secondary winding *S* is connected with a resistor *R* and capacitor *C* in series. The voltage developed across the capacitor *C* is amplified and then applied across the *Y*-plates of *CRO*.

The current passing through the primary winding is measured with an ammeter. It directly gives the value of *H* in the specimen. The voltage drop across the capacitor *C* is a measure of the magnetic induction *B*. Let dB/dt be the rate of change of magnetic induction. The voltage across the secondary winding at any instant *t* is given by

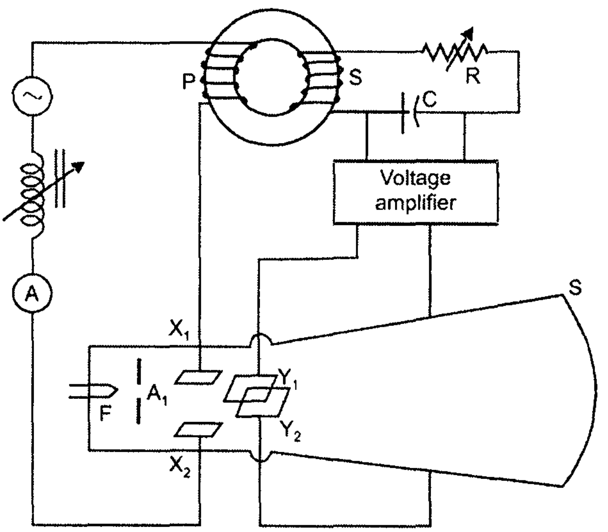


Fig. 13.34. *B* – *H* curve with *CRO*.

$$V_S = K\,dB/dt,$$

where *K* is a constant which depends on the number of primary and secondary turns and the dimensions of the ring. Since *R* and *C* are in series, the instantaneous current $i = V_S/(R + 1/\omega C)$. If *R* is kept very high, the capacitive inductance can be neglected. Thus we may write $i = V_S/R$.

The voltage across the capacitor *C* may thus be written as

$$V_C = \frac{1}{C}\int i\,dt = \frac{1}{C}\int \frac{V_S}{R}dt = \frac{K}{CR}\int \frac{dB}{dt}dt = \frac{K}{CR}B\,. \qquad ..(126)$$

Thus the voltage across the capacitor *C* and thus across the *Y* plates of a *CRO* varies directly as the magnetic induction with in the core of the specimen. Thus *H* and *B* act across the *X* and *Y*-plates respectively. A well defined *B* – *H* curve will be seen on the screen of the oscilloscope.

Hysteresis Loss : In an unmagnetized specimen, the directions of magnetization in different domains are different so that net magnetization is zero. When this specimen is placed in an external magnetic field, these domains are rotated so that the magnetization is aligned with the field direction. The work is thus done and the energy is taken by the specimen under this process. When the field is removed the domain boundaries do not move completely back to their original positions and the specimen retains some magnetism. Thus the energy is not recovered during demagnetization and there is always a definite loss of energy of each hysteresis loop. *This energy loss is known as hysteresis loss.*

Let *m* be the magnetic moment of any one domain which makes an angle θ with the direction of magnetizing field *H*. The components of *m* along and ⊥ to *H* are $m\cos\theta$ and $m\sin\theta$ respectively. The summation of all such terms of all the domains in volume of the specimen will be

$$\sum_n m\cos\theta \text{ and } \sum_n m\sin\theta,$$

where summation is taken over all the *n* domains, which are in unit volume. As the magnetization is along **H**, hence from the definition of **M**, the intensity of magnetization, we have

$$\Sigma m \cos\theta = M \text{ and } \Sigma m \sin\theta = 0$$

$$\therefore \qquad dM = d[\Sigma m \cos\theta] = -\Sigma m \sin\theta \, d\theta.$$

As the restoring couple acting on the domain of moment m, when it is inclined at an angle θ to the **H**– direction is $mB \sin\theta$ ($= \mu_0 mH \sin\theta$). Hence the work done when it moves through a small angle $d\theta$ is

$$dW' = \mu_0 mH \sin\theta \, d\theta.$$

$\therefore$ Work done per unit volume of the material $= \Sigma \mu_0 mH \sin\theta \, d\theta$. ...(127)

If the intensity of magnetization M is increased to $M + dM$ by increasing H to $H + dH$, then the work done per unit volume as M increases (or the dipoles are rotated more along H, *i.e.*, θ decreases) is given as

$$dW = -\Sigma \mu_0 m (H + dH) \sin\theta \, d\theta. \qquad ...(128)$$

As dH and $d\theta$ both are very small, hence by assuming H as constant, we get

$$dW = -\Sigma \mu_0 mH \sin\theta \, d\theta = -\mu_0 H \, \Sigma m \sin\theta \, d\theta = \mu_0 H dM.$$

$\therefore$ Work done per unit volume for a complete cycle of magnetization

$$W = \mu_0 \oint H dM. \qquad ...(129)$$

As $B = \mu_0 (H + M)$, hence for constant magnetizing field

$$dB = \mu_0 dM.$$

Therefore, $$W = \oint H dB. \qquad ...(130)$$

Let us consider $M - H$ curve (or $B - H$ curve). At point near Q on the curve (Fig. 13.35) PQ represents H and PR represents a small change in M, *i.e.*, dM (or dB). Hence the area $PQSR$ will represent HdM (or HdB), the work on the material per unit volume. Thus the work done on the material per unit volume during the path ab will be the sum of all such areas, *i.e.*, area $OabJO$. When the material is taken from b to c, or the magnetizing field is decreased to zero, the energy equal to the area $bJcb$ is restored back per unit volume of the material. Thus the horizontally shaded areas represent the work done by the material per unit volume, whilst the areas which are also vertically shaded represent the work done on the material per unit volume. Hence the excess of work done on the material over that done by it, per unit volume, is represented by the area enclosed within the hysteresis loop $abcdefa$.

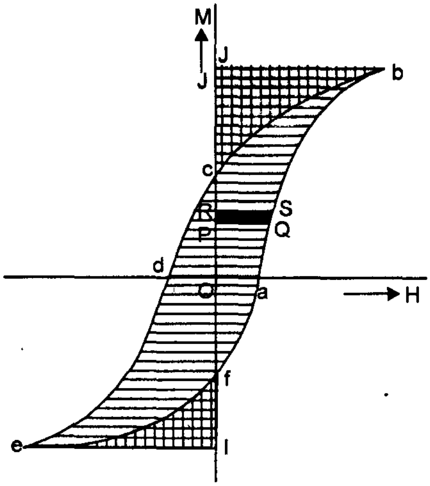


Fig. 13.35. *M-H. loop.*

$\therefore$ *Work done per unit volume per cycle* $= \mu_0 \times$ *area of the* $M - H$ *loop*

$= $ *Area of* $B - H$ *loop.*

A rough idea of the magnitude of hysteresis losses may be obtained from an empirical formula of *Steinmetz.*

Work done or the energy loss per unit volume per cycle

$$W = \eta (B_{max})^K, \qquad ...(131)$$

where B_{max} is the maximum magnetic induction in the loop, η, the *Steinmetz coefficient,* is a constant varying from material to material. The exponent K varies between 1.4 and 1.8 for the various ferromagnetic materials, most accepted value is 1.6.

13.16 SOFT AND HARD MAGNETIC MATERIALS

The magnetic properties of a ferromagnetic substance can be obtained from the size and shape of the Hysteresis loop. For a few materials the hysteresis curves are shown in Fig. 13.36. Curves (*a*), (*b*), (*c*) and (*d*) are respectively for the specimen of soft iron, steel, cobalt and nickel materials. The study of these curves gives the following informations :

(*i*) *Susceptibility :* The susceptibility, the intensity of magnetization per unit magnetizing field, (*i.e.*, M/H) is greater for soft materials than for hard materials.

(*ii*) *Permeability :* The permeability, the magnetic induction per unit magnetizing field (*i.e.*, B/H) is greater for soft materials than for hard materials.

(*iii*) *Retentivity :* When a magnetic specimen is first magnetized and then the magnetizing field is reduced to zero, the specimen retains intensity of magnetization (or the magnetic induction), this is known as the *retentivity* (or *the remanent induction*). It is greater for soft materials than for hard materials.

(*iv*) *Coercivity :* To demagnetize the magnetic specimen completely a *–ve* field is required. The value of reverse field H required to reduce the intensity of magnetization to zero is known as the *coercivity*. It is less for soft materials than for hard materials.

(*v*) *Hysteresis Loss :* The area of the hysteresis loop and hence hysteresis loss per unit volume per cycle is less for soft materials than for hard materials.

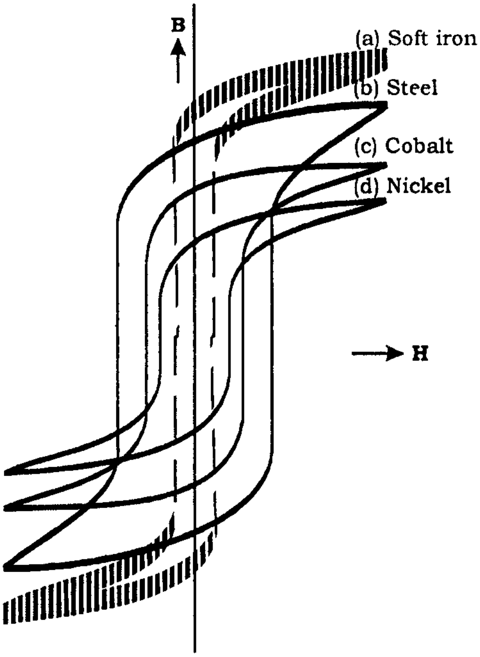


Fig. 13.36. Hysteresis loops.

Choice of Magnetic Materials : The choice of magnetic materials for different uses can be decided from the hyteresis curve of a specimen of the material.

(1) *Permanent Magnets :* The materials for a permanent magnet should have : (a) high retentivity (so that the magnet is strong) and (b) high coercivity (so that the magnetizing is not wiped out by stray magnetic fields, mechanical ill-treatment or by temperature change). As the material in this case is never put to cyclic changes of magnetization, hence hysteresis is immaterial. From the view point of these facts, *steel is more suitable for the construction of permanent magnets than soft iron.* The fact that the retentivity of iron is little greater than that of steel is outweighted by the much smaller value of its coercivity. Recently a number of alloys, having large values of coercivity have been developed for the purpose of construction of permanent magnets. The very suitable alloy of highest coercivity is named as *vicalloy* (vanadium, iron and cobalt).

(2) *Electromagnets :* The materials for the construction of electromagnets should have (*a*) high initial permeability, (*b*) low hysteresis loss and (*c*) maximum magnetic induction **B** with comparatively small value of magnetizing field. From the view point of these facts *the soft iron is an ideal material for this purpose.*

(3) *Cores of transformers and chokes, Armatures of Dynamos and motors, and for telephone Diaphragms* : As the magnetic material used in these cases is subjected to cyclic changes. Thus the essential requirements for the selection of the material are : (*a*) high initial permeability to obtain large flux density **B** for low values of magnetizing field **H**, (*b*) low hysteresis loss to prevent the breakdown of insulation of the windings as less dissipation of energy produces a small heating effect, and (*c*) high specific resistance to reduce eddy current losses. *Soft iron is better than steel for these purposes.* Soft iron has initial permeability about 250. The permeability is greatly increased by alloying it with 4% silicon. It is very useful and is known as *transformer steel.* There are some other alloys of iron and nickel, known as *permalloys. Mumetal (Ni 76%, Cu 5%, Fe 17.5%, Cr 1.5%)* has an initial relative permeability between 10,000 to 100,000. *Radiometal (Ni 48%, Fe 48.5%, Cu 3%, Mn 0.5%)* has an initial relative permeability between 2,000 to 15,000.

13.17 ANTIFERROMAGNETISM AND FERRITES

According to the Heisenberg theory of ferromagnetism the change from parallel to antiparallel spin alignment of neighbouring atoms is associated with the change in electrostatic energy. If this energy change favours parallel alignment and of sufficient magnitude the material composed of these atoms is called a *ferromagnet. For ferromagnetic materials the exchange interaction energy* $I_{ex} > 0$, *i.e.*, parallel spins have lower magnetic energy than antiparallel spins. In certain materials such as *chromium* and *manganese,* I_{ex} is negative, *i.e.*, antiparallel spins have lower magnetic energy than parallel spins. As a result, the spins of neighbouring atoms are found to be oriented in opposite directions and the total magnetization is equal to zero. This phenomenon is called *antiferromagnetics*.

In antiferromagnetics, the exchange interaction vector I_{ex} is directed against the magnetization vector **M**. *Thus the exchange magnetic induction* $\mathbf{B}_{ex} = -\mu_0 I_{ex}'\mathbf{M}$. Carrying out calculations as discussed in article 13.14, we get

$$\textit{Susceptibility}\ \chi_a = \frac{C}{T+I_{ex}'C} = \frac{C}{T+\textcircled{H}}, \qquad \text{...(132)}$$

where $\textcircled{H} = I_{ex}'\,C$ is the Curie-Weiss temperature. The temperature of transition to the antiferromagnetic state is other than the Curie-Weiss temperature and is called the *Neel temperature* T_N. Below this temperature, the total spontaneous magnetization of an antiferromagnetic is equal to zero since the spins become aligned in an alternation configuration, as shown in Fig. 13.37(*b*). Above this temperature, the spin directions are random.

The alternating spin configuration in chromium like materials can be experimentally proved by scattering of neutrons from a crystal of material. The degree of scattering depends on whether the neutron spin is parallel or antiparallel to the scatterer. It is observed that the interference pattern of the scattered neutrons changes when the temperature changes from above the Neel temperature to below the Neel temperature (*i.e.*, the spin state changes from random state to alternate state).

It may so happen that atoms have spontaneous magnetizations in opposite directions (spin up and spin down), but with different intensities [Fig. 13.37(*c*)] and thus the material has a net, nonzero magnetic moment in one of the directions. Such a material is called *ferrimagnet* or *ferrite*. The phenomenon is called *ferrimagnetism*. The ferrites have properties similar to those of ferromagnetics and may be assumed to have a combination of the ferromagnetic and antiferromagnetic alignment. The simplest ferrites are oxides represented by the chemical formula $MO\ Fe_2\ O_3$, where M is a divalent metal ion such as *Co, Ni, Mn, Cu, Mg, Zn, Cd* or divalent iron.

The ferrites are of considerable technical importance because : (*a*) They have relatively large saturation magnetization, (*b*) Their resistivities fall in the range from 1 to 10^4 Ωm, (*c*) They are poor conductors of electricity. This property is valuable in high frequency applications where the eddy current losses in conducting materials pose serious problems.

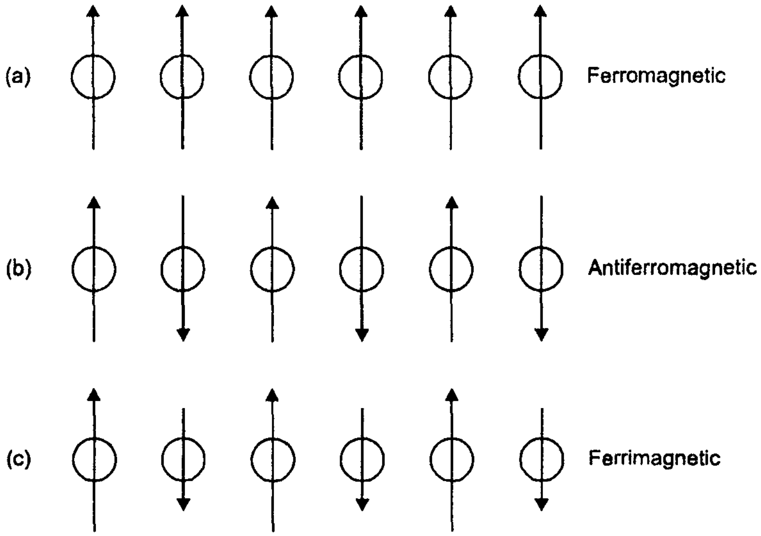


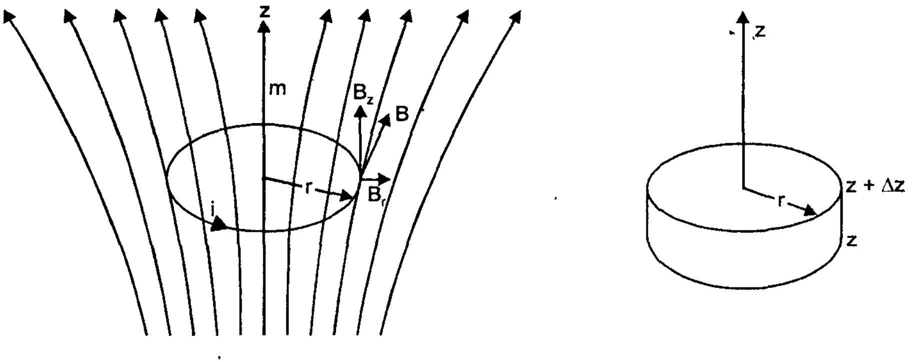
Fig. 13.37. Schematic representation of atomic spins.

13.18 MAGNETIC DIPOLE IN A NON-UNIFORM MAGNETIC FIELD

Consider a small circular current loop of radius r carrying a current i and placed in a non-uniform magnetic field **B**. Let us assume for simplicity that the field is symmetric about the z-axis and decreases as we proceed in the z-direction. In this problem, we want to find the force on the current loop due to the external field. Since the net force on the current loop due to its own field is zero, we therefore ignore its own field and consider **B** as the total field under consideration.

At every point on the current loop [Fig. 13.38(*a*)] the magnetic field **B** has two components, B_r in the plane of loop directed radially outwards, and B_z in the z-direction. On account of the symmetry of loop about the z-axis, the force due to the B_z component is zero on the current loop and the net force is due to the radial component B_r only, which is given as

$$\mathbf{F} = -\mathbf{k}\, iB_r\, \Sigma dl = -\mathbf{k}\, 2\pi r\, i\, B_r. \qquad ...(133)$$



(*a*) (*b*)

Fig. 13.38. Current loop in a non-uniform magnetic field.

The negative sign shows that the direction of the force is along the negative z-direction.

The value of B_r can be easily evaluated by using Gauss's law, *i.e.*, div $\mathbf{B} = 0$ or the net amount of flux of magnetic field out of any volume is everywhere zero. Consider a small cylinder of radius r and height Δz. The outward flux from the curved side surface is $2\pi r\ \Delta z\ B_r$ and the net outward flux from the end surfaces is $\pi r^2\ [-B_z\ (z) + B_z\ (z + \Delta z)]$. Therefore

$$\oint_S \mathbf{B}\cdot d\mathbf{S} = 2\pi r\ \Delta z B_r + \pi r^2\ [-B_z\ (z) + B_z\ (z + \Delta z)] = 0.$$

or

$$B_r = -\tfrac{1}{2} r[-B_z(z)+B_z(z+\Delta z)]/\Delta z.$$

Since $B_z(z)$ and $B_z(z + \Delta z)$ are the magnetic fields at z and $z + \Delta z$ respectively. To the first approximation the bracket term may thus be written as $(\partial B_z/\partial z)\Delta z$.

$$\therefore \qquad B_r = -\frac{r}{2}\frac{\partial B_z}{\partial z}.$$

Since B_z decreases as z increases, B_r is therefore positive. Substituting the value of B_r in equation (133), we get

$$\mathbf{F} = -\mathbf{k}2\pi r i\left(-\frac{r}{2}\frac{\partial B_z}{\partial z}\right) = \mathbf{k}\pi r^2 i\frac{\partial B_z}{\partial z} = \mathbf{k}m\frac{\partial B_z}{\partial z}. \qquad ...(134)$$

Here $m = \pi r^2\ i$ is the magnetic moment of the current loop.

In the above derivation, we have taken the special case of a circular loop. Since the force is found to depend on the dipole moment 'm' of the current loop not on its shape and the magnetic dipole moment $\mathbf{m}$ is along the z-direction in the above case, hence the equation for the general case may be written as

$$\mathbf{F} = (\mathbf{m}\cdot\nabla)\,\mathbf{B}. \qquad ...(135)$$

Thus we see that the force acts in the direction of increasing magnetic field, when the dipole moment is parallel to $\mathbf{B}$, in the direction of decreasing magnetic field when the dipole moment is antiparallel to $\mathbf{B}$. If the external field $\mathbf{B}$ is uniform this force is zero.

13.19 MEASUREMENT OF SUSCEPTIBILITY

A sample of the given material placed in an inhomogeneous magnetic field, will tend to move to the region of small magnetic intensity if it is diamagnetic and to the region of higher magnetic intensity if it is paramagnetic. Thus a force $\mathbf{F}$ acts on the sample placed in a non uniform magnetic field. The work is done by this force which is equal to the change in potential energy caused by the displacement of the sample. The susceptibility of the given material is measured by measuring the force on the substance in an inhomogeneous magnetic field. If instead of a permanent dipole, we have a specimen of volume Δv and of susceptibility χ, the magnetic moment of the specimen will be

$$\mathbf{m} = \mathbf{M}\ \Delta v = \chi\ \mathbf{H}\ \Delta v = (1/\mu_0)\ \chi\ \mathbf{B}\ \Delta v.$$

$$\therefore \qquad \mathbf{F} = (\mathbf{m}\cdot\nabla)\,\mathbf{B} = (1/\mu_0)\ \chi\ \Delta v\ (\mathbf{B}\cdot\nabla)\ \mathbf{B}$$

and

$$\Delta F_z = \frac{\chi}{2\mu_0}\Delta v\frac{\partial}{\partial z}(B^2).$$

If the specimen has a susceptibility χ_1 and is immersed in a medium with susceptibility χ_2, the force

$$\Delta F_z = \frac{1}{2\mu_0}(\chi_1-\chi_2)\Delta v\frac{\partial}{\partial z}(B^2).$$

If the specimen is made into a long cylinder of uniform cross section A, the vertical force in the z-direction on an element $A\ \Delta z$ is

$$\Delta F_z = \frac{1}{2\mu_0}(\chi_1-\chi_2)A\frac{\partial}{\partial z}(B^2)\Delta z$$

$$\therefore \qquad F_z = \frac{1}{2\mu_0}(\chi_1-\chi_2)A\int\frac{\partial}{\partial z}(B^2)dz$$

$$= (\chi_1-\chi_2)A\,(B_1^2-B_2^2)/2\mu_0, \qquad \text{...(137)}$$

where B_1 is the field at the lower end of the specimen and B_2 that at the upper end. Above relation can be employed for determining the susceptibility of a dia-or paramagnetic material (solid or liquid) as given below :

***(a)* Susceptibility of solids (Gouy' method) :** This method was first employed by Gouy. In this method the specimen is taken in the form of a thin rod and suspended from one arm of a sensitive microbalance in the magnetic field between the wedge shaped pole pieces of an electromagnet (Fig. 13.39). The pole pieces are sufficiently wide apart and the magnetic field is uniform horizontally in the central part of the field where the specimen is suspended. Since the vertical dimension of the pole pieces is small, the magnetic field will fall of rapidly in the vertical direction. Due to this change in magnetic field along the specimen rod, the specimen will experience a force along its length (*i.e.*, vertical direction), upward or downward depending on whether the specimen is dia-or paramagnetic respectively.

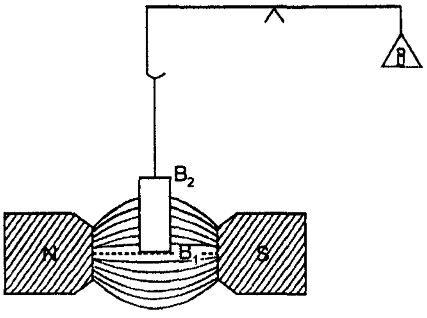


Fig. 13.39. Gouy's Method.

In this experiment, the specimen is first balanced by putting weights on the other arm of the balance. The magnetic field is switched on and the balance is again made by adjusting the weights. The weights m added or removed in the second case give us the force experienced by the specimen due to the magnetic field. As the magnetic field falls off rapidly in the z-direction, neglecting B_2 and comparing the forces we get

$$mg = A\,(\chi_s-\chi_a)\,B_1^2/2\mu_0. \qquad \text{...(138)}$$

The magnetic field B_1 can be measured by the fluxmeter. Thus by putting the susceptibility of air χ_a and other known quantities in the above equation, the χ_s of the given specimen is calculated.

***(b)* Susceptibility of liquids (Quincke method) :** Quincke devised a method for determining susceptibilities of acqueous solutions, many liquids and liquified gases. This method is particularly useful in investigating the variation in the susceptibility of acqueous solution of iron salt with concentration. *The susceptibility, in such solutions is found to be proportional to the iron content of the solution.*

A carefully clean glass tube of about 2cm in diameter is drawn out in an air gas flame so that part of it is in the form of a long narrow capillary of uniform bore. This thin part is bent to the form of U-tube as shown in Fig. 13.40. The U-tube is placed so that its plane is vertical and its narrow limb passes between the poles of an electromagnet. The tube is partially filled, so that the surface of the liquid in the narrow limb is near the centre of the field. When the field is switched on, the liquid meniscus rises or falls.

If the surface tension effects are negligible and Z is the vertical distance between the levels of the menisci when the field is applied, then

Change in hydrostatic pressure = Pressure due to magnetic field.

In our present problem, field B_2 at the end B is zero and B_1 at the end A is the field applied B, $\chi_1 = \chi_s$ the susceptibility of solution and and $\chi_2 = \chi_a$ that of air. Hence we have

$$g(\rho - \sigma) Z = (\chi_s - \chi_a) B^2/2\mu_0. \quad \text{...(139)}$$

Here ρ is the density of the solution and σ that of air. Let α be the cross sectional area of the narrow limb and β that of the wide limb, z the observed rise of the meniscus in the narrow tube and z_0 the fall in the wide tube. As total volume remains constant, hence

$$\beta z_0 = \alpha z, \text{ or } z_0 = (\alpha/\beta) z.$$

$$\therefore \quad Z = z + z_0 = z\,(1 + \alpha/\beta).$$

Hence $(1/2\mu_0)(\chi_s - \chi_a) B^2 = g\,(\rho - \sigma)\, z\,(1 + \alpha/\beta)$. ...(140)

With the help of this equation susceptibility of any solution can be calculated. Unknown field in the narrow gap can be calculated as a reverse process, if the susceptibility is known.

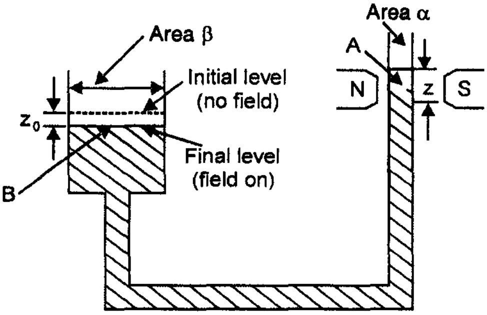


Fig. 13.40. Quincke Method.

(c) Curie Balance Method : This method was first used by Curie in 1895 for the measurement of susceptibility of weakly magnetic substances. The specimen is placed in a glass tube which is suspended from a light arm carrying a balance weight and a copper vane. The *Cu*-vane moves in the gap of a permanent magnet and serves to damp the oscillations of the arm by inducing eddy currents in the vane. The light arm is suspended by a silk or quartz fibre. The moving system also carries a mirror to measure deflection accurately. A permanent magnet of ring shape is mounted on the one end of an arm, which may be rotated so that the air gap of the magnet may be moved toward or away from the tube containing the specimen.

This method depends on measuring the force on the substance in an inhomogeneous magnetizable material of susceptibility χ and volume Δv which is given by Eq.(136). The pole tips of an electromagnet is adjusted so that $\partial B_x^2/\partial z$ is large while $\partial B_y^2/\partial z$ and $\partial B_z^2/\partial z$ are very small. Special shapes of pole tips are designed to make $\partial B_x^2/\partial z$ uniform over a large volume so that different specimens may be placed in the same position. If the specimen of susceptibility χ_1 is immersed in a medium with susceptibility χ_2, then the force

$$F_z = (\chi_1 - \chi_2)\, \Delta v\, (\partial B^2/\partial z)/2\mu_0.$$

This is measured in terms of the rotation of the torsion head, which is proportional to the torque due to this force.

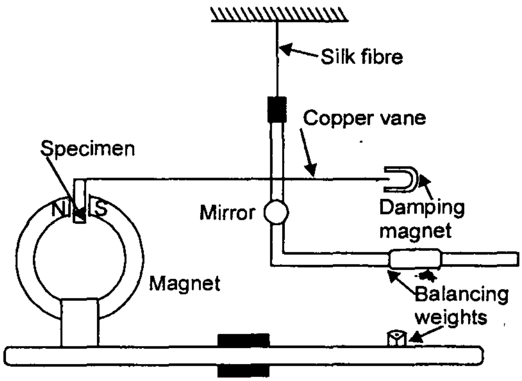


Fig. 13.41. Curie balance method.

First of all by using the glass tube alone the magnet is rotated until maximum deflection is obtained. The specimen is then placed in the tube and the maximum deflection θ_1 of the moving system is observed. The specimen is then replaced by some material whose susceptibility is known (distilled water is generally used for this purpose) and a deflection θ_2 is observed. If θ_0 was the deflection with the glass tube only, then

$$\frac{\text{Susceptibility of specimen 1}}{\text{Susceptibility of specimen 2}} = \frac{\chi_1}{\chi_2} = \frac{m_2(\theta_1-\theta_0)}{m_1(\theta_2-\theta_0)}, \qquad \text{...(141)}$$

where m_1 and m_2 are the masses of the specimen and the standard material respectively.

13.20 THE STERN GERLACH EXPERIMENT

Stern and Gerlach in 1922 performed an atomic beam experiment and demonstrated that the directions of vector magnetic moment of an atom placed in an external magnetic field were quantized with respect to the field applied.

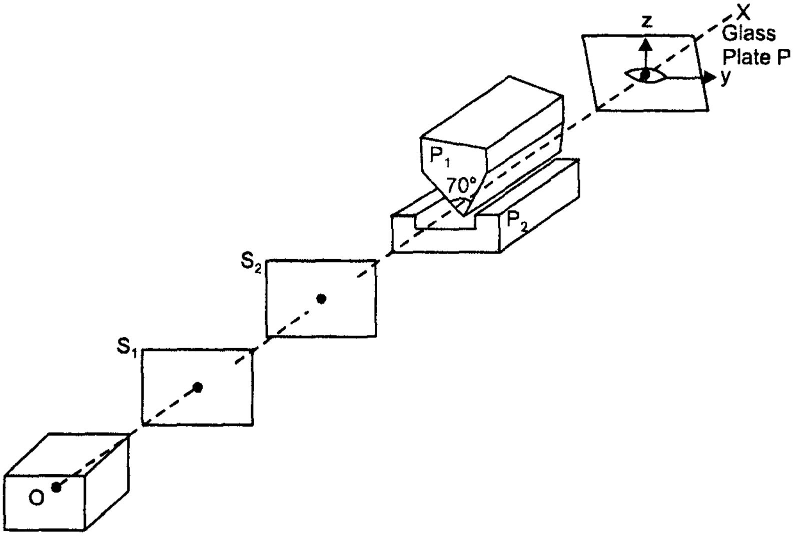


Fig. 13.42. The Stern-Gerlach Experiment.

The Stern-Gerlach atomic beam experiment is illustrated in Fig. 13.42. A strap-shaped beam of silver atoms was formed by evaporating silver in a heated oven O and allowing atoms from the vapour to stream out through collimating slits S_1 and S_2. This beam was allowed to pass along and very close to the knife edge pole piece P_1 of the magnet and condensed on the glass plate P. This specially shaped magnet had produced the non-homogeneous magnetic field, the field was much stronger near the edge than elsewhere. Thus the $\partial B_z/\partial z$ was large near the knife edge. The silver atoms were not deflected by the Lorentz force, as they were electrically neutral.

The silver atom has a single electron outside the closed shells. The orbital angular momentum of an electron thus produces a magnetic dipole moment. On account of it, the silver atoms may be considered as small magnets. Suppose that the beam of atoms having magnetic dipole of strength **m** travels in the direction of x-axis across a nonuniform magnetic field parallel to the z-axis. Then atoms which have a component of their magnetic moment in the direction of the field are deflected by a force $F_z = m_z\, \partial B_z/\partial z$. The amount of the deflection depends on the length of the atom's path and on its

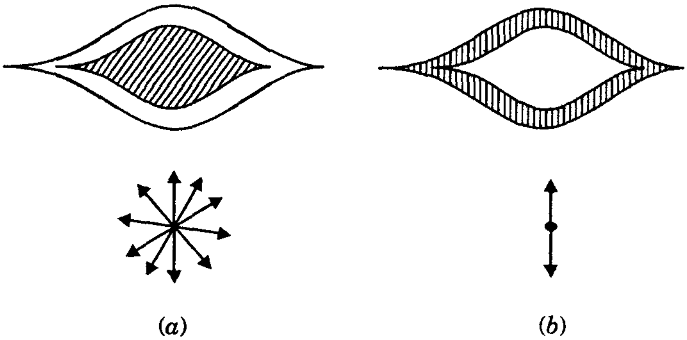


Fig. 13.43. Pattern of Silver deposite.

velocity. The measurement of the distribution of silver deposited on the glass plate will give the value of the m_z.

Classically, each atom will enter the field with its magnetic axis inclined at some angle ϕ to the field and executed a Larmour precession about the field. Since all the values of ϕ occur among the atoms, thus a continuous distribution of m_z values will be expected and atoms will be drawn out into a continuous band [Fig. 13.43(*a*)]. The observed silver deposite is shown in Fig. 13.43(*b*). It reveals only two components of m_z, one parallel to **B** other antiparallel, thus gives direct proof that the directions of **m** with respect to the magnetic field are quantized. Thus may be explained quantum mechanically as : The silver atom is normally in a $2S_{1/2}$ state (g = 2), thus the half the atoms have $M_J = \frac{1}{2}$ and a magnetic moment in the direction of the field of $g\ M_J$ = 1 Bohr magneton, and the other half have $M_J = -\frac{1}{2}$ and magnetic moment –1 Bohr magneton. The silver beam is thus broken up into separate beams and forms on the target a series of distinct spots one for each possible value of M_J.

13.21 MAGNETOSTRICTION

The *Magnetostriction* is the phenomenon which refers to the changes in the dimensions of a ferromagnetic material due to the magnetising field. The longitudinal and transverse effects when the specimen is magnetised in a specific direction were first observed by Joule, hence are called *Joule effects*. If the magnetic specimen exhibits a bending effect due to the magnetising field, such an effect is called *Guillemin effect*. If the specimen material is twisted when subjected to circular and longitudinal magnetising field, such an effect is called *Wiedemann effect*. There are changes in volume due to the magnetising field and thus the modulii of elasticity of a ferromagnetic material also change.

The reverse effect, known as *Villari effect*, is also seen as a change in magnetic induction due to (*i*) changes in the longitudinal and transverse mechanical stresses and (*ii*) the bending and twisting of ferromagnetic material when placed in a magnetic field.

The Joule effect may be demonstrated easily by placing an iron rod inside a solenoid. On passing a current through the solenoid, it is found that the rod lengthens first, then decreases and becomes shorter than in the unmagnetised condition. The change in length dl of the rod of length l is observed by pressing one end of the rod against a rigid support and by putting the other end on a bent glass tube to the vertical limb of which is attached a mirror. As the length of the rod increases the limb of the tube rolls along and rotates the mirror. The rotation in the mirror is measured with the help of lamp and scale arrangement. The variation of dl/l with the magnetising field B is shown in Fig. 13.44 for *Fe*, *Co* and *Ni*.

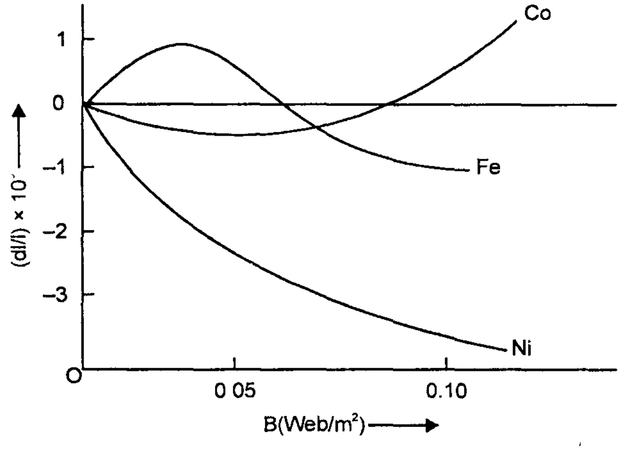


Fig. 13.44. Magnetostriction effects.

If a small rod of strongly magnetostriction alloy (54% *Fe*, 36%*Ni* and 10% *Cr*) is placed in a solenoid and an alternating current is allowed to pass. For a particular length of the rod, it is possible to produce resonance. The experimental arrangement of this type is called *magnetostriction oscillator* and is used in echo sounding devices.

13.22 MAGNETIC CIRCUIT

A magnetic circuit is the closed path taken by the magnetic flux set up in an electrical machine or apparatus by a magnetizing force which is a current coil or a permanent magnet. It consists chiefly of some ferromagnetic material. The magnetic circuits may be classified as : (*i*) undivided and (*ii*) divided. The type of circuit to be prepared depends mainly on the purpose and design of the electrical apparatus.

The undivided magnetic circuit carries one and the same flux across its all parts. Its simplest example is the toroid or Rowland ring in which the magnetic flux is confined to the region inside the toroidal winding.

The magnetic flux is generally different in different limbs of the divided magnetic circuit. Divided circuits may be classified as symmetrical and unsymmetrical. The former is made up of identical undivided components carrying one and the same flux in all of their parts. In the latter case these conditions are not met. The magnetic circuits of various electromagnets are the examples of divided circuits.

Let us now discuss few simple cases :

(*a*) Rowland Ring : Consider a Rowland ring of ferromagnetic material magnetised by a current *i* in a closely fitting toroidal winding of total turns *N*. We know from Ampere's circuital law, that

$$\oint \mathbf{H} \cdot dl = Ni.$$

As the line integral of electric field **E** over a circuital path is the electromotive force (emf), hence by analogy, the closed line integral of **H** is termed as *magnetomotive* force (mmf).

$$\therefore \qquad \text{mmf} = \oint \mathbf{H} \cdot dl = Ni.$$

If *A* is the cross-sectional area of the ring specimen, then the magnetic flux

$$\Phi = BA = \mu HA \text{ or } H = \Phi/\mu A.$$

$$\therefore \qquad \text{mmf} = \oint H dl = \oint \Phi dl/\mu A = \Phi \oint dl/\mu A = Ni. \qquad ...(142)$$

Φ is taken outside the integral as we consider it to be essentially constant at all points in the circuit. The Eq. (142) shows that it is analogous to the equation for a series current circuit (*i.e., Ohm's law*).

$$emf = \text{Current} \times \text{Resistance} = i \oint \rho (dl/A). \qquad ...(143)$$

The Eqs. (142) and (143) suggest that

(*a*) The magnetomotive force ($\oint \mathbf{H} \cdot dl$) is analogous with emf ($\oint \mathbf{E} \cdot d\mathbf{l}$)

(*b*) The magnetic flux Φ is analogous with current *i* in Ohm's law.

(*c*) The magnetic resistance, known as magnetic reluctance ($\oint dl/\mu A$) is analogous with electric resistance ($\oint \rho dl/A$), provided $1/\mu$ corresponds with resistivity ρ.

$$\therefore \qquad \text{mmf} = \text{Flux } \Phi \times \text{Reluctance } R$$

or

$$\text{Total flux } \Phi = \frac{\text{mmf}}{\text{Reluctance}} = \frac{Ni}{\oint dl/\mu A}. \qquad ...(144)$$

This equation is often referred to as the ohm's law of magnetic circuit. It is also known as Rosanquet's or Rawland's law.

However, this analogy is only in the form of the expressions, as there are following points which show dissimilarity between magnetic and electrical circuits :

(*a*) In an electric circuit the electric current is a flow of electrons (or ions), but in magnetic circuit there is no flow of such particles.

(*b*) In an electric circuit, energy is expended as long as the current flows, but in a magnetic circuit energy is expended in establishing the magnetic flux and not in maintaining it.

(*c*) At a given temperature, the resistivity ρ is independent of the current strength, while the corresponding quantity l/μ in reluctance varies with magnetic flux Φ.

If the magnetic circuit is made up of a number of components, *i.e.*, various types of ferromagnetic materials and possibly air gaps, then the total reluctance

$$\oint \frac{dl}{\mu A} = \frac{l_1}{\mu_1 A_1} + \frac{l_2}{\mu_2 A_2} + \frac{l_3}{\mu_3 A_3} + \ldots = \sum_i \frac{l_i}{\mu_i A_i}. \qquad \ldots(145)$$

This indicates that the reluctance of a magnetic circuit is the sum of the reluctances of its parts, similar to the resistance of an electric circuit which is the sum of its component resistances.

$$\therefore \qquad mmf = \Phi \times \Sigma\,(l/\mu A). \qquad \ldots(146)$$

This formula for magnetic circuit is used by engineers for calculating the number of ampere turns in the winding required to produce a desired flux density in electromagnets, motors or in dynamos.

Ring with a small air gap : Consider an iron ring of mean radius R, permeability μ and with a small airgap of length l_g. Let the area of cross section of the ring be A, the introduction of the airgap into the magnetic circuit (ring) does not affect the reluctance of the iron path if the air gap length l_g is negligible in comparison to the iron core length $(2\pi R - l_g)$.

The reluctance of the iron core = $(2\pi R - l_g)/\mu A$.

The reluctance of the air gap = $l_g/\mu_0 A$.

$\therefore$ Total reluctance of the magnetic circuit = $\dfrac{2\pi R - l_g}{\mu A} + \dfrac{l_g}{\mu_0 A}$.

If the current i is passing through toroidal winding of N turns, the flux in the circuit

$$\Phi = \frac{\text{mmf}}{\text{Reluctance}} = \frac{Ni}{\dfrac{2\pi R - l_g}{\mu A} + \dfrac{l_g}{\mu_0 A}}. \qquad \ldots(147)$$

Since the relative permeabilities of the ferromagnetic materials are about 10^3 to 10^4 μ_0, the reluctance of the air gap therefore increases to ten to hundred times depending upon the ratio of $l_g/(2\pi R - l_g)$. Thus the magnetomotive force needed to drive the flux through the magnetic circuit with a gap is greater than that required in the absence of the gap. On account of this reason, the air gap should be made as small as possible whenever it is indispensable for normal operation of the electrical apparatus.

Because of the large ratio of μ/μ_0, the lines of **B** show a fringing effect at the air gap, thus making the effective cross section of the gap larger than its iron core face. For the ring of circular cross section of the radius r, the radius of an air gap is taken as the sum of the actual radius r and the width of the airgap l_g.

For the rectangular cross-section of sides a and b, the each dimension is assumed to be increased by the width of the air gap, thus the area of the gap

$$A = (a + l_g)(b + l_g) \text{ for rectangular cross section}$$

$$A = \pi (r + l_g)^2 \text{ for circular cross section.}$$

(b) Electromagnet : Let us consider an electromagnet, the vertical portions of the iron core are wound with coils of wire carrying a current i. Most of the lines of the **B**-field are confined to the core and the air gap. The **H** field due to the current i in the coils ($\mathbf{H}_C$) is everywhere clockwise in the core, but not have same magnitude at all points. The poles set up a demagnetizing field $\mathbf{H}_P$ which at all points in the core is opposite to $\mathbf{H}_C$. The $\mathbf{H}_P$ at any point is the vector sum of the fields $\mathbf{H}_N$ and $\mathbf{H}_S$ due to north and south poles at this point.

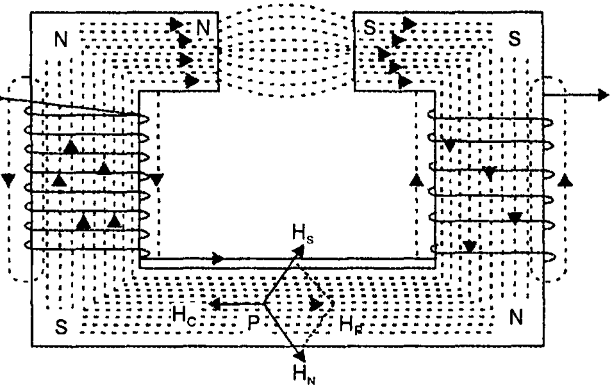


Fig. 13.45. Lines of **H** in an electromagnet.

An exact analytic solution of the problem is very tedious. For this let us neglect any leakage of flux. If the flux Φ is the same over all cross sections of the air gap. If the core is of same cross section A_1 at all points and the air gap is of larger cross-section A_2, then flux densities in the core and the gap are

$$B_1 = \Phi/A_1 \text{ and } B_2 = \Phi/A_2. \qquad ...(147)$$

The corresponding values of H are

$$H_1 = B_1/\mu_1 = \Phi/\mu_1 A_1 \text{ and } H_2 = B_2/\mu_2 = \Phi/\mu_2 A_2. \qquad ...(148)$$

If l_1 and l_2 are the path lengths in the core gap and the air gap respectively, then

$$\oint \mathbf{H} \cdot dl = H_1 l_1 + H_2 l_2 = \Phi[l_1/\mu_1 A_1 + l_2/\mu_2 A_2].$$

As **H** is the vector sum of $\mathbf{H}_C$ and $\mathbf{H}_P$, hence

$$\oint \mathbf{H} \cdot dl = \oint \mathbf{H}_C \cdot d\mathbf{l} + \oint \mathbf{H}_P \cdot d\mathbf{l} = Ni + 0,$$

where N is the total number of turns in the winding.

$$\therefore \qquad \Phi = \frac{Ni}{(l_1/\mu_1 A_1) + (l_2/\mu_2 A_2)}. \qquad ...(149)$$

Thus the magnetic flux in the air gap is equal to the mmf Ni divided by the sum of the reluctances of the iron yoke and of the air gap. This is a general relation. It shows that *reluctances in a magnetic circuit add in the same way as the conductances in an electric circuit.*

(c) Magnetic Circuit with a Permanent Magnet : The magnetomotive force (mmf) in the circuit is produced by current winding as well as by permanent magnet. Consider a magnetic circuit consisting of a permanent magnet of length l_m, two soft iron side arms of total length l_s and an airgap of length l_g as shown in Fig. 11.46. Applying Ampere's law to the circuit gives

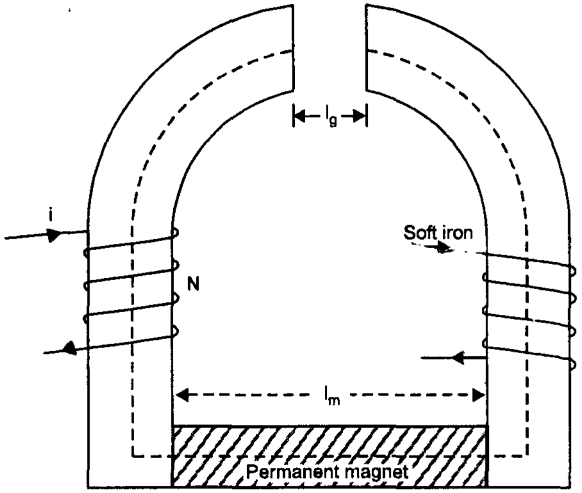


Fig. 13.46. Magnetic circuit with a permanent magnet.

$$H_m l_m + H_s L_s + H_g l_g = Ni, \qquad \text{...(150)}$$

where H_m, H_s and H_g are the magnetic fields in the permanent magnet, soft iron side arms and the airgap respectively. If we represent the cross sectional areas of these as A_m, A_s andA_g. The arms have a winding of N-turns and current i.

Since the magnetic flux Φ is continuous throughout the entire circuit, therefore $\Phi = B_m A_m = \mu_0 H_g A_g = \mu_s H_s A_s$, where μ_0 and μ_s are the permeabilities of the air and the soft iron respectively. Thus the relation (150) may be written as

$$H_m l_m + \frac{B_m A_m}{\mu_s A_s} l_s + \frac{B_m A_m}{\mu_0 A_g} l_g = Ni$$

$$\text{or} \quad H_m l_m + B_m A_m \left(\frac{l_s}{\mu_s A_s} + \frac{l_g}{\mu_0 A_g} \right) = Ni$$

$$\text{or} \qquad B_m = -\frac{H_m l_m}{RA_m} + \frac{Ni}{RA_m} \qquad \text{...(151)}$$

where the reluctance of the circuit $R = l_s/\mu_s A_s + l_g/\mu_0 A_g$.

If the air gap is not too narrow because of large ratio μ_s/μ_0, the effect of the soft iron arms can be neglected and thus we have

$$B_m = -\frac{\mu_0 l_m A_g}{l_g A_m} H_m + \frac{\mu_0 A_g}{l_g A_m} Ni. \qquad \text{...(152)}$$

If there is no current coil, $i = 0$ and we have

$$B_m = -(\mu_0 l_m A_g / l_g A_m) H_m. \qquad \text{...(153)}$$

The curve between B_m and H_m is a linear curve. The Eq. (151) or (152) is plotted along with the hysteresis curve of the magnet on the same curve. The point of intersection of the two curves gives the operating point of the magnet.

EXERCISES

Example 1. *Determine the magnetic moment of a triangular loop of Fig. 13.47.*

Since the loop lies in a plane having point (2, 0, 0), (0, 2, 0) and (0, 0, 2). The equation of the plane on which this loop lies can be obtained using the general equation

$$Ax + By + Cz = (A^2 + B^2 + C^2)^{1/2}$$

$$\text{as} \qquad x + y + z = 2.$$

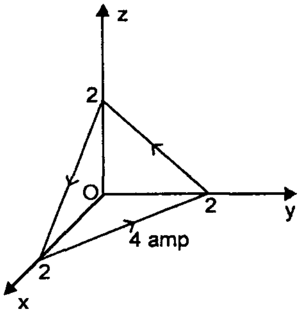


Fig. 13.47. Example 1.

If we define this plane surface by a function $f = x + y + z - 2$.

Its outward normal $\hat{n} = \pm \dfrac{\nabla f}{|\nabla f|} = \pm \dfrac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}}$.

From the direction of current, we choose plus sign. Therefore the magnetic moment of the loop

$$\begin{aligned}\mathbf{m} &= IS\hat{n} = 4 \times \tfrac{1}{2}(2\sqrt{2})(2\sqrt{2})\sin 60° \times \frac{\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z}{\sqrt{3}} \\ &= 8\,(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\, Am^2.\end{aligned}$$

Example 2. *A circular wire loop of radius a is rotating uniformly with angular velocity ω about a diameter PQ as shown in Fig. 13.48. At its centre and lying along this diameter is a small magnet of total magnetic moment M. What is the induced emf between the point P(or Q) and a point on the loop midway between P and Q ?*

At a point distance r from the centre O, the magnetic field established by the small magnet of magnetic moment M is

$$\mathbf{B} = \frac{\mu_0}{4\pi}\left[\frac{3(\mathbf{M}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{M}}{r^3}\right]$$

where $\mathbf{M} = M\mathbf{a}_z$. Consider the mid point C of arc PQ. The velocity of a point on arc QC is $\mathbf{v} = \omega a \sin\theta\, \mathbf{a}_\phi$. The induced emf between the points Q and C along the arc QC is given by

$$\mathcal{E}_{QC} = \int_Q^C (\mathbf{v} \times \mathbf{B}) . d\mathbf{l},$$

where $d\mathbf{l} = a\, d\theta\, \mathbf{a}_\theta$.

As $\mathbf{a}_z = \cos\theta\, \mathbf{a}_r - \sin\theta\, \mathbf{a}_\theta$, $\mathbf{a}_\phi \times \mathbf{a}_r = \mathbf{a}_\theta$, $\mathbf{a}_\phi \times \mathbf{a}_\theta = -\mathbf{a}_r$, we have

$$\mathbf{v} \times \mathbf{B} = \frac{\mu_0 \omega\, a\, M \sin\theta}{4\pi a^3}(2\cos\theta\, \mathbf{a}_\theta + \sin\theta\, \mathbf{a}_r)$$

$$\therefore \quad \mathcal{E}_{QC} = \frac{\mu_0 M \omega}{4\pi a}\int_0^{\pi/2} 2\cos\theta \sin\theta\, d\theta = \frac{\mu_0 M\omega}{4\pi a}$$

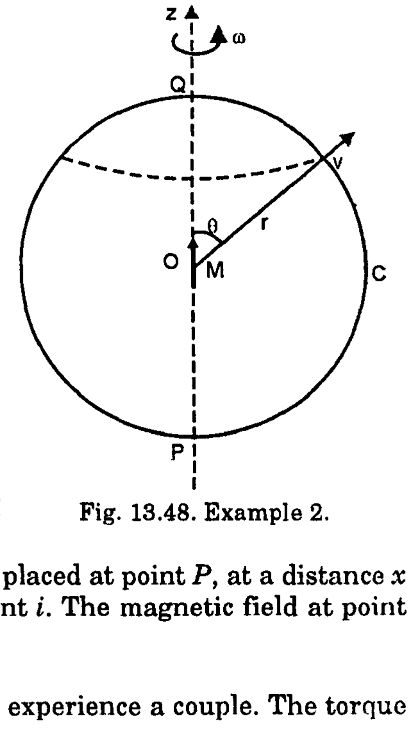


Fig. 13.48. Example 2.

Example 3. *Calculate the force acting on a dipole placed on the axis of the current carrying loop of wire.*

Consider a small magnet (dipole) of magnetic moment m placed at point P, at a distance x along the axis of a loop of wire of radius a, carrying a current i. The magnetic field at point P due to this current loop is

$$B = \mu_0 i a^2 / 2\,(a^2 + x^2)^{3/2}.$$

It is along the axis of the coil. The magnet will therefore experience a couple. The torque due to this couple is

$$\tau = mB\sin\theta = \frac{\mu_0 i a^2 m}{2(a^2 + x^2)^{3/2}}\sin\theta.$$

This couple will tend to reduce θ. If the centre of the magnet is fixed at P and the magnet is free to rotate about P, then the equation of motion is

$$I\frac{d^2\theta}{dt^2} + \frac{\mu_0 i a^2 m \sin\theta}{2(a^2 + x^2)^{3/2}} = 0\,.$$

or $$\frac{d^2\theta}{dt^2}+\frac{\mu_0 i a^2 m}{2I(a^2+x^2)^{3/2}}\sin\theta=0,$$

where I is the moment of inertia of the dipole. If θ is small, then $\sin\theta = \theta$ and the above equation may be written as

$$\frac{d^2\theta}{dt^2}+\omega^2\theta=0$$

where $\omega^2 = \mu_0 i a^2 m/2I(a^2+x^2)^{3/2}$.

It is the equation of S. H. M. If i is known, the measurement of time period gives us m.

If **m** lies along the axis of the loop. In this case no couple acts on the magnet, there is a force F only which is along the axis and is of value

$$F = m\frac{\partial B}{\partial x}=-\frac{3\mu_0 i a^2 m x}{2(a^2+x^2)^{5/2}}.$$

Example 4. *Two small magnets are kept at a distance r from each other, their magnetic moments* $\mathbf{M}_1$ *and* $\mathbf{M}_2$ *are inclined at angles* θ_1 *and* θ_2 *with the line joining them. If r is greater than the size of each magnet, calculate : (a) the torque acting on each of them, (b) the forces experienced by each, and (c) the mutual potential energy of two magnets.*

We know that the potential energy of magnetic dipole of magnetic moment M and placed in a magnetic field **B** is

$$U = -\mathbf{M}\cdot\mathbf{B} = -MB\cos\theta.$$

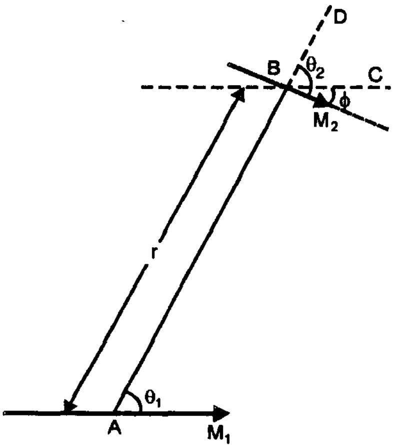


Fig. 13.49. Example 4.

Magnetic field **B** at point B due to magnetic dipole at A is having two components $2\mu_0 M_1\cos\theta_1/4\pi r^3$ along AB or **r** and $\mu_0 M_1\sin\theta_1/4\pi r^3$ along normal to **r**. If this field **B** is assumed as the resultant of B_1 along BC and B_2 along BD, then

$$B_1\cos\theta_1+B_2 = \frac{2\mu_0 M_1\cos\theta_1}{4\pi r^3}$$

and $$B_1\sin\theta_1 = -\frac{\mu_0 M_1\sin\theta_1}{4\pi r^3}$$

$$\therefore \quad B_1 = -\frac{\mu_0 M_1}{4\pi r^3} \text{ and } B_2=\frac{3\mu_0 M_1\cos\theta_1}{4\pi r^3}$$

Hence in vector form, magnetic field

$$\mathbf{B} = -\frac{\mu_0\mathbf{M}_1}{4\pi r^3}+\frac{3\mu_0(\mathbf{M}_1\cdot\mathbf{r})\,\mathbf{r}}{4\pi r^5}.$$

Thus the potential energy of the magnet M_2 in the field **B** due to magnet M_1 is given by

$$U = -\mathbf{M}_2\cdot\mathbf{B}=\frac{\mu_0\mathbf{M}_2\cdot\mathbf{M}_1}{4\pi r^3}-\frac{3\mu_0(\mathbf{M}_1\cdot\mathbf{r})(\mathbf{M}_2\cdot\mathbf{r})}{4\pi r^5}.$$

When the magnets are coplaner, this becomes

$$U = -\mu_0 M_1 M_2[\cos\phi - 3\cos\theta_1\cos\theta_2]/4\pi r^3.$$

Torque exerted by dipole M_1 on M_2 is

$$\tau_{12} = \mathbf{M}_2 \times \mathbf{B} = \mathbf{M}_2 \times \left[-\frac{\mu_0 \mathbf{M}_1}{4\pi r^3} + \frac{3\mu_0 (\mathbf{M}_1.\mathbf{r})\,\mathbf{r}}{4\pi r^5}\right]$$

$$= -\frac{\mu_0 \mathbf{M}_2 \times \mathbf{M}_1}{4\pi r^3} + \frac{3\mu_0 (\mathbf{M}_1 \cdot \mathbf{r})\,(\mathbf{M}_2 \times \mathbf{r})}{4\pi r^5}$$

In the special case of coplaner magnets, we get

$$\tau_{12} = -\frac{\mu_0 M_1 M_2 \sin(\theta_2 - \theta_1)}{4\pi r^3} + \frac{3\mu_0 M_1 M_2 \cos\theta_1 \sin\theta_2}{4\pi r^3}$$

$$= \frac{\mu_0 M_1 M_2}{4\pi r^3}[-\sin\theta_2 \cos\theta_1 + \cos\theta_2 \sin\theta_1 + 3\cos\theta_1 \sin\theta_2]$$

$$= \mu_0 M_1 M_2 \,[2\cos\theta_1 \sin\theta_2 + \cos\theta_2 \sin\theta_1]/4\pi r^3.$$

It must be ⊥ to the plane containing $\mathbf{M}_2$ and $\mathbf{B}$.

The torque acting on M_1 by M_2 is not same as that on M_2 by M_1. The expression can be similarly obtained and is given as

$$\tau_{21} = \mu_0 M_1 M_2 \,[2\sin\theta_1 \cos\theta_2 + \sin\theta_2 \cos\theta_1]/4\pi r^3.$$

Let us compute the force exerted by M_1 on M_2 tending to move the latter without rotation. Suppose $\mathbf{M}_2$ undergoes a displacement $\delta\mathbf{r}$. The force is given as

$$\mathbf{F}_{12}\cdot\delta\mathbf{r} = -\delta U = -\delta[\mu_0 \mathbf{M}_1\cdot\mathbf{M}_2/4\pi r^3 - 3\mu_0 (\mathbf{M}_1\cdot\mathbf{r})\,(\mathbf{M}_2\cdot\mathbf{r})/4\pi r^5]$$

$$= \frac{3\mu_0 \mathbf{M}_1\cdot\mathbf{M}_2}{4\pi r^4}\delta r + \frac{3\mu_0(\mathbf{M}_1\cdot\delta\mathbf{r})\,(\mathbf{M}_2\cdot\mathbf{r}) + 3\mu_0(\mathbf{M}_1\cdot\mathbf{r})\,(\mathbf{M}_2\cdot\delta\mathbf{r})}{4\pi r^5} - \frac{15\mu_0(\mathbf{M}_1\cdot\mathbf{r})\,(\mathbf{M}_2\cdot\mathbf{r})\delta r}{4\pi r^6}$$

or $\mathbf{F}_{12} = (\alpha\mathbf{r} + \beta\mathbf{M}_1 + \gamma\mathbf{M}_2)/4\pi r^5$,

where $\alpha = 3\mu_0[(\mathbf{M}_1\cdot\mathbf{M}_2) - 5(\mathbf{M}_1\cdot\mathbf{r})\,(\mathbf{M}_2\cdot\mathbf{r})r^{-2}]$,

$\beta = 3\mu_0\,(\mathbf{M}_2\cdot\mathbf{r})$ and $\gamma = 3\mu_0\,(\mathbf{M}_1\cdot\mathbf{r})$.

Example 5. *Two small magnets lie in the same plane and make angles θ_1 and θ_2 with the line joining their centres. Show that the line of action of the resultant force between them divides the line of centre in the ratio.*

($\tan\theta_2 + 2\tan\theta_1$) : ($\tan\theta_1 + 2\tan\theta_2$).

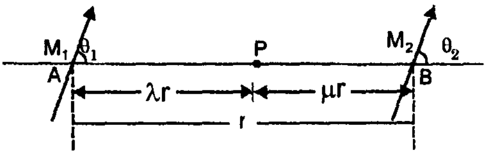


Fig. 13.50. Example 5.

Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be the magnetic moments of the small magnets, placed at point A and B. Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are coplaner, hence $\mathbf{F}_{12}$ and τ_{12} also lie in the plane of $\mathbf{M}_1$ and $\mathbf{M}_2$ and thus reduce to a single force. If the line of action of this force cuts AB at P, such that $\mathbf{AP} = \lambda\mathbf{r}$ and $\mathbf{BP} = -\mu\mathbf{r}$. Changing of the origin from A to P follows that the couple at P must vanish.

$$\therefore \quad \tau_{12} + \mu\mathbf{r} \times \mathbf{F}_{12} = 0.$$

Similarly $\tau_{21} - \lambda\mathbf{r} \times \mathbf{F}_{21} = \tau_{21} + \lambda\mathbf{r} \times \mathbf{F}_{12} = 0$

$$\therefore \quad \lambda\tau_{12} = \mu\tau_{21}.$$

Using relations for torque from example 4, we get

$$\frac{\lambda}{\mu} = \frac{2\sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1}{2\sin\theta_2\cos\theta_1 + \sin\theta_1\cos\theta_2} = \frac{2\tan\theta_1 + \tan\theta_2}{2\tan\theta_2 + \tan\theta_1}$$

Example 6. *A coil of 200 turns and 2cm^2 area of cross-section carries a current of 10 amp. If it is kept in such a way that its magnetic moment* **M** *is parallel to the horizontal component of earth's magnetic field* **H**. *Calculate the radius of the horizontal circle around the coil at which the total horizontal field is everywhere radially directed. Given $H = 3 \times 10^{-5}$ weber/meter^2.*

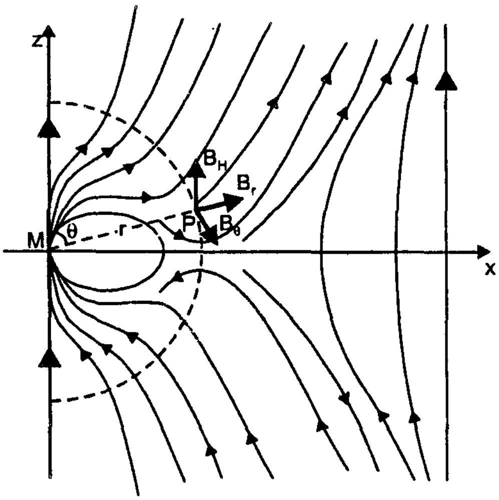


Fig. 13.51. Example 6.

Let the x-z plane be the horizontal plane and let the horizontal component of earth magnetic field $B_H (= H)$ is parallel to the +ve z-direction. Magnetic field at point P due to the magnetic moment **M** is

$$B_r = 2\mu_0 M \cos\theta / 4\pi r^3$$

and $$B_\theta = \mu_0 M \sin\theta / 4\pi r^3.$$

If the magnetic field is radial at point P, then

$$B_\theta = B_H \sin\theta$$

or $$\mu_0 M \sin\theta / 4\pi r^3 = B_H \sin\theta$$

$$\therefore \quad r = R = [\mu_0 M / 4\pi B_H]^{1/3}$$

In our problem $M = iAn = 10 \times 2 \times 10^{-4} \times 200 = 0.4$

$$\therefore \quad r = [4\pi \times 10^{-7} \times 0.4 / 4\pi \times 3 \times 10^{-5}]^{1/3} = 0.11 \text{ m}.$$

Example 7. *Prove that the radius of the orbital motion of an electron around its nucleus remains same, whether the magnetic field is applied or not in the direction $\perp$ to its plane of motion.*

We know that in the absence of magnetic field, the electrostatic force between the nucleus and electron is balanced by the centripetal force. Thus for an electron of mass m_e, radius r and linear velocity v, we have

$$\text{Electrostatic force } F = m_e v^2/r = m_e r\omega^2.$$

When a magnetic field is applied the angular velocity changes to $\omega + \omega_L$ (Here sign of ω_L will depend on the direction of **B** with respect to the original motion). In addition to it a new force e (**v** × **B**) is applied to balance the new angular velocity. Thus in the presence of magnetic field

$$F = m_e r(\omega + \omega_L)^2 + evB = m_e r(\omega + \omega_L)^2 + er(\omega + \omega_L)B$$
$$= m_e r\omega^2 + m_e r\omega_L^2 + 2m_e r\omega\omega_L + er\omega B + er\omega_L B.$$

As $\omega_L = -eB/2m_e$, hence third and fourth terms cancel, leaving $F = m_e r\omega^2$ + terms consisting ω_L and ω_L^2. As $\omega_L << \omega$, hence we have same value of electrostatic force for equilibrium, or the radius of the electron orbit remains same. The magentic field only causes change in velocity.

Example 8. *A sphere of radius r carries a charge q, distributed uniformly over its surface. If it is rotating about an axis of sphere with angular velocity ω, calculate the magnetic moment of the sphere.*

The magnetic moment of a charge q of mass m and of angular momentum L is given by the equation

$$M = qL/2m.$$

The angular momentum $L = mvr = mr^2\,\omega$

$$\therefore \qquad M = qr^2\omega/2.$$

Let us consider the given sphere as consisting of narrow discs. The area of one disc = $2\pi r \sin\theta\, r d\theta$. As charge is uniformly distributed over the surface, hence surface charge density

$$\sigma = q/4\pi r^2.$$

$\therefore$ Charge on the disc = $2\pi r^2 \sin\theta\, d\theta\sigma = \frac{1}{2} q \sin\theta\, d\theta$.

The magnetic moment due to this disc

$$dM = (\tfrac{1}{2} q \sin\theta\, d\theta)(r\sin\theta)^2\,\omega/2 = \tfrac{1}{4} q\omega r^2 \sin^3\theta\, d\theta.$$

$\therefore$ Total magnetic moment of the whole sphere

$$= \int_0^\pi \tfrac{1}{4} q\omega r^2 \sin^3\theta\, d\theta = \tfrac{1}{4} q\omega r^2 (4/3) = \tfrac{1}{3} q\omega r^2.$$

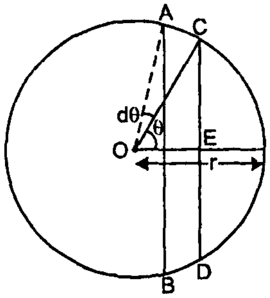


Fig. 13.52. Example 8.

Example 9. *In the Rowland ring of mean circumferential length 40 cm 2.0 amp current is passing through the winding of 400 turns. Magnetic induction measured is 1.0 weber/m². Calculate (a) the magnetic intensity, (b) the magnetization, (c) magnetic susceptibility, (d) the equivalent surface current and (e) the magnetic coefficient.*

The magnetic intensity H is independent of the core material and is given by

$$H = ni\,(400/40 \times 10^{-2}) \times 2 \text{ amp} = 2000 \text{ amp/m}.$$

The magnetization M is zero, if core is not present. In the presence of the core

$$M = \left(\frac{B - \mu_0 H}{\mu_0}\right) = \frac{1.0 - 4\pi \times 10^{-7} \times 2 \times 10^3}{4\pi \times 10^{-7}}$$

$$= 7.94 \times 10^5 \text{ amp/m}.$$

The magnetic susceptibility $\chi_m = \dfrac{M}{H} = \dfrac{7.94 \times 10^5}{2 \times 10^3} = 397.$

The equivalent surface current $i_s = j^s \times$ length of the ring

or $\qquad i_s = 7.97 \times 10^5 \times 0.40 = 3.188 \times 10^5$ amp.

The magnetic coefficient or the relative permeability

$$k = 1 + \chi_m = 1 + 397 = 398.$$

Example 10. *The magnetic field of 20 CGS units (emu or oersted) produces a flux of 2400 CGS units (emu or maxwell) in a bar of iron of cross section 0.2 cm². Calculate the permeability and susceptibility of the bar.*

$$B = \frac{\Phi}{A} = \frac{2400 \times 10^{-8}}{0.2 \times 10^{-4}} = 1.20 \text{ weber/m}^2.$$

The permeability of the bar material

$$\mu = \frac{B}{H} = \frac{1.2}{20/4\pi \times 10^{-3}} = 7.54 \times 10^{-4} \text{ henry/meter.}$$

The magnetic susceptibility and the permeability of a material are related with each other as

$$\mu = \mu_0(1+\chi_m)$$

or $$\chi_m = \mu/\mu_0 - 1 = \frac{7.54\times10^{-4}}{4\pi\times10^{-7}} - 1 = 600 - 1 = 599 \text{ mks units.}$$

Example 11. *An iron rod 20 cm long 1 cm in diameter and of a permeability 1000 is placed inside a solenoid 1 meter long wound uniformly with 600 turns. If the current of 0.5 ampere is passed through the solenoid, find the magnetic moment of the rod.*

We know that B, H and M are related with each other as

$$B = \mu_0(H+M).$$

Since $B = \mu H = \mu_r\mu_0 H$, where μ is the permeability of the material and μ_r its relative permeability. Hence

$$M = \frac{B-\mu_0 H}{\mu_0} = \frac{\mu_0\mu_r H - \mu_0 H}{\mu_0} = (\mu_r - 1)\,H.$$

In a solenoid of n turns per unit length carrying a current i, the magnetic field intensity H is given by the relation

$$H = ni.$$

$$\therefore \quad M = (\mu_r - 1)\,ni.$$

The substitution of the numerical values gives

$$M = (1000-1)\times600\times0.5 = 299700 \text{ amp/m.}$$

As the intensity of magnetization is defined as the magnetic moment per unit volume, therefore the magnetic moment of the rod

$$m = M\times\text{volume} = M\times\pi r^2 l$$
$$= 299700\times3.14\times(0.5\times10^{-2})^2\times0.20 = 4.71 \text{ amp. } m^2.$$

Example 12. *A solid toroid of iron has a core radius 1 cm and a toroid of 10 cm. A winding of 800 turns around the toroid carries a steady current of 10 amp.If the maximum magnetization of iron arises from the spin alignment of two electrons in each atom with $m_0 = 2m_B$, and the magnetic induction measured in the iron is 1.85T. Determine (i) the magnetization M, (ii) the saturation value of M, (iii) the magnetization current i_m and (iv) the values of μ, χ and k.*

In the toroid, the magnetic field intensity is uniform and is given by

$$H = ni = (N/2\pi R)\,i = (800/2\times3.14\times0.10)\,10$$
$$= 1.274\times10^4 \text{ amp/m.}$$

The intensity of magnetization M is given by

$$M = \frac{B}{\mu_0} - H = \frac{1.85}{4\times3.14\times10^{-7}} - 1.274\times10^4$$
$$= 1.473\times10^6 - 1.274\times10^4 = 1.46\times10^6 \text{ amp/m.}$$

The maximum magnetization is given by

$$M_m = m_0 N = 2\,m_B\,(N_0\rho/A)$$

For iron, the density ρ = 7860 kg/m^3, atomic mass A = 55.9 gm/mol = 0.0559 kg/mol, Avogadro number $N = 6.023\times10^{23}$ mol^{-1}. Since $m_B = 9.27\times10^{-24}$ amp/m^2. Therefore

$$M_m = 2\times(9.27\times10^{24})\,(6.023\times10^{23}\times7860/0.0559)$$
$$= 1.57\times10^6 \text{ amp/m.}$$

Thus $M/M_m = 1.46 \times 10^6/1.57 \times 10^6 = 0.93,$

i.e. the magnetization is 93% of the saturation value.

The surface magnetization current $i_m = Ml = M\,(2\pi R)$

$$= 1.46 \times 10^6 \times 2 \times 3.14 \times 0.10 = 9.17 \times 10^5 \text{ amp.}$$

Since the total coil current is 800 turns × 10amp = 8×10^3 amp. Thus we see that magnetization current is in the same direction as that of coil current.

The permeability of the iron $\mu = B/H = 1.85/1.274 \times 10^4$

$$= 1.46 \times 10^{-4} \text{ T. m/amp.}$$

The relative permeability $k = \mu/\mu_0 = 1.46 \times 10^{-4}/4 \times 3.14 \times 10^{-7}$

$$= 115.6.$$

∴ Magnetic susceptibility $\chi = k - 1 = 114.6.$

Example 13. *Prove that the tangential component of* **H** *is discontinuous across a current sheet and its normal component is continuous.*

The surface of a closely wound solenoid constitutes a surface current sheet. The line integral of **H** around the contour or the closed loop is equal to the total current passing through the loop, passing through the surface ΔS (say). If the enclosed current has a volume density j_v, then we will have

$$\oint \mathbf{H}\cdot dl = \int \mathbf{j}_v\cdot d\mathbf{S}.$$

If the contour (or loop) is of rectangular shape of width very small h and length Δl. The ends contributions are negligible and the line integral is mainly due to long sides.

$$\therefore \quad (H_{1t} - H_{2t})\,\Delta l = j_v\,\Delta S = j_v\,h\,(\Delta S/h) = j_v\,h\,\Delta l = j_S \Delta l.$$

or $$H_{1t} - H_{2t} = j_S.$$

Thus the tangential component of **H** is discontinuous across a current sheet by an amount equal to the surface current density. It is same as that of **D** at a polarization current sheet.

One can prove very well that the normal component of **H** is continuous across the sheet as the normal component of **B** is continuous and $\mathbf{H} = \mathbf{B}/\mu_0$.

Example 14. *An iron ring of mean circumference 60 cm and cross sectional area 8 sq cm is wound uniformly with a primary coil of 240 turns. A secondary coil containing 10 turns of insulated wire is wound over the primary and is connected to a ballistic galvanometer, the total resistance of the secondary circuit being 200 ohms. When a current of 2.0 amp. flowing in the primary is reversed, the galvanometer gives a throw of 60 scale divisions. When a capacitor of capacity 2.0μF charged to potential of 25 volts is discharged through the galvanometer the throw is 40 scale divisions. Calculate (a) the magnetic field in the primary, (b) the magnetic inductance in the iron and (c) the permeability of iron.*

The magnetic field intensity H in the primary is given by

$$H = n_1 i = (N_1/2\pi r)\, i$$

$$= (240/60 \times 10^{-2}) \times 2.0 = 800 \text{ amp./meter.}$$

The μ is the permeability of iron, the magnetic induction in it is given by

$$B = \mu H = 800\mu \; wb/m^2.$$

When the current in the primary is reversed the charge passing in a secondary circuit of total number of turns N and having a resistance R is given by

$$q = \frac{2NBA}{R} = \frac{2 \times 10 \times 800\mu \times 8 \times 10^{-4}}{200} = 6.4 \times 10^{-2} \mu \text{ coul.}$$

If K is the ballistic constant of the galvanometer, the deflection θ due to this charge is given by the relation

$$q = K\theta.$$

$$\text{or} \qquad 6.4 \times 10^{-2}\mu = K \times 60. \qquad \ldots(i)$$

When a capacitor of capacity C is charged to a potential of V volts, the charge q is given by

$$q = CV = K\theta',$$

$$\text{or} \qquad 2.0 \times 10^{-6} \times 25 = K \times 40 \quad \text{or} \quad K = 1.25 \times 10^{-6} \text{ coul/scale division.}$$

Substituting this value of K in equation (i), we get

$$6.4 \times 10^{-2}\mu = 1.25 \times 10^{-6} \times 60$$

$$\text{or} \qquad \mu = 11.72 \times 10^{-4}$$

$$\therefore \qquad B = 800\mu = 800 \times 11.72 \times 10^{-4}$$

$$= 0.9375 \text{ web./m}^2.$$

Example 15. *A conducting loop of area A and total resistance R is suspended by a torsion spring of constant k in a uniform magnetic field* $\mathbf{B} = B\mathbf{j}$. *The loop is in the y-z plane at equilibrium and can rotate about the z-axis with moment of inertia I. Assuming that the torsion spring is non-conducting and the self inductance of the loop is negligible. Find the equation of motion for the loop, if displaced by a small angle* θ_0 *from the equilibrium.*

The magnetic flux passing through the loop $\Phi = BA \sin\theta$, where θ is the angle between the plane of the loop and the magnetic field B. The induced emf and the current are given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = -BA\cos\theta \cdot \frac{d\theta}{dt}, \quad i = \frac{\mathcal{E}}{R} = -\frac{BA\cos\theta}{R} \cdot \frac{d\theta}{dt}.$$

The magnetic moment of the loop $m = iA = -(BA^2 \cos\theta/R)\, d\theta/dt$. Thus the magnetic torque on the loop $\tau = mB\cos\theta = -(B^2A^2\cos^2\theta/R)\,\dot{\theta}$.

Twisting torque $= k\theta$.

Both torques will resist the rotation of the loop. Thus

$$I\frac{d^2\theta}{dt^2} + \frac{B^2A^2\cos^2\theta}{R}\frac{d\theta}{dt} + k\theta = 0.$$

Since $\theta << \theta_0$ which is already small. Therefore $\cos^2\theta \simeq 1$.

$$\therefore \qquad I\frac{d^2\theta}{dt^2} + \frac{B^2A^2}{R}\frac{d\theta}{dt} + k\theta = 0.$$

This is the differential equation. Let its solution be $\theta = A_0 e^{\alpha t}$. Therefore $d\theta/dt = A_0 \alpha e^{\alpha t}$ and $d^2\theta/dt^2 = A_0 \alpha^2 e^{\alpha t}$.

$$\therefore \qquad A_0 e^{\alpha t}\,[I\alpha^2 + (B^2A^2/R)\,\alpha + k] = 0.$$

As the first term can not be zero, therefore $I\alpha^2 + (B^2A^2/R)\,\alpha + k = 0$.

$$\text{or} \qquad \alpha = \frac{-(B^2A^2/R) \pm [(B^2A^2/k)^2 - 4Ik]^{1/2}}{2I} = -\frac{B^2A^2}{2IR} \pm j\sqrt{\frac{k}{I} - \frac{B^4A^4}{4I^2R^2}}$$

$$= -\beta \pm j\gamma.$$

Thus the general solution of the equation of motion

$$\theta = e^{-\beta t}[A_1 \cos \gamma t + A_2 \sin \gamma t]$$

Since at $t = 0$, $\theta = \theta_0$ and $\dot{\theta} = 0$, therefore

$$A_1 = \theta_0 \text{ and } A_2 = (\beta/\gamma)A_1 = (\beta/\gamma)\,\theta_0.$$

$$\therefore \quad \theta = \theta_0 e^{-\beta t}[\cos \gamma t + (\beta/\gamma) \sin \gamma t]$$

Example 16. *Calculate the hourly loss of energy in the iron core of a transformer, the hysteresis loop of which is equivalent in area to 2500 ergs/cc. Given frequency 50 cycles/sec; density of iron 7.5 gm/cc and weight of the core 10 kg.*

The volume of the core $= M/\rho = 10/7.5 \times 10^3 = 1.33 \times 10^{-3}\ m^3$.

The energy loss in the iron core per cycle per unit volume

$$= 2500 \text{ ergs/cc} = 2500 \times 10^{-1} \text{ joule/m}^3.$$

The energy loss in the iron core per cycle

$$= 250 \times 1.33 \times 10^{-3} \text{ joule} = 33.25 \times 10^{-2} \text{ joule.}$$

Number of cycles in an hour $= 60 \times 60 \times 50 = 180000$.

∴ The hourly loss of energy in the iron core

$$= 33.25 \times 10^{-2} \times 180000 = 5.985 \times 10^4 \text{ joules.}$$

Example 17. *When 23.15 gm of $NiCl_2$ is dissolved in 100 gm water, the density of the solution is 1255 kg/m^3. Show that the maximum height of a column of the solution which can be supported by magnetic force, when one surface of the column is in a uniform magnetic induction of 1 web/m^2 is 3.0 mm. Given that the susceptibilities of 1 kg of $NiCl_2$ and water are 0.438×10^{-6} and -0.009×10^{-6} respectively; volume susceptibility of air $= 0.4 \times 10^{-6}$.*

Since 23.15 gm. of $NiCl_2$ is dissolved in 100 gm. of H_2O, the mass susceptibility of the solution is given by

$$(\chi_{sol})_{mass} = \chi_1 \times \frac{m_1}{m_1 + m_2} + \chi_2 \frac{m_2}{m_1 + m_2}$$

$$= 0.438 \times 10^{-6} \times \frac{23.15}{123.15} + (-0.009 \times 10^{-6}) \times \frac{100}{123.15}$$

$$= 7.5028 \times 10^{-8}$$

$$\therefore (\chi_{sol})_{\text{vol}} = (\chi_{sol})_{mass} \times \rho = 7.5028 \times 10^{-8} \times 1255$$

$$= 9.416 \times 10^{-5}.$$

The column of the solution is balanced by a magnetic force, if

$$g(\rho - \sigma)h = \tfrac{1}{2}\mu_0(\chi_{sol} - \chi_a)H^2 = \tfrac{1}{2}(\chi_{sol} - \chi_a)\frac{B^2}{\mu_0}$$

or

$$h = (\chi_{sol} - \chi_a)B^2/2\,\mu_0 g(\rho - \sigma).$$

Since the air density σ is negligible in comparison to ρ, the density of the solution, hence the substitution of numerical values gives

$$h = (9.416 \times 10^{-5} - 0.4 \times 10^{-6}) \times 1^2/2 \times (4 \times 3.14 \times 10^{-7} \times 9.81 \times 1255)$$

$$= 3.03 \times 10^{-3} \text{ meter} = 0.303 \text{ cm.}$$

Example 18. *In a toroidal winding of N turns surrounding a ferromagnetic specimen a narrow gap of width d has been cut. Calculate the value of magnetic field both in the gap and in the material.*

Let us consider a toroid of mean radius r and of circular cross-section with a radius a (<< r). The tangential component of **H** is continuous across the boundary separating the toroid material and the air region just out side. **B** is $\mu_0\mathbf{H}$ and $\mu\mathbf{H}$ outside and inside the toroid respectively. Using Ampere's circuital law for **H**, we have

$$\oint \mathbf{H}\cdot d\mathbf{l} = 2\pi r H = Ni \quad \text{or} \quad H = Ni/2\pi r.$$

If a small section of thickness d is removed from the toroid, the normal component of **B** must be continuous, hence **B** normal to the faces has the same value in the interior of the toroid. The field H is thus B/μ_0 in the gap and B/μ in the toroid. If we represent these fields as H_g and H_t respectively, then

$$\oint \mathbf{H}\cdot d\mathbf{l} = H_t(2\pi r - d) + H_g d = Ni$$

$$\text{or} \quad \frac{B}{\mu}(2\pi r - d) + \frac{B}{\mu_0}d = Ni \quad \text{or} \quad B = \frac{Ni\mu\mu_0}{2\pi r\mu_0 + (\mu - \mu_0)d}.$$

$$\therefore \quad H_t = \frac{B}{\mu} = \frac{Ni\mu_0}{2\pi r\mu_0 + (\mu - \mu_0)d}$$

$$\text{and} \quad H_g = \frac{B}{\mu_0} = \frac{Ni\mu}{2\pi r\mu_0 + (\mu - \mu_0)d}.$$

In our present problem the current $i = 0$ and the **B** and **H** are due to permanent magnetization. Hence

$$H_t(2\pi r - d) + H_g d = 0$$

$$\text{or} \quad H_t = -H_g d/(2\pi r - d).$$

Thus the field inside the toroid must be oppositely directed to the field in the gap. As H and B are in the same direction as the **B** is continuous into the toroid, hence in the toroid B and H_t are oppositely directed.

Example 19. *A circular ring of mild steel having a mean diameter 20 cm and a cross sectional area of 2.5 sq cm is wound with 500 turns of wire. Calculate (i) the reluctance of the magnetic circuit and (ii) the current necessary to produce a flux of 25000 lines. (μ for mild steel = 800 in CGS units).*

The total reluctance of the magnetic circuit is given by

$$R = \sum \frac{l}{\mu A} = \frac{2 \times 3.14 \times 10 \times 10^{-2}}{(800 \times 4 \times 3.14 \times 10^{-7})(2.5 \times 10^{-4})}$$

$$= 25 \times 10^5 \text{ amp/web.}$$

The total magnetic flux linked with the magnetic circuit

$$\Phi = \frac{\text{m.m.f.}(Ni)}{\text{Reluctance}} \quad \text{or current } i = \frac{\Phi \times \text{Reluctance}}{N}$$

$$\therefore \quad i = (25000 \times 10^{-8})(25 \times 10^5)/500$$

$$= 1.25 \text{ amp.}$$

Example 20. *Find the total reluctance and permeance between the ends of the (i) series connected and (ii) parallel connected rectangular iron blocks. Assuming that B is uniform throughout the block and normal to the ends. Given that μ_r = 500 and 2000 for the block materials.*

Consider that the size of each block is $l \times b \times t$. The reluctance R of each block

$$R = \frac{l}{\mu A} = \frac{l}{\mu_r \mu_0 \, b \times t} \qquad ...(i)$$

The permeance of each block = $1/R = \mu_r \mu_0 \, bt/l$.

If the blocks are connected in series, the total reluctance

$$R = R_1 + R_2 = \frac{l}{\mu_0 bt}\left(\frac{1}{\mu_{r_1}} + \frac{1}{\mu_{r_2}}\right) = \frac{l}{\mu_0 \, bt}\left(\frac{1}{500} + \frac{1}{2000}\right)$$

$$= l/400 \, \mu_0 \, bt. \qquad ...(ii)$$

Thus the total permeance of the system = $400 \, \mu_0 \, bt/l$.

If the blocks are connected in parallel, the total permeance of the system is given by

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} = 500 \frac{\mu_0 bt}{l} + \frac{2000 \, \mu_0 bt}{l} = \frac{2500 \, \mu_0 bt}{l}$$

∴ Total reluctance $R = l/2500 \, \mu_0 \, bt$. ...(iii)

Thus the reluctance in parallel decreases, while in series increases.

Example 21. *Calculate the current required in the winding of the toroidal ring of mean length 50 cm and of cross sectional area 4 cm² to produce a flux of 4 × 10^{-4} weber in the ring. Given : the no. of turns of wire is 200 and permeability of iron at this flux density is 65 × 10^{-4} web/amps. m. If an air gap of 0.1 cm length is cut in the ring, find the value of current required to maintain the same flux.*

$$\text{Flux density } B = \frac{\Phi}{A} = \frac{4 \times 10^{-4}}{4 \times 10^{-4}} = 1 \, web/m^2.$$

$$\text{Hence the reluctance } R = \frac{l}{\mu A} = \frac{0.5}{65 \times 10^{-4} \times 4 \times 10^{-4}}$$

$$= 1.92 \times 10^5 \text{ amp/web.}$$

∴Magnetomotive force (mmf.) = Flux × Reluctance

$= 4 \times 10^{-4} \times 1.92 \times 10^5 = 76.9$ amp turn

If the ring is wound with 200 turns, the current required is

= 76.9/200 = 0.385 amp.

If the air gap 0.1 cm in length is cut in the ring, the reluctance of the air gap is

$$R = \frac{l}{\mu_0 A} = \frac{10^{-3}}{4\pi \times 10^{-7} \times 4 \times 10^{-4}}$$

$$= 1.99 \times 10^6 \text{ amp/weber.}$$

As 0.1 cm is negligible in comparison with the total length of the ring, hence the reluctance of the ring will be same as before.

$\therefore$ Total reluctance of the entire circuit $= 1.92 \times 10^5 + 1.99 \times 10^6$

$= 21.82 \times 10^5$ amp/weber.

Hence the number of amperes required to maintain the same flux is $4 \times 10^{-4} \times 21.82 \times 10^5 = 872.8$ amp turns and the corresponding current is $872.8/200 = 4.36$ amp.

Example 22. *Design a magnet to produce a field of 10^4 gauss in a 0.1 m gap having an area of 1m × 2m. Assume very high permeability iron. Calculate the power required and the weight of the necessary Cu (resistivity $\rho = 2 \times 10^{-6}$ ohm-cm, density $d = 8$ gm/cm^3 and maximum current density $j_{max} = 1000$ amp/cm^2). Calculate the force of attraction between the poles of the magnet.*

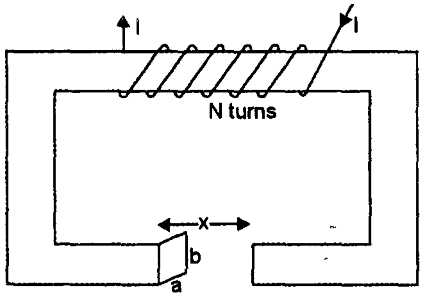


Fig. 13.53. Example 22.

Let L be the periphery of a cross section of the magnet parallel to the plane of the diagram as shown in Fig. 13.53. From Ampere's circuital law

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \frac{B}{\mu_0} x + \frac{B}{\mu_0 \mu_r}(L - x) = NI\,.$$

Since μ_r is very high, therefore the second term may be neglected

$$\therefore \quad N = Bx/\mu_0 I = Bx/\mu_0 jA,$$

where A is the area of cross section of the Cu-wire.

The power dissipated in the wire = Power required P

$$\therefore \quad P = I^2 R = I^2 \rho \frac{2N(a+b)}{A} = \frac{2j\rho(a+b)Bx}{\mu_0}$$

$$= 2 \times 1000 \times 10^4 \times 2 \times 10^{-8}(1+2) \times 0.1/4\pi \times 10^{-7} = 9.5 \times 10^4 \text{ W}$$

The necessary weight of Cu $= 2(a+b)NAd = 2(a+b)Bxd/\mu_0 j$

$= 2(1+2) \times 1 \times 0.1 \times 8 \times 10^3/4\pi \times 10^{-7} \times 1000 \times 10^4$

$= 380$ kg.

The force of attraction between the poles $F = B^2 ab/2\mu_0 = 8 \times 10^5\ N$.

Example 23. *Design a magnetic circuit that produces a magnetic field of 1.5 tesla in air gap 1 cm long and 48 cm^2 in area.*

Assuming a magnetic circuit, energized by a pair of permanent magnets. As the ends of the permanent magnet act as either sources or sinks of **H**, hence the flux **H** over a closed surface is not necessarily equal to zero. Since there are neither free currents nor displacement currents, hence

$$\oint \mathbf{H} \cdot d\mathbf{l} = H_s L_s + H_g L_g - H_m L_m = 0,$$

where the subscripts s, g and m refer respectively to the soft iron, the air gap and the permanent magnets. L_s, L_g and L_m are the lengths in respective regions. As the reluctance of the iron is much smaller than of the air gap, hence $H_s L_s$ is negligible and $H_m L_m \approx H_g L_g$.

If the leakage flux is negligible, the flux Φ is same over any cross section.

$$A_s B_s = A_g B_g = A_m B_m.$$

As for an alloy Alnico, the energy product $H_m B_m$ is maximum at $B_m = 0.80$ tesla and $H_m =$

3.6×10^4 amp turn/meter. Hence the area of cross section of the permanent magnet

$$A_m = \frac{A_g B_g}{B_m} = \frac{48 \times 10^{-4} \times 1.5}{0.80} = 9 \times 10^{-3} \text{ meter}^2.$$

Length of the magnetic region $L_m = H_g L_g / H_m = B_g L_g / \mu_0 H_m$

$$= \frac{1.5 \times 10^{-2}}{4\pi \times 10^{-7} \times 3.6 \times 10^4} = 0.33 \text{ meter.}$$

Thus we require two permanent magnets each of about 16 cm long and of 11 cms dia. If the magnets of suitable shape are not available, smaller magnets may be used in series and parallel combinations.

Example 24. *The middle limb of the symmetric magnetic circuit of cast steel (Fig. 13.54) carries a coil of 500 turns. Find the coil current if a magnetic flux of 2 × 10^{-3} web is produced in the left and right hand limbs of the magnetic circuit.*

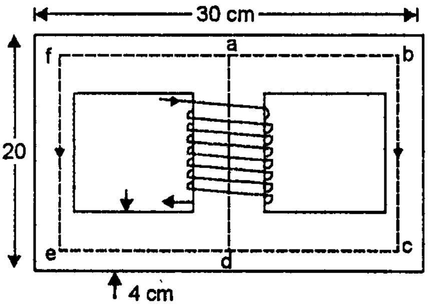


Fig. 13.54. Example 24.

From the geometry of Fig. 13.54, we see that the circuit may be considered as built of two magnetically equivalent closed paths *abcd* and *afeda*. The section *ad* = l_1 = 16 cm is in common to both paths of the circuit. The path length l_2 of the left hand path *afed* = 42 cm and the path length l_3 of the right hand path *abcd* = 42 cm. Thus we have

$$l_1 = 16 \text{ cm}, l_2 = l_3 = 42 \text{ cm},$$

$$S_1 = 6 \times 5 = 30 \text{ cm}^2 \text{ and } S_2 = S_3 = 4 \times 5 = 20 \text{ cm}^2.$$

$$\therefore \quad B_2 = \Phi_2/S_2 = 2 \times 10^{-3}/20 \times 10^{-4} = 1 \text{ tesla.}$$

The magnetic flux in the middle limb will be twice of that in the side limbs, *i.e.*, $\Phi_1 = 2\Phi_2$.

$$\therefore \quad B_1 = \Phi_1/S_1 = 2 \times 2 \times 10^{-3}/30 \times 10^{-4} = 1.33 \text{ tesla.}$$

From the magnetization curve of cast steel, we know that H_2 = 920 amp/m and H_1 = 1725 amp/m. By Ampere's circuital law

$$H_1 l_1 + H_2 l_2 = Ni$$

or $1725 \times 0.16 + 920 \times 0.42 = 500i$, or $i = 662.4/500 = 1.328$ amp.

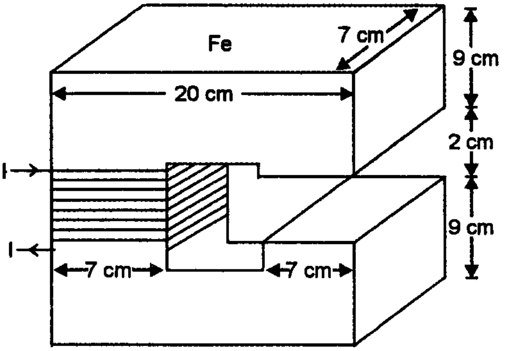


Fig. 13.55. Example 25.

Example 25. *A C-magnet with all dimensions in cm is shown in Fig. 13.55. The relative permeability of the soft iron yoke is 3000. If a current I = 1 amp. is to produce a field of about 100 gauss, how many turns of wire are required ?*

Consider a cross section of the magnet parallel to the plane of the paper and denote its periphery by *L*. As the normal component of **B** is continuous, the magnetic intensity in the gap is B/μ_0, while that inside the magnet is $B/\mu_0\mu_r$, where μ_r is the relative permeability of the iron. From Ampere's circuital law, we have

$$\oint_L \mathbf{H}\cdot d\mathbf{l} = NI$$

$$\frac{B}{\mu_0}d + \frac{B}{\mu_0\mu_r}(4l-d) = NI, \quad \text{or} \quad N = \frac{B}{\mu_0 I}\left[d + \frac{1}{\mu_r}(4l-d)\right]$$

In this problem, $l = 20$ cm, $d = 2$ cm, $\mu_r = 3000$ and $B = 100$ gauss

$$\therefore \quad N = \frac{100\times10^{-4}}{4\pi\times10^{-7}\times1}\left[0.02 + \frac{1}{3000}(4\times0.20-0.02)\right]$$

$$= 161 \text{ turns.}$$

ORAL QUESTIONS

1. Why cannot one obtain an isolated north or south pole by breaking a magnet into two pieces?
2. Why is diamagnetism, unlike paramagnetism, relatively temperature independent?
3. Is the saturated magnetization for a paramagnetic substance very much different from that for a ferromagnetic substance of the same size?
4. A hollow cylinder is uniformly magnetized in the direction parallel to its axis. How are B and H affected by the introduction of a cylinder of ferromagnetic substance inside the former?
5. Name the corresponding quantities in magnetostatics, of the following quantities in electrostatics; E, D and P.
6. Name the corresponding quantities in magnetic circuit of the following quantities in electric circuit : Current, Current density, Conductivity, Electromotance, Resistance, Conductance.
7. Why is the magnetic susceptibility of diamagnetic substances negative?
8. What are the main differences in the boundary conditions for electrostatic fields and magnetostatic fields?
9. Why does a magnet attract small pieces of iron?
10. How can you distinguish two identical bars, one is magnet and other is not? Please note that you are not allowed to suspend either bar or to use any other apparatus.
11. Two iron bars attract each other, no matter in which combinations their ends are brought near each other. What do you conclude about their magnetization?
12. Why do you use substance in the form of an anchor ring while going to draw B-H curve?
13. How can you demagnetise magnetic substance completely?
14. How can you explain hysteresis loss?
15. Which physical properties are important to note when choosing the material for electromagnets.
16. Why is the absolute permeability not used for calculating field strength in the steel?
17. Out of two identical iron bars, one is magnetized. How can the two be identified without using any other device.
18. One magnetic dipole is placed in the magnetic field produced by another magnetic dipole. The distance between the dipoles r is quite large compared to the size of the dipoles. Which of the following is incorrect ? (*a*) force has the non central nature, (*b*) force falls as r^{-4}, (*c*) force is always attractive if both the dipoles are parallel irrespective of their location.
19. Which of these materials requires the least value of magnetic field strength to magnetize it ? (*a*) Nickel, (*b*) Silver, (*c*) Tungsten. (*a*)
20. Which of the following is not true of ferromagnetic materials ? (*a*) They have a large χ_m, (*b*) fixed value of μ_r, (*c*) energy loss is proportional to the area of the hysteresis loop. (*b*)
21. Reluctance in a magnetic circuit is analogous to (*a*) resistance in a *d-c* circuit, (*b*) volume of water in a hydraulic circuit, (*c*) voltage in an a-c circuit. (*a*)
22. The magnetic field is increasing through a copper plate. The eddy currents (*a*) help the field increase, (*b*) slow down the increase and (*c*) do nothing. (*b*)
23. One magnetic dipole is placed in the magnetic field produced by another magnetic dipole. The distance r between these dipoles is quite large. Which of the following is incorrect? (*a*) The force observed has the noncentral nature, (*b*) The force is always attractive if both dipoles are parallel irrespective of their location, (*c*) force falls as r^{-4}. (*b*)

PROBLEMS

1. A cylindrical bar magnet of radius a and length $2l$ is uniformly magnetised along its length with intensity I. Show that at a point on the axis a distance x from the centre of the magnet $\Phi_{\text{axis}} = \frac{1}{2}I\{(a^2 + (x - l)^2]^{1/2} + [a^2 + (x + l)^2]^{1/2} + 2l\}$.

2. Two similar permanent magnets A and B, having magnetic moments M are arranged in such a way that their axes are mutually ⊥ to each other. They are separated by a distance $(x_1 + x_2)$, large compared with their lengths. A compass needle is placed a distance x_1 from one magnet and x_2 from the other. The needle makes an angle θ with the direction of first magnet. Prove that the ratio $x_1/x_2 = (2 \cot \theta)^{-1/2}$.

3. An annealed iron ring of mean length 15 cm is wound with a toroidal coil of 100 turns. Determining the magnetic induction in the ring when the current in the winding is (a) 0.1 amp. and (b) 0.2 amp. *(0.25 weber/m², 0.95 weber/m²)*

4. Two magnetic molecules lie in one plane, their axes making an angle θ and θ′ with the line joining the molecules. Show that the couple on the second molecule due to first will vanish, if $\tan \theta + 2 \tan \theta' = 0$.

5. Two small magnets of moment M are placed at the corners, A, B of a square $ABCD$ with their axes in the directions AB *and* BA respectively. If two other small magnets of moment M' free to turn about their centres at C, D take up positions of equilibrium with their axes along AC *and* BD. Prove that the ratio of M to M' is $2 : \left(\sqrt{2} - 1\right)$.

6. A specimen of unmagnetised iron of volume 20 cc is placed in a magnetic field of 50 oersteds : If the magnetic moment produced in it is 6000 cgs units, find the flux density in it.

(0.3818 web/m²)

7. A magnetic field of 40 emu produces a flux of 2500 emu in an iron rod of 0.5 sq. cm. area of cross section. Calculate magnetic permeability and susceptibility of the rod material.

(1.57 × 10⁻⁴ henry/m; 124 mks units)

8. The susceptibility of a gm mole of helium gas is -2.4×10^{-11} (SI units). Show that this corresponds to a value for the mean square distance of each electron in the helium atom of $1.22\, a_0^2$, where a_0 is the radius of the first Bohr orbit in the hydrogen atom and having value 0.528×10^{-10}m.

9. Rayleigh found that at low values of the magnetic induction the hysteresis loop with tips at B_0, H_0 and $-B_0$, $-H_0$ is described by the equation

$$B = \mu H + \frac{1}{2}a\,(H_0^2 - H^2) \text{ upper half of loop}$$

and

$$B = \mu H - \frac{1}{2}a\,(H_0^2 - H^2) \text{ lower half of loop.}$$

Show that the energy loss per cycle represented by the area of the loop in each unit volume of the specimen is $(4/3)\, aH_0^3$.

10. Calculate a value for the loss of energy due to hysteresis when 5 kg of iron are subjected for 1 hr to cyclic magnetic changes of frequency 25 cycles/sec, if the hysteresis loop represents a loss of 2.5×10^3 erg cm^{-3} cycle^{-1}. The density of iron = 7.8 gm cm^{-3}. *(1.44 × 10³ joule)*

11. An iron rod of density 7.7 and specific heat 0.1 is subjected to magnetization at 50 cycle/sec. If the area enclosed by M-H curve for the specimen is 5×10^4 erg, find the rise in temperature per minute. *(277.6 K)*

12. The armature core of a generator, made of iron, has a hyteresis B-H loop of area 5×10^4 units. The core is cylindrical of length 40 cm and radius 10 cm. If the armature rotates at 100 revolutions per sec, find the rate of rise of temperature (assuming no loss due to radiation). Given the density of iron as 7.7 and specific heat 0.11.

13. An electron in an atom of hydrogen circulates in the orbit of radius 0.51×10^{-10} meter. Calculate the change in magnetic moment for this electron, if a magnetic field of induction 2.0 weber/m² acts at right angles to the plane of the orbit. *(3.7 × 10⁻²⁹ amp/m²)*

14. The earth has a magnetic dipole moment of 6.4 × 10^{21} amp. m^2. (*a*) What current would have to be set up in single turn of wire going around earth at its magnetic equator, if we wished to set up such a dipole? *(5 × 10^7 amp)*

15. The dipole moment associated with an atom of iron in an iron bar is 1.8 × 10^{-23} amp. m^2. Assume that all the atoms in the bar of length 5 cm and of cross sectional area 1 cm^2 have their dipole moments aligned. Calculate the dipole moment of the bar and the torque required to hold this magnet at right angles to an external field of 1500 gauss? *(7.5 amp m^2, 11 newton meter)*

16. Electric charge q is uniformly distributed over the volume of a sphere of radius a. Calculate the equivalent magnetic moment if this sphere spins about its diameter with an angular velocity ω. *($q\omega\mu_0 a^2/5$)*

17. An iron ring 10 cm mean radius and 5 sq. cm cross section has an air gap 1 cm wide. The ring is bound uniformly with a coil of 900 turns. A flux of 10^4 maxwells is required in the air gap. If K for iron is 2000, find the strength of current required. *(1.82 amp)*

18. A magnetic flux of 2 × 10^{-3} web is to be produced in a ring-shaped steel core of radius 2 cm. If the radius of the ring is 10 cm. Calculate (*i*) the current needed in a magnetizing coil of 100 turns to maintain the given flux, (*ii*) the permeability of the core steel. (*iii*) the number of coil turns if an air gap of 2mm is cut in the core. *(31.4 amp, 3.2 × 10^{-4} Tm/amp, 181)*

19. Nickel has a density of 9.1 × 10^{28} atoms/m^3, an atomic magnetic moment of 9.28 × 10^{-24} amp. m^2 and a Curie temperature of 631 K. Determine the saturation magnetization and γ for nickel. *(8.44 × 10^5 amp/m, 2654)*

20. A solid contains weakly interacting paramagnetic atoms of spin half and a magnetic moment 9.27 × 10^{-24} joule/tesla. Calculate the temperature upto which one has to cool the solid in the presence of a magnetic field of 3 tesla to have 75% of the atoms with their spins parallel to the field. *(3.67 K)*

21. A long solenoid with a radius of 4 cm carries a current of 1 amp in a coil that has a wrapping density of 200 turns per meter. A cylinder of radius 2 cm and made from hot rolled iron sheet (for which B = 0.20 T at H = 200 A/m is placed along the axis of the solenoid. Determine the field vectors **H**, **B** and **M** inside the cylinder, outside the solenoid and in between the cylinder and the solenoid surface. Compare the magnetic flux in the iron core with that in between core and solenoid.
(H = 200 A/m, 0, 200 A/m; B = 0.2T, 0, 2.51 × 10^{-4} T; M = 1.59 × 10^5 A/m; 0, 0;
Φ = 2.51 × 10^{-4} Wb, 9.46 × 10^{-7} Wb)

22. Two very long cylindrical magnets, each of length L and diameter d are placed next to each other and separated by a small gap of length l_g (<< L, d). If the each magnet has uniform magnetization **M**, calculate the magnetic field **H** and **B** in the gap. [$\mu_0 M (1 - d^2/8L^2 - l_g/d)$]

23. A large number (N) of non-interacting magnetic dipoles each of magnetic dipole moment (μ) are contained in a large box which is placed in an external uniform magnetic field (**B**). Explain with relevant thoery, how the average magnetic moment perunit volume of the system varies with the variation of temperature? *(magnetization I decreases with increasing τ)*

24. A U-shaped electromagnet is designed to lift a 400 kg mass. The iron yoke (μ_r = 3000) has a cross-section of 40 cm^2 and mean length of 50 cm, and the air gaps are each 0.1 mm long (yoke and mass). Calculate the number of turns in the coil when the excitation current is 1 amp. *(162)*

25. A toroid of square cross-section with r_1 = 2 cm, r_2 = 3 cm, and height a = 1 cm has a two part core. The part I is silicon-steel of mean length 7.9 cm and the part II is nickel-iron alloy of mean length 7.9 cm. Find the flux that is produced from an mmf of 17.38 amp. *(10^{-4} Wb)*

26. A thin circular disc of radius R is uniformly charged with surface charge density σ. The disc rotates about its axis with a uniform angular velocity ω. A small magnetic dipole of moment μ is located at point $P(a, 0, 0)$ on the axis of the disc. (*a*) obtain the expression for the magnetic moment of the disc and (*b*) obtain the expression for the force on the dipole placed at P given that $a >> R$.

14 MAXWELL'S EQUATIONS AND ELECTROMAGNETIC WAVES

14.1 INTRODUCTION

We have mainly restricted our discussions to static or time invariant **E** and **B** fields. The electrostatic field **E** can exist in the absence of magnetic field **H** (or **B**), a capacitor with a static charge q is an example. The magnetostatic field **H** (or **B**) can exist in the absence of electric field, a conductor with a constant current has a magnetic field **H** without an **E**-field. We shall examine situations where electric and magnetic fields are dynamic or time varying. In such a case the electromagnetic fields are interdependent *i.e.*, **H** cannot exist without an **E**-field nor can **E** exist without a corresponding **H**-field. In other words, a time varying electric field necessarily involves a corresponding time varying magnetic field. The time varying fields are usually due to accelerated charges or time varying currents. Although much valuable information can be derived from static field theory. The time variable fields are involved in the experiments of Faraday and Hertz and the theoretical analysis of Maxwell's equations. Two major concepts have been introduced : (*i*) the electromotive force based on Faraday's experiments and (*ii*) displacement current resulted from Maxwell's hypothesis. As a result of these concepts, Maxwell's equations and the boundary conditions for static electromagnetic fields are modified to account for the time variation of the fields. The Maxwell's equations should be regarded as the base for the study of electromagnetic waves.

14.2 DISPLACEMENT CURRENT

The basic laws of electricity and magnetism, which we have discussed in the preceding chapters can be summarized in differential form by the four equations.

Gauss's law (Electrostatics) $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0,$ ..(1)

Gauss' law (magnetism) $\nabla \cdot \mathbf{B} = 0,$ ...(2)

Faraday's law of Induction $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t,$ ...(3)

Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J},$ ...(4)

These equations are written for fields in vacuum, in the presence of electric charge of density ρ, and electric current of density **J**. These are in macroscopic form and in S.I. units. In the presence of simple dielectrics, ε_0 is replaced by ε in Eq. (1). If simple ferromagnetic materials are present, μ_0 is replaced by μ in the Eq. (4).

Let us recall that all but Faraday's law were derived from steady-state observations. Maxwell while studying the electromagnetic laws, mentioned above, noticed that there was some thing strange about Eq. (4). As the divergence of a curl is always zero, hence Eq. (4) becomes

$$\text{div curl } \mathbf{B} = \text{div}(\mu_0 \mathbf{J}) = \mu_0 \text{div } \mathbf{J} = 0. \qquad ...(5)$$

Equation of continuity div $\mathbf{J} + \partial\rho/\partial t = 0$ shows that $\partial\rho/\partial t$ must thus be zero, *i.e.* the total flux of current out of any closed surface is zero. But this certainly cannot in general be zero because we know that the charges can be moved from one place to another.

Maxwell proposed that Eq. (4) was not correct for moving charges and suggested that for changing charge distributions or variable charge densities

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J} + ? \qquad ...(6)$$

To know this unknown, Maxwell suggested that similar to the electric field due to a changing magnetic field (Faraday's law of induction) there would be a magnetic field due to the changing electric field, *i.e.*,

$$\text{curl } \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t}. \qquad ...(7)$$

Here constant term $1/c^2$ or $\mu_0\varepsilon_0$ corresponds to the system of units in S.I. units. Using this term as missing term, we get

$$\text{curl } \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t = \mu_0 (\mathbf{J} + \mathbf{J}_d). \qquad ...(8)$$

Maxwell called this added term $\mathbf{J}_d$, *the displacement current density.*

To check this equation, take divergence of both sides

$$\begin{aligned} \text{div (curl } \mathbf{B}) &= \text{div } (\mu_0 \mathbf{J}) + \text{div } [\mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t] \\ &= \mu_0 \text{ div } \mathbf{J} + \mu_0 \text{ div } (\partial \mathbf{D}/\partial t] \\ &= \mu_0 (\text{div } \mathbf{J} + \partial\rho/\partial t). \qquad (\text{as div } \mathbf{D} = \rho) \end{aligned}$$

As div (curl **B**) is zero, hence right hand side is also zero, which is same as obtained by virtue of the continuity condition.

Maxwell used an indirect argument to back up his idea for the requirement of extra term in Ampere's law. In the case of a current carrying wire, one may consider a flat surface *A* and the sack-shaped surface *A'* with the common closed path, as shown in Fig. 14.1 (*a*). Since the surface integral of the current density **J** is the total current threading the path of the line integral. Thus the current passing through the surfaces *A* and *A'* is the same, as required by Kirchhoff's law.

Now consider a different situation, when the current deposits charge on the plates of a capacitor. Fig. 14.1(*b*) shows two surfaces *A* and *A'* with the same loop as this common boundary. A current *I* crosses surface *A*, while no current crosses surface *A'*. Thus we have

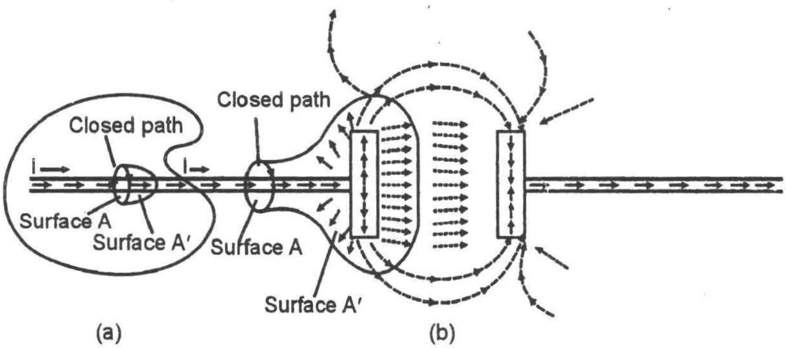


Fig. 14.1. Conduction current (continuous arrows) and displacement current (dotted arrows)

$$\oint_A \mathbf{B}\cdot d\mathbf{A} = \mu_0 I \quad \text{and} \quad \oint_{A'} \mathbf{B}\cdot d\mathbf{A} = 0. \qquad \text{...(9)}$$

These relations thus predict two contradictory values for the surface integral of **B**. We therefore conclude that the Ampere's law in original form is in error. Maxwell noted that even if no current passes through surface A', there is a changing electric field through it. As the charge builds up on the capacitor plates, the electric field and the electric flux is developed in the space between the plates. If q is the charge at any instant t on the plates, each of area A, then the electric field in the gap

$$E = q/\varepsilon_0 A.$$

$$\therefore \qquad \frac{dE}{dt} = \frac{1}{\varepsilon_0 A}\frac{dq}{dt} = \frac{1}{\varepsilon_0 A} I'$$

or

$$I' = \varepsilon_0 A \frac{dE}{dt}. \qquad \text{...(10)}$$

Hence the current density in the gap of the plates

$$= I'/A = \varepsilon_0 dE/dt.$$

or

$$\mathbf{J}_d = \varepsilon_0 d\mathbf{E}/dt.$$

Thus the variation of **E** with time produces an additional current of current density $\mathbf{J}_d$ in space, called *displacement current density*. In the case of a parallel plate air capacitor, the displacement current in the space is such as shown by dotted arrows, while conduction current I in the connecting wire and plates is such as shown by continuous arrows in the Fig. 14.1. Thus Eq. (6) becomes

$$\text{curl}\,\mathbf{B} = \mu_0\mathbf{J} + \mu_0\mathbf{J}_d = \mu_0(\mathbf{J} + \mathbf{J}_d).$$

This is the Maxwell's curl equation for magnetic fields or the Ampere's circuital law for time varying conditions.

14.2 MAXWELL'S EQUATIONS

Faraday's law shows that electricity and magnetism are fundamentally connected. The introduction of displacement current by James Clerk Maxwell enhances this connection. Thus we have a complete, consistent set of fundamental laws for electricity and magnetism, known as *electromagnetic field equations or Maxwell's equations*. These are the generalized forms of electromagnetic equations in time varying conditions.

In the absence of dielectric or magnetic materials, Maxwell's equations are :

(*i*) **Gauss's law for electric field,** which may be written as

$$\text{div}\,\mathbf{E} = \rho/\varepsilon_0 \quad \text{or} \quad \oint_S \mathbf{E}\cdot d\mathbf{S} = \frac{1}{\varepsilon_0}\int_\tau \rho\, d\tau. \qquad ...(11)$$

It states the relation between the electric field and the charges that produced it. In static situations, it is equivalent to Coulomb's law and relates the electric flux through a closed surface to the charge enclosed. This law is actually a generalization of Coulomb's law. Although Coulomb's law is valid for static charge only, Gauss's law holds even if the charges are in arbitrary motion, *i.e.* if the electric field varies with time.

(*ii*) **Gauss's law for magnetic fields,** which may be written as

$$\text{div}\,\mathbf{B} = 0 \quad \text{or} \quad \oint_S \mathbf{B}\cdot d\mathbf{S} = 0. \qquad ...(12)$$

It states that there are no magnetic charges or magnetic monopole which would generate a magnetic field in the same way as electric charges create an electric field. The magnetic field lines have therefore *no sources or sinks* to start or end on, as electric field lines start on positive charges and end on negative charges. They are either closed or go to infinity. Even for the bar magnet, lines of **B** exist inside as well as outside the magnetic material. The relation (12) holds even for time dependent magnetic fields.

(*iii*) **Faraday's law of induction,** which may be written as

$$\text{curl}\,\mathbf{E} = -\,\partial\mathbf{B}/\partial t \ \text{or} \ \oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{\partial}{\partial t}\oint_S \mathbf{B}\cdot d\mathbf{S} = -\frac{\partial\Phi_B}{\partial t}. \qquad ...(13)$$

It states the relation between the induced electric field generated by a changing magnetic flux. It shows that a varying magnetic field acts as one of the possible sources of an electric field. As discussed earlier, the minus sign is very important, which represents the fact that the induced electric field would give rise to an induced current that opposes the change in the magnetic flux. Thus we see that an electric field is set up in a region of space where the magnetic field is changing with time.

(*iv*) **Generalized Ampere's law,** which may be written as

$$\text{curl}\,\mathbf{B} = \mu_0\,[\mathbf{J} + \varepsilon_0\,\partial\mathbf{E}/\partial t]$$

or

$$\oint_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 \oint_S \left(\mathbf{J} + \varepsilon_0 \frac{\partial\mathbf{E}}{\partial t}\right)\cdot d\mathbf{S} \qquad ...(14)$$

$$= \mu_0 I + \mu_0\varepsilon_0 \frac{\partial\Phi_E}{\partial t}. \qquad ...(15)$$

It states the relation between a magnetic field and the current that gives rise to the field. It shows that the magnetic field is produced by an electric current or by a changing electric field (or flux). The second term representing the rate of change of electric field flux is known as displacement current distribution, introduced by Maxwell. Thus we see that the conduction currents and displacement currents are two possible sources of a magnetic field.

In the Gaussian system of units, Maxwell's equations may be written as

$$\nabla\cdot\mathbf{E} = 4\pi\rho, \qquad \nabla\cdot\mathbf{B} = 0,$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t}. \qquad ...(16)$$

The first two equations show that the net flux of lines of **E** or **B** is zero throughout any arbitrary volume of no source. They show that lines of **E** and **B** are continuous. The third equation shows that the electric field due to the variation of **B** is not only a statement of Faraday's law but a property of space independent of the presence of a conducting loop of wire. The fourth equation shows the new concept of magnetic field generation by displacement current.

Maxwell's equations are the basic equations for all electromagnetism. They are fundamental as they are valid even relativistically. Now we cannot treat electric and magnetic fields separately if they vary with time. As a changing magnetic field produces an electric field and a changing electric field produces a magnetic field. An important outcome of this fact is the production of electromagnetic waves. The symmetry between electric and magnetic field is not perfect, because magnetic monopole does not exist. Faraday's law contains no term like the $\mu_0 i$ term in Ampere's law, because there is no free magnetic charge to form a magnetic current. The symmetry is perfect in the vacuum, where there is no electric charges, such as

$$\text{div}\,\mathbf{E} = 0,\ \text{div}\,\mathbf{B} = 0,\ \text{curl}\,\mathbf{B} = \mu_0\varepsilon_0\,\partial\mathbf{E}/\partial t \text{ and curl}\,\mathbf{E} = -\partial\mathbf{B}/\partial t. \qquad ...(17)$$

In the presence of matter, the Maxwell's equations are modified by replacing ε_0 by ε and μ_0 by μ. Except for ferromagnetic materials, μ is very close to μ_0 and the replacing of μ_0 by μ does not affect much.

The electromagnetic properties of material media may be taken into account through relations

$$\mathbf{D} = \varepsilon\,\mathbf{E},\ \mathbf{B} = \mu\,\mathbf{H} \quad \text{and} \quad \mathbf{J} = \sigma\,\mathbf{E}, \qquad ...(18)$$

known as *constitutive relations.* They express the relations between field vectors and currents, which take into account the properties of a material medium. Dielectric properties, which are phenomenologically described by polarization, are taken into account in permittivity ε. Magnetic properties, which are phenomenologically described by magnetization, are taken into account in permeability μ. The conducting properties of the medium are included in conductivity σ.

Maxwell's equation are applicable under the following conditions :

(1) Material bodies are at rest relative to the electromagnetic field,

(2) Material constants ε, μ and σ may depend on coordinates, but should be independent of time and field vectors.

(3) There should be no ferromagnetic substance and permanent magnets in the field region.

14.3 ELECTROMAGNETIC WAVES

Two Maxwell's equations, Faraday's law and the generalized Ampere's law show that when the electric and magnetic fields are time dependent, they influence each other and are said to be *coupled.* As a consequence of this coupling, the electric and magnetic fields transport energy over very large distances. The coupled fields produce travelling Waves called *electromagnetic waves.* The existence of EM-waves was first investigated by Heinrich Hertz, who succeeded in generating and detecting radio waves. Which are sometimes called Hertian waves in his honour.

The EM-waves include light rays, radio-waves, TV-signals and radar beams. All forms of EM-energy have fundamental characteristics : (*i*) They travel at high velocity, (*ii*) They assume the properties of waves in travelling, and (*iii*) They radiate outward from a source.

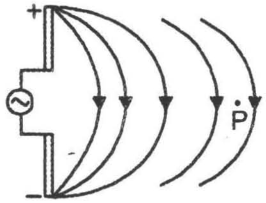


Fig. 14.2. Electric field closed to oscillating charge.

Before proceeding towards mathematics, let us see the qualitative features. For this purpose, consider an electric dipole antenna attached to an alternating voltage generator. When the upper end of the antenna is +vely charged, the electric field close to the antenna will be in –ve y-direction as shown in Fig. 14.2. After a half cycle the oscillator will have reversed the charges at the ends and the electric field at any point P will be in the +ve y-direction. If the charge on one end oscillates sinusoidally with frequency f as

$$q = q_0 \sin 2\pi ft, \qquad \text{...(19)}$$

then the electric field at point P will be given as

$$E_y = -E_0 \sin 2\pi ft. \qquad \text{...(20)}$$

The –ve sign occurs, as electric field is in downward when upper end is +ve.

The electric field is assumed to be the disturbance sent out by the dipole source as a wave on a string produced by an oscillating source. At a point P in the path of the wave, the electric field reverses periodically as the wave passes through the point. The frequency of the oscillating electric field at P is the same as the frequency of the source.

If the point P is at some large distance x from the antenna, one can expect the field to take a time t_0 $(= x/v)$ to reach this point. Here v is the velocity with which the field travels from the antenna to point P. The field at point P will be that existed at the antenna at time $t - t_0$, not that at time t. Thus the field at time t at point P is given by

$$\begin{aligned} E_y &= -E_0 \sin 2\pi f(t - t_0) = -E_0 \sin [2\pi f(t - x/v)], \\ &= -E_0 \sin (\omega t - x/\bar{\lambda}) = E_0 \sin (\kappa x - \omega t), \end{aligned} \qquad \text{...(21)}$$

where κ is called the wave number and is related to the wavelength λ as $\kappa = 2\pi/\lambda = 1/\bar{\lambda}$.

At a large distance, E_0 will vary very slowly with x and is constant approximately. Hence at *any particular instant*, the field along the x-axis appears as sinusoidal wave. As successive portions of the electric field (or wave) move through point P, the field will oscillate sinusoidally up and down in x–y plane thus giving rise to the alternating electric field **E** associated with an electromagnetic wave.

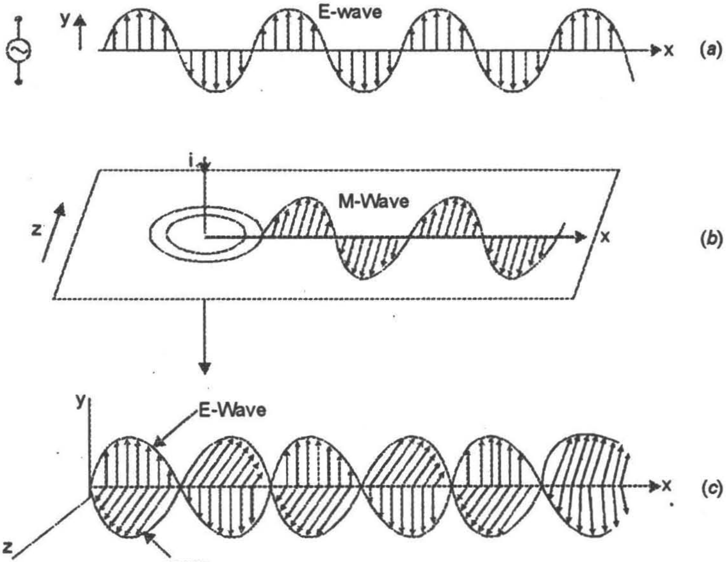


Fig. 14.3. Production of *E*-wave, *M*-wave and *EM*-wave.

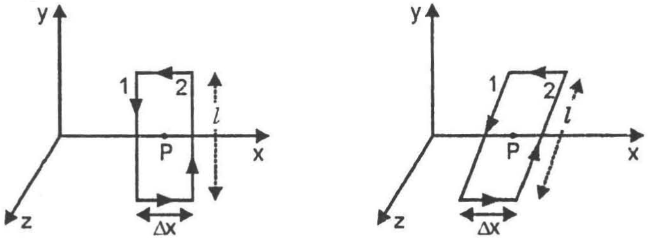
The electric field wave is nearly a plane wave, its intensity does not change much with distance. It is a transverse wave, because the vibrating quantity (**E** – vector) is directed perpendicular to the direction of propagation of the wave. The wave requires no material medium for its transmission and thus can exist in vacuum or empty space.

The antenna also generates magnetic field waves. As the charges at the end of the antenna change, there results an alternating current in the antenna. Because a magnetic field circles a current, an oscillating magnetic field is thus produced. The magnetic field wave also travels out along the x-axis as a transverse wave. The magnetic field is in the z-direction, if the electric field is in the y-direction. The E-wave, M-wave and the combination EM-wave are shown in Fig. 14.3. The E-M wave is a monochromatic, plane, linearly polarized wave traveling in a vacuum. Since the current that produces EM-wave must change with time. Thus the charges that produce it must be accelerating. The important point is that the electric field remains parallel to the current direction and the magnetic field remains perpendicular both to the electric field and the direction of propagation.

14.4 EM-WAVES FROM MAXWELL'S EQUATIONS

Let us consider only the case of plane waves with the direction of propagation along the x-axis, where the electric and magnetic fields are directed transverse to the direction of travel.

Consider a rectangle of infinitesimal size in xy plane [Fig. 14.4 (a)]. As the electromagnetic wave passes over it, the magnetic flux Φ_B through it will change. It will give rise to induced electric fields around the rectangle, which are the electric component of the traveling wave. As the wave is moving to the right through the rectangle, the flux Φ_B is decreasing with time. If the rectangle is assumed to be a conducting loop, a counter clockwise induced current would appear in it. This current would produce a magnetic field within the rectangle, that would oppose the decrease in Φ_B. Since there is no conducting loop, but the net induced electric field E acts counter clockwise around the rectangle (E at right edge is greater than at the left edge), *hence the electric field configuration is as induced by the changing magnetic field.*



(a) (b)

Fig. 14.4. Applying Maxwell's equations to the rectangles.

As **E** is directed in the $+y$ direction, hence $\mathbf{E}\cdot d\mathbf{l}$ will be zero over the ends of length Δx. if we go around the rectangle counter clockwise, we get from Maxwell's Eq. (13)

$$\int_2 E_y dl - \int_1 E_y dl = -\frac{\partial B_z}{\partial t} l\Delta x.$$

(As $dS = l\ \Delta x$ is along z-direction). Assuming E_y as E_{y1} and E_{y2} along paths 1 and 2 respectively, we have

$$(E_{y2} - E_{y1})l = -l\Delta x(\partial B_z/\partial t).$$

Since Δx is such a small distance, we can assume E_y to change linearly with x in this region. Thus we can write

$$E_{y2} = E_{y1} + (\partial E_y/\partial x)\ \Delta x \quad \text{or} \quad E_{y2} - E_{y1} = (\partial E_y/\partial x)\Delta x.$$

$$\therefore \qquad \left(\frac{\partial E_y}{\partial x}\right)\Delta x l = -l\Delta x\left(\frac{\partial B_z}{\partial t}\right) \text{ or } \frac{\partial E_y}{\partial x} = -\frac{\partial B_z}{\partial t}. \qquad \text{...(22)}$$

In evaluating $\partial E_y/\partial x$, it is assumed that t is constant, because Fig. 14.3 is an *instantaneous snap shot.* In evaluating $\partial B_z/\partial t$, it is assumed that x is constant because the time rate of change of B at a particular place is required. The minus sign in this equation is appropriate, as E_y is increasing with x and B_z is decreasing with time t.

Now consider a rectangle in xz plane [(Fig. 14.4 (b)] for which flux Φ_E is decreasing with time as the wave moves through it. This change will induce a magnetic field at points around the periphery of the rectangle. This induced field is the magnetic component of the electromagnetic wave. As clear from Eq. (15), the closed line integral of **B** around the rectangle, if we proceed in counter clockwise direction, is negative, while it is positive for **E**. *Thus we see that the lines of induced fields* **E** *and* **B** *circulate in opposite directions, even though fluxes* Φ_B *and* Φ_E *respectively are changing with time in the same way.*

Using similar operations for Eq. (15), here too the $\mathbf{B}\cdot d\,\mathbf{l}$ will be zero over the ends of the closed path in xz plane, Fig. 14.4 (b). Hence Eq. (15) with no conduction current becomes

$$\int_1 B_z dl - \int_2 B_z\, dl = \mu_0\varepsilon_0 \frac{\partial}{\partial t}\int E_y l\, \Delta x.$$

Assuming B_z as B_{z1} and B_{z2} along paths 1 and 2 respectively, we have

$$(B_{z1} - B_{z2})\, l = \mu_0\varepsilon_0\, l\Delta x(\partial E_y/\partial t).$$

If B_z changes with x linearly with in this small distance Δx, then we have

$$B_{z_2} = B_{z_1} + (\partial B_z/\partial x)\,\Delta x. \quad \text{or} \quad B_{z_2} - B_{z_1} = (\partial B_z/\partial x)\,\Delta x$$

$$\therefore \qquad -\left(\frac{\partial B_z}{\partial x}\right)\Delta x l = \mu_0\varepsilon_0\, l\,\Delta x\frac{\partial E_y}{\partial t}, \text{ or } \frac{\partial B_z}{\partial x} = -\mu_0\varepsilon_0\frac{\partial E_y}{\partial t}. \qquad \text{...(23)}$$

Differentiating Eq. (22) with respect to x and Eq. (23) with respect to t and subtracting, we get

$$\partial^2 E_y/\partial x^2 = \mu_0\varepsilon_0\,(\partial^2 E_y/\partial t^2). \qquad \text{...(24)}$$

Similarly we can get an equation involving B_z only after differentiating Eqs. (22) and (23) in opposite order and then subtracting, as

$$\partial^2 B_z/\partial x^2 = \mu_0\varepsilon_0\,(\partial^2 B_z/\partial t^2). \qquad \text{...(25)}$$

Comparison with the equation for a transverse wave on a string shows immediately that Eqs. (24) and (25) represent the differential equations that imply wave propagation. The velocity of propagation along x- direction is given by

$$v = 1/(\mu_0\,\varepsilon_0)^{1/2}. \qquad \text{...(26)}$$

Substitution of numerical values $\mu_0 = 4\pi \times 10^{-7}$ N-sec^2/C^2 and $\varepsilon_0 = 8.85 \times 10^{-12}$ C$^2/N - m^2$, gives $v = 2.998 \times 10^8$ m/sec, which is the measured velocity of light c. This shows that light may be electromagnetic radiation.

To show the relationship between the magnitude of the electric and magnetic vectors in the wave, let us differentiate Eq. (21) with respect to t. This gives

$$\partial E_y/\partial t = -E_0\,\omega\cos(\kappa x - \omega t) \qquad \text{...(27)}$$

Substituting this value in Eq. (23), we get

$$\partial B_z/\partial x = \mu_0\varepsilon_0 E_0 \omega \cos(\kappa x - \omega t).$$

On integrating with respect to x, we get

$$B_z = \mu_0 \varepsilon_0 E_0 v \sin(\kappa x - \omega t) = \mu_0 \varepsilon_0 v E_y \qquad ...(28)$$

or

$$B_z = (\mu_0 \varepsilon_0)^{1/2} E_y = E_y/v = E_y/c. \qquad ...(29)$$

At any point in space the E and B *are related by*

$$E_y/B_z = E/B = v\ (= c). \qquad ...(30)$$

The relation (28) may be written as

$$B_z = B_0 \sin(\kappa x - \omega t). \qquad ...(31)$$

It has the spatial and time dependence. Here the amplitude of the magnetic field

$$B_0 = \mu_0 \varepsilon_0 v E_0 = E_0/v = E_0/c. \qquad ...(32)$$

Thus we see that the ratio of electric and magnetic field amplitudes in an electromagnetic wave is independent of the currents that set up the original wave and that determine the fields and their direction of propagation.

Since $B_0 = \mu_0 H_0$, and H_0 has dimensions of A/m, while E_0 has dimensions of V/m. Therefore E_0/H_0 has dimensions of ohm. The quantity E_0/H_0 is called the impedance of free space and is denoted by Z_0. Its value is 377Ω.

Electric and Magnetic fields are in phase. It is clear from Eqs. (21) and (31) that the electric field and magnetic field are exactly in the same phase, *i.e.* when the electric field is maximum, the magnetic field is also a maximum; when one is zero, the other is zero and so forth. Fig. 14.3(*c*) also shows that *E*– and*M*– waves are transverse and in the same phase.

Unpolarized plane waves are made up of a superposition of polarized plane waves, with the random distribution of **E** and **B** in the *yz*- plane. The electric and magnetic vectors of each component of these plane waves are perpendicular to each other.

Transverse Nature of Plane Waves. We now show that for a plane wave travelling in the x direction the electric and magnetic fields vectors are limited in the y-z plane only. Maxwell's first equation tells us that net flux of **E** is zero through any arbitrary volume consisting zero charge distribution. Since, **E** does not vary with y or z, hence the net flux through the faces $dx\,dz$ and $dx\,dy$ is zero. Therefore we get

$$\oint \mathbf{E}\cdot d\mathbf{S} = \int_{xy} \mathbf{E}\cdot d\mathbf{S} + \int_{yz} \mathbf{E}\cdot d\mathbf{S} + \int_{xz} \mathbf{E}\cdot d\mathbf{S} = 0$$

or

$$\int_{yz} \mathbf{E}\cdot d\mathbf{S} = 0. \qquad ...(33)$$

It is only possible, if E_x does not vary with x. Similarly we may prove that B_x does not vary with x. We have already seen that the electric field and magnetic field are perpendicular to each other, or $\mathbf{E}\cdot\mathbf{B} = 0$. These results show that the wave has no field component in the direction of travel, or the **E** and **B** are *transverse to the direction of propagation.*

14.4 PROPAGATION OF ELECTROMAGNETIC ENERGY

Since E and B fields in the electromagnetic wave involve stored energy, the propagation of electromagnetic wave thus implies the transmission of energy. The rate of energy transmitted across a unit area perpendicular to the direction of flow is called *Poynting vector,* or *energy* *propagation vector S*. In optics this is known as *intensity*. To understand the meaning of **S**, a volume surrounded by a closed surface is considered. The rate of change of energy with in the volume is thus defined as

$$\frac{\partial U}{\partial t} = -\oint \mathbf{S} \cdot d\mathbf{A} \qquad ...(34)$$

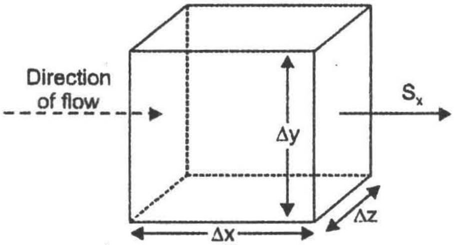


Fig. 14.5. Energy flow through elementary volume $\Delta x\ \Delta y\ \Delta z$.

Here we have used $d\mathbf{A}$ for surface area to avoid confusion with **S**, the poynting vector. The negative sign is used because the integral is negative when energy is entering the volume.

Consider an elementary parallelopiped with sides Δx, Δy, Δz as shown in Fig. 14.5. If the flow of electromagnetic wave is along x- direction, the flow of energy into and out of volume will be through the $\Delta y\ \Delta z$ faces only and **S** will be along x-direction. Thus

$$\oint \mathbf{S} \cdot d\mathbf{A} = (S_x \Delta y \Delta z)_{x+\Delta x} - (S_x\ \Delta y\ \Delta z)_x = [(S_x)_{x+\Delta x} - (S_x)_x]\ \Delta y \Delta z$$

$$= (\partial S_x / \partial x)\ \Delta x \Delta y \Delta z. \qquad ...(35)$$

Since the energy flow is due to E_y and B_z, hence from the previous results we have the relation for energy in this volume as

$$U = \left[\tfrac{1}{2}\varepsilon_0\ E_y^{\ 2} + \tfrac{1}{2}(\mu)_0^{-1}\ B_z^2\right] \Delta x\ \Delta y\ \Delta z$$

$$\therefore \qquad \frac{\partial U}{\partial t} = \left[\varepsilon_0 E_y\ \frac{\partial E_y}{\partial t} + \mu_0^{-1}\ B_z\ \frac{\partial B_z}{\partial t}\right] \Delta x\ \Delta y\ \Delta z$$

Using relations (22) and (23), we get

$$\frac{\partial U}{\partial t} = -\frac{1}{\mu_0}\left[E_y\ \frac{\partial B_z}{\partial x} + B_z\ \frac{\partial E_y}{\partial x}\right] \Delta x\ \Delta y\ \Delta z$$

$$= -\frac{1}{\mu_0}\frac{\partial}{\partial x}(E_y B_z)\ \Delta x\ \Delta y\ \Delta z = -\oint \mathbf{S} \cdot d\mathbf{A}$$

$$\therefore \qquad \frac{1}{\mu_0}\frac{\partial}{\partial x}(E_y B_z)\ \Delta x\ \Delta y\ \Delta z = \frac{\partial S_x}{\partial x}\ \Delta x\ \Delta y\ \Delta z$$

or

$$\frac{1}{\mu_0}\frac{\partial}{\partial x}(E_y B_z) = \frac{\partial S_x}{\partial x}.$$

Integration with respect to x gives

$$S_x = E_y\, B_z / \mu_0. \qquad ...(36)$$

Here the constant of integration has been neglected.

For the general case it can be written as

$$\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0 \text{ joules/sec. } m^2\ (= \text{watt}/m^2). \qquad ...(37)$$

This shows that the direction of energy flow is perpendicular to the plane containing **E** and **B**. In this equation *E and B* are the instantaneous values. In the case of sinusoidally varying fields, we have

$$\mathbf{S}_{av} = \frac{1}{\mu_0}\mathbf{E}_{eff} \times \mathbf{B}_{eff} = \frac{1}{\mu_0}\frac{\mathbf{E}_0}{\sqrt{2}} \times \frac{\mathbf{B}_0}{\sqrt{2}}$$

$$= (\mathbf{E}_0 \times \mathbf{B}_0)/2\,\mu_0. \qquad \text{...(38)}$$

Law of Conservation of Energy for the Electromagnetic Field. By the law of conservation of energy the Joules heat is evolved at the expense of the energy of the fields, since there are no other sources of energy. We know an expression for the Joule heat evolved in unit volume in unit time as

$$Q/V = \mathbf{J}\cdot\mathbf{E} = \sigma E^2.$$

Hence for a volume V enclosed by a surface A, we have

$$Q = \int_V \mathbf{J}\cdot\mathbf{E}\,dV.$$

Substituting the value of **J** from Eq. (14), we get

$$Q = \int_V \left[\frac{1}{\mu_0}\text{curl}\,\mathbf{B} - \varepsilon_0\frac{\partial \mathbf{E}}{\partial t}\right].\,\mathbf{E}dV.$$

Using the vector identity div (**A** × **B**) = **B**·curl **A** – **A**·curl **B** and Eq. (13) for curl **E**, we get

$$Q = -\frac{1}{\mu_0}\int_V \text{div}\,(\mathbf{E}\times\mathbf{B})\,dV - \int_V\left[\varepsilon_0\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\right]dV \qquad \text{...(39)}$$

We know the self-evident equalities

$$\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) = \frac{1}{2}\frac{\partial}{\partial t}(E^2)$$

and

$$\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t} = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = \frac{1}{2}\frac{\partial}{\partial t}(B^2)$$

$$\therefore \int_V\left[\varepsilon_0\mathbf{E}\cdot\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\mu_0}\mathbf{B}\cdot\frac{\partial \mathbf{B}}{\partial t}\right]dV = \frac{1}{2}\frac{\partial}{\partial t}\int_V\left(\varepsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)dV = \frac{dU}{dt}.$$

Here we have taken out *dt* as *V* is independent of time. Using Gauss's theorem, the first term of right hand of Eq. (39) becomes

$$\frac{1}{\mu_0}\int_V \text{div}\,(\mathbf{E}\times\mathbf{B})\,dV = \frac{1}{\mu_0}\int_A (\mathbf{E}\times\mathbf{B}).\,d\mathbf{A} = \int_V \mathbf{S}\cdot\mathbf{dA}.$$

Hence Eq. (39) reduces to

$$Q = -\int_V \mathbf{S}\cdot\mathbf{dA} - dU/dt, \qquad \text{...(40)}$$

i.e. Joule Heat = flow of energy through the surface A enclosing the volume minus the energy flow of electromagnetic field within the volume V.

The Eq. (40) may be written as

$$-\frac{dU}{dt} = Q+\int_A \mathbf{S}\cdot d\mathbf{A}=\int_V \mathbf{J}\cdot\mathbf{E}\,dV+\frac{1}{\mu_0}\int_A(\mathbf{E}\times\mathbf{B}).\,d\mathbf{A} \qquad ...(41)$$

This expresses the law of conservation of energy. It states that *a change in the electromagnetic field energy in a volume occurs at the expenses of the work of conduction currents in this volume and the energy flux through the surface bounding the volume.*

If the electro-magnetic field energy remains unchanged, then

$$Q = -\int_A \mathbf{S}\cdot d\mathbf{A}=-\frac{1}{\mu_0}\int_A(\mathbf{E}\times\mathbf{B})\cdot d\mathbf{A}, \qquad ...(42)$$

i.e., all the work performed in a closed volume is at the expense of the electromagnetic energy flux through the surface bounding the volume. In terms of energy density u, equation (41) may be written as

$$-\int \frac{\partial u}{\partial t}dV = \int_V \mathbf{J}\cdot\mathbf{E}\,dV+\int_V \nabla\cdot\mathbf{S}\,dV$$

or $$\int_V\left(\frac{\partial u}{\partial t}+\mathbf{J}\cdot\mathbf{E}+\nabla\cdot\mathbf{S}\right)dV = 0,$$

Which yields the *differential form* of the *principle of conservation of energy.*

$$\mathbf{J}\cdot\mathbf{E} = -\partial u/\partial t-\nabla\cdot\mathbf{S}. \qquad ...(43)$$

This is known as **Poynting theorem.**

Electric and Magnetic Contributions : *Since* the energy density stored in an electromagnetic field is given by

$$u = \tfrac{1}{2}\,\varepsilon_0 E^2+B^2/2\mu_0. \qquad ...(44)$$

Using the relations (21) and (31) for monochromatic plane wave, we get

$$u = \tfrac{1}{2}\,\varepsilon_0\,(E_0^{\,2}+c^2 B_0^{\,2})\sin^2(\kappa x-\omega t). \qquad ...(45)$$

Because $E_0 = cB_0$, the energy contained in an electromagnetic wave is shared equally between the magnetic field and the electric field. Thus we may write

$$u = \varepsilon_0 E^2 = B^2/\mu_0. \qquad ...(46)$$

The oscillations in the electromagnetic wave are so rapid that we may consider the average of energy density over the period of one oscillation. Thus

$$u_{av} = \langle u\rangle = \tfrac{1}{2}\varepsilon_0 E_0^{\,2} = B_0^{\,2}/2\mu_0, \qquad ...(47)$$

as the average of the sine-squared factor over one period is one half.

For a monochromatic plane wave

$$\begin{aligned}\mathbf{S} &= (\mathbf{E}\times\mathbf{B})/\mu_0 = c\varepsilon_0\,E_0^{\,2}\sin^2(\kappa x-\omega t)\,\mathbf{i}\\ &= c\,u\,\mathbf{i}. \qquad ...(48)\end{aligned}$$

Here we see that the energy density and hence the magnitude of the Poynting vector varies with time. The average value of **S** over one cycle of electromagnetic wave is called the *intensity* I of the radiation. Thus the intensity of em-wave is the average power per unit area transported by the em-wave.

$$\therefore \qquad I = \langle S \rangle = c \langle u \rangle = \tfrac{1}{2}\, c\, \varepsilon_0\, E_0^2. \qquad \text{...(49)}$$

Momentum of Electromagnetic Wave : Electromagnetic fields not only carry energy, they also carry momentum. To see this qualitatively, consider an electromagnetic wave propagating along x direction with its electric and magnetic field vectors along y- and z-axes respectively. When such a wave impinges on a charge $+q$, the charge experiences a force. If at any instant **E**-vector is along $+y$ direction, a force q **E** *will also act along* $+y$– direction. Thus the charge accelerates and moves with some velocity v in the $+y$ direction. The force $q\mathbf{v} \times \mathbf{B}$ due to **B** vector (which must act in the $+z$ – direction) acts along $+x$ direction and thus pushes the charge in that direction. When the electric field vector reverses sign, the electric force on the charge acts in the $-y$-direction and the velocity component is thus along $-y$-direction. The magnetic field vector has also reversed sign, thus the magnetic force continues to act in the x-direction. Thus for a complete cycle, the average of force $q\mathbf{E}$ is zero, but the force in the x-direction is always positive. The charge has an increased momentum in the x-direction. From conservation of momentum, this momentum is transferred by the em-wave.

The momentum density (the linear momentum per unit volume) of an electromagnetic wave,

$$p = S/c^2 = u/c \qquad \text{...(50)}$$

$$\therefore \qquad \langle p \rangle = \langle u \rangle / c, \qquad \text{...(51)}$$

Radiation Pressure : When the electromagnetic waves are incident on the surface of a substance, they are absorbed or reflected and thus transfer momentum to the substance. The rate at which momentum is transferred per unit area is known as *radiation pressure.*

If all the incident wave is absorbed, the whole of the momentum carried by the wave is transferred to the absorbing surface. For the radiations falling normally on a surface of area A in a time interval Δt, the

$$\text{Impulse} = \text{Momentum density} \times \text{volume}$$

$$F\,\Delta t = (u/c)\,(Ac\,\Delta t) = uA\,\Delta t.$$

$$\therefore \qquad \text{Rediation pressure } P_{\text{rad}} = F/A = u. \qquad \text{...(52)}$$

The radiation pressure will be $2u$, if the electromagnetic wave is totally reflected.

14.5 ELECTROMAGNETIC WAVES IN MATTER

If the vacuum is replaced by matter, all equations given above must be modified by using μ and ε for μ_0 and ε_0 respectively. If the medium is homogeneous and isotropic and for which the dielectric constant k and relative magnetic permeability k_m are constant, then the speed of electromagnetic wave in medium is given by

$$v = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{1}{\sqrt{\varepsilon_0\mu_0 k k_m}} = \frac{c}{\sqrt{k k_m}} = \frac{c}{n}, \qquad \text{...(53)}$$

where c is the speed of light in vacuum and n is the refractive index of the medium. In most dielectrics, $k_m \simeq 1$ and $k > 1$. Thus the speed of electromagnetic wave in the medium is always less than that in vacuum. In a perfect conductor, there can be no electric field. Hence the em-wave can not penetrate a perfect conductor and thus totally reflected. Thus we see that the polished metal surfaces are good reflectors to em-waves. However, these waves can penetrate real conductors and the electric currents produced dissipate some of the wave energy.

In a medium,

energy density $u = \frac{1}{2}\varepsilon E^2 + B^2/2\mu$

Poynting vector $\mathbf{S} = (\mathbf{E}\times\mathbf{B})/\mu$

Intensity of wave $I = \frac{1}{2}\varepsilon c\, E_0^{\,2}$. ...(54)

The quantities k and k_m are also represented as ε_r and μ_r, such that $\varepsilon = \varepsilon_r\varepsilon_0$ and $\mu = \mu_r\,\mu_0$. There are following possible media :

(1) Free space ($\sigma = 0$, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$)

(2) Non conducting media : (*a*) dielectrics without losses ($\sigma = 0$, $\varepsilon = \varepsilon_r\,\varepsilon_0$, $\mu = \mu_r\,\mu_0$)

(*b*) Dielectrics with losses ($\sigma \neq 0$, $\varepsilon = \varepsilon_r\,\varepsilon_0$, $\mu = \mu_r\,\mu_0$)

(3) *Good conductors* ($\sigma \simeq \infty$, $\varepsilon = \varepsilon_0$, $\mu = \mu_r\,\mu_0$).

Before we consider wave motion in these different media, it is appropriate to derive the wave equations in a linear medium using Maxwell's equations.

14.6 MAXWELL'S EQUATIONS AND WAVE EQUATION

Let us find the equation for electromagnetic wave propagating in a linear medium using Maxwell's equations. In a linear medium, we have $\mathbf{B} = \mu\,\mathbf{H}$, $\mathbf{D} = \varepsilon\mathbf{E}$ *and* $\mathbf{J} = \sigma\mathbf{E}$. The quantities permittivity ε, permeability μ and conductivity σ are assumed to be constants independent of position and time variables. Under these conditions, Maxwell's equations may be written as

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon \quad ...(i) \qquad \nabla\cdot\mathbf{B} = 0 \quad (ii)$$

$$\nabla\times\mathbf{E} = -\,\partial\mathbf{B}/\partial t \quad ...(iii) \qquad \nabla\times\mathbf{B} = \mu\mathbf{J} + \mu\varepsilon\partial\mathbf{E}/\partial t \quad ...(iv)$$

Substituting $\mathbf{J} = \sigma\mathbf{E}$ in the equation (*iv*), we get

$$\nabla\times\mathbf{B} = \mu\,\sigma\,\mathbf{E} + \mu\,\varepsilon\,\partial\mathbf{E}/\partial t.$$

Taking the curl of this equation, we get

$$\nabla\times(\nabla\times\mathbf{B}) = \mu\,\sigma\,(\nabla\times\mathbf{E}) + \mu\varepsilon\,\frac{\partial}{\partial t}\,(\nabla\times\mathbf{E}) = -\,\mu\sigma\,\frac{\partial\mathbf{B}}{\partial t} - \mu\varepsilon\,\frac{\partial^2\mathbf{B}}{\partial t^2}. \quad ...(55)$$

Using the vector identity

$$\nabla\times(\nabla\times\mathbf{A}) = \nabla\,(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A},$$

we get $$\nabla\,(\nabla\cdot\mathbf{B}) - \nabla^2\mathbf{B} = -\,\mu\sigma\,\partial\mathbf{B}/\partial t - \mu\varepsilon\,\partial^2\,\mathbf{B}/\partial t^2$$

or $$\nabla^2\mathbf{B} - \mu\varepsilon\,\partial^2\mathbf{B}/\partial t^2 - \mu\sigma\,\partial\mathbf{B}/\partial t = 0. \quad ...(56)$$

In order to solve Maxwell's equations for $\mathbf{E}$, we take the curl of equation (*iii*) as

$$\nabla\times\nabla\times\mathbf{E} = \nabla\times\left(-\frac{\partial\boldsymbol{B}}{\partial t}\right) = -\frac{\partial}{\partial t}(\nabla\times\mathbf{B}) = -\frac{\partial}{\partial t}\left[\mu\,\sigma\,\mathbf{E} + \mu\,\varepsilon\,\frac{\partial\mathbf{E}}{\partial t}\right]$$

Using the vector identity $\nabla\times(\nabla\times\mathbf{E}) = \nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E}$, we get

$$\nabla(\nabla\cdot\mathbf{E}) - \nabla^2\mathbf{E} = -\,\mu\sigma\,\frac{\partial\mathbf{E}}{\partial t} - \mu\varepsilon\,\frac{\partial^2\mathbf{E}}{\partial t^2}. \quad ...(57)$$

Let us restrict to a charge free medium for which

$\nabla\cdot\mathbf{D} = 0$ or $\nabla\cdot\mathbf{E} = 0$. Thus Eq. (57) reduces to

$$\nabla^2\mathbf{E} - \mu\,\varepsilon\,\partial^2\mathbf{E}/\partial t^2 - \mu\sigma\,\partial\mathbf{E}/\partial t = 0. \qquad ...(58)$$

We thus obtain partial three dimensional second order (in space and time) differential equations each for **E** and **B** separately.

14.7 PROPAGATION OF EM-WAVES IN NONCONDUCTING MEDIUM

For a non-conducting medium, such as dielectrics,

$\sigma = 0$, then Eqns. (56) and (58) reduce to

$$\nabla^2\mathbf{B} - \mu\varepsilon\,\partial^2\mathbf{B}/\partial t^2 = 0, \qquad ...(59)$$

and

$$\nabla^2\mathbf{E} - \mu\varepsilon\,\partial^2\mathbf{E}/\partial t^2 = 0. \qquad ...(60)$$

These are called the undamped, three dimensional and homogeneous wave equations.

An electromagnetic wave is said to be a plane wave if the field vectors of the wave have the same magnitude at all points of any plane perpendicular to the direction of propagation of wave. A wave is called *monochromatic* if the time variation of the field vectors of the wave obeys a harmonic law with a certain constant frequency, since frequency corresponds perceptually to colour.

Thus for a plane electromagnetic wave propagating along the x-axis, the field vectors of the wave have the form

$$\mathbf{E}(x, t) = \mathbf{E}(x)\,e^{-i\omega t},\ \mathbf{B}(x, t) = \mathbf{B}(x)\,e^{-i\omega t}. \qquad ...(61)$$

The wave is called *homogeneous* if the constant phase surfaces coincide with the surfaces of constant amplitude.

Equations (59) and (60), show that the field vectors satisfy the wave equation, where the velocity of the wave

$$v = 1/\sqrt{\mu\varepsilon}. \qquad ...(62)$$

Substituting the value of $\mathbf{E}(x, t)$ in Eq. (60), we get

$$d^2\,\mathbf{E}(x)/dt^2 + \kappa^2\,\mathbf{E}(x) = 0, \qquad ...(63)$$

where $\kappa^2 = \omega^2\,\mu\,\varepsilon$ or $\kappa = \omega/v$ = wave number. The general solution of this equation is given by

$$\mathbf{E}(x) = \mathbf{E}_0 e^{i\kappa x} + \mathbf{E}_0' e^{-i\kappa x}, \qquad ...(64)$$

where $\mathbf{E}_0$ and $\mathbf{E}_0'$ are constants. Substituting this value of $\mathbf{E}(x)$ in Eq. (61), we get

$$\mathbf{E}(x, t) = \mathbf{E}_0\,e^{i(\kappa x - \omega t)} + \mathbf{E}_0' e^{-i(\kappa x + \omega t)}. \qquad ...(65)$$

This relation represents plane wave in both directions along the x-axis, the first term for the wave along $+x$– axis, and the second term for the wave along $-x$ axis. Thus for the wave propagating in the positive direction of the x-axis, we have

$$\mathbf{E}(x, t) = \mathbf{E}_0 e^{i(\kappa x - \omega t)}. \qquad ...(66)$$

Similarly the solution for **B** can be found to be

$$\mathbf{B}(x, t) = \mathbf{B}_0 e^{i(\kappa x - \omega t)}. \qquad ...(67)$$

The relations (66) and (67) show the plane, monochromatic and homogeneous waves propagating along $+x$–direction in a homogeneous dielectric without a change in their amplitudes. Let us now discuss a few properties.

Phase Velocity : The velocity of motion of an equiphase surface is called the phase velocity. Thus the differentiation of constant phase with respect to time gives

$$\kappa x - \omega t = \text{const or } \kappa\,dx/dt - \omega = 0.$$

$$\therefore \qquad \text{Phase velocity } v_{Ph} = \frac{dx}{dt} = \frac{\omega}{\kappa} = \frac{1}{\sqrt{\varepsilon\mu}} = \frac{c}{\sqrt{kk_m}}. \qquad ...(68)$$

Let **r** be the radius vector of a point on such an equiphase plane. We have $\boldsymbol{\kappa}\cdot\mathbf{r} = \kappa x$ and

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0\, e^{i(\boldsymbol{\kappa}\cdot\mathbf{r} - \omega t)};\ B\,(\mathbf{r}, t) = \mathbf{B}_0\, e^{i(\boldsymbol{\kappa}\cdot\mathbf{r} - \omega t)} \qquad ...(69)$$

where κ is in the direction of propagation of em-wave.

Properties of Waves : Maxwell's equation div **E** = 0 gives

$$\nabla\cdot\mathbf{E} = i\,\boldsymbol{\kappa}\cdot\mathbf{E} = 0. \qquad ...(70)$$

It shows that the electric field vector **E** of the wave is perpendicular to κ, *i.e.*, perpendicular to the direction of propagation of wave.

Similarly, Maxwell's equation div **B** = 0, gives

$$\nabla\cdot\mathbf{B} = i\,\boldsymbol{\kappa}\cdot\mathbf{B} = 0, \qquad ...(71)$$

i.e., vector **B** is perpendicular to κ, *i.e.*, perpendicular to the direction of propagation.

It shows that **E** and **B** are perpendicular to the direction of propagation of wave, *i.e.*, the em-waves are *transverse*.

Maxwell's equation $\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$ gives

$$\boldsymbol{\kappa}\times\mathbf{E} = \omega\mathbf{B}. \qquad ...(72)$$

Let $\hat{\kappa}$ be the unit vector in the direction of wave propagation. Then we have

$$\boldsymbol{\kappa} = \hat{\kappa}\ \omega/v$$

$$\therefore \qquad \hat{\kappa}\times\mathbf{E} = v\,\mathbf{B}. \qquad ...(73)$$

It further shows that **E** and **B** are mutually perpendicular to each other and are also perpendicular to $\hat{\kappa}$. Further **E** and **B** are in *phase*. Their *amplitudes* are related by

$$B_0 = (\kappa/\omega)\,E_0 = E_0/v. \qquad ...(74)$$

14.8 PROPAGATION OF EM-WAVES IN A CONDUCTING MEDIUM

We consider a homogeneous medium (μ = const, ε = const), charge free and external current free. The currents existing in the material are induced only by the electromagnetic waves itself, thus we have $\mathbf{J} = \sigma\mathbf{E}$. As discussed earlier, the electromagnetic fields in such medium are given as

$$\nabla^2\mathbf{E} - \mu\varepsilon\,\partial^2\mathbf{E}/\partial t^2 - \mu\sigma\,\partial\mathbf{E}/\partial t = 0 \qquad ...(58)$$

$$\text{and} \qquad \nabla^2\mathbf{B} - \mu\varepsilon\,\partial^2\mathbf{B}/\partial t^2 - \mu\sigma\,\partial\mathbf{B}/\partial t = 0. \qquad ...(56)$$

For a plane monochromatic wave, we take $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}(\mathbf{r})\,e^{-i\omega t}$, hence Eq. (58) may be written as

$$\nabla^2\mathbf{E} + (\mu\varepsilon\omega^2 + i\,\mu\sigma\omega)\,\mathbf{E} = 0$$

$$\text{or} \qquad \nabla^2\mathbf{E} + n_0^2\,\frac{\omega^2}{c^2}\left(1 + i\frac{\sigma}{\varepsilon\omega}\right)\mathbf{E} = 0, \qquad ...(75)$$

where $n_0 = \sqrt{kk_m}$ = refractive index of the medium.

The quantity $n_0\,(1 + i\,\sigma/\varepsilon\omega)^{1/2}$ = complex refractive index, n_c, which may be written as

$$n_c = n_0\,(1 + i\,\sigma/\varepsilon\omega)^{1/2} = n + i\,m, \qquad ...(76)$$

where n and m are real quantities, called the *optical constants.*

We consider a plane wave propagating in x-direction. Therefore we take $|\mathbf{E}|$ independent of the coordinates y and z, ∇^2 may be replaced by $\partial^2/\partial x^2$. Thus Eq. (75) reduces to

$$\frac{d^2\mathbf{E}}{dx^2} + \frac{n_0^{\,2}\omega^2}{c^2}\left(1+\frac{i\sigma}{\varepsilon\omega}\right)\mathbf{E} = 0. \qquad ...(77)$$

In the absence of the conducting properties of the material (*i.e.*, $\sigma = 0$), this equation is same as Eq. (63) and has the solution $\mathbf{E} = \mathbf{E}_0 e^{i\kappa x}$. Thus, we see that for the conducting medium, the wave number κ will be complex.

Let us assume the solution of Eq. (77), as

$$\mathbf{E}(x) = \mathbf{E}_0 e^{i\gamma x}, \qquad ...(78)$$

where γ is a *constant complex wave number,* which is independent of x and $\mathbf{E}_0$. Substituting this value of $\mathbf{E}(x)$ in the Eq. (77), we get

$$\gamma^2 = \frac{n_0^2\,\omega^2}{c^2}\left(1+\frac{i\sigma}{\varepsilon\omega}\right). \qquad ...(79)$$

This relation is known as *dispersion relation* as it relates the propagation constant and the angular frequency ω. The complex quantity γ may be written as

$$\gamma = \kappa + i\,s. \qquad ...(80)$$

$$\therefore \qquad \kappa + i\,s = \frac{n_0\omega}{c}\left(1+\frac{i\sigma}{\varepsilon\omega}\right)^{1/2}$$

$$\text{or} \qquad \kappa^2 - s^2 + 2i\,\kappa s = \frac{n_0^2\,\omega^2}{c^2}\left(1+\frac{i\sigma}{\varepsilon\omega}\right)$$

$$\therefore \qquad \kappa^2 - s^2 = \frac{n_0^2\,\omega^2}{c^2} \text{ and } 2\kappa s = \frac{n_0^2\,\omega\sigma}{c^2\varepsilon}.$$

On solving, we get

$$\kappa = \frac{n_0\omega}{\sqrt{2}\,c}\left[\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2}+1\right]^{1/2}$$

$$\text{and} \qquad s = \frac{n_0\omega}{\sqrt{2}\,c}\left[\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2}-1\right]^{1/2} \qquad ...(81)$$

Thus the electric field vector takes the form

$$\mathbf{E} = \mathbf{E}_0\, e^{i\,(\kappa + is)x}\, e^{-i\omega t} = \mathbf{E}_0 e^{-sx}\, e^{i(\kappa x - \omega t)}. \qquad ...(82)$$

Similarly, we get the relation, for magnetic field vector $\mathbf{B}$ as

$$\mathbf{B} = \mathbf{B}_0\, e^{-sx}\, e^{-i(\kappa x - \omega t)}. \qquad ...(83)$$

The comparison of equations (76) and (81) gives

$$\kappa = \frac{\omega}{c}\,n \text{ and } s = \frac{\omega}{c}\,m. \qquad ...(84)$$

The quantity n is interpreted as the *refractive index* of the medium. It depends on the conductivity σ of the medium and the frequency ω of the wave. The Eq. (82) shows that the amplitude of the wave decreases during its propagation, *i.e.,* the em-wave propagating in a conducting medium is *damped.* The distance in which the electric field vector of the wave decreases to $1/e$ of its initial value is called the *penetration depth* Δ of the wave into a conducting medium. This distance is also called *skin depth*. Thus

$$\Delta = \frac{1}{s} = \frac{c}{\omega m} = \frac{\sqrt{2}\,c}{n_0\omega}\left[\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2} - 1\right]^{-1/2} \qquad \text{...(85)}$$

The quantity s is known as the *attenuation constant,* which also depends on σ and ω. A rapid attenuation of electromagnetic waves in a conducting medium results into Joule's heat.

$$\text{The phase velocity } v_{ph} = \frac{\omega}{\kappa} = \left[\frac{2}{\mu\varepsilon\,[1+(1+\sigma^2/\varepsilon^2\omega^2)^{1/2}]}\right]^{1/2}$$

This shows that the phase velocity in a conducting medium is smaller than its value in a non-conducting medium.

The *wavelength* in a conducting medium

$$\lambda = \frac{2\pi}{\kappa} = \frac{2\pi}{\omega}\left[\frac{2}{\mu\varepsilon\,[1+(1+\sigma^2/\varepsilon^2\omega^2)^{1/2}]}\right]^{1/2}.$$

is smaller in comparison with the wavelength in a non-conducting medium.

Maxwell's equation $\nabla \times \mathbf{E} = -\partial\mathbf{B}/\partial t$ gives

$$\gamma\hat{\kappa} \times \mathbf{E} = \omega\mathbf{B}$$

or

$$\frac{n_0\omega}{c}\left(1+\frac{i\sigma}{\varepsilon\omega}\right)^{1/2} E_0 e^{-sx}\, e^{i(\kappa x-\omega t)} = \omega B_0 \cdot e^{-sx}\, e^{i(\kappa x-\omega t)}$$

$\therefore$

$$B_0 = \frac{n_0}{c}\left(1+\frac{i\sigma}{\varepsilon\omega}\right)^{1/2} E_0 = \frac{n_c}{c}E_0 = \frac{\gamma}{\omega}E_0. \qquad \text{...(86)}$$

This equation shows that the electric and magnetic field vectors are out of phase, although they are perpendicular to the direction of propagation of wave.

The ratio of E_0/H_0, the intrinsic impedance of the medium is a complex quantity dependent of frequency.

$$\mathbf{Z}_0 = \frac{c}{n_0\,[1+i\,\sigma/\varepsilon\omega]^{1/2}} \qquad \text{..(87)}$$

Above relations show that the ratio B_0/E_0 in the conducting medium is larger than that in a non-conducting medium.

The phase of $\mathbf{Z}_0$ may be calculated by expressing it in complex polar form.

$$\mathbf{Z}_0 = |\,\mathbf{Z}_0\,|e^{-i\phi/2}$$

$$\text{where } |\,\mathbf{Z}_0\,| = \frac{c/n_0}{[1+\sigma^2/\varepsilon^2\omega^2]^{1/4}} \text{ and } \tan\phi = \frac{\sigma}{\varepsilon\omega}. \qquad \text{...(88)}$$

The electric and magnetic fields vectors of a monochromatic plane wave in a conducting medium are shown in Fig. 14.6.

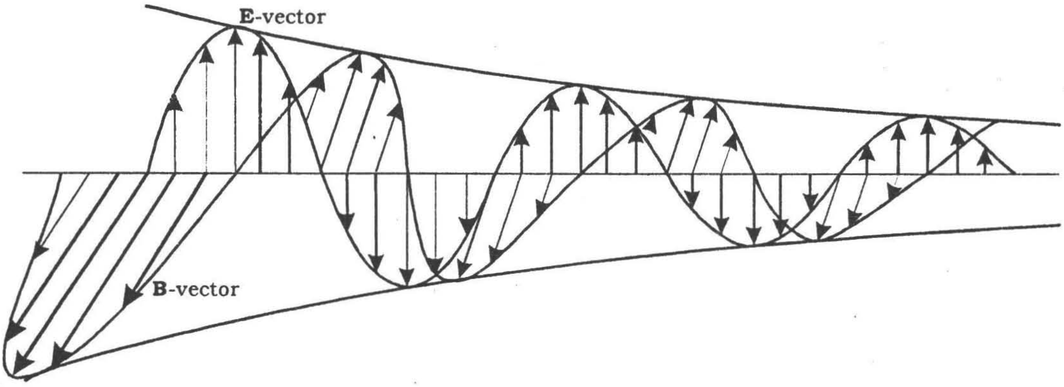


Fig. 14.6. E and B vector in a conductor.

$$\therefore \qquad \text{Skin depth } \Delta = \frac{c}{n_0}\left(\frac{2\varepsilon}{\omega\sigma}\right)^{1/2} = \left(\frac{2}{k_m \mu_0 \sigma\omega}\right)^{1/2}$$

Since $k_m \simeq 1$, therefore $\Delta = (2/\mu_0\, \sigma\omega)^{1/2}$. ...(90)

This relation shows that Δ does not depend on the dielectric properties of the good conductors, however it depends on the frequency and conductivity. It shows that waves of higher frequencies do not penetrate as much as those of lower frequencies. Since for $\sigma/\varepsilon\omega >> 1$, $\tan\phi$ becomes very large, hence $\phi \to \pi/2$. Thus the phase difference between **E** and **B** fields in a perfect conductor is $\pi/4$.

Good Insulators (or weakly conducting medium) : The conductivity of good insulators is very small and $\sigma/\varepsilon\omega < 1$ even at very low frequencies.

$$\therefore \qquad \kappa = \frac{n_0\omega}{c} = \sqrt{k\, k_m}, \quad s = \frac{n_0\omega}{2c}\cdot\frac{\sigma}{\varepsilon\omega} = \frac{1}{2}\sqrt{kk_m}\,\frac{\sigma}{\varepsilon\omega} \qquad ...(91)$$

$$\text{Thus skin depth } \Delta = \frac{2\varepsilon\omega}{\sigma\sqrt{kk_m}} = \frac{2\varepsilon\omega}{\sigma\sqrt{\kappa}}. \qquad ...(92)$$

Since $\sigma/\varepsilon\omega < 1$, hence $s < \kappa$, the *wave propagates many wave lengths without much loss, the material thus acts as transparent.*

Since, $\sigma/\varepsilon\omega < 1$, $\tan\phi = \phi = \sigma/\varepsilon\omega$, therefore the phase angle between **E** and **B** fields is

$$\phi/2 = \sigma/2\varepsilon\omega. \qquad ...(93)$$

One can prove easily that in a conductor, the energy is not equally shared between the electric and magnetic fields. The magnetic contribution predominates. In a good conductor, the energy is almost magnetic.

14.9 POLARIZATION OF PLANE WAVES

We can produce transverse and longitudinal waves in a string. Sound waves are longitudinal. Electromagnetic waves are transverse. There are two dimensions perpendicular to any given line of propagation. Correspondingly, transverse waves occur in two independent states, vibrations up and down (vertical direction), left and right (horizontal direction). If the electromagnetic wave propagates along x-direction, the electric vibrations may be along y- or z-directions, *i.e.*,

$$\mathbf{E}_y = E_{oy}\, e^{i(\kappa x - \omega t)}\,\mathbf{j} \text{ and } \mathbf{E}_z = E_{oz}\, e^{i(\kappa x - \omega t)}\,\mathbf{k}. \qquad ...(94)$$

If the electric field of a plane wave is aligned along a fixed direction in space, it is said to be *linearly polarized.* The linearly polarized plane wave may thus be considered a superposition of two waves, of same amplitude and phase one vertically polarized and other horizontally polarized.

$$\therefore \quad \mathbf{E}\,(x, t) = E_0 \cos\phi\, e^{i(\kappa x - \omega t)}\,\mathbf{j} + E_0 \sin\phi\, e^{i(\kappa x - \omega t)}\,\mathbf{k}$$

$$= E_0\, e^{i(\kappa x - \omega t + \phi)}\,\hat{n}\,. \qquad ...(95)$$

where $\hat{n}$ is the unit *polarization vector* and ϕ is the *polarization angle.* Above equation represents linearly or plane polarized wave, because the displacement is parallel to a fixed vector along $\hat{n}$.

Other types of polarization, such as *elliptically* and *circularly* polarizations, can be constructed by the superposition of two plane polarized waves, which are not in the same phase. Let us consider electric vectors of two linearly polarized waves, $\mathbf{E}_1$ and $\mathbf{E}_2$, both moving in the x-direction, polarized in the y- and z-directions respectively and are in different phases. The sum of these vectors is given by

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 = (E_{oy}e^{i\phi_y}\,\mathbf{j} + E_{oz}e^{i\phi_z}\mathbf{k})\, e^{i(\kappa x - \omega t)}$$

$$= E_{oy}e^{i(\kappa x - \omega t + \phi_y)}\,\mathbf{j} + E_{oz}e^{i(\kappa x - \omega t + \phi_z)}\mathbf{k}\,. \qquad ...(96)$$

The corresponding real wave is

$$\mathbf{E} = E_{oy} \cos(\omega t - \kappa x - \phi_y)\,\mathbf{j} + E_{oz} \cos(\omega t - \kappa x - \phi_z)\,\mathbf{k}. \qquad ...(97)$$

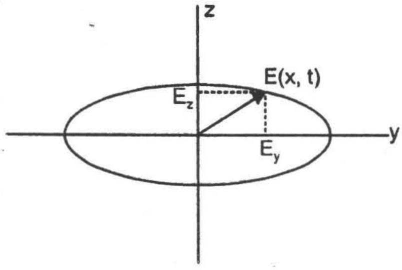


Fig. 14.7. Elliptical polarization.

If $\phi_y \neq \phi_z$, the resultant waves [Eqs. (96) and (97)] are said to be elliptically polarized [Fig. 14.7)]

A special case of polarization is when $\phi_y = \phi$ and $\phi_z = \phi \pm \pi/2$. Then at a plane $x = x_0$, we have

$$\mathbf{E} = E_{oy} \cos(\omega t - \kappa x_0 - \phi)\,\mathbf{j} + E_{oz} \cos(\omega t - \kappa x_0 - \phi \mp \pi/2)\,\mathbf{k}$$

$$= E_{oy} \cos(\omega t - \kappa x_0 - \phi)\,\mathbf{j} \pm E_{oz} \sin(\omega t - \kappa x_0 - \phi)\,\mathbf{k}$$

$$= E_y\,\mathbf{j} \pm E_z\,\mathbf{k}. \qquad ...(98)$$

Hence we have

$$(E_y/E_{oy})^2 + (E_z/E_{oz})^2 = 1. \qquad ...(99)$$

This is the equation of an ellipse with major and minor axes E_{oy} and E_{oz} respectively. The electric vector $\mathbf{E}$ moves clockwise or counter clockwise depending upon the sign (+) or (–) in

$$\phi_z = \phi \pm \pi/2.$$

Circular polarization is special case of elliptical polarization where $E_{oy} = E_{oz} = E_0$. Thus for circularly polarized wave we have

$$\mathbf{E} = E_0 \cos(\omega t - \kappa x_0 - \phi)\,\mathbf{j} \pm E_0 \sin(\omega t - \kappa x_0 - \phi)\,\mathbf{k}$$

$$= E_y\,\mathbf{j} \pm E_z\,\mathbf{k}. \qquad ...(100)$$

$$\therefore \quad E_y^{\,2} + E_z^{\,2} = E_0^{\,2}. \qquad ...(101)$$

This is the equation of a circle of radius E_0. The Eq. (101) shows that the magnitude of **E** remains constant, although its direction rotates. One can prove easily that a sum of two circularly polarized waves of opposite rotation (helicity) produces a linearly polarized wave.

14.10 BOUNDARY CONDITIONS ON THE FIELDS

We have already discussed previously the behaviour of the static electric and magnetic fields across the boundary. The boundary conditions on the electromagnetic fields governed by Maxwell's equations can be deduced from these equations.

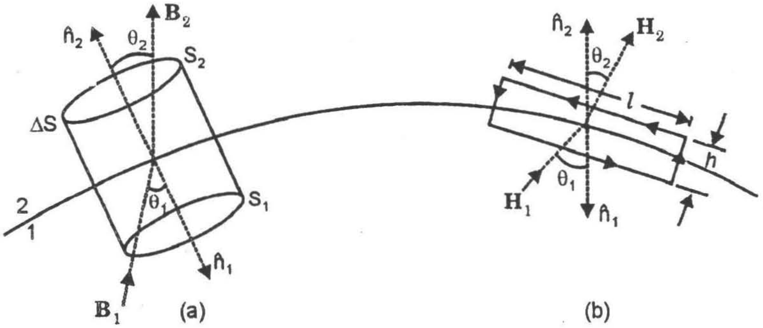
To derive the boundary conditions for **D** and **B** at the interface between two media (dielectric or conducting), we consider a pill box shaped surface at the interface, as shown in Fig. 14.8 (*a*). Maxwell's divergence equations $\nabla \cdot \mathbf{B}(t) = 0$ and $\nabla \cdot \mathbf{D}(t) = \rho_f(t)$, are exactly of the same form as the static equations, except that **B**, **D** and ρ_f are time dependent. We may deduce that

$$B_{1n}(t) = B_{2n}(t) \qquad ...(102)$$

and

$$D_{2n}(t) - D_{1n}(t) = \sigma_f(t), \qquad ...(103)$$

where $\sigma_f(t)$ is the surface density of charge on the interface.



(a) (b)

Fig. 14.8. (*a*) A pill box shaped surface, (*b*) A rectangular path at the interface.

The Maxwell's curl equations $\nabla \times \mathbf{H} = [\mathbf{J}_f + \partial \mathbf{D}/\partial t]$ and $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ are different from the static relations $\nabla \times \mathbf{H} = \mathbf{J}_f$ and $\nabla \times \mathbf{E} = 0$. To deduce the boundary conditions for the tangential components, consider a rectangular loop of length *l* and width *h* partially immersed in medium 1. Integrating the curl equations for **H** over an area ΔS enclosed by the rectangular loop, we get,

$$\int_{\Delta S} (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_{\Delta S} \mathbf{J}_f \cdot d\mathbf{S} + \int_{\Delta S} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

or

$$\oint \mathbf{H} \cdot d\mathbf{l} = \int_{\Delta S} \mathbf{J}_f \cdot d\mathbf{S} + \int_{\Delta S} \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}.$$

Since the width *h* is very small and the integral of $\partial \mathbf{D}/\partial t$ over ΔS vanishes if $\partial \mathbf{D}/\partial t$ is bounded. Thus

$$(\mathbf{H}_2 - \mathbf{H}_1) \cdot \mathbf{l} = \mathbf{J}_f \cdot \Delta \mathbf{S} = I_f = (\mathbf{K}_f \times \hat{n}).\mathbf{l}$$

where $\mathbf{K}_f$ is the free surface current density and I_f is the free current passing through the loop. Thus we have

$$\mathbf{H}_{2t}(t) - \mathbf{H}_{1t}(t) = \mathbf{K}_f(t) \times \hat{n}. \qquad ...(104)$$

Thus the tangential components of **H** are discontinuous by an amount proportional to the surface current density.

A similar integration of $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ over the rectangular loop provided $\partial \mathbf{B}/\partial t$ is bounded, gives

$$E_{1t}(t) = E_{2t}(t). \qquad \text{...(105)}$$

Above boundary conditions can be expressed in terms of **E** and **B** alone as

$$B_{1n}(t) = B_{2n}(t); \quad \varepsilon_2 E_{2n}(t) - \varepsilon_1 E_{1n}(t) = \sigma_f(t)$$

$$\frac{1}{\mu_2}\mathbf{B}_{2t}(t) - \frac{1}{\mu_1}\mathbf{B}_{1t}(t) = \mathbf{K}_f \times \hat{n}; \quad \mathbf{E}_{1t}(t) = \mathbf{E}_{2t}(t) \qquad \text{...(106)}$$

Some Special Cases

(*i*) *Dielectric-dielectric interface :* When both the media have zero conductivity, then the free charge or free current at the interface is zero and we have

$$B_{1n}(t) = B_{2n}(t); \varepsilon_1 E_{1n}(t) = \varepsilon_2 E_{2n}(t)$$

$$\mathbf{B}_{1t}(t)/\mu_1 = \mathbf{B}_{2t}(t)/\mu_2; \mathbf{E}_{1t}(t) = \mathbf{E}_{2t}(t). \qquad \text{...(107)}$$

(*ii*) *Conductor dielectric interface :* If the medium 1 is highly conducting, then E_{1n} is zero. Thus we have

$$\varepsilon_2 E_{2n}(t) = D_{2n}(t) = \sigma_f(t). \qquad \text{...(108)}$$

Since the electric field in a highly conducting medium is zero, the **H**-field will therefore be zero. Thus we have

$$\mathbf{H}_{1t}(t) = 0 \text{ and } B_{1n}(t) = \mu_1 H_{1n}(t) = 0$$

$$\therefore \quad \mathbf{H}_{2t}(t) = \mathbf{K}_f \times \hat{n}. \qquad \text{...(109)}$$

14.11 REFLECTION AND TRANSMISSION FROM A DIELECTRIC INTERFACE

Consider a plane surface *AB* dividing a space into two regions 1 and 2, with different dielectric constants k_1 and k_2 respectively (Fig. 14.9). If a plane wave with wave vector κ and frequency ω is incident on the surface from the side of region 1, part of this wave will reflect into region 1 and part will refract and transmit into region 2. If the refracted and reflected waves have wave vectors κ' and κ'' respectively, then the electric field vectors for these three waves are

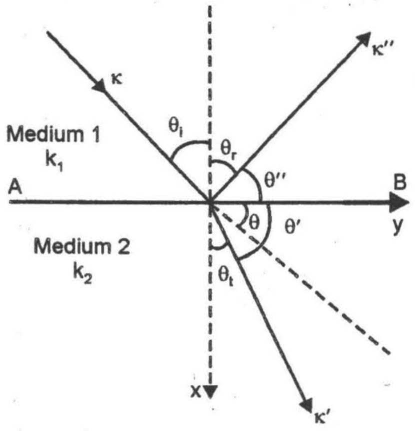


Fig. 14.9. Reflection and refraction of em-wave.

incident wave

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0 e^{i(\kappa \cdot \mathbf{r} - \omega t)}, \qquad \text{...(110)}$$

$$\mathbf{B}(\mathbf{r}, t) = (\varepsilon_1 \mu_1)^{1/2}\, \hat{\kappa} \times \mathbf{E}_0 e^{i(\kappa \cdot r - \omega t)} \qquad \text{...(111)}$$

Refracted wave

$$\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}_0{}' e^{i(\kappa' \cdot \mathbf{r} - \omega' t)} \qquad \text{...(112)}$$

$$\mathbf{B}'(\mathbf{r}, t) = (\varepsilon_2 \mu_2)^{1/2}\, \hat{\kappa}' \times \mathbf{E}_0{}' e^{i(\kappa' \cdot \mathbf{r} - \omega' t)} \qquad \text{...(113)}$$

Reflected wave

$$\mathbf{E}''(\mathbf{r}, t) = \mathbf{E}_0{}'' e^{i(\kappa'' \cdot \mathbf{r} - \omega'' t)} \qquad \text{...(114)}$$

and

$$\mathbf{B}''(\mathbf{r}, t) = (\varepsilon_1 \mu_1)^{1/2}\, \hat{\kappa}'' \times \mathbf{E}_0{}'' e^{i(\kappa'' \cdot \mathbf{r} - \omega'' t)} \qquad \text{...(115)}$$

where $\hat{\kappa}$, $\hat{\kappa}'$ and $\hat{\kappa}''$ are the unit vectors along the propagation vectors κ, κ' and κ'' respectively. The corresponding angles with respect to the normal to the interface are θ_i, θ_t and θ_r.

There are two possible cases : (*a*) The component of the wave polarized in the plane of incidence (the *p*-polarization). (*b*) The component polarized normal to the plane of incidence (the *s*-polarization).

In the present case the plane of incidence is *xy*- plane. The plane surface *AB* is in *yz*- plane. In the case of linear materials, the frequencies of the three waves are identical, can be proved as follows :

Frequency of wave does not change under reflection and refraction. Let us use the boundary condition for the waves polarized in the plane of incidence (*p*-polarization) : *the tangential components of the electric field vector remain continuous at the boundary of two media.*

$$E_{0t}e^{i\kappa\cdot\mathbf{r}}e^{-i\omega t} + E''_{ot}e^{i\kappa''\cdot\mathbf{r}}e^{-i\omega'' t} = E'_{ot}e^{i\kappa'\cdot\mathbf{r}}\,e^{-i\omega' t}. \qquad ...(116)$$

Differentiating both sides of the equation with respect to t, we have

$$-i\,\omega\,E_{ot}\,e^{i\kappa\cdot r}\,e^{-i\omega t} - i\omega''\,E''_{ot}\,e^{i\kappa''\cdot r}\,e^{-i\omega'' t} = -i\omega'\,E'_{ot}e^{i\kappa'\cdot r}\,e^{-i\omega' t}. \qquad ...(117)$$

Substituting the value of E''_{ot} from Eq. (116) in Eq. (117) and solving, we get

$$E_{ot}e^{i\kappa\cdot r}\,(\omega - \omega'')\,e^{-i\omega t} = E'_{ot}\,e^{i\kappa'\cdot r}\,(\omega' - \omega'')\,e^{-i\omega' t}. \qquad ...(118)$$

This equation is satisfied for all values of t, which is possible only if the exponential factors are equal, *i.e.*,

$$\omega = \omega'.$$

Similarly, by eliminating E'_{ot} from Eqs. (116) and (117) and with the same argument, we have

$$\omega = \omega''$$

$$\therefore \qquad \omega = \omega' = \omega''. \qquad ...(119)$$

We will first discuss normal incidence and then oblique incidence below.

A–Normal Incidence. Consider an electromagnetic wave of frequency ω propagating along the x–axis, normal to the plane interface of two linear dielectric materials of dielectric constants k_1 and k_2 and refractive indices n_1 and n_2 (Fig. 14.10). The amplitude $\mathbf{E}_0$ is polarized along the y-axis. The reflected wave travels in the negative x-direction, while the refracted or transmitted wave travels in the x-direction. We thus have for the

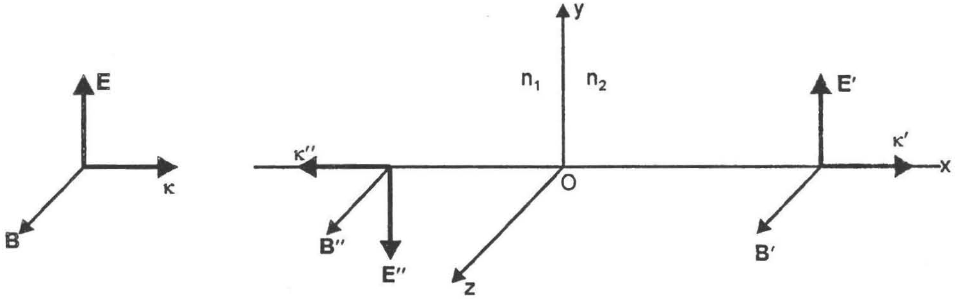


Fig. 14.10. Reflection and refraction at normal incidence.

Incident wave

$$E_y = E_0e^{i(\kappa x - \omega t)},\ E_x = E_z = 0.$$

and $$B_z = (\varepsilon_1\mu_1)^{1/2} E_y = (\varepsilon_1\mu_1)^{1/2} E_0 e^{i(\kappa x - \omega t)}. \quad ...(120)$$

Refracted wave

$$E_y' = E_0' e^{i(\kappa' x - \omega t)};\ E_x' = E_z' = 0$$

and $$B_z' = (\varepsilon_2\mu_2)^{1/2} E_y' = (\varepsilon_2\mu_2)^{1/2} E_0' e^{i(\kappa' x - \omega t)};\ B_x = B_y = 0. \quad ...(121)$$

Reflected wave : The reflected wave travels in the negative x-direction. Since **E**, **B** and κ' are mutually perpendicular vectors, hence for the reflected wave either **E** or **B** is reversed. For the p-polarization, **E** reverses sign and we have

$$E_y'' = -E_0'' e^{-i(\kappa'' x + \omega t)};\ E_x'' = E_z'' = 0 \quad ...(122)$$

and $$B_z'' = -(\varepsilon_1\mu_1)^{1/2} E_y'' = (\varepsilon_1\mu_1)^{1/2} E_0'' e^{-i(\kappa'' x + \omega t)},\ B_x'' = B_y'' = 0. \quad ...(123)$$

Here we have make use of the relation $\mathbf{B} = \pm (\varepsilon\mu)^{1/2} \kappa \times \mathbf{E}$, where positive sign is for the incident and transmitted waves and negative sign is for the reflected wave. The frequencies are assumed to be equal to satisfy the boundary conditions at $x = 0$ at all times.

Since the normal components of the fields vanish, only the boundary conditions for the tangential components are to be considered.

1. The continuity of tangential component of the electric field at $x = 0$ gives

$$E_0 - E_0'' = E_0'. \quad ...(124)$$

2. The continuity of the tangential components of **H** at $x = 0$ gives

$$H_0 + H_0'' = H_0'. \quad ...(125)$$

For non-magnetic media $\mu_1 = \mu_2 = \mu_0$, the above relation may be written as

$$B_0 + B_0'' = B_0' \quad \text{or} \quad (\varepsilon_1)^{1/2} E_0 + (\varepsilon_1)^{1/2} E_0'' = (\varepsilon_2)^{1/2} E_0'.$$

or $$n_1(E_0 + E_0'') = n_2 E_0'. \quad ...(126)$$

On solving Eqs. (124) and (126), we get

$$E_0'' = \frac{n_2 - n_1}{n_2 + n_1} E_0 \text{ and } E_0' = \frac{2n_1}{n_2 + n_1} E_0. \quad ...(127)$$

If we define $n_2/n_1 = n_{12}$ = refractive index of medium 2 with respect to medium 1, then we have

$$E_0'' = \frac{n_{12} - 1}{n_{12} + 1} E_0 \text{ and } E_0' = \frac{2}{n_{12} + 1} E_0. \quad ...(128)$$

This pair of equations is known as *Fresnel's equations* for plane polarized waves. If we define

$$E_0''/E_0 = r_{12} \text{ and } E_0'/E_0 = t_{12},$$

then we have

$$r_{12} = \frac{n_{12} - 1}{n_{12} + 1} \text{ and } t_{12} = \frac{2}{n_{12} + 1}, \quad ...(129)$$

where r_{12} is the *Fresnel's reflection coefficient* and t_{12} is the *Fresnel's transmission coefficient*. Since the polarization is parallel to the plane of incidence, we may write r_{12} and t_{12} as r_{12p} and t_{12p} respectively. It may the proved that $r_{12p} = -r_{12s}$ and $t_{12p} = t_{12s}$.

The intensity of the wave is described by the magnitude of the Poynting vector as

$$S = \frac{E_y B_z}{\mu} = \frac{E_y}{\mu} (\varepsilon\mu)^{1/2} E_y = \left(\frac{\varepsilon}{\mu}\right)^{1/2} E_y^2.$$

Since **E** and **B** vary harmonically, the average values

$$< E^2 > \ = \ \tfrac{1}{2} E_0^{\ 2} \text{ and} < B^2 > \tfrac{1}{2} B_0^{\ 2}$$

$$\therefore \qquad S \ = \ \tfrac{1}{2} (\varepsilon/\mu)^{1/2} \, E_0^{\ 2}. \qquad \ldots(130)$$

The mean intensities per period for the incident, refracted and reflected waves are

$$S_0 \ = \ \tfrac{1}{2} (\varepsilon_1/\mu_1)^{1/2} \, E_0^{\ 2} \qquad \ldots(131)$$

$$S_0' \ = \ \frac{1}{2}\left(\frac{\varepsilon_2}{\mu_2}\right)^{1/2} E_0'^{2} = \frac{1}{2}\left(\frac{\varepsilon_2}{\mu_2}\right)^{1/2} \frac{4}{(n_{12}+1)^2} E_0^{\ 2} \qquad \ldots(132)$$

and

$$S_0'' \ = \ \frac{1}{2}\left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2} E''^2_0 = \frac{1}{2}\left(\frac{\varepsilon_1}{\mu_1}\right)^{1/2} \left(\frac{n_{12}-1}{n_{12}+1}\right)^2 E_0^{\ 2}. \qquad \ldots(133)$$

These equations are also known as *Fresnel's formulae*.

Let us define S_0'/S_0 as *transmittance* T and S_0''/S_0 as *reflectance* R. Thus we have

$$R \ = \ \left(\frac{n_{12}-1}{n_{12}+1}\right)^2 = r_{12}^{\ 2} \text{ and } T = \frac{4n_{12}}{(n_{12}+1)^2} = t_{12}^{\ 2} \qquad \ldots(134)$$

This gives

$$R + T \ = \ 1.$$

Relations (131), (132) and (133) show that

$$S_0 \ = \ S_0' + S_0'', \qquad \ldots(136)$$

i.e. the *conservation of energy* is satisfied in the precess of reflection and refraction at the boundary separating dielectrics.

B–Oblique Incidence : Let us consider a more general case of oblique incidence, in which the incoming wave meets the boundary. The Fresnel's coefficients are found to be different in two possible cases of polarization (p- and s-polarizations).

***p*-polarization :** The incident, reflected and refracted waves are shown in Fig. 14.11. The frequencies of these wave are identical for the linear media. The electric and magnetic field vectors may be written as :

Incident wave

$$\mathbf{E}(\mathbf{r}, t) \ = \ \mathbf{E}_0 \, e^{i(\kappa \cdot \mathbf{r} - \omega t)} \qquad \ldots(137)$$

$$\mathbf{B}(\mathbf{r}, t) \ = \ (\varepsilon_1\mu_1)^{1/2} \, \hat{\kappa} \times \mathbf{E}_0 \, e^{i(\kappa \cdot \mathbf{r} - \omega t)} \qquad \ldots(138)$$

Refracted wave

$$\mathbf{E}'(\mathbf{r}, t) \ = \ \mathbf{E}_0' e^{i(\kappa' . \mathbf{r} - \omega t)} \qquad \ldots(139)$$

$$\mathbf{B}'(\mathbf{r}, t) \ = \ (\varepsilon_2\mu_2)^{1/2} \, \hat{\kappa}' \times \mathbf{E}_0' \, e^{i(\kappa' . \mathbf{r} - \omega t)} \qquad \ldots(140)$$

Reflected wave

$$\mathbf{E}''(\mathbf{r}, t) \ = \ \mathbf{E}_0'' \, e^{i(\kappa'' \cdot \mathbf{r} - \omega t)} \qquad \ldots(141)$$

$$\mathbf{B}''(\mathbf{r}, t) \ = \ (\varepsilon_1\mu_1)^{1/2} \, \hat{\kappa}'' \times \mathbf{E}_0'' \, e^{i(\kappa'' . \mathbf{r} - \omega t)}, \qquad \ldots(142)$$

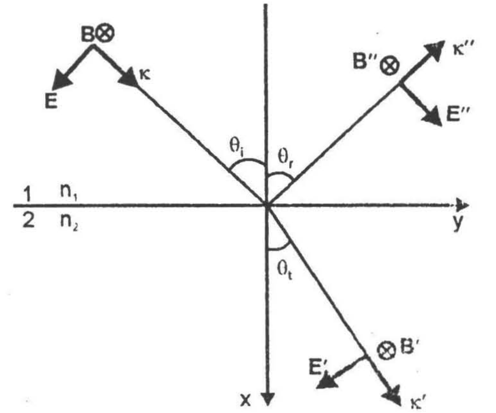


Fig. 14.11. Reflection and refraction at oblique incidence.

where $\hat{\kappa}, \hat{\kappa}'$ and $\hat{\kappa}''$ are the unit vectors along the propagation vectors $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}''$ respectively. The corresponding angles with respect to the normal to the interface are θ_i, θ_t and θ_r.

Let us apply boundary conditions : (1) The tangential component of electric field at the interface ($x = 0$) is continuous, *i.e.*,

$$E_0 \cos\theta_i \, e^{i(\boldsymbol{\kappa}\cdot\mathbf{r})} e^{-i\omega t} - E_0'' \cos\theta_r \, e^{i(\boldsymbol{\kappa}''\,\mathbf{r})} e^{-i\omega t} = E_0' \cos\theta_t \, e^{i(\boldsymbol{\kappa}'\cdot\mathbf{r})} e^{-i\omega t}. \quad ...(143)$$

where $\mathbf{r}$ is the radius vector in the plane of interface.

(2) The tangential component of **H**-field at the interface ($x = 0$) is continuous, *i.e.*, $H + H'' = H'$,

or $$(\varepsilon_1/\mu_1)^{1/2} E_0 \, e^{i(\boldsymbol{\kappa}\cdot\mathbf{r})} e^{-i\omega t} + (\varepsilon_1/\mu_1)^{1/2} E_0'' \, e^{(\boldsymbol{\kappa}''\cdot\mathbf{r})} e^{-i\omega t} = (\varepsilon_2/\mu_2)^{1/2} E_0' e^{i(\boldsymbol{\kappa}'\cdot\mathbf{r})} e^{-i\omega t}$$

Since the media are non-magnetic, therefore $\mu_1 = \mu_2 = \mu_0$. Thus above relation reduces to

$$n_1 E_0 e^{i(\boldsymbol{\kappa}\cdot\mathbf{r})} + n_1 E''_0 \, e^{i(\boldsymbol{\kappa}''\cdot\mathbf{r})} = n_2 E_0'' \, e^{i(\boldsymbol{\kappa}'\cdot\mathbf{r})}. \quad ...(144)$$

Because the boundary conditions must hold at all points on the interface and for all times, the *exponential functions must be equal.* This is called *phase matching.*

$$\text{Evidently } \boldsymbol{\kappa}\cdot\mathbf{r} = \boldsymbol{\kappa}'\cdot\mathbf{r} = \boldsymbol{\kappa}''\,.\,\mathbf{r} \text{ at } x = 0. \quad ...(145)$$

If a vector $\hat{n}$ is a unit vector normal to the interface and is along positive x–direction, hence the vector identity $\mathbf{r} = -\hat{n} \times (\hat{n} \times \mathbf{r})$ gives

$$\boldsymbol{\kappa}\cdot\mathbf{r} = -\boldsymbol{\kappa}\cdot[\hat{n} \times (\hat{n} \times \mathbf{r})] = -(\boldsymbol{\kappa} \times \hat{n})\cdot(\hat{n} \times \mathbf{r})$$

$$[\because \mathbf{A}\cdot(\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \times \mathbf{B})\cdot\mathbf{C}]$$

Hence Eqn. (137) becomes

$$(\boldsymbol{\kappa} \times \hat{n})\cdot(\hat{n} \times \mathbf{r}) = (\boldsymbol{\kappa}' \times \hat{n})\cdot(\hat{n} \times \mathbf{r}) = (\boldsymbol{\kappa}''\cdot\hat{n})\cdot(\hat{n} \times \mathbf{r})$$

or $$\boldsymbol{\kappa} \times \hat{n} = \boldsymbol{\kappa}' \times \hat{n} = \boldsymbol{\kappa}'' \times \hat{n}. \quad ...(146)$$

This shows that $\boldsymbol{\kappa}$, $\boldsymbol{\kappa}'$ and $\boldsymbol{\kappa}''$ are coplaner *i.e.* the incident, refracted and reflected waves lie in the same plane. The above relation also gives that

$$\kappa \sin\theta_i = \kappa' \sin\theta_t = \kappa'' \sin\theta_r. \quad ...(147)$$

The wave numbers κ, κ' and κ'' may be written in terms of velocities of the incident, refracted and reflected waves respectively, as

$$\kappa = \omega/v, \; \kappa' = \omega/v', \; \kappa'' = \omega/v''$$

Since the incident and reflected waves are propagated in the same medium, we may write

$$v = v'', \text{ therefore } \kappa = \kappa''.$$

This along with Eq. (147) gives

$$\sin\theta_i = \sin\theta_r \text{ or } \theta_i = \theta_r, \quad ...(148)$$

i.e., the angle of incident is equal to the angle of reflection.

Equation (147) may be written as

$$\sin\theta_i/\sin\theta_t = \kappa'/\kappa = v/v' = n_2/n_1, \quad ...(149)$$

i.e., the ratio of sine of the angle of incident to the sine of the angle of refraction is equal to the ratio of the refractive indices of the second to the first or the refractive index of the second medium with respect to the first n_{12}. *This is called the law of refraction or Snell's law.*

Relations (146), (148) and (149) are fundamental laws of geometrical optics.

Using the phase matching condition (145), the relations (143) and (144) reduce to

$$E_0 \cos \theta_i - E_0'' \cos \theta_r = E_0' \cos \theta_t \text{ and } n_1 E_0 + n_1 E_0'' = n_2 E_0' \qquad \text{...(150)}$$

Solving these relations and using relation $\theta_i = \theta_r$, we get

$$r_{12p} = \frac{E_0''}{E_0} = \frac{n_2 \cos\theta_i - n_1 \cos\theta_t}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{n_{12}\cos\theta_i - \cos\theta_t}{n_{12}\cos\theta_i + \cos\theta_t} \qquad \text{...(151)}$$

and

$$t_{12p} = \frac{E_0'}{E_0} = \frac{2n_1 \cos\theta_i}{n_2 \cos\theta_i + n_1 \cos\theta_t} = \frac{2\cos\theta_i}{n_{12}\cos\theta_i + \cos\theta_t}, \qquad \text{...(152)}$$

where r_{12p} and t_{12p} are called *Fresnel's reflection and transmission* coefficients respectively. Above relations are called *Fresnel's equations.*

We can eliminate the refractive indices from above expressions using Snell's law as :

$$r_{12p} = \frac{\tan(\theta_i - \theta_t)}{\tan(\theta_i + \theta_t)} \text{ and } t_{12p} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)}. \qquad \text{...(153)}$$

s-Polarization : In this case the electric field of the incident wave is normal to the plane of incidence and therefore **B**–field is in the plane of incidence as shown in Fig. 14.12. The incident, reflected and refracted waves have the same frequencies. In s–polarization, the magnetic field vector changes upon reflection instead of that of electric field vector in p–polarization. Thus the continuity of the tangential components of **E** and **H** gives

$$E_0 e^{i(\kappa\cdot\mathbf{r})} e^{-i\omega t} + E_0'' e^{i(\kappa''\cdot\mathbf{r})} e^{-i\omega t} = E'_0 e^{i(\kappa'\cdot\mathbf{r})} e^{-i\omega t}$$

and $(\varepsilon_1/\mu_1)^{1/2} E_0 \cos\theta_i e^{i(\kappa\cdot\mathbf{r})} e^{-i\omega t} - (\varepsilon_1/\mu_1)^{1/2} E''_0 \cos\theta_r e^{i(\kappa''\cdot\mathbf{r})} e^{-i\omega t}$

$$= (\varepsilon_2/\mu_2)^{1/2} E_0' \cos\theta_t e^{i(\kappa'\cdot\mathbf{r})} e^{-i\omega t}$$

The phase matching gives the same relation $\kappa \cdot \mathbf{r} = \kappa'' \cdot \mathbf{r} = \kappa' \cdot \mathbf{r}$. Hence the laws of reflection and refraction also hold good. The phase matching reduces the above equations to

$$E_0 + E_0'' = E_0' \qquad \text{...(154)}$$

and

$$(\varepsilon_1/\mu_1)^{1/2} [E_0 \cos\theta_i - E''_0 \cos\theta_r] = (\varepsilon_2/\mu_2)^{1/2} E_0' \cos\theta_t.$$

For non-magnetic media, $\mu_1 = \mu_2 = \mu_0$, therefore,

$$n_1(E_0 \cos\theta_i - E_0'' \cos\theta_r) = n_2 E_0' \cos\theta_t. \qquad \text{...(155)}$$

Solving Eqs. (146) and (147) and using the relation $= \theta_r$, we get

$$= \frac{E_0''}{E_0} = \frac{n_1 \cos\theta_i - n_2 \cos\theta_t}{n_1 \cos\theta_i + n_2 \cos\theta_t} = \frac{\cos\theta_i - n_{12}\cos\theta_t}{\cos\theta_i + n_{12}\cos\theta_t} \qquad \text{...(156)}$$

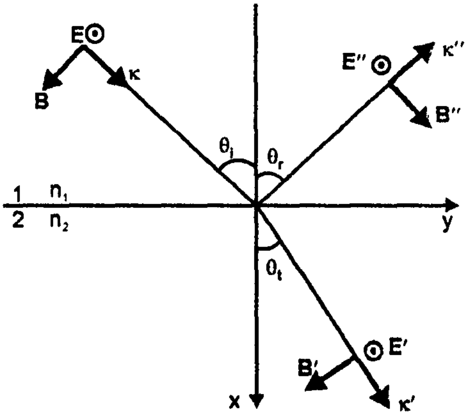


Fig. 14.12. Oblique incidence (*s-polarization*).

and $$t_{12s} = \frac{E_0'}{E_0} = \frac{2n_1\cos\theta_i}{n_1\cos\theta_i + n_2\cos\theta_t} = \frac{2\cos\theta_i}{\cos\theta_i + n_{12}\cos\theta_t}, \qquad ...(157)$$

where r_{12s} and t_{12s} are called the ***Fresnel reflection and transmission coefficients*** respectively. Above relations are called Fresnel's equations in the case of s-polarization.

Eliminating refractive indices using Snell's law, we get

$$r_{12s} = \frac{\sin(\theta_t - \theta_i)}{\sin(\theta_t + \theta_i)} \text{ and } t_{12s} = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_t + \theta_i)} \qquad ...(158)$$

We can prove easily that $R_s + T_s = 1$. ...(159)

Let us now consider some *limiting cases :*

1. *Normal incidence :* At normal incidence, $\theta_i = \theta_t = 0$, therefore

$$r_{12p} = -r_{12s} = \frac{n_{12} - 1}{n_{12} + 1} \quad \text{and} \quad t_{12p} = t_{12s} = \frac{2}{n_{12} + 1}. \qquad ...(160)$$

Since the reflectance R and transmittance T depend on r_{12}^2 and t_{12}^2 respectively, therefore are same for s and p both.

2. *Grazing angle incidence :* In the grazing angle incidence $\theta_i = \pi/2$, therefore we have

$$r_{12p} = r_{12s} = -1 \text{ and } t_{12p} = t_{12s} = 0. \qquad ...(161)$$

Hence $R_p = R_s = 1$ and $T_p = T_s = 0$. ...(162)

Since for air to glass ($n_2 = 1.5$), $R_p = R_s = 0.04$ at normal incidence, thus we see that the *reflectivity of a dielectric surface increases as the angle of incidence varies from normal to grazing.*

14.12 CRITICAL ANGLE

We have seen that $R_p = R_s = 1$ for the grazing angle incidence. From Eqs. (142) and (147) we see that perfect reflection also occurs for $\theta_t = \pi/2$ as well as for $\theta_i = \pi/2$. The incident angle for which $\theta_t = \pi/2$ is called the *critical angle*. From Snell's law, the critical angle θ_c is given by

$$\sin\theta_c/\sin\pi/2 = n_2/n_1 \text{ or } \sin\theta_c = n_2/n_1.$$

Since $\sin\theta_c$ can not be greater than 1, therefore n_2 must be smaller than n_1 for the real value of θ_c. Thus for the wave incident on the interface from the region of higher refractive index and when $\theta_t = \pi/2$, the reflectance is unity and the intensity of the reflected wave is the same as the intensity of incident wave. This phenomenon is called *total internal reflection.* From Snell's law

$$\frac{\sin\theta_i}{\sin\theta_t} = \frac{n_2}{n_1} = \sin\theta_c$$

or $$\sin\theta_t = \sin\theta_i/\sin\theta_c. \qquad ...(164)$$

Thus for the incident angles greater than critical angle (*i.e.*, $\theta_i > \theta_c$), $\sin\theta_t > 1$. This condition can not be satisfied if θ_t is a real angle. In this case reflectance for both p and s polarization is found to be unity.

14.13 BREWSTER ANGLE

The electromagnetic waves emitted by the ordinary sources are unpolarized, *i.e.*, the direction of polarization is symmetric around the direction of propagation. We can produce polarized light by reflection and refraction.

Relations (153) and (158) show that the reflectance is zero if (*i*) $\theta_i = \theta_t$, but it is possible when $n_1 = n_2$ or two media are optically indistinguishable.

(*ii*) $\theta_i + \theta_t = \pi/2$. In this case $r_{12p} = 0$ or the amplitude of the p–polarized reflected wave is zero, but $r_{12s} \neq 0$. Hence the reflected waves will be purely polarized normal to the plane of incidence.

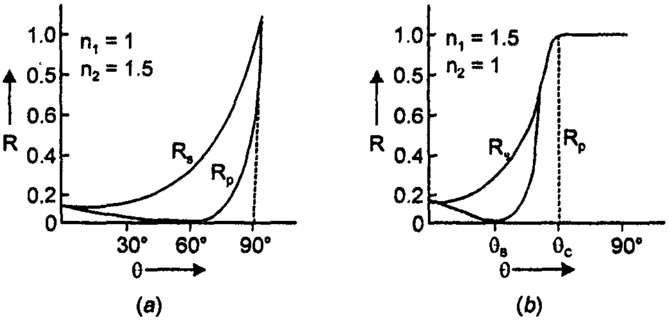
The angle of incident for which r_{12p} vanishes is called Brewster's angle, after the discoverer of this phenomenon. It is represented as θ_B or ϕ. Thus from Snell's law

$$n_1 \sin\theta_B = n_2 \sin\theta_t = n_2 \sin(\pi/2 - \theta_B) = n_2 \cos\theta_B$$

or
$$\tan\theta_B = n_2/n_1. \qquad \text{...(165)}$$

This relation is known as Brewster's law.

The values of reflectance for all values of angle of incidence at air glass interface with $n_1 = 1$ and $n_2 = 1.5$ are shown in Fig. 14.13(*a*). The Brewster's angle $\theta_B = 56°$ for this case. No total internal reflection occurs. The reflectance for p–polarization is generally lower than that for s–polarization. This property is used in polaroid sun glasses.



(*a*) (*b*)

Fig. 14.13. Reflectance of p–and–s polarization.

If $n_2 < n_1$, the total internal reflection occurs for angle of incidence equal to and greater than θ_c, the critical angle, thus $R_p = R_s = 1$ for $\theta_i \geq \theta_c$. For the wave from glass to air $\theta_B = 34°$ and $\theta_c = 42°$. The values for R for various angles of incidence is shown in Fig. 14.13(*b*).

The Brewster angle effect is widely used in the various applications, such as the construction of lasers, study of structure of atoms and properties of surfaces.

14.14 REFLECTION AND TRANSMISSION AT A CONDUCTING INTERFACE

In this section we consider the case where one of the media is conducting and other is dielectric. Let the boundary between a dielectric and conductor be in y–z plane. We will treat only the case of normal incidence as discussed earlier.

Normal Incidence : For the monochromatic plane wave traveling in the x–direction and polarized in the y–direction from a nonconducting linear medium of permittivity ε_1 and refractive index n_1 to the conducting medium of permittivity ε_2, conductivity σ_2 and complex refractive index n_c, we have

Incident wave

$$E_y = E_0 e^{(\kappa x - \omega t)}, E_x = E_z = 0;\; B_z = (\varepsilon_1 \mu_1)^{1/2} E_0 e^{i(\kappa x - \omega t)} \qquad \text{...(166)}$$

Refracted wave

$$E_y' = E_0' e^{i(\gamma x - \omega t)}, E_x' = E_z' = 0; B_z' = \frac{\gamma}{\omega} E_0' e^{i(\gamma x - \omega t)} \qquad \text{...(167)}$$

where the wave number γ is a complex quantity (Eq. 80). Its imaginary part results in an attenuation of the wave (Eq. 82).

Reflected wave

$$E_y'' = -E_0'' e^{-i(\kappa x - \omega t)}, E_x'' = E_z'' = 0; B_z'' = (\varepsilon_1 \mu_1)^{1/2} E_0'' e^{-i(\kappa x - \omega t)}. \qquad \text{...(168)}$$

The treatment of this problem is very similar to that of non-conducting media discussed in section 14.11. The boundary conditions satisfied by the fields are the same in both cases : the *tangential components of* **E** *and* **H** *fields are continuous at* $x = 0$, *i.e.,*

$$E_0 - E_0'' = E_0' \text{ and } H_0 + H_0'' = H_0'.$$

For non-magnetic media $\mu_1 = \mu_2 = \mu_0$ and the second relation may be written as

$$B_0 + B_0'' = B_0'.$$

On solving these equations, we get

$$E_0'' = \frac{n_c - n_1}{n_c + n_1} E_0 \text{ or } r_{12} = \frac{n_c - n_1}{n_c + n_1} \qquad ...(169)$$

and $$E_0' = \frac{2n_1}{n_c + n_1} E_0 \text{ or } t_{12} = \frac{2n_1}{n_c + n_1}. \qquad ...(170)$$

Since $n_c = n_0 (1 + i\, \sigma/\varepsilon\omega)^{1/2} = n + im$ is a complex refractive index, hence r_{12} and t_{12} will be the complex quantities, may be represented as $\mathbf{r}_{12}$ and $\mathbf{t}_{12}$ respectively. Substituting $n_c = n + im$ in the above relations, we get

$$\mathbf{r}_{12} = \frac{(n - n_1) + im}{(n + n_1) + im} \text{ and } \mathbf{t}_{12} = \frac{2n_1}{(n + n_1) + im}. \qquad ...(171)$$

Because of the complex nature of reflection and transmission coefficients or *Fresnels coefficients,* the reflected and transmitted waves are expected to have phase shifts relative to the incoming wave. Let us assume

$$\mathbf{r}_{12} = |\mathbf{r}_{12}| e^{i\phi_r} \text{ and } \mathbf{t}_{12} = |\mathbf{t}_{12}| e^{i\phi_t} \qquad ...(172)$$

We thus get

$$|\mathbf{r}_{12}| = \left[\frac{(n - n_1)^2 + m^2}{(n + n_1)^2 + m^2}\right]^{1/2}, \tan\phi_r = \frac{2n_1 m}{n^2 - n_1^2 + m^2} \qquad ...(173)$$

and $$|\mathbf{t}_{12}| = \frac{2n_1}{[(n + n_1)^2 + m^2]^{1/2}}, \tan\phi_t = -\frac{m}{n + n_1}. \qquad ...(174)$$

Thus we see that the reflected and transmitted waves are phase shifted relative to incident wave.

$$\therefore \quad \text{Reflectance } R = |\mathbf{r}_{12}\mathbf{r}_{12}^*| = |\mathbf{r}_{12}|^2 = \frac{(n - n_1)^2 + m^2}{(n + n_1)^2 + m^2}. \qquad ...(175)$$

and transmittance $T = 1 - R = \dfrac{4n\, n_1}{(n + n_1)^2 + m^2}.$ ...(176)

Since the transmitted wave is eventually absorbed in the conducting medium, hence T is referred as absorption A.

$$\therefore \quad T = A \text{ and } R = 1 - A. \qquad ...(177)$$

For a good conductor, such as metals $\sigma_2/\varepsilon_2\, \omega >> 1$. Thus Eq. (81) gives

$$\kappa = s = \frac{n_0 \omega}{\sqrt{2}\, c}\sqrt{\frac{\sigma_2}{\varepsilon_2 \omega}} \text{ or } n = m = n_0 \sqrt{\frac{\sigma_2}{2\varepsilon_2 \omega}}. \qquad ...(178)$$

For normal incidence from air on a conducting medium $n_1 = 1$. Thus for a good conducting materials for which $n = m >> 1$,

$$A \approx \frac{2}{m} = 2\sqrt{\frac{2\varepsilon_2\omega}{\sigma_2}},$$

$$\therefore \qquad R = 1 \sim \frac{2}{m} = 1 - 2\sqrt{\frac{2\varepsilon_2\omega}{\sigma_2}}. \qquad ...(179)$$

Thus we see that $A \to 0$ and $R \to \infty$ as the conductivity of the conducting material increases. The relation (171) is known as *Hagen–Rubens formula*. For the conductors, this relation is valid at frequencies as high as 10^{15} *Hz*. For the moderately good conductors, it is accurate at frequencies below the microwave frequencies.

14.15 OPTICAL DISPERSION IN MATERIALS

In the preceding sections, we have seen that the propagation of electromagnetic waves through a linear material depends on three properties of the material; the *permittivity* ε, *permeability* μ and *conductivity* σ. If ε and μ are constant quantities, then the speed of waves in a non-conductor $v = 1/\sqrt{\varepsilon\mu}$ and the refractive index $n = c/v$. It has been observed experimentally that n is a function of frequency ω. When the speed of wave varies with its frequency, the medium is called *dispersive* and the phenomenon is called *dispersion*.

Dispersion in Nonconductors : The electrons in a non conductor are attached with the molecules by binding forces. The actual value requires quantum mechanical treatment. One may assume the electron as attached to the end of spring of force constant K. If y is the displacement of electron from equilibrium, the binding force is given by

$$F_{\text{binding}} = -Ky = -m\omega_0^2 y, \qquad ...(180)$$

where ω_0 is the natural frequency of electron oscillation and m is the electron mass.

When the electron oscillates, some damping force will act which depends on the velocity dy/dt. Therefore

$$F_{\text{damping}} = -m\gamma\, dy/dt, \qquad ...(181)$$

where γ is known as *damping constant*.

The electron experiences a driving force due to the electromagnetic wave, which is given by

$$F_{\text{driving}} = eE = eE_0 e^{-i\omega t}, \qquad ...(182)$$

where ω is the angular velocity of *em* – wave and e is the charge of the electron. Newton's second law thus gives

$$\begin{aligned} md^2y/dt^2 &= F_{\text{total}} = F_{\text{binding}} + F_{\text{damping}} + F_{\text{driving}} \\ &= -m\omega_0^2 y - m\gamma\, dy/dt + eE_0 e^{-i\omega t} \end{aligned}$$

$$\text{or} \qquad \frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = (e/m)\,E_0 e^{-i\omega t}. \qquad ...(183)$$

This describes the electron as damped harmonic oscillator driven at frequency ω. The theory described below is thus known as *Drude-Lorentz Theory*.

In the steady state, the system oscillates at the driving frequency, thus the solution of Eq. (183) may be assumed as $y = Ae^{-i\omega t}$. ...(184)

Substituting the values of y, and its derivates in Eq. (183), we get

$$(-\omega^2 - i\omega\gamma + \omega_0^2)Ae^{-i\omega t} = (e/m)\,E_0 e^{-i\omega t}$$

or

$$A = \frac{e/m}{(\omega_0^2 - \omega^2) - i\,\omega\gamma}\,E_0. \qquad ...(185)$$

Thus the amplitude of the electron motion is proportional to the driving field. It is a function of driving frequencies and is large for $\omega = \omega_0$. In the absence of damping, resonance amplitude is infinite.

The dipole moment associated with the motion of the electron is given by

$$p = ey = \frac{e^2/m}{(\omega_0^2 - \omega^2) - i\,\omega\gamma}\,E_0 e^{-i\omega t}. \qquad ...(186)$$

The imaginary term in the denominator indicates that **p** is out of phase with **E**, lagging behind by an angle $\tan^{-1}[\omega\gamma/(\omega_0^2 - \omega^2)]$, which tends to zero when $\omega << \omega_0$ and rises to π when $\omega >> \omega_0$.

In actual practice, differently situated electrons with in a molecule exhibit different natural frequencies and damping coefficients. If a fraction f_j of electrons has frequency ω_j and damping constant γ_j per molecule and there are N molecules per unit volume, the polarization **P** is given by

$$\mathbf{P} = \frac{Ne^2}{m}\left[\sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2) - i\omega\gamma_j}\right]\mathbf{E}. \qquad ...(187)$$

Here summation is taken over all groups of electrons having same values of ω and γ. The fraction f_j is also called oscillator strength. For this $\Sigma f_j = 1$.

The complex polarization **P** is proportional to the complex field **E** and we have the complex electrical susceptibility as

$$\mathbf{P} = \varepsilon_0 \chi_E \mathbf{E},$$

$$\therefore \qquad \chi_E = \frac{Ne^2}{m\varepsilon_0}\left[\sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2) - i\omega\gamma_j}\right]. \qquad ...(188)$$

The complex permittivity $\varepsilon = \varepsilon_0(1 + \chi_E)$ and relative permittivity or dielectric constant

$k = 1 + \chi_E$. Thus we have

$$k = 1 + \frac{Ne^2}{m\varepsilon_0}\left[\sum_j \frac{f_j}{(\omega_{0j}^2 - \omega^2) - i\omega\gamma_j}\right] \qquad ...(189)$$

It shows that k is also a complex quantity and frequency dependent. Relation (189) is known as *dispersion relation* for the non conductors.

In the dispersive medium the wave equation for the given frequency is given by

$$\nabla^2\mathbf{E} = \varepsilon\mu_0 \partial^2\mathbf{E}/\partial t^2. \qquad ...(190)$$

The ε is a complex function of ω. Hence the wave number κ of the electromagnetic wave will also be a complex quantity.

Let $\kappa = \kappa_r + \kappa_i$, where κ_r and κ_i be the real and imaginary parts of κ respectively. Thus the solution of the Eq. (190) becomes

$$\mathbf{E}(x, t) = E_0 e^{-\kappa_i x} e^{i(\kappa_r x - \omega t)}. \qquad ...(191)$$

Evidently κ_i measures the attenuation of the wave. As the intensity is proportional to the square of amplitude. Thus the quantity

$$\alpha = 2\kappa_i \qquad \text{...(192)}$$

is called the ***absorption coefficient***. The wave velocity $v = \omega/\kappa_r$ and therefore the refractive index of the medium

$$n = (c/\omega)\,\kappa_r, \qquad \text{...(193)}$$

For gases, χ_E is very small, we can thus write

$$\kappa = \frac{\omega}{c}\sqrt{\frac{\varepsilon}{\varepsilon_0}} = \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j}{(\omega_{0j}{}^2 - \omega^2) - i\omega\gamma_j}\right]$$

$$= \frac{\omega}{c}\left[1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j\left\{(\omega_{0_j}^2 - \omega^2) - i\omega\gamma_j\right\}}{(\omega_{0_j}{}^2 - \omega^2)^2 - \omega^2\gamma_j{}^2}\right]$$

$$\therefore \qquad n = \frac{c}{\omega}\kappa_r = 1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j(\omega_{0j}{}^2 - \omega^2)}{(\omega_{0j}{}^2 - \omega^2)^2 + \omega^2\gamma_j{}^2} \qquad \text{...(194)}$$

and

$$\alpha = 2\kappa_i = \frac{Ne^2\omega^2}{m\varepsilon_0 c}\sum_j \frac{f_j\gamma_j}{(\omega_{0j}{}^2 - \omega^2)^2 + \omega^2\gamma_j{}^2}. \qquad \text{...(195)}$$

The refractive index n and the absorption coefficient α are plotted on the same graph (Fig. 14.14)). The n rises gradually with increasing frequency, as observed in optics. In the immediate neighbourhood of the resonance, the refractive index is found to drop sharply. Because of this typical behaviour, it is called *anomalous dispersion*. In the anomalous dispersion region ($\omega_1 < \omega < \omega_2$), the imaginary part and thus the absorption coefficient α has a peak. The material acts as practically opaque in this region. Thus we see that there are successive regions of transparency, absorption, high reflectance and transparency as the frequency increases from zero.

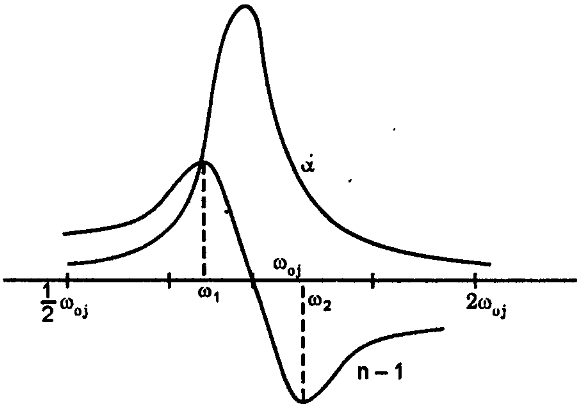


Fig. 14.14. Variation of α and $n - 1$ with the frequency ω.

Far from the resonances, *i.e.*, when $|\omega_{0j} - \omega| >> \gamma_j$, the damping can be ignored and Eq. (194) reduces to

$$n = 1 + \frac{Ne^2}{2m\varepsilon_0}\sum_j \frac{f_j}{\omega_{0j}{}^2 - \omega^2}. \qquad \text{...(196)}$$

For most substances the natural frequencies ω_{0j} are scattered all over the spectrum for transparent materials, the nearest resonances lie in the ultra violet region. Thus for $\omega < \omega_{0j}$

$$\frac{1}{\omega_{0j}{}^2 - \omega^2} = \frac{1}{\omega_{0j}{}^2}\left(1 - \frac{\omega^2}{\omega_{0j}{}^2}\right)^{-1} = \frac{1}{\omega_{0j}{}^2} + \frac{\omega^2}{\omega_{0j}{}^4}.$$

$$\therefore \qquad n = 1 + \left(\frac{Ne^2}{2m\varepsilon_0} \sum_j \frac{f_j}{\omega_{0j}^{\;2}} \right) + \omega^2 \left(\frac{Ne^2}{2m\varepsilon_0} \sum_j \frac{f_j}{\omega_{0j}^{\;4}} \right).$$

In terms of wave length in vacuum ($\lambda = 2\pi c/\omega$)

$$n = 1 + A\,(1 + B/\lambda^2). \qquad ...(197)$$

This is known as *Cauchy's relation*. Here A is called *coefficient of refraction* and B is called the *coefficient of dispersion*. This relation is useful for the refractive index of a transparent material. It applies reasonably well to most *gases* in the optical region.

Wood's experiment for Anomalous dispersion : R. W. Wood demonstrated experimentally the anomalous dispersion of sodium vapour. The experimental arrangement is shown in Fig. 14.15(*a*). T is a steel tube which is partially evacuated through an opening. The tube is attached with water cooled glass windows at its ends. Pieces of sodium are placed along the bottom of the tube. When the tube is heated at the bottom, the sodium gets vaporized. The sodium vapour diffuses upwards. Its density thus gradually decreases upwards. The vapour in a tube optically acts as equivalent to a prism whose thickness increases downwards.

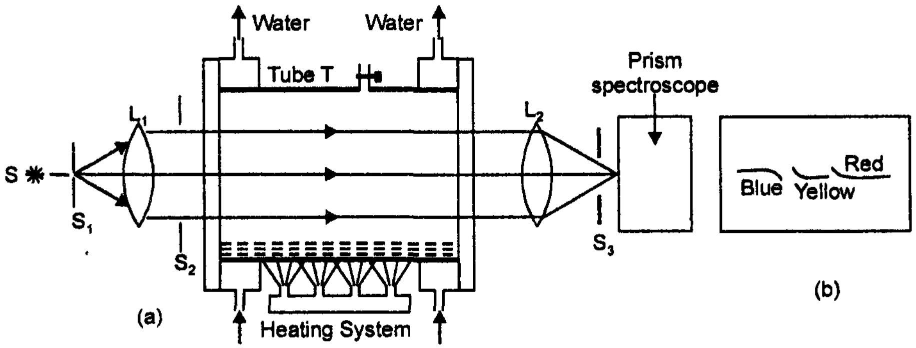


Fig. 14.15(*a*). Wood's experimental arrangement, (*b*) the anomalous dispersion of sodium vapour.

White light from source S is collimated by lens L_1 and slits S_1 and S_2. The slit S_2 selects a portion of the tube across which the vapour density gradient is uniform. The light is refracted by the sodium vapour prism and is then focussed to the prism spectroscope.

When the heating system is off, a sharp white image of S_1 is formed across the slit S_3, which produces a narrow horizontal continuous spectrum in the focal plane of spectroscope. When the heating system is one, the sodium is vaporised, the prismatic action of the vapour causes the dispersion in the images of S_1 along the length of the slit S_3. The refractive index of the vapour is nearly unity for wavelengths remote from D-lines and the horizontal continuous spectrum is uneffected. When the D-lines are approached from the violet side, the refractive index of the vapour becomes less than one and the light is deviated upwards. As the spectroscope inverts the image of the slit, we see downward shift of the spectrum due to decrease in refractive index. On the other hand, an upward shift of the spectrum on the long wavelength side of D-lines is due to increase in refractive index. The spectrum observed by Wood, as shown in Fig. 14.15(*b*), shows the anomalous dispersion in the neighbourhood of D_1 and D_2 lines of sodium due to strong selective absorptions at these lines.

Dispersion in Solids and Liquids : In the dilute media, such as gases, the mutual interaction between the particles is negligible and the driving field **E** is same as the electric field **E** of the wave. But in solid or liquid non-conducting materials, the molecular field E_m is found to be different as derived in chapter 6. It is given by the relation

$$\mathbf{E}_m = \mathbf{E} + L\mathbf{P}/\varepsilon_0 = \mathbf{E} + L\,(k-1)\,\mathbf{E}. \qquad \text{...(198)}$$

For nonpolar dielectric $L = 1/3$ and for metal $L = 0$

In this case Eq. (183) may be written as

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0{}^2 y = \left(\frac{e}{m}\right)E_m. \qquad \text{...(199)}$$

We assume that E_m and P like E depend on position and time sinusoidally or in complex form. Hence polarization **P** is given by the relation

$$\mathbf{P} = \frac{Ne^2}{m}\left[\sum_j \frac{f_j}{(\omega_{0j}{}^2 - \omega^2) - i\omega\gamma_j}\right]\mathbf{E}_m$$

$$\varepsilon_0\chi_E\,\mathbf{E} = \frac{Ne^2}{m}\left[\sum_j \frac{f_j}{(\omega_{0j}{}^2 - \omega^2) - i\omega\gamma_j}\right][1 + L(k-1)]\mathbf{E}$$

or

$$\frac{k-1}{1+L(k-1)} = \frac{Ne^2}{m\varepsilon_0}\left[\sum_j \frac{f_j}{(\omega_{0j}{}^2 - \omega^2) - i\omega\gamma_j}\right] \qquad \text{...(200)}$$

For non polar dielectric

$$\frac{k-1}{k+2} = \frac{Ne^2}{3m\varepsilon_0}\left[\sum_j \frac{f_j}{(\omega_{0j}{}^2 - \omega^2) - i\omega\gamma_j}\right]. \qquad \text{...(201)}$$

This is the generalization of Clausius-Mossotti relation. It can be solved for the real and imaginary parts of k.

Free Electrons in Conductors and Plasmas : The free electrons in conductors are not bound to any particular atom or molecule, therefore binding force is zero. Thus Eq. (183) reduces to

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} = \left(\frac{e}{m}\right)E_0\,e^{-i\omega t} \qquad \text{...(202)}$$

It has a solution

$$y = \frac{-(e/m)}{\omega^2 + i\omega\gamma}E_0\,e^{-i\omega t}. \qquad \text{...(203)}$$

In this case $L = 0$ and Eq. (200) reduces to

$$k - 1 = \frac{Ne^2}{m\varepsilon_0}\sum_j \frac{f_j}{-\omega^2 - i\omega\gamma_j} = \frac{-Ne^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega(\omega + i\gamma_j)} \qquad \text{...(204)}$$

The real and imaginary parts are given by

$$k_r - 1 = -\frac{Ne^2}{m\varepsilon_0}\sum_j \frac{f_j}{\omega^2 + \gamma_j^2} \qquad \text{...(205)}$$

and

$$k_i - 1 = \frac{Ne^2}{m\varepsilon_0}\sum_j \frac{f_j\gamma_j}{\omega(\omega^2 + \gamma_j^2)}. \qquad \text{...(206)}$$

The current that the electrons generate is given by

$$J = Ne\, n_f dy/dt,$$

where n_f is the number of free electrons per molecule. In a complex form the current density

$$\mathbf{J} = \frac{Nn_f e^2/m}{\gamma - i\omega}\mathbf{E}. \qquad ...(207)$$

It gives a complex conductivity $\sigma = Nn_f e^2/m\ (\gamma - i\omega)$. It is independent of ω at low frequencies.

For a **dilute plasma** (ionized gas) the damping is negligible and σ is purely imaginary

$$\sigma = i(Nn_f e^2/m\omega). \qquad ...(208)$$

Inserting this value of conductivity into the general formula for the wave number (Eq. 75), we get

$$\kappa^2 = \mu\varepsilon\omega^2 + i\,\mu\,\sigma\omega = \left(\omega^2 - \frac{Nn_f e^2}{m\varepsilon}\right)\mu\varepsilon.$$

Since $\mu \simeq \mu_0$ and $\varepsilon \simeq \varepsilon_0$ in a plasma, therefore

$$\kappa^2 = \frac{1}{c^2}(\omega^2 - \omega_p{}^2), \qquad ...(209)$$

where

$$\omega_p = e\sqrt{Nn_f/m\varepsilon_0}. \qquad ...(210)$$

is the *plasma frequency*. For $\omega > \omega_p$, the wave number κ is real and waves propagate without attenuation at a wave speed

$$v = \omega/\kappa = c\,[1 - (\omega_p/\omega)^2]^{-1/2} \qquad ...(211)$$

The refractive index $n = [1 - (\omega_p/\omega)^2]^{-1/2}$. ...(212)

For frequencies $\omega < \omega_p$, κ is purely imaginary and the waves are attenuated and given by

$$\mathbf{E}\,(x, t) = \mathbf{E}_0\, e^{-(1/c)(\omega_p{}^2 - \omega^2)^{1/2}\, x} e^{-i\omega t}. \qquad ...(213)$$

Thus the plasma is opaque to waves of frequencies less than ω_p and transparent to those above ω_p.

14.6 ELECTROMAGNETIC CAVITIES

In the oscillating mass spring system the *K. E.* is stored in the moving mass and *P. E.* is in the spring, *thus the two forms of energy appear in separate parts of the system.* Such a system is called the system with *lumped elements*. In other oscillating systems such as acoustic cavity resonator (organ pipe), the *K. E.* and *P.E.* are not separated but are present throughout the volume of the cavity. Such a cavity is called the oscillating system with *distributed elements*.

A simple *L* – *C* circuit is an example of lumped elements, as the two kinds of energy are stored at different placed (*L* and *C*) in this case. In modern practice, the *electromagnetic cavity* resonator replaces the lumped parameter *LC* resonant circuit. It is a *distributed electromagnetic oscillator*. A metallic enclosure acts as an electromagnetic resonator, when a system of oscillating electric and magnetic fields such as a magnetron is set up in it. The oscillations can be sustained within the enclosure with a very small expenditure of power, the power required to compensate the power loss in the cavity walls. The resonant frequencies are far higher than the acoustic frequencies in the cavities of the same size, because of this fact velocity of em-waves in free space is much greater than the velocity of sound in air. In the fundamental mode (or lowest frequency) the angular frequency of oscillations for the cavity of length l is given by

$$\omega = 2\pi\nu = 2\pi v/\lambda = 2\pi v/2l = \pi v/l. \qquad ...(214)$$

Electromagnetic cavities are used as the resonant circuit in high frequency tubes, (*e.g.*, klystrons), for bandpass filters, for wavemeters and for a number of other applications.

Consider a close cylinder of radius a and length l, whose walls are made of copper or some other good conductor. If it is connected to a source of electromagnetic radiation through a small hole in its side walls, the electric and magnetic lines of force are set up in the cavity. These vary with time. The electric lines are the horizontal lines originating on positive charges at one end of the cylinder and terminating on negative charges at the other end. At any point in the cavity the energy stored per unit volume in the electric field is $\frac{1}{2}\varepsilon_0 E^2$. The magnetic lines form circles about the cylinder axis. It is maximum when the electric field is zero. At any point in the cavity the energy stored in the magnetic field per unit volume is $B^2/2\mu_0$. Thus the energy is shuttled back and forth between the electric and magnetic fields as in the LC circuit. These fields no longer occupy completely separate regions of space hence it is the case of distributed elements.

It is easy to prove that the angular frequency of oscillation for the electromagnetic cavity in the fundamental mode is given by

$$\omega = 1.19\, c/a, \qquad (\textit{cavity of radius } a) \qquad ...(215)$$

The structure of the interior fields depends on the geometry or shape of the cavity.

14.17 TRANSMISSION LINES

Transmission line is a device to transmit high frequency electromagnetic energy from a given source (*generator*) to the load (*antenna*). It consists of more uniform and parallel conductors. The cross sections of several common forms of transmission lines are shown in Fig. 14.16. (*a*) shows the two wire line, (*b*) the shielded two wire line and (*c*) the coaxial line, (*d*) plane line, (*e*) wire above conducting plane, and (*f*) microstrip line. Coaxial cables are routinely used in electrical laboratories and in connecting TV sets to TV antennas. Microstrip lines are particularly important in integrated circuits where metallic strips connecting electronic elements are deposited on the dielectric substrates.

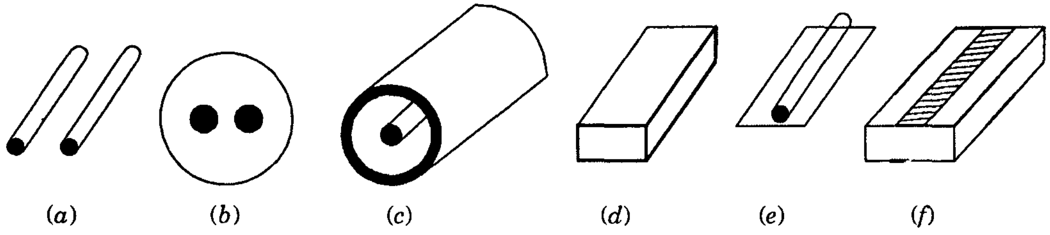

(*a*) (*b*) (*c*) (*d*) (*e*) (*f*)

Fig. 14.16. Various forms of transmission lines.

Consider a coaxial cable. If a potential difference is applied suddenly between the central and the outer conductors of the cable, it does not appear instantaneously all along the line but propagate, with a finite speed. For resistanceless line this speed is exactly equal to c, the velocity of light. At a distance l along the line, the potential difference rises suddenly at a time given by $t = l/c$ from zero to the emf applied. At any instant $t(= t_1)$, the signal has not yet reached points where $x > ct_1$. If a sinusoidal output of an electromagnetic oscillator is applied the sinusoidal wave will be propagated. If the oscillator frequency is 50 cycles/sec, the wavelength of the wave propagated will be

$$\lambda = 3 \times 10^8/50 = 6 \times 10^6 \text{ meters.}$$

It is very high. Frequencies in the radio or the microwave range are much higher and the wavelengths are smaller.

We deal with lumped elements if the wavelength is much larger than the dimensions of the system and with distributed elements otherwise. Therefore a trasmission line 60 meters long will be a lumped system for electromagnetic radiation at 50 cycles/sec ($\lambda = 6 \times 10^6 m$) but a distributed system at 10^8 cycles/sec (λ = 3m). In a lumped system the analysis is carried out in terms of parameters *L*, *C* and *R*. In a distributed system the analysis is carried out in terms of the fields.

The distributed parameters of a transmission line are :

(1) *Distributed resistance* : The wires of a transmission line are of some conducting material, hence have resistance. It is distributed uniformly along their entire lengths.

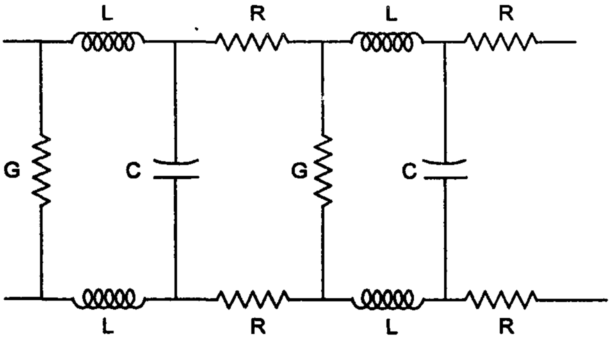


Fig. 14.17. Equivalent Circuit.

(2) *Distributed inductance* : The magnetic field developed about the wires of transmission line opposes the change in line current. Hence the inductance, which is distributed uniformly along the entire length of the wire, is present in series with line resistance.

(3) *Distributed conductance* : The dielectric between the wires is not perfect, a conduction current will flow between the wires. It is measured in terms of shunt conductance.

(4) *Distributed capacitance* : The two parallel conducting lines are separated by a dielectric, hence form a capacitor. It shunts the line.

Therefore a trasmission line is equivalent to a circuit with resistances and inductances in series and conductances and capacitances in parallel (Fig. 14.17).

In the coaxial cable, the current moves along the inner conductor in one direction and returns along the other conductor in opposite direction. The space between the inner and outer conductors is filled with a dielectric and an electric field acts between these conductors. The lines of force of the magnetic field are concentric circles with their centres on the axis of the cable. Since there is no loss of energy during transmission along it, hence div **E** = 0. The direction of motion of the electromagnetic energy remains the same if the current in the cable is reversed.

Transmission Line Equations : The em-wave propagates through a two conducting transmission line. From Fig. 14.17, we expect the equivalent circuit of a portion of the line to be as shown in Fig. 14.18. The model is in terms of the line parameters *R*, *L*, *G* and *C* and is called the *L*-type equivalent circuit. By applying Kirchhoff's voltage law, we get

$$V(z, t) = R\Delta z I(z,t) + L\Delta z \cdot \frac{\partial I(z,t)}{\partial t} + V(z+\Delta z, t)$$

or $$-\frac{V(z+\Delta z,t)-V(z,t)}{\Delta z} = RI(z,t)+L\frac{\partial I(z,t)}{\partial t}$$

Using the limit $\Delta z \to 0$, we get

$$\frac{-\partial V(z,t)}{\partial z} = RI(z,t)+L\frac{\partial I(z,t)}{\partial t} \qquad ..(216)$$

Applying Kirchhoff's current law, we get

$$I(z,\ t) = I(z+\Delta z,\ t) + \Delta I$$

$$= I(z+\Delta z,t)+G\Delta z\,V(z+\Delta z,t)+C\,\Delta z\frac{\partial V(z+\Delta z,t)}{\partial t}$$

or $$\frac{I(z+\Delta z,t)-I(z,t)}{\partial z} = GV(z+\Delta z,t)+C\frac{\partial V(z+\Delta z,t)}{\partial t}.$$

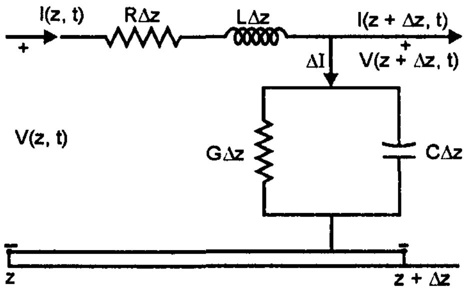


Fig. 14.18. *L*-type equivalent circuit.

As $\Delta z \to 0$, Thus the above equation becomes

$$\frac{\partial I(z,t)}{\partial z} = GV(z,t)+C\frac{\partial V(z,t)}{\partial t}. \qquad ...(217)$$

If we assume harmonic time dependence of source, such as $V(z,\ t) = R_e\left[V_s(z)e^{i\omega t}\right]$, $I(z,\ t) = R_e\left[I_s(z)e^{i\omega t}\right]$, where $V_s(z)$ and $I_s(z)$ are the phasor forms of $V(z,\ t)$ and $I(z,\ t)$ respectively. Thus equations (216) and (217) become

$$\frac{-dV_s}{dz} = (R+i\omega L)I_s\,,\ \frac{-dI_s}{dz}=(G+i\omega C)V_s. \qquad ...(218)$$

$$\therefore \qquad \frac{d^2V_s}{dz^2} = (R+i\omega L)\,(G+i\omega C)\,V_s.$$

or $$d^2V_s/dz^2-\gamma^2V_s = 0. \qquad ...(219)$$

where $$\gamma=\alpha\pm i\beta = \sqrt{(R+i\omega L)(G+i\omega C)}$$

Similarly, we get

$$d^2I_s/dz^2 - \gamma^2I_s = 0. \qquad ...(220)$$

Thus we have equations (219) & (220), the wave equations for voltage and current similar in form to the wave equations for plane waves. Thus γ is the propagation constant and α the attenuation constant and β the phase constant. The characteristic impedance Z_0 of the line is the ratio of positively travelling voltage wave to current wave at any point on the line.

$$Z_0 = \frac{V_0^+}{I_0^+}=\frac{R+i\omega L}{\gamma}=\sqrt{\frac{R+i\omega L}{G+i\omega C}}=R_0+iX_0.$$

Special cases : (*a*) *Lossless line,* when the conductors of the line are perfect ($\sigma_c = \simeq \infty$) and dielectric medium separating them is loss less ($\sigma \simeq 0$). Therefore $R = 0 = G$. For such a line

$$\alpha = 0,\ \gamma = i\beta = i\omega\sqrt{LC}$$

$$u = \omega/\beta=1/\sqrt{LC}=f\lambda$$

and $$Z_0 = R_0=\sqrt{L/C}.$$

(*b*) *Distorsionless line :* When the attenuation constant α is frequency independent and the phase constant β is linearly dependent on frequency. Therefore $R/L = G/C$. For a distorsionless line $\alpha = \sqrt{RG}, \beta = \omega\sqrt{LC}, Z_0 = R_0 = \sqrt{R/G} = \sqrt{L/C}$. and $u = \omega/\beta = 1/\sqrt{LC} = f\lambda$.

At high frequencies, the transmission line begins to radiate electromagnetic energy and acts as an aerial. At very high frequencies large losses of energy occur in the insulation of conductors. Therefore transmission lines are not suitable for transmitting energy at high frequencies (λ ~ centimeter).

14.18 WAVE GUIDES

At the high frequencies ($\lambda < 10$cm) the losses in a transmission line become quite large. It is convenient to use hollow conducting tube for the propagation of electromagnetic energy at such frequencies. The wave is confined to move in the direction of the tube. Such a wave is called a *quided wave* and the hollow conducting tube as *wave guide*. In the wave guide the wave is altered from a plane wave, as is required in order that the electric and magnetic boundary conditions at the surface of the conductor be satisfied. There are many different modes (patterns) of electromagnetic waves that can satisfy the boundary conditions and propagate energy down a guide. There are two most commonly used types : *rectangular and circular*. The geometry and size of a guide allows only one mode of transmission for a limited range of frequencies.

We know that *electric field must always be perpendicular to the surface of a conductor. As the magnetic field is always perpendicular to the electric field in e.m. waves, hence must be parallel to the surface of the conductor.* Therefore the lines of **B** form closed loops in planes parallel to the broad face of the rectangular guide. The electric field configuration is similar to that in a polarized plane wave. Lines of **B** and **E** for wave guide of rectangular cross section are shown in Fig. 14.18, (*a*) is showing a side view of the lines of **E** and (*b*) is showing the top view of the lines of **B**.

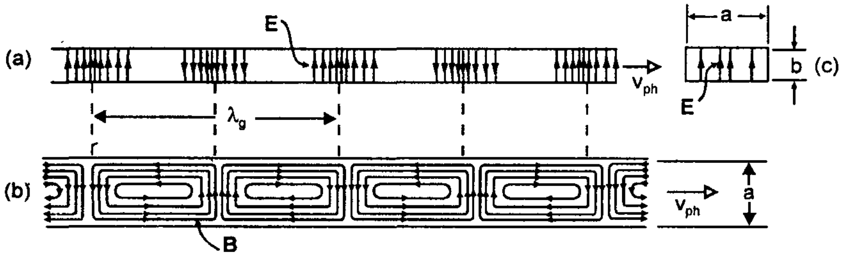


Fig. 14.18. Lines of *E* and *B* of rectangular wave guide.

Maxwell's equations, subject to the boundary conditions at the conductor surface, can determine the configuration of electromagnetic field inside the guide.

For a wave guide of perfect conductor, **E** = 0 and therefore **B** = 0 inside the material. Hence the boundary conditions at the inner wall are $\mathbf{E}_{\parallel} = 0$ and $\mathbf{B}_{\perp} = 0$. For the waves confined to the interior of a hollow cylindrical pipe with propagation along x-axis (axis of the pipe), **E** and **B** fields have the general form

$$\mathbf{E}\,(x, y, z, t) = \mathbf{E}_0\,(y, z)\, e^{i(\kappa x - \omega t)} \qquad \text{...(216)}$$

and
$$\mathbf{B}\,(x, y, z, t) = \mathbf{B}_0\,(y, z)\, e^{i\,(\kappa x - \omega t)} \qquad \text{...(217)}$$

These must satisfy Maxwell's equations in the interior of the wave guide :

$$\nabla \cdot \mathbf{E} = 0,\ \nabla \times \mathbf{E} = -\,\partial \mathbf{B}/\partial t,\ \nabla \cdot \mathbf{B} = 0 \text{ and } \nabla \times \mathbf{B} = \mu\varepsilon\, \partial \mathbf{E}/\partial t \text{....(218)}$$

μ and ε are the permeability and permittivity respectively of the medium filled inside the cylinder. For a hollow pipe $\mu = \mu_0$, $\varepsilon = \varepsilon_0$ and $\mu_0\varepsilon_0 = 1/c^2$.

Let us write $\mathbf{E}_0 = E_x\,\mathbf{i} + E_y\,\mathbf{j} + E_z\,\mathbf{k};\ \mathbf{B}_0 = B_x\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$,

where each of the components is a function of y and z. Putting this into Maxwell's equation (218) , we get

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = i\,\omega B_x;\ \frac{\partial E_x}{\partial z} - i\kappa E_z = i\omega B_y;\ i\kappa E_y - \frac{\partial E_x}{\partial y} = i\omega B_z.$$

$$\frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = -\frac{i\omega}{c^2}E_x;\ \frac{\partial B_x}{\partial z} - i\kappa B_z = -\frac{i\omega}{c^2}E_y;\ i\kappa B_y - \frac{\partial B_x}{\partial z} = -\frac{i\omega}{c^2}E_z. \qquad \text{...(219)}$$

On solving for E_y, E_z, B_y and B_z, we get

$$E_y = \frac{i}{(\omega/c)^2 - \kappa^2}\left(\kappa\frac{\partial E_x}{\partial y} + \omega\frac{\partial B_x}{\partial z}\right),$$

$$E_z = \frac{i}{(\omega/c)^2 - \kappa^2}\left(\kappa\frac{\partial E_x}{\partial z} - \omega\frac{\partial B_x}{\partial y}\right)$$

$$B_y = \frac{i}{(\omega/c)^2 - \kappa^2}\left(\kappa\frac{\partial B_x}{\partial y} - \mu\varepsilon\,\omega\frac{\partial E_x}{\partial z}\right),$$

$$B_z = \frac{i}{(\omega/c)^2 - \kappa^2}\left(\kappa\frac{\partial B_x}{\partial z} + \mu\varepsilon\,\omega\frac{\partial E_x}{\partial y}\right), \qquad \text{...(220)}$$

Inserting these values in the divergence equations (218), we get

$$\left[\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\omega^2}{c^2} - \kappa^2\right]\left[\begin{matrix} E_x \\ B_x \end{matrix}\right] = 0. \qquad \text{...(221)}$$

If $E_x = 0$, we call these *transverse electric* (*TE*) waves. If $B_x = 0$, they are called *transverse magnetic* (*TM*) waves. If both $E_x = 0$ and $B_x = 0$, we call the waves as *TEM waves*. It can be proved that *TEM*-waves cannot occur in a hollow wave guide.

TE – Waves in a Rectangular Wave Guide : Consider a wave guide of rectangular cross section, height a and width b. Since $E_x = 0$ for *TE*-mode, the solution of second part of Eq. (221) along with the boundary condition $B_\perp = 0$ may be written as

$$B_x\,(y, z) = Y(y)\,Z(z)$$

$$\therefore \qquad Z\frac{\partial^2 Y}{\partial y^2} + Y\frac{\partial^2 Z}{\partial z^2} + \left[\left(\frac{\omega}{c}\right)^2 - \kappa^2\right]YZ = 0. \qquad \text{...(222)}$$

Since y and z dependent terms must be constant, therefore

$$\frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} = -\kappa_y^2 \text{ and } \frac{1}{Z}\frac{\partial^2 Z}{\partial z^2} = -\kappa_z^2 \qquad \text{...(223)}$$

$$\therefore\ -\kappa_y^2 - \kappa_z^2 + (\omega/c)^2 - \kappa^2 = 0 \text{ or } \kappa^2 = (\omega/c)^2 - \kappa_y^2 - \kappa_z^2. \qquad \text{...(224)}$$

The general solution of Eq. $\partial^2 Y/\partial y^2 + \kappa_y^2 Y = 0$ is

$$Y = A_1 \sin(\kappa_y y) + B_1 \cos(\kappa_y y).$$

The boundary condition requires that B_y and hence from Eq. (220) $dY/\partial y$ vanishes at $y = 0$ and $y = a$. Thus $A_1 = 0$ and $\kappa_y = m\pi/a$, where m is an integer 0, 1, 2, ...

$$\therefore \qquad Y = B_1 \cos(m\pi y/a).$$

With the similar procedure for Z, we get

$$Z = B_2 \cos(n\pi z/b).$$

Thus for TE_{mn} mode, we have

$$B_x = B_0 \cos(m\pi y/a)\cos(n\pi z/b) \qquad \text{...(225)}$$

and the wave number κ is given by

$$\kappa = \sqrt{(\omega/c)^2 - \pi^2[(m/a)^2 + (n/b)^2]} \qquad \text{...(226)}$$

If

$$\omega < c\,\pi[(m/a)^2 + (n/b)^2]^{1/2} = \omega_{mn}, \qquad \text{...(227)}$$

the wave number κ will be imaginary and instead of the wave, exponentially attenuated fields are observed. Due to this reason ω_{mn} is called the *cut off frequency*. Its lowest value is for the mode T_{10}.

$$\omega_{10} = c\pi/a. \qquad \text{...(228)}$$

In a wave guide, we consider two kinds of velocity. The one is *phase velocity* v_{ph} *or wave velocity* v, the velocity with which the wave pattern along the guide travel. It is obtained by measuring the distance between peaks in the wave (λ_g). The other is the velocity with which energy or information carrying signals travel along the guide. This is *called group velocity* v_{gr}. It is given by $v_{gr} = d\omega/d\kappa$.

The phase velocity of propagation along the wave guide axis is c at the critical frequency, but at higher frequencies

$$v_{ph} = \frac{\omega}{\sqrt{\omega^2/c^2 - (m\pi/a)^2 - (n\pi/b)^2}} = \frac{c}{\sqrt{1 - (\omega_{mn}/\omega)^2}} \qquad \text{...(229)}$$

It is greater than c.

$$\lambda_g = \frac{v_{ph}}{f} = \frac{2\pi}{\sqrt{\omega^2/c^2 - (m\pi/a)^2 - (n\pi/b)^2}} \qquad \text{...(230)}$$

λ_g is related to λ in free space by the equation

$$\lambda_g = \lambda[1 - (\lambda/2a)^2]^{-1/2} \qquad \text{...(231)}$$

where a is the broad dimension of the guide.

The group velocity $\quad v_{gr} = d\omega/d\kappa = c\sqrt{1 - (\omega_{mn}/\omega)^2} \qquad \text{...(232)}$

Relation (231) shows that only waves with wave length smaller than twice the transverse dimension of the wave guide can propagate.

These relations are identically the same for *TM* mode as those for *TE mode*. Mode of transmission is represented by subscripts m and n, where m is always taken across wide dimension of the guide and n is taken along narrow dimension. In the TM mode, the field components are zero for $m = n = 0$, hence the lowest cut off frequency will occur for $m = n = 1$. This particular wave is called TM_{11} wave. In the TE-mode, the field components do not vanish if either m or n (but not both) is zero. Thus TE_{10} and TE_{01} waves are possible. Since $a > b$, hence the TE_{10} wave has a lower cut off frequency than the TE_{01} wave. The mode TE_{00} cannot occur in rectangular wave guide, as for this $\kappa = \omega/c$ and thus equations (220) are indeterminate.

At a first glance v_{ph}/c shows a direct contradiction to relativity theory. Actually the relativistic limitation on velocities ($v \leq c$) refers to mass or energy transformation, hence we must compare v_{gr} which is always smaller than or equal to c in general

$$v_{ph}\, v_{gr} = c^2. \qquad ...(233)$$

14.19 RADIO WAVE PROPAGATION

Radio waves mean that band of electromagnetic energy which covers the frequency range from a few kc/s to a few Mc/s. Propagation of radio waves from the transmitting to the receiving antenna may take place in one or more of the following ways :

(1) Ground wave or surface wave propagation : This is of practical importance at low frequency broadcasting and can exist when the transmitting and receiving antennas are close to the earth surface. The curvature of the earth gives an advantage to the surface wave for a long distance. Surface waves can bend round the curvature of the earth just as the electromagnetic waves in a wire can bend round its curvature.

Surface wave does not mean that most of the wave energy is confined to the earth's surface. Actually a greater part of energy is in the air above the surface. But in this particular form of propagation, the wave is guided along the earth's surface and around its bends and curvatures in a manner analogous to that in a guided wire wave.

(2) Space wave propagation : In this type of propagation the wave energy travels from transmitting to receiving antenna either straight through air (or space) or being reflected at ground. The space wave propagation is useful mainly for very high frequencies (> 30 Mc/s). It is limited by earth's curvature.

(3) Sky wave propagation : In this type of propagation the wave is reflected back to earth from ionized regions in the upper atmosphere (ionosphere). This region starts about 80 kilometers above the earth surface. The sky waves are of greater importance for very long distance radio communication. These waves are of medium and high frequencies (short waves) and are in the frequency range of 2 to 30 Mc/sec.

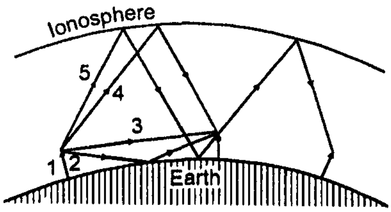


Fig. 14.19. Surface waves (1), Space waves (2, 3), Sky waves (4, 5).

Since the propagation of space and surface waves are limited by the curvature of the earth, hence are useless for large distances communication. Sky wave propagation is the only way for propagation over long distances (> 1000 km). By multiple reflections of the waves with the ionosphere and earth surface, it is possible for a wave of appropriate frequency to travel to any distance round the earth. The waves are also reflected from different parts of the ionized regions and the received signal is the vector sum of all components. This ionized region is produced in the upper atmosphere chiefly by ultra violet radiation from the sun. At such a height the pressure of the gas is very low and the ultra violet radiation from the sun is very intense to produce ionization. The gases at different altitudes differ in their tendancy to become ionized, thus different layers of ions of different densities are formed at different heights.

EXERCISES

Example 1. *A 40 pF air gap capacitor is charged by a 80 V battery through a 2Ω resistor. The plates of the capacitor are circular each of area 100 cm². At the instant the battery is connected, calculate (1) the displacement current between the plates, (b) the rate of change of electric field between the plates and (c) the magnetic field induced between the plates if E is uniform between the plates and zero beyond the edges of the plates.*

(*a*) We know that in *RC*-circuit the charge on the capacitor at an instant is given by $Q = C\mathcal{E}(1 - e^{-t/RC})$.

At the instant the battery is connected, *i.e.*, $t = 0$

$$\left(\frac{dQ}{dt}\right)_{t=0} = \left[\frac{C\mathcal{E}}{RC}e^{-t/RC}\right]_{t=0} = \frac{\mathcal{E}}{R} = \frac{80}{2} = 40\,A.$$

This is the rate at which charge accumulates on the plates of the capacitor. It is equal to the conduction current. As the definition of displacement current, the displacement current is also equal to 40 *A*.

(*b*) We know that $I_D = \varepsilon_0 A\, dE/dt$, therefore

$$\frac{dE}{dt} = \frac{I_D}{\varepsilon_0 A} = \frac{40}{8.85\times 10^{-12}\times 10^{-2}} = 4.52\times 10^{14}\,V/m.$$

(*c*) The conduction current is zero between the plates, therefore from Ampere's law we get

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\varepsilon_0 d\Phi_E/dt.$$

We consider a circle of radius *r*, centered at the centre of the plate. For $r \le R$,

$$B\cdot 2\pi r = \mu_0\varepsilon_0\frac{d}{dt}(\pi r^2 E) = \mu_0\varepsilon_0\pi r^2\frac{dE}{dt}$$

or

$$B = \frac{1}{2}\mu_0\varepsilon_0 r\frac{dE}{dt}. \qquad (r \le R)$$

Given that $\mathbf{E} = 0$ for $r > R$, therefore for the circular path beyond the edges of the plates

$$B\cdot 2\pi r = \mu_0\varepsilon_0\frac{d}{dt}(\pi R^2 E) = \mu_0\varepsilon_0\pi R^2\frac{dE}{dt}$$

or

$$B = \frac{\mu_0\varepsilon_0 R^2}{2r}\frac{dE}{dt}. \qquad (r > R)$$

Thus we see that *B* is maximum at the edges $r = R$.

Since $R = (A/\pi)^{1/2} = 5.6$ cm, therefore

$$B_{\max} = \frac{\mu_0\varepsilon_0 R}{2}\cdot\frac{dE}{dt} = \tfrac{1}{2}\times(4\pi\times 10^{-7})(8.85\times 10^{-12})(5.6\times 10^{-2})(4.52\times 10^{14})$$

$$= 1.4\times 10^{-4}\,T.$$

Example 2. *Prove why the extra term in Ampere's law had not been discovered previously by using an example of a current $i = i_0 \sin 2\pi ft$ flowing through a long straight wire that has a resistance of 100Ω per meter of length.*

The potential across the unit length of wire due to a current i is $100\, i = 100\, i_0 \sin 2\pi ft$ volts/m. along the wire. As a result the electric field will exist parallel to the wire $E = 100\, i_0 \sin 2\pi ft$. volt/m.

Consider a circle of radius r around the current carrying wire. Ampere's law gives

$$\oint \mathbf{B}\cdot d\mathbf{l} = \mu_0\left[i+\varepsilon_0\frac{d\Phi_E}{dt}\right]$$

or

$$B\,2\pi r = \mu_0\left[i_0\sin 2\pi ft+\varepsilon_0\frac{d}{dt}(100\pi r^2 i_0 \sin 2\pi ft)\right]$$

$$= \mu_0\left[i_0\sin 2\pi ft+200\pi^2\varepsilon_0 r^2 f\, i_0\cos 2\pi ft\right]$$

For $r = 1$ cm and $f = 50Hz$, we get

$$B = (\mu_0/2\pi)\, 10^2[i_0 \sin 2\pi ft + 8.85 \times 10^{-11}\, i_0 \cos 2\pi ft].$$

This shows that the second term is negligible in comparison with the first. It was therefore not detected experimentally. The second term becomes sizable and quite important at higher frequencies. (~*MHz*).

Example 3. *A parallel plate capacitor ($A = 6cm^2$ and $d = 2mm$) has a voltage $50 \sin 10^3 t$ V applied to its plates. Calculate the displacement current given $k = 2$.*

Since $D = \varepsilon E = \varepsilon V/d$, therefore $J_d = \partial D/\partial t = (\varepsilon/d)\, dV/dt$.

$$\therefore \quad I_d = J_d\cdot A = \frac{\varepsilon A}{d}\frac{dV}{dt} = \frac{k\varepsilon_0 A}{d}\cdot\frac{dV}{dt}$$

$$= \frac{2\times 8.85\times 10^{-12}\times 6\times 10^{-4}}{2\times 10^{-3}}\times 50\times 10^3\cos 10^3 t$$

$$= 2.655 \times 10^{-7} \cos 10^3 t \text{ amp.}$$

Example 4. *The conductors of a coaxial cable with solid inner and thin outer cylinders of radii a and b respectively are connected to a battery and resistor. In the region between a and b, find (a) the electric field, (b) the magnetic field, (c) the poynting vector and (d) the power flow in terms of V, R, a and b.*

We can prove as in the earlier chapters that the electric field in the region between a and b is given by

$$\mathbf{E} = [V/r \log_e (b/a)]\, \mathbf{a}_r$$

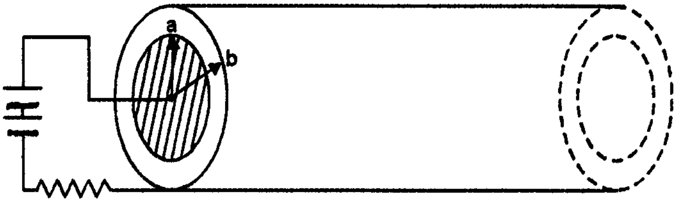


Fig. 14.20. Example 4.

Similarly the magnetic field $\mathbf{B} = (\mu_0\, i/2\pi r)\,\mathbf{a}_\theta$

As the current $i = V/R$, therefore $\mathbf{B} = (\mu_0 V/2\pi rR)\,\mathbf{a}_\theta$

$$\therefore \text{ Poynting Vector} \quad \mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{V}{r\log_e(b/a)}\mathbf{a}_r \times \frac{V}{2\pi rR}\mathbf{a}_\theta$$

$$= [V^2/2\pi r^2 R \log_e b/a]\,\mathbf{a}_z$$

$$\text{Therefore the power flow } dU/dt = \int \mathbf{S}\cdot d\mathbf{A} = \int_a^b \frac{V^2}{2\pi R\log_e\frac{b}{a}}\frac{1}{r^2}\cdot 2\pi r dr = \frac{V^2}{R}.$$

Example 5. *Electric field vector* $\mathbf{E}$ *of an em-wave travelling in vacuum is given by*

$$E(x, t) = 5j\cos(10^7 t + \kappa x)\ V/m.$$

Find (a) the direction of propagation, (b) state of polarization, (c) wave number and (d) the energy associated with this wave.

The electric vector of *em*-wave in vacuum is expressed as

$$\mathbf{E}(r, t) = \mathbf{E}_0 \exp[\pm i(\kappa\mathbf{r} - \omega t)]$$

where $\kappa\cdot\mathbf{r}$ defines the direction of propagation, $\mathbf{E}_0$ defines the polarization direction. On comparing with the given equation

$$\mathbf{E}(x, t) = 5\mathbf{j}\cos(10^7 t + \kappa x)\ V/m, \text{ we get}$$

(*a*) Wave travels in -*x*-direction,

(*b*) It is polarized in *y*-direction,

(*c*) $\omega = 10^7$, therefore wave number.

$$\kappa = \omega/c = 10^7/3\times10^8 = 1/30\ \text{m}^{-1}$$

(*d*) Energy associated with the wave

$$U = \varepsilon_0|E_0|^2 = \varepsilon_0 5^2 = 25\,\varepsilon_0 \text{ joule.}$$

Example 6. *The average energy per second reaching the earth from the sun is about 1350 J/sec.m^2. Assume this is a single em-wave and calculate the maximum values of E and B fields for sunlight at the earth.*

The average energy per unit area per sec $S_{av} = \varepsilon_0 c E_0^2/2$

$$\therefore \quad E_0 = \sqrt{\frac{2S_{av}}{\varepsilon_0 c}} = \sqrt{\frac{2\times1350}{8.85\times10^{-12}\times3\times10^8}}$$

$$= 1.01\times10^3\ V/m.$$

The maximum value of **B**– field

$$B_0 = \frac{E_0}{c} = \frac{1.01\times10^3}{3\times10^8} = 3.4\times10^{-6}\ T.$$

Example 7. *Will a small piece of ice weighing 1 μg be blown out of the solar system ? Given that radiant power emitted by the sun = 3.9×10^{26} W, $M_{sun} = 2\times10^{30}$ kg, $G = 6.67\times10^{-11}$ Nm^2/kg^2 and $\rho_{ice} = 0.9$ gm/cm^3.*

Assume for simplicity that the ice sphere moves only in a radial direction with respect to the centre of the sun and it is at a distance *r* from the sun.

The gravitational force on the ice sphere of mass $m = G\,m\,M_s/r^2$. It is toward the sun (*inward*).

If P_0 is the radiant power emitted by the sun, the intensity of radiation at a distance r from the sun

$$<S> = P_0/4\pi r^2.$$

Assuming that the ice piece is a sphere of radius a and that it absorbs all radiant energy incident upon it. The energy it absorbs per unit time

$$dU/dt = <S>\pi a^2 = P_0\,a^2/4r^2.$$

The force exerted by the radiation that the ice piece absorbs

$$F_{rad} = \frac{1}{c}\frac{dU}{dt} = \frac{P_0a^2}{4cr^2}(outward)$$

Net radial force acting on the ice piece

$$F = -G\,M_s\,m/r^2 + P_0\,a^2/4cr^2\ (outward)$$

Ice piece will be blown away from the sun, it $F > 0$ or $P_0\,a^2/4c > GM_sm$

or $$a^2/m > 4c\,GM_s/P_0.$$

For a ice sphere $$m = \frac{4}{3}\pi a^3\rho \text{ or } a = \left(\frac{3m}{4\pi\rho}\right)^{1/3}$$

$$\therefore \quad \frac{a^2}{m} = \left(\frac{3m}{4\pi\rho}\right)^{2/3}\frac{1}{m} = \left(\frac{9}{16\pi^2\rho^2 m}\right)^{1/3} = \left(\frac{9}{16\times10\times0.81\times10^6\times10^{-9}}\right)^{1/3}$$

$$= 4.1m^2/\text{kg}.$$

$$\frac{4cGM_s}{P_0} = \frac{4\times3\times10^8\times6.67\times10^{-11}\times2\times10^{30}}{3.9\times10^{26}} = 410m^2/\text{kg}.$$

Since $a^2/m < 4cGM_s/P_0$, hence ice piece will not blow away from the sun.

Example 8. *In a medium characterized by $\sigma = 0$, $\rho = 0$, $\mu_r = 1$ and $k = 4$, $\mathbf{E} = 25\sin(10^8t - \beta z)\,\mathbf{a}_y$ V/m. Calculate β and $\mathbf{H}$.*

In this case Maxwell's equations become

$$\nabla\cdot\mathbf{D} = \varepsilon\nabla\cdot\mathbf{E} = 0 \text{ or } \partial E_y/\partial y = 0$$

$$\nabla\cdot\mathbf{B} = \mu\nabla\cdot\mathbf{H} = 0$$

$$\nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t = -\mu\partial\mathbf{H}/\partial t, \text{ or } \mathbf{H} = -\frac{1}{\mu}\int(\nabla\times\mathbf{E})dt.$$

$$\nabla\times\mathbf{H} = \sigma\mathbf{E} + \varepsilon\frac{\partial\mathbf{E}}{\partial t} = \varepsilon\frac{\partial\mathbf{E}}{\partial t}, \text{ or } \mathbf{E} = \frac{1}{\varepsilon}\int(\nabla\times\mathbf{H})dt.$$

Since $\mathbf{E}$ is along y-axis, therefore

$$\nabla\times\mathbf{E} = \begin{vmatrix}\mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ 0 & E_y & 0\end{vmatrix} = \frac{\partial E_y}{\partial z}\mathbf{a}_x + \frac{\partial E_y}{\partial x}\mathbf{a}_z$$

$$= 25\,\beta\cos(10^8t - \beta z)\,\mathbf{a}_x$$

and $$\mathbf{H} = -\frac{25\beta}{\mu}\int \cos(10^8 t - \beta z)dt\,\mathbf{a}_x = \frac{-25\beta}{\mu \times 10^8}\sin(10^8 t - \beta z)\,\mathbf{a}_x \qquad ...(i)$$

Since **H** is along x-direction and independent of x, therefore

$$\nabla\cdot\mathbf{H} = \partial H_x/\partial x = 0 \text{ as required by Maxwell's second eqn.}$$

$$\nabla\times\mathbf{H} = -\frac{\partial H_x}{\partial z}\mathbf{a}_y - \frac{\partial H_x}{\partial y}\mathbf{a}_z = \frac{25\beta^2}{\mu \times 10^8}\cos(10^8 t - \beta z)\mathbf{a}_y$$

$$\therefore \qquad \mathbf{E} = \frac{25\beta^2}{\varepsilon\mu \times 10^8}\int \cos(10^8 t - \beta z)dt\,\mathbf{a}_y = \frac{25\beta^2}{\varepsilon\mu \times 10^{16}}\sin(10^8 t - \beta z)\,\mathbf{a}_y \qquad ...(ii)$$

On comparison with the given value of **E**, we get

$$\beta^2/\mu\varepsilon \times 10^{16} = 1 \text{ or } \beta = 10^8(\mu\varepsilon)^{1/2} = 2 \times 10^8/c = 2/3.$$

Therefore $$\mathbf{H} = \pm\frac{25 \times (2/3)}{4\pi \times 10^{-7} \times 10^8}\sin(10^8 t - \beta z)\mathbf{a}_x$$

$$= \pm\frac{5}{12\pi}\sin(10^8 t \pm \frac{2}{3}z)\mathbf{a}_x \text{ amp}/m.$$

Example 9. *A plane monochromatic plane polarized em-wave is travelling eastward. The wave is polarized with* **E** *directed vertically up and down alternately. Calculate* **E, B** *and* **S** *provided that the amplitude of the electric field strength is 0.05 V/m and the frequency is 6 MHz.*

Let the em-wave be propagated along the x-axis with its direction of polarization along y-axis. Thus $E_x = E_z = 0$. The magnetic field vector **B** is ⊥ to the direction of propagation of wave and that of **E**. Therefore **B** will be confined along the z-direction. Thus $B_x = B_y = 0$.

The frequency of wave $f = 6 \times 10^6\ Hz$. Therefore we have

$$\lambda = c/f = 3 \times 10^8/6 \times 10^6 = 50 \text{ m}.$$

Thus the electric field is given by

$$E_y = E_0 \sin\frac{2\pi}{\lambda}(ct - x) = 0.05 \sin[2\pi \times 6 \times 10^6 t - 2\pi x/50]$$

$$= 0.05 \sin(3.77 \times 10^6 t - 0.126\,x)\ V/m$$

or $$\mathbf{E} = 0.05 \sin(3.77 \times 10^6 t - 0.126x)\boldsymbol{j} \text{ volt/m}.$$

Since **B** is ⊥ to *E* and the direction of propagation of wave, therefore

$$\mathbf{B}_z = E_y/c = 1.67 \times 10^{-10}\sin(3.77 \times 10^6 t - 0.126\,x)$$

or $$\mathbf{B} = 1.67 \times 10^{-10}\sin(3.77 \times 10^6 t - 0.126\,x)\,\mathbf{k}\ Wb/m^2.$$

The Poynting vector **S** is given by

$$\mathbf{S} = \frac{\mathbf{E}\times\mathbf{B}}{\mu_0} = \frac{E_y\mathbf{j}\times B_z\mathbf{k}}{\mu_0} = \frac{E_yB_z}{\mu_0}\boldsymbol{i}.$$

Thus **S** will have only x-component. Its magnitude is given by

$$S = \frac{0.05 \times 1.67 \times 10^{-10}}{4\pi \times 10^{-7}}\sin^2(3.77 \times 10^6 t - 0.126\,x)$$

$$= 6.65 \times 10^{-6}\sin^2(3.77 \times 10^6 t - 0.126x)\ W/m^2$$

$$S_{av} = \langle S \rangle = 3.325 \times 10^{-6}\ W/m^2.$$

Example 10. *Electromagnetic waves in free space are given by*

$$\boldsymbol{E}(x, y, z, t) = \boldsymbol{E}_0(x, y)e^{i(\kappa z - \omega t)};\ \boldsymbol{B}(x, y, z, t) = \boldsymbol{B}_0(x, y)e^{i(\kappa z - \omega t)}.$$

where $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ are in the x, y plane. Find the relation between κ and ω, as well as the relation between $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$. Show that $\boldsymbol{E}_0$ and $\boldsymbol{B}_0$ satisfy the equations for electrostatics and magnetostatics in free space.

$$\nabla \times \mathbf{E} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_x & E_y & E_z \end{vmatrix} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ E_{ox} & E_{oy} & 0 \end{vmatrix} e^{i(\kappa z - \omega t)}$$

$$= [-i\kappa E_{oy}\,\mathbf{a}_x + i\kappa E_{ox}\,\mathbf{a}_y + (\partial E_{oy}/\partial x - \partial E_{ox}/\partial y)\,\mathbf{a}_z]\, e^{i(\kappa z - \omega t)}$$

$$= [i\kappa\,\mathbf{a}_z \times \mathbf{E}_0 + \nabla \times \mathbf{E}_0]\, e^{i(\kappa z - \omega t)}$$

Similarly
$$\nabla \times \mathbf{B} = [i\kappa\,\mathbf{a}_z \times \mathbf{B}_0 + \nabla \times \mathbf{B}_0]\, e^{i(\kappa z - \omega t)}$$

In free space, $\sigma = 0$, and Maxwell's equations are

$$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \text{ and } \nabla \times \mathbf{B} = \mu_0 \varepsilon_0\, \partial \mathbf{E}/\partial t.$$

$$\therefore \quad i\kappa\,\mathbf{a}_z \times \mathbf{E}_0 + \nabla \times \mathbf{E}_0 = i\omega \mathbf{B}_0(x, y) \text{ or } i\kappa\,\mathbf{a}_z \times \mathbf{E}_0 = i\omega \mathbf{B}_0(x, y) - \nabla \times \mathbf{E}_0$$

and
$$i\kappa\,\mathbf{a}_z \times \mathbf{B}_0 = -(i\omega/c^2)\,\mathbf{E}_0 - \nabla \times \mathbf{B}_0.$$

As $\nabla \times \mathbf{E}_0$ and $\nabla \times \mathbf{B}_0$ have only z-components while $\mathbf{a}_z \times \mathbf{E}_0$ and $\mathbf{a}_z \times \mathbf{B}_0$ are in the xy plane, we require

$$\nabla \times \mathbf{E}_0 = 0 \text{ and } \nabla \times \mathbf{B}_0 = 0$$

$$\therefore \quad \mathbf{a}_z \times \mathbf{E}_0(x, y) = (\omega/\kappa)\,\mathbf{B}_0(x, y) \qquad \ldots(i)$$

and
$$\mathbf{a}_z \times \mathbf{B}_0(x, y) = -(\omega/\kappa c^2)\,\mathbf{E}_0(x, y) \qquad \ldots(ii)$$

Taking the vector product of $\mathbf{a}_z$ and eqn. (1), we get

$$\mathbf{a}_z \times (\mathbf{a}_z \times \mathbf{E}_0) = -\mathbf{E}_0 = (\omega/\kappa)\,\mathbf{a}_z \times \mathbf{B}_0.$$

Its substitution in eqn. (*ii*), gives

$$\omega^2/\kappa^2 c^2 = 1 \text{ or } \kappa = \omega/c$$

Equations (*i*) and (*ii*) relate $\mathbf{E}_0$ and $\mathbf{B}_0$ and show that $\mathbf{E}_0$, $\mathbf{B}_0$ and $\mathbf{a}_z$ are mutually perpendicular. Their amplitude are related by

$$E_0(x, y) = (\omega/\kappa)\, B_0(x, y) = c\, B_0(x, y).$$

Maxwell equations $\nabla\cdot\mathbf{E} = 0$ and $\nabla\cdot\mathbf{B} = 0$, give

$$\nabla\cdot\mathbf{E}_0 = 0 \text{ and } \nabla\cdot\mathbf{B}_0 = 0,$$

i.e., the $\mathbf{E}_0$ and $\mathbf{B}_0$ satisfy the equations for electrostatics and magnetostatics.

Example 11. *State the Maxwell's equations without dielectrics or magnetic materials. Justify your answers in all of the following cases : (a) If the signs of all the source charges are reversed, what happens to **E** and **B**? (b) If the system is space inverted. What happens to ρ, **j**, **E** and **B**? (c) If the system is time inverted, what happens to ρ, **j**, **E** and **B**.*

Maxwell's equations in the absence of dielectrics and magnetic materials are given as

$$\nabla\cdot\mathbf{E} = \rho/\varepsilon_0 \quad ; \quad \nabla\times\mathbf{E} = -\partial\mathbf{B}/\partial t$$

$$\nabla\cdot\mathbf{B} = 0 \quad ; \quad \nabla\times\mathbf{B} = \mu_0\mathbf{j} + \mu_0\varepsilon_0\,\partial\mathbf{E}/\partial t.$$

(*a*) Under charge inversion $e \to -e$, $\rho \to \rho' = -\rho$, $\mathbf{j} \to \mathbf{j}' = -\mathbf{j}$.

Thus we have $\nabla \to \nabla' = \nabla$, $\partial/\partial t \to \partial/\partial t' = \partial/\partial t$, under such transformations, Maxwell's equations remain the same, *i.e.*,

$$\nabla'\cdot\mathbf{E}' = \rho'/\varepsilon_0 \quad ; \quad \nabla'\times\mathbf{E}' = -\partial\mathbf{B}'/\partial t'$$

$$\nabla'\cdot\mathbf{B}' = 0 \quad ; \quad \nabla'\times\mathbf{B}' = \mu_0\mathbf{j}' + \mu_0\varepsilon_0\,\partial\mathbf{E}'/\partial t'.$$

Since $\rho' = -\rho$, therefore Maxwell's first equations give $\mathbf{E}'(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t)$

Substituting this in the fourth Maxwell's equation, we get

$$\nabla'\times\mathbf{B}' = \nabla\times\mathbf{B}' = -\mu_0\mathbf{j} - \mu_0\varepsilon_0\,\partial\mathbf{E}/\partial t.$$

Hence $\mathbf{B}'(\mathbf{r}, t) = -\mathbf{B}(\mathbf{r}, t)$

(*b*) Under space inversion $\mathbf{r} \to \mathbf{r}' = -\mathbf{r}$, $\nabla \to \nabla' = -\nabla$

$$\partial/\partial t \to \partial/\partial t' = \partial/\partial t \quad , \quad e \to e' = e$$

$$\therefore \quad \rho(\mathbf{r}, t) \to \rho'(\mathbf{r}, t) = \rho \quad , \quad \mathbf{j} \to \mathbf{j}' = \rho'\mathbf{u}' = -\rho\mathbf{u} = -\mathbf{j}.$$

where $\mathbf{u}$ is the velocity of the charges in an elementary volume.

As Maxwell's equations remain the same under this transformation, we have

$$\mathbf{E}'(\mathbf{r}, t) = -\mathbf{E}(\mathbf{r}, t) \ ; \quad \mathbf{B}'(\mathbf{r}, t) = \mathbf{B}(\mathbf{r}, t)$$

(*c*) Under time inversion, $\partial/\partial t' \to \partial/\partial t = -\partial/\partial t$, we have

$\nabla \to \nabla' = \nabla$ and $e \to e' = e$. Thus $\rho' = \rho$, $\mathbf{j}' = -\rho\mathbf{u} = -\mathbf{j}$ and we have from covariance of Maxwell's equations that

$$\mathbf{E}'(\mathbf{r}, t) = \mathbf{E}(\mathbf{r}, t), \mathbf{B}'(\mathbf{r}, t) = -\mathbf{B}(\mathbf{r}, t).$$

Example 12. *A plane electromagnetic wave of intensity I falls upon a glass plate with refractive index n. The wave vector is at right angles to the surface (normal incidence). Show that the coefficient of reflection at normal incidence is given by* $R = (n-1)^2/(n+1)^2$ *for a single interface.*

Let the incident, refracted and reflected electromagnetic waves be represented by

$$\mathbf{E} = \mathbf{E}_0 e^{i(\kappa\cdot\mathbf{r}-\omega t)}; \mathbf{E}' = \mathbf{E}'_0 e^{i(\kappa'\cdot\mathbf{r}-\omega t)}; \mathbf{E}'' = \mathbf{E}''_0 e^{i(\kappa''\cdot\mathbf{r}-\omega t)} \qquad ...(1)$$

As the permeability of glass is very nearly equal to that of vacuum, *i.e.*, $\mu = \mu_0$ and the refractive index of glass $n = (\varepsilon/\varepsilon_0)^{1/2}$.

The plane electromagnetic wave can be decomposed into two polarized components with mutually perpendicular planes of polarization. Let the interface is along x-direction and the wave propagation is along z-axis. The incident wave can be decomposed into two polarized components with $\mathbf{E}$ parallel to x and y-directions. As $\mathbf{E}$, $\mathbf{H}$ and κ form a right hand set with components

Incident	E_x, H_y	(x-polarization)	E_y, H_x	(y-polarization)
Refracted	E_x', H_y'	(x-polarization)	$E_y', -H_x'$	(y-polarization)
Reflected	$E_x'', -H_y''$	(x-polarization)	E_y'', H_x''	(y-polarization)

The boundary condition that the tangential components are continuous,

$$E_x + E_x'' = E_x' \qquad ...(2)$$

and

$$H_y - H_y'' = H_y' \qquad ...(3)$$

For a plane wave $\sqrt{\mu}\,|\mathbf{H}| = \sqrt{\varepsilon}\,|\mathbf{E}|$. With $\mu \simeq \mu_0$ and $(\varepsilon/\varepsilon_0)^{1/2} = n$, relation (3) becomes

$$E_x - E_x'' = n\,E_x' \qquad \text{...(4)}$$

Thus relations (2) and (4) give

$$E_x'' = [(1-n)/(1+n)]\,E_x. \qquad \text{...(5)}$$

The same result holds for y-polarization. Hence for normal incidence, we get

$$E'' = [(1-n)/(1+n)]\,E. \qquad \text{...(6)}$$

Intensity of a wave is given by the poynting vector S or N over one period.

$$\mathbf{N} = R_e\,\mathbf{E} \times R_e\,\mathbf{H} = \tfrac{1}{4}(\mathbf{E}\times\mathbf{H} + \mathbf{E}^* \times \mathbf{H}^* + \mathbf{E}\times\mathbf{H}^* + \mathbf{E}^*\times\mathbf{H})$$

The first two terms of R.H.S. contain the time factor $e^{\pm 2i\omega t}$, which vanish on taking average over one period. Hence the intensity

$$I = \langle N \rangle = \frac{1}{2}R_e\,|\mathbf{E}\times\mathbf{H}^*| = \frac{1}{2}EH^* = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,E_0^2$$

$\therefore$ Coefficient of Reflection $R = \dfrac{E_0''^2}{E_0^2} = \left(\dfrac{1-n}{1+n}\right)^2$.

Example 13. *What is the attenuation distance for a plane wave propagating in a good conductor.*

For the ohmic conducting medium of permeability μ, permittivity ε and conductivity σ, the general wave equation is given by

$$\nabla^2\mathbf{E} - \mu\varepsilon\,\partial^2\mathbf{E}/\partial t^2 - \mu\sigma\partial\mathbf{E}/\partial t = 0$$

For plane electromagnetic waves of angular frequency ω, $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(\mathbf{r})\,e^{-i\omega t}$, the above equation becomes

$$\nabla^2\mathbf{E}_0 = \mu\varepsilon\omega^2(1 + i\sigma/\omega\varepsilon)\,\mathbf{E}_0 = 0.$$

Consider the plane wave incident on the conductor along the inward normal, along z-axis. In the conductor it is represented as

$$\mathbf{E} = \mathbf{E}_0\,e^{i(\kappa z - \omega t)}$$

The wave vector has magnitude $\kappa = \dfrac{\omega}{v} = \omega(\mu\varepsilon)^{1/2}\left(1 + \dfrac{i\sigma}{\omega\varepsilon}\right)^{1/2}$

Let $\kappa = \alpha + i\beta$, we have $\alpha^2 - \beta^2 + 2i\,\alpha\beta = \omega^2\mu\varepsilon\,(1 + i\sigma/\omega\varepsilon)$

$$\therefore \qquad \alpha^2 - \beta^2 = \omega^2\mu\varepsilon \text{ and } \alpha\beta = \omega\mu\sigma/2$$

For a good conductor, *i.e.*, for $\sigma/\varepsilon\omega >> 1$, we have

$$\alpha = \beta = \sqrt{\omega\mu\sigma/2}$$

In the conductor, we thus have

$$\mathbf{E} = \mathbf{E}_0\,e^{-\beta z}\,e^{i(\alpha z - \omega t)}$$

The attenuation length δ is the distance the wave travels for its amplitude to reduce to e^{-1} of its initial value.

Thus $$\delta = 1/\beta = \sqrt{2/\omega\mu\sigma}.$$

Example 14. *A cylindrical resistor of radius a and length l has resistivity ρ. A current I enters in it through the leads which are cylindrical wires of radius b. If the leads have zero resistivity and the voltage across the resistor is V, calculate (a) the Poynting vector at the surface* *of the resistor, (b) the Poynting vector at the surface of wire, (c) the energy flow in the resistor and (d) the energy flow in the wire.*

The electric field at the surface of the resistor, carrying a current I is given by

$$E = \frac{V}{l} = \frac{IR}{l} = \frac{I(\rho l/A)}{l} = \frac{\rho I}{\pi a^2} \qquad \textit{(along the length)}$$

The magnetic field at the surface of the resistor is given by Ampere's law

$$\oint \mathbf{B}\cdot d\mathbf{l} = B\, 2\pi a = \mu_0 I$$

or $$B = \mu_0 I/2\pi a. \qquad \textit{(around the surface)}$$

$\therefore$ Poynting vector $\mathbf{S} = \mathbf{E} \times \mathbf{B}/\mu_0$

or $$S = (\rho I/\pi a^2)\,(\mu_0 I/2\pi a)/\mu_0 = \rho I^2/2\pi^2 a^3.$$

Therefore energy flux through the cylindrical surface of the resistor per unit time $(\rho I^2/2\pi^2a^3)\,(2\pi a l) = \rho l I^2/\pi a^2$.

The electric field at the surface of the perfect conducting leads ($\rho = 0$) is zero. Therefore the Poynting vector at the surface of the wire is also zero, although B at the wire surface = $\mu_0 I/2\pi b$.

$\therefore$ Energy flow through the leads (in the wire) = 0.

The rate of Joule heat produced = $VI = I^2 R = I^2\, \rho l/\pi a^2$.

Thus we see that the rate at which energy flows into the resistor through its cylindrical surface is equal to the rate at which the Joule heat is produced.

Since no energy flows through the wires, the energy dissipated in the resistor can be thought of as enterng through the **E**-and **B**-field vectors.

Example 15. *A pulsed source of microwave produes bursts of 20 GHz radiation, each burst lasts 1 ns. A parabolic reflector of radius r = 6 cm is used to produce a parallel beam of radiation. The average output power of the source during each pulse is 25 kW. Calculate (a) the wavelength of radiation, (b) total energy of each pulse, (c) average energy density and average value of Poynting vector during each pulse, (d) the average momentum density and average radiation pressure on totally absorptive surface if placed ⊥ to the direction of propagation, and (e) the amplitudes of electric and magnetic field vectors.*

The wavelength of the radiation $\lambda = \dfrac{c}{\nu} = \dfrac{3 \times 10^8}{20 \times 10^9} = 1.5$ cm

Total energy delivers per pulse U = (25kW) × 10^{-9} sec = 2.5×10^{-5} J.

Length of the pulse $l = c\,\Delta t = 3 \times 10^8 \times 10^{-9} = 0.3$ m.

$$\therefore \quad \text{Average energy density } \langle u \rangle = \frac{U}{\pi R^2 l} = \frac{2.5 \times 10^{-5}}{3.14 \times (6 \times 10^{-2})^2 \times 0.3}$$

$$= 7.37 \times 10^{-3}\ J/m^3$$

The average value of poynting vector $\langle S \rangle = \langle u \rangle c = 2.21 \times 10^6\ W/m^2$.

The average momentum density

$$\langle p \rangle = \frac{\langle u \rangle}{c} = \frac{7.37 \times 10^{-3}}{3 \times 10^8} = 2.46 \times 10^{-11}\ Js/m^4.$$

Since the total momentum contained in a volume $Al = \pi R^2\, c\, \Delta t$ will be absorbed by the perpendicular surface during time Δt. The force F exerted by the wave on an area A of the surface is given by

Impulse ($F\,\Delta t$) = Momentum density (u/c) × volume ($Ac\Delta t$)

or $F = uA$

$\therefore$ Radiation pressure $p_{rad} = F/A = u$.

or Average radiation pressure $<p_{rad}> = <u> = 7.37 \times 10^{-3}\,Pa$.

The amplitude of the electric field vector

$$E_0 = \sqrt{\frac{2<S>}{\varepsilon_0 c}} = \sqrt{\frac{2<u>}{\varepsilon_0}} = \sqrt{\frac{2\times 7.37\times 10^{-3}}{8.85\times 10^{-12}}} = 4.08\times 10^4\,V/m.$$

The amplitude of the magnetic field vector $B_0 = E_0/c = 1.36 \times 10^{-4}\,T$.

Example 16. *Find the force with which photons having a flux density S of electromagnetic energy act on a perfectly reflecting sphere of radius r.*

Consider an area $d\sigma$ on the sphere at any point P with polar coordinates (r, θ). We know that for the electromagnetic waves incident on the surface along the normal and completely absorbed, the pressure exerted is $p = S/c$. If the waves are completely reflected, the momentum change is doubled and the pressure $p = 2S/c$.

In the present case, the force acting on an area $d\,\sigma$ due to the waves falling horizontally is given by

$$dF = [2S/c)\cos\theta)]\,d\,\sigma.$$

It will be directed along **PO,** *i.e.*, towards the centre of the sphere (Fig. 14.20). In view of the axial summetry of pressure distribution, the only non zero component force will be that in the direction of the initial flow of photons. Thus the component of $d\mathbf{F}$ along x direction.

$$\begin{aligned} dF_x &= -(2S/c)\,d\,\sigma\cos^2\theta \\ &= -(2S/c)\,[2\pi\,(r\sin\theta)\,rd\,\theta]\cos^2\theta \\ &= -(4\pi S/c)\,r^2\cos^2\theta\sin\theta d\,\theta. \end{aligned}$$

Hence the total force along x-axis

$$F_x = -4\pi\frac{S}{c}r^2\int_0^{\pi/2}\cos^2\theta\sin\theta\,d\,\theta = \frac{4\pi}{3}\frac{S}{c}r^2\,.$$

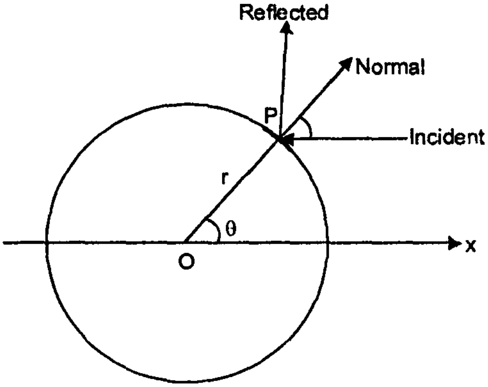


Fig. 14.20. Reflection at the spherical surface.

This is 4/3 times greater than that when the entire energy of flow is absorbed by the sphere.

Example 17. *The regions x < 0 and x > 0 are filled with material of permeabilities 1.5 μ_0 and 5 μ_0 respectively. The magnetic fields in these regions are respectively $10\,\hat{x} + 2.4\,\hat{y}$ and $10\,\hat{x} + 25\,\hat{y} - 17.5\,\hat{z}$ in teslas. Determine the current distribution at x = 0.*

If $\mathbf{K}_f$ be the surface current density at the boundary surface. The boundary condition for the tangential components

$$\mathbf{H}_{2t}(t) - \mathbf{H}_{1t}(t) = \mathbf{K}_f(t)\times\hat{n}$$

may be written as

$$\left(\frac{\mathbf{B}_{2t}}{\mu_2} - \frac{\mathbf{B}_{1t}}{\mu_1}\right) = \mathbf{K}_f\times\hat{x}$$

or

$$\left(\frac{25\hat{y} - 17.5\hat{z}}{5\mu_0} - \frac{2.4\hat{y}}{1.5\mu_0}\right) = \mathbf{K}_f\times\hat{x}$$

or

$$(5\hat{y} - 3.5\hat{z} - 1.6\hat{y})/\mu_0 = \mathbf{K}_f\times\hat{x}$$

or $$\mathbf{K}_f = (3.5\hat{y} + 3.4\hat{z})/\mu_0 \text{ amp/}m.$$

Example 18. *A light wave falls obliquely at an angle θ_i on a plane interface between two dielectric media of refractive indices n_1 and n_2. If the wave is linearly polarized with the direction of polarization being at 45° to the plane of incidence, calculate the rotation in polarization of the transmitted wave.*

If the interface plane is the *yz* plane and the plane of incidence is *xy*, as shown in Fig. 14.10. Since the direction of polarization being at 45° to the plane of incidence, hence the amplitudes of the incident *p* - and *s*-polarization are equal and are given by

$$E_p = E_0/\sqrt{2} \text{ and } E_s = E_0/\sqrt{2}.$$

The amplitudes of transmitted waves are given by

$$E_p' = t_{12p} E_p = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)\cos(\theta_i - \theta_t)} \cdot \frac{E_0}{\sqrt{2}}$$

and $$E_s' = t_{12s} E_s = \frac{2\cos\theta_i \sin\theta_t}{\sin(\theta_i + \theta_t)} \cdot \frac{E_0}{\sqrt{2}}.$$

Thus the angle ϕ that the polarization of the transmitted wave makes with the plane of incidence is given by

$$E_s'/E_p' = \cos(\theta_i - \theta_t) = \tan\phi \text{ or } \phi = \tan^{-1}[\cos(\theta_i - \theta_t)]$$

The polarization has undergone a rotation $\delta\,\theta = \phi - \pi/4$.

Example 19. *A dielectric slab of refractive index n_2, thickness d and with plane parallel faces lies betwen two nonpermeable media of refractive indices n_1 and n_2. A linearly polarized wave is normally incident on the layered interface. Find the conditions for highest and lowest reflectivities.*

If the direction of propagation is assumed to be along *x*-axis. As the wave strikes the $x = 0$ interface (Fig. 14.21), the fraction of the intensity will be reflected. Thus in the region $x < 0$, incident and reflected both waves exist. The transmitted wave suffers also reflection at $x = d$ interface. Thus in the region $0 < x < d$,both counter propagating waves exist in the region. In the region $x > d$, only transmitted wave exists. Thus **E** and **B** fields in these regions are as :

$x < 0$ $$\mathbf{E}_1 = \mathbf{E}_{01}\, e^{i(\kappa_1 x - \omega t)} - \mathbf{E}_{01}''\, e^{-i(\kappa_1 x + \omega t)}$$

$$\mathbf{B}_1 = \mathbf{B}_{01}\, e^{i(\kappa_1 x - \omega t)} + \mathbf{B}_{01}''\, e^{-i(\kappa_1 x + \omega t)}$$

where$\kappa_1 = n_1\,\omega/c$ and $B_1 = n_1 E_1/c$.

$0 < x < d$ $$\mathbf{E}_2 = \mathbf{E}_{02}\, e^{i(\kappa_2 x - \omega t)} - \mathbf{E}_{02}''\, e^{-i(\kappa_2 x + \omega t)}$$

$$\mathbf{B}_2 = \mathbf{B}_{02}\, e^{i(\kappa_2 x - \omega t)} - \mathbf{B}_{02}''\, e^{-i(\kappa_2 x + \omega t)}$$

$$x > d\mathbf{E}_3 = \mathbf{E}_{03}\, e^{i(\kappa_3 x - \omega t)} \text{ and } \mathbf{B}_3 = \mathbf{B}_{03}\, e^{-i(\kappa_3 x - \omega t)}$$

Using the boundary conditions, we get

$$E_{01} - E_{01}'' = E_{02} - E_{02}'', n_1(E_{01} + E_{01}'') = n_2(E_{02} + E_{02}'')$$

$$E_{02}\, e^{in_2\omega d/c} - E_{02}''\, e^{-in_2\omega d/c} = E_{03}\, e^{in_3\omega d/c}$$

and $$n_2(E_{02}\, e^{in_2\omega d/c} + E_{02}''\, e^{in_2\omega d/c}) = n_3 E_{03}\, e^{n_3\omega d/c}$$

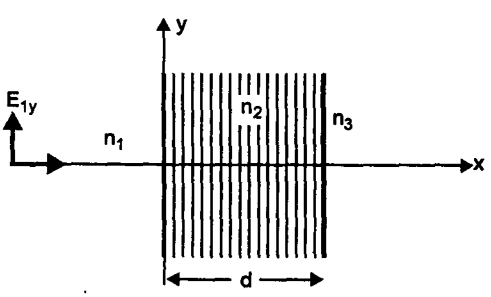


Fig. 14.21. Reflection from a dielectric slab at normal incidence.

After solving these equations, we get

$$\frac{E_{01}}{E_{03}} = \frac{1}{2}\left[\left(1+\frac{n_3}{n_1}\right)\cos\frac{n_2\omega d}{c} - i\left(\frac{n_2}{n_1}+\frac{n_3}{n_2}\right)\sin\frac{n_2\omega d}{c}\right]e^{in_3\omega d/c}$$

The transmitted intensity is given by

$$\frac{{E_{01}}^2}{{E_{03}}^2} = \frac{1}{4}\left(1+\frac{n_3}{n_1}\right)^2\cos^2\left(\frac{n_2\omega d}{c}\right)+\frac{1}{n}\left(\frac{n_2}{n_1}+\frac{n_3}{n_2}\right)^2\sin^2\left(\frac{n_2\omega d}{c}\right)$$

If, the medium n_1 is air, n_3 is an optical system and n_2 is the layer of thickness d, such that the reflected wave is minimum for a wavelength λ. This coating is called *anti reflection coating*. For normal reflection the reflectivity is minimum for $n_2 d = (2m + 1)\lambda/4$. For $m = 0$, we get

$$R_{\min} = (n_1 n_3 - n_2^2)^2/(n_1 n_3 + n_2^2)^2.$$

Reflectivity is maximum for $n_2 d = m\,\lambda_0/2$. Thus

$$R_{\max} = (n_1 - n_3)^2/(n_1 + n_3)^2.$$

Example 20. *A parallel wire transmission line is constructed of copper wire (dia 0.162", $\sigma_c = 58\ Ms/m$) with a separation of 12" in air. Find the per-meter values of L, C, G, dc resistance and ac-resistance at 1 MHz.*

The parallel line wire has a radius $a = \frac{1}{2}\times 0.162'' = 2.06\times 10^{-3}\,m$ and the separation $d = 12'' = 0.305\ m$. Since $d >> a$, the four geometrical factors are given by

$$GFL = \log_e(d/a) = 5.0;\ GFC = 1/GFL = 0.2$$

$$GFR_d = 2/a^2 = 4.71\times 10^5\ \text{m}^{-2};\ GFR_a = 2/a = 971\ \text{m}^{-1}.$$

For air as dielectric medium, $\mu_d = \mu_0,\ \varepsilon_d = \varepsilon_0,\ \sigma_d = 0$ for copper $\mu_c = \mu_0$. Therefore

$$L = \frac{\mu_d}{\pi}(GFL) = \frac{\mu_0}{\pi}\times 5.0 = 2.0\,\mu H/m$$

$$C = \pi\,\varepsilon_d\,(GFC) = \pi\,\varepsilon_0\times 0.2 = 5.56\,pF/m$$

$$R_d = \frac{1}{\pi\sigma_c}(GFR_d) = \frac{4.71\times 10^5}{\pi\times 58\times 10^6} = 2.58\times 10^{-3}\,\Omega/m$$

Skin depth $\delta = 1/\sqrt{\pi f \mu_c\,\sigma_c} = 66\,\mu m$

$$R_a = \frac{1}{2\pi\delta_c\,\sigma_c}(GFR_a) = \frac{1\times 971}{2\pi\times 66\times 10^{-6}\times 58\times 10^6} = 4.04\times 10^{-2}\,\Omega/m.$$

$$G = C\sigma_d/\varepsilon_d = 0\,S/m.$$

Example 21. *Two meter long transmission line operating at $\omega = 10^6$ rad/sec has $\alpha = 8$ dB/m, $\beta = 1$ rad/m and $Z_0 = 60 + j30$ ohm. If the line is connected to a source of emf 10V and impedance 40Ω and terminated by a load of 20 + j40Ω, calculate (a) the input impedance, (b) the sending end current and (c) the current at the middle of the line.*

Since α the attenuation constant of the medium is measured in nepers/m (Np/m) or decibels/m, where 1 $Np = 20\log_{10} e = 8.686\ dB$.

$$\therefore \qquad \alpha = 8\ dB/m = 8/8.686 = 0.921\ Np/m.$$

$$\text{and} \qquad \gamma = \alpha + j\beta = 0.921 + j$$

Since $$\tan(x+jy) = \frac{\sinh 2x}{\cosh 2x+\cos 2y} + j\frac{\sin 2y}{\cosh 2x+\cos 2y}$$

$$\therefore \qquad \tanh \gamma l = \tanh(1.84+j2) = 1.033 - j\,0.03929.$$

Input impedance $$Z_{in} = Z_0\left(\frac{Z_L + Z_0\tanh\gamma l}{Z_0 + Z_L\tanh\gamma l}\right)$$

$$= (60+j30)\left[\frac{20+j40+(60+j30)(1.033-j0.03929)}{60+j30+(20+j40)(1.033-j0.03929)}\right]$$

$$= 60.2667 + j28.7166\Omega.$$

The sending end current $I\,(Z=0) = I_0 = V_g/(Z_{in}+Z_g)$

$$= \frac{10}{60.2667+j28.7166+40} = 0.0922 - j\,0.0264$$

$$= 95.9\underline{|-15.98^\circ} mA.$$

The current at the middle of the line

$$I_s\,(Z=l/2) = \frac{V_0^+}{Z_0}e^{-\gamma l/2} - \frac{V_0^-}{Z_0}e^{\gamma l/2}$$

where $$V_0^+ = \tfrac{1}{2}(V_0 + Z_0 I_0) \text{ and } V_0^- = \tfrac{1}{2}(V_0 - Z_0 I_0).$$

and $$V_0 = Z_{in} I_0 = (60.2667 + 28.7166)\,(0.0922 - j\,0.0264)$$

Example 22. *A rectangular wave guide has a broad wall dimension of 2 cm and is fed by a 10 GHz carrier from a coaxial cable. Determine whether a TE_{10} mode of wave will be propagated. If so, calculate its guide wave length and phase and group velocities.*

We know that for TE_{10} mode the critical wavelength

$$\lambda_c = 2a = 2 \times 2 = 4\text{cm}.$$

The free space wavelength $$\lambda = \frac{v}{f} = \frac{c}{f} = \frac{3\times10^8}{10\times10^9} = 3\text{cm}.$$

Since the critical wavelength λ_c for TE_{10} mode is greater than the free space wavelength λ, therefore TE_{10} wave will propagate.

The wavelength of the guide wave λ_g for the TE_{10} mode

$$= \frac{\lambda}{\sqrt{1-(\lambda/\lambda_c)^2}} = \frac{3}{\sqrt{1-(3/4)^2}} = 4.536\text{cm}.$$

$\therefore$ Phase velocity $$v_{ph} = \lambda_g f = 4.536 \times 10^{-2} \times 10 \times 10^9 = 4.536 \times 10^8 \text{ m/sec}$$

Group velocity $$v_{gr} = c\frac{\lambda}{\lambda_g} = 3\times10^8 \times \frac{3}{4.536} = 1.984\times10^8 \text{ m/sec.}$$

Example 23. *In a hollow copper box of dimensions 2 cm × 3 cm × 1 cm. How many modes are there in the range $4/\sqrt{5} \le \lambda \le 8/\sqrt{13}$ cm? Find also the wavelengths.*

For the cavity resonator of wavelength of the stationary wave mode m, n, p is given by

$$\lambda_{m,n,p} = \frac{2}{(m/a)^2+(n/b)^2+(p/h)^2} = \frac{2}{\sqrt{m^2/4+n^2/9+p^2}}.$$

If $4/\sqrt{5}\text{ cm} \leq 2/\sqrt{m^2/4+n^2/9+p^2} \leq 8/\sqrt{13}\text{ cm}$

we have $$\frac{13}{16} \leq \frac{m^2}{4}+\frac{n^2}{9}+p^2 \leq \frac{5}{4}.$$

As the integers m, n, p must be either 0 or positive with $mn + np + pm \neq 0$, we have

$$m = 1, n = 3; m = 2, n = 1; \text{ for } p = 0$$

$$m = 1, n = 0; m = 0, n = 1, \text{ for } p = 1.$$

Each set of m, n, p corresponds to a *TE* and a *TM*-mode. Hence in the wavelength range $4/\sqrt{5} \leq \lambda \leq 8/\sqrt{13}$ cm, there are eight resonant modes 2 for each (1, 3, 0), (2, 1, 0), (1, 0, 1) and (0, 1, 1).

The wavelengths of the four double modes are $4/\sqrt{5}, 6/\sqrt{10}, 4/\sqrt{5}, 6/\sqrt{10}$ cm respectively. However there are only two different resonant wavelengths.

Example 24. *An electromagnetic wave is propagating in the TE-mode in the rectangular wave guide. The walls of the wave guide are conducting and the inside is vacuum. Find the cut off frequency in this mode ? If the inside is filled with a material with dielectric constant ε, how does the cut off frequency change ?*

Let the wave guide is along z-axis and the sides of cross section are a along x-axis and b along y-axis. In *TE*-mode, $E_z = 0$ and $H_z \neq 0$. The transverse waves in the wave guide are standing waves, while the z-component is a traveling wave.

Let m and n be the numbers of half waves in the x-and y-directions respectively. The wave numbers of the standing waves are

$$k_x = m\pi/a, \; k_y = n\pi/b.$$

The wave number of the travelling wave $k_z^2 = k^2 - (k_x^2 + k_y^2)$, where $k^2 = \mu\varepsilon\omega^2$.

If the inside of the wave guide is vacuum, we have $k^2 = \mu_0 \varepsilon_0 \omega^2$,

$$\therefore \quad k_z^2 = \mu_0 \varepsilon_0 \omega^2 - [(m\pi/a)^2 + (n\pi/b)^2]$$

If $k_z^2 < 0$, k_z is purely imaginary and the travelling wave $\sim e^{ik_z z}$ becomes exponentially attenuating, *i.e.*, no propagation. Hence the cut off frequency

$$\omega_{mn} = \frac{\pi}{\sqrt{\mu_0 \varepsilon_0}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

If the wave guide is filled with a dielectric, the results can be obtained by replacing $\varepsilon_0 \to \varepsilon$ and $\mu_0 \to \mu$,

$$\therefore \quad \text{Cut off frequency} = \frac{\pi}{\sqrt{\mu\varepsilon}}\sqrt{\left(\frac{m}{a}\right)^2+\left(\frac{n}{b}\right)^2}$$

ORAL QUESTIONS

1. Why is the quantity $\varepsilon_0\, \partial E/\partial t$ referred to as a displacement current ?
2. Give the symmetries that appear between two groups of two equations of Maxwell's.
3. If a monopole exists, which of the Maxwell's equations (in free space) will be modified ? (2, 3)
4. The concept of diplacement current was a major contribution attributed to (*a*) Faraday; (*b*) Maxwell; (*c*) Coulomb; (*d*) Lorentz. (*b*)
5. If $E_s = 20\, e^{j5x}\, \mathbf{a}_y$, which of the following is not a correct representation of **E**. (*a*) $Re\, (\mathbf{E}_0\, e^{-j\omega t})$. (*a*) $I_m\, (\mathbf{E}_0\, e^{j\omega t})$, (*c*) $20 \cos (\omega t + j\, 5x)\, \mathbf{a}_y$. (*c*)
6. Which one is correct for electromagnetic waves ? (*a*) **B** is $\perp$ to $\kappa \times \mathbf{E}$, (*a*) **B** is $\parallel$ to $\kappa \times \mathbf{E}$, (*c*) $\mathbf{B} = \kappa \times \mathbf{E}$. (*b*)
7. The electric field component of a wave in free space is given by $\mathbf{E} = 15 \sin (10^7 t + \kappa z)\, \mathbf{a}_y$. V/m. Which of the following are correct statements : (*a*) The wave propagates along $\mathbf{a}_y$, (*b*) wavelength $\lambda = 188.5\ m$, (*c*) wave amplitude = 15 V/m, (*d*) The wave attenuates in travelling. (*b, c*)
8. In a certain medium, the electric field component of the wave is given by $\mathbf{E} = 10 \cos (10^8 t - 3\,z)\, \mathbf{a}_y$. What type of medium is it?
   (*a*) Free space, (*b*) Perfect dielectric, (*c*) Perfect conductor, (*d*) Lossless dielectric. (*d*)
9. Is it correct that the em-wave travel faster in conductors than in dielectrics. (No)
10. An electromagnetic wave consists of two distinct types of electric and magnetic field vibrations. Where the vibration is in time or in space ?
11. An electromagnetic wave travels along the x-axis with its electric vector in the y-direction. What effect does the wave have on a loop of wire placed (*a*) in xy-planed and (*b*) in xz-plane ?
12. Is **E** or **B** field vector parallel to the direction of propagation of em-wave ? (No)
13. Which energy changes into Joule's heat when the electric current flows in a conductor ?
14. What is the structure of a plane wave ? At which velocity does it propagate in a vacuum ?
15. What does the momentum density of an electromagnetic wave depend on ?
16. What do you mean by spin depth ?
17. What happens to the frequency of reflected or refracted em-wave from a dielectric interface ?
18. What is Brewster's angle ? What are its applications ?
19. Which of the following statement is not true of the line parameters *R, L, G* and *C* ?
   (*a*) *R* and *L* are series elements, (*b*) *G* and *C* are shunt elements, (*c*) Both *R* and *G* depend on the conductivity of the conductors forming the line. (*c*)
20. For a lossy transmission line, the characteristic impedance does not depend on
   (*a*) Operating frequency of the line, (*b*) Length of the line, (*c*) Conductivity of the conductors. (*b*)
21. The positive pulse is sent into a transmission line which is short circuited at the other end. The pulse reflected back (*a*) is positive, (*b*) is negative, (*c*) does not exist. (*b*)
22. The dominant mode for rectangular wave guide is (*a*) TE_{11}, (*b*) TE_{10}, (*c*) TM_{11}, (*d*) TE_{101}. (*b*)

PROBLEMS

1. A conductor with cross sectional area of 10 cm^2 carries a conduction current $0.2 \sin 10^9 t\ mA$. Calculate the magnitude of the displacement current density, given that $\sigma = 2.5 \times 10^6 S/m$ and $\varepsilon_0 = 6$.
2. An audio oscillator with $V = 100 \sin 1.2 \times 10^5\, t$ is connected across the parallel plate capacitor. If the plates are circular each of radius 40 cm and are separated by a distance of 10cm, (*a*) Calculate the displacement current between the plates, (*b*) *B*-field at a distance r = 38 cm from the centre line of the plates. *[0.53 cos (1.2 × 10⁵t) mA, 2.5 × 10⁻¹⁰ cos (1.2 × 10⁵t)T]*

3. A parallel plate capacitor with circular plates of radius a is connected to a source of constant emf E. The distance between the plates varies slowly according to the harmonic law $d = d_0 + \Delta \sin \omega t$. Find the magnetic field produced between the plates by displacement currents.

   $[\mu_0 \varepsilon_0 E \omega a \Delta \cos \omega t / 2d^2]$

4. A New Delhi radio station is 100 km from Khurja. It radiates an average power of 50 kW. Assuming spherical radiation, find the value of **S** at Khurja. Calculate the values of E_0 and B_0 at an antenna located at Khurja. $[0.4 \times 10^{-6}\ W/m^2,\ 1.7 \times 10^{-2}\ V/m,\ 5.7 \times 10^{-11} T]$

5. A solar energy flux on the earth orbit is approximately equal to 1400 W/m^2. Find the radius of a black spherical particle of density 5 gm/cm^2 for which the light pressure in the inter planetary space is equal to the solar attraction. Given that $M_{sun} = 2 \times 10^{30}$ kg, $G = 6.7 \times 10^{-11}$ Nm^2/kg^2 and r the distance between the earth and the sun is 150 × 10^6 km. $[0.5 \times 10^{-7}\ m]$

6. A piece of paper of area 100 cm^2 and mass 1 gm is suspended in a horizontal position by using the radiation pressure. The paper is black. How many watts must the light produce to hold the paper against the pull of gravity ? What would happen to the paper ?

   [2.9 × 10^6 *W, Paper would burn*]

7. Show that in a source free region ($\mathbf{J} = 0$, $\rho_v = 0$), Maxwell's equations can be reduced to two.

8. In a certain region with $\sigma = 0$, $\mu = \mu_0$ and $\varepsilon = 6.25\ \varepsilon_0$, the magnetic field of an em-wave is given by $\mathbf{H} = 0.6 \cos \beta x \cos 10^8 t\ \mathbf{a}_z\ A/m$. Find β and **E** using Maxwell's equations.

   $(0.833,\ 100.5 \sin \beta x \sin 10^8 t\ \boldsymbol{a}_y\ V/m)$

9. The electric field in air is given by $\mathbf{E} = \rho t\, e^{-\rho-t}\, \mathbf{a}_\phi$ V/m, find **B** and **J**.

   $[(2-\rho)(1+t)\, e^{-\rho-t}\, \boldsymbol{a}_z\ Wb/m^2,\ (1+t)(3-\rho)(4\pi)^{-1}\, 10^7\, e^{-\rho-t}\, \mathbf{a}_\phi A/m^2]$

10. The magnetic field vector of an em-wave has an rms value of 10^{-8} T. Find the intensity of the wave. How much energy is transported per minute through a 1.0 m^2 area ?

    $[2.4 \times 10^{-2}\ W/m^2;\ 1.44 J]$

11. The intensity of the em-wave from a distant 1.2 MHz radio station is 3 × 10^{-11} W/m^2. Find the expressions for the electric and magnetic fields in this region.

    $[1.5 \times 10^{-4} \sin (7.5 \times 10^6 t - 0.025\, x)\ V/m;\ 5.0 \times 10^{-13} \sin (7.5 \times 10^6\, t - 0.025 x) T]$

12. A wire of radius 1mm whose resistance per unit length is 3 × 10^{-3} Ω/m carries a current of 25.1 A. Find the values of **H**, **B** and **S** at a point very near the surface of the wire.

    [4 × 10^3 *A/m tangent to circle around wire,* 7.53 × 10^{-2} *V/m parallel to the wire, 301 W/m*^{2} ⊥ *to the wire*]

13. The regions $z < 0$ and $z > 0$ are filled with materials of permeabilities 5 μ_0 and $2\mu_0$ respectively. The magnetic field in the region $z > 0$ is $\mathbf{B}_2 = (75\hat{x} + 40\hat{z})\ \mu_0$ teslas. If a current sheet of surface density $-10\hat{y}$ A/m is located in the plane $z = 0$, calculate the magnetic field in region 1.

    $[\mu_0 (50\,\hat{x} + 40\,\hat{z}) T]$

14. A plane polarized em-wave travelling in a dielectric medium of refractive index n is reflected at normal incident from the surface of a conductor. Find the phase change in electric vector if the refractive index of the conductor $n_2 = n\,(1 + i\,\rho)$. $[\tan^{-1}(2/\rho)]$

15. A plane polarized electric wave $\mathbf{E} = \mathbf{E}_0 \exp\, [i\{\omega t - \omega n\, (\boldsymbol{\kappa}\cdot\mathbf{r})/c\}]$

    Derive **E** and **H** from Maxwell's equations and obtain an expression for the refractive index n in terms of ω, ε, μ and σ.

$$n = \sqrt{\frac{\mu\varepsilon}{2\mu_0\varepsilon_0}} \left[\sqrt{\left(1+\sigma^2/\varepsilon^2\omega^2\right)^{1/2} + 1} - i\sqrt{(1+\sigma^2/\varepsilon^2\omega^2)^{1/2} - 1} \right]$$

16. A linearly polarized electromagnetic wave of frequency ω is incident from a region of refractive index n_1 onto a plane interface with a material of refractive index n_2 ($< n_1$). If the incidence angle $\theta_i > \theta_c$, the critical angle. Show that the average transmitted Poynting vector vanishes. If the direction of the polarization of the wave to be at 45° to the plane of incidence, calculate the angle of incidence for which phases of the p-and s-polarizations are equal. *[$\sin^{-1} n_2/n_1$ or $\pi/2$]*

17. An electromagnetic wave of amplitude $\mathbf{H}_0 = 3\,\hat{x}$ (A/m) and frequency $\omega = 400\,\pi \times 10^6$ radians/sec is normally incident on a foil of silver of thickness 15 μm and conductivity $\sigma_c = 62 \times 10^6\ \Omega^{-1}m^{-1}$. Determine the field amplitudes just after it enters the foil, just after it exists the foil and just before it exists the foil. *[6, 3.14×10^{-4}, 0.21 A/m]*

18. In a free space $\mathbf{E}(\mathbf{x}, t) = 10^3 \sin(\omega t - \beta x)\,\mathbf{a}_y$ V/m. Obtain $\mathbf{H}(\mathbf{x}, t)$. Determine the propagation constant γ with the given frequency $f = 95.5$ MHz. *($j2.0\ m^{-1}$)*

19. A telephone line has $R = 0.03\ \Omega/m$, $L = 0.1\ mH/m$, $G = 0$ and $C = 20\ n\,F/m$. At the frequency of 1 kHz, calculate (a) characteristic impedance of the line, (b) propagation constant and (c) the phase velocity. *[$70.75\underline{|-1.367^\circ}\ \Omega$, $2.121 \times 10^{-4} + j8.888 \times 10^{-3}/m$, $7.069 \times 10^5\ m/sec.$]*

20. The transmission line operating at 500 MHz has $Z_0 = 40\Omega$, $\alpha = 0.04\ Np/m$, $\beta = 1.5$ rad/m. Find the distributed line parameters R, L, G and C. *[1.6 Ω/m, 19.1 nH/m, 10^{-3} S/m, 11.94 pF/m]*

21. A standard wave guide has the following dimensions : Outer walls 1″ × 0.5″, wall thickness 0.05 ″. Calculate the highest frequencies which can be trasmitted in the TE_{01}, TE_{10}, TE_{11}, and TM_{11} modes. *[6.56, 13.1, 14.8, 16.2 GHz]*

22. A coaxial cable with the dimensions a =0.5 mm, $b = 3\ mm$ and $t = 0.4\ mm$ is filled with a dielectric material having $\varepsilon_r = 2.0$, $\sigma_d = 10\ \mu\,S/m$. The conductors have $\sigma_c = 50\ MS/m$. Calculate the per meter values of L, C, G, R_d and R_a at 50 MHz.

    [0.358 $\mu\,H/m$, 62 pF/m, 35.1 $\mu\,S/m$, 0.03 Ω/m, 0.743 Ω/m]

23. Eastimate the number of distinct standing light which can exist between frequencies 1.0×10^{15} Hz and 1.2×10^{15} Hz in a cavity of volume 1 cm^3. *(2.26×10^{14})*

    Resonant frequency f for a cubical cavity resonator of sides a is

    $$4\pi^2 f^2 = (\pi^2/\mu\,\varepsilon\,a^2)\,(m^2 + n^2 + p^2).$$

    Each set of positive integers m, n, p with $m^2 + n^2 + p^2 \le r^2 = 4\,a^2f^2/v^2$ corresponds to a frequency $\le f(r)$. Therefore the number of modes with frequencies $\le f(r)$

    $$N = \frac{1}{8}\cdot\frac{4}{3}\pi r^3 = 4\pi f^3 V/3v^3.$$

    Each set of m, n, p corresponds to two modes of same frequency

    $$\therefore \qquad N = 8\,\pi f^3\,V/3\,v^2.$$

24. Find the radius and guide wavelength in an air dielectric cylindrical wave guide for the dominant mode at $f = 30$ GHz $= 1.5\,f_c\ TE_{11}$. Repeat the calculations with a lossless dielectric of $\varepsilon_r = 2.2$.

    [0.44 cm, 1.34 cm; 0.296 cm, 0.903 cm]

15 RADIATION EMISSION

15.1 HERTZ'S EXPERIMENT

Even though Maxwell's theory of electromagnetic waves was published in 1864, it was Heinrich Hertz who in 1887 actually demonstrated the production and detection of electromagnetic waves. Maxwell's theory remained collapsed during this period of 23 years, because there was no technology at that time to create AC currents of enough high frequency to provide detectable radiation.

Fig. 15.1 represents the experimental arrangement for the production of electromagnetic waves. It consists of two large spheres S and S' attached to two large metal plates P and P' respectively. The spheres are connected to an induction coil. Hertz found that when there is dielectric breakdown in air due to high potential difference between the spheres, sparks occur between the spheres. The electromagnetic wave is generated with its electric vector parallel to the gap. To detect such waves, Hertz used a circular wire with a gap and placed it near the spark gap. The electromagnetic waves reaching the gap of the open wire (detector) had an electric field strong enough to establish a high potential difference between the gap $G\,G'$ and cause a spark. The sensitivity of spark is maximum when the detector gap and source gap are parallel and is minimum or zero when they are at right angles to each other. It concludes that the em-waves are transverse. The frequencies of em-waves studied by Hertz are quite different from the frequencies of the visible light.

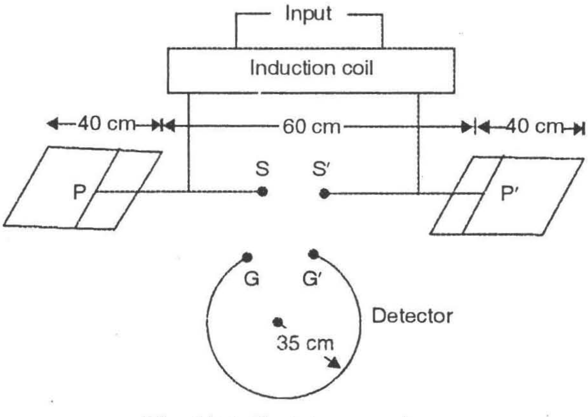


Fig. 15.1 Hertz' apparatus.

Maxwell's equations accurately describe the waves of all frequencies. The collection of electromagnetic waves of all frequencies is known as *electromagnetic spectrum*. The term *electromagnetic radiation* refers to the entire spectrum of em-waves including Gamma rays, X-rays, ultraviolet, visible, infrared radiations, microwaves and radio waves.

15.2 DIPOLE RADIATION

We know that accelerating charges produce electromagnetic waves. We refer to the systems in which accelerating charges initiate em-waves as broadcasting antennas and the systems in which we detect the response of charges to the fields of em-waves as *receiving antennas.Dipole antenna* is the simplest antenna formed by a charge that moves back and forth in harmonic motion along a line. The radiation emitted is called dipole radiation. The intensity of em-wave from a point source decreases as $1/r^2$. Actually there are no point like sources of electromagnetic radiation. The sun is not a point, but really a large collection of dipole antennas whose orientation is random, because charges with in it move and accelerate. The electric field that observer would see at an angle θ to the antenna direction is proportional to sin θ. Thus the intensity of radiation emitted by a dipole antenna along the direction of θ is proportional to $\sin\theta/r^2$.

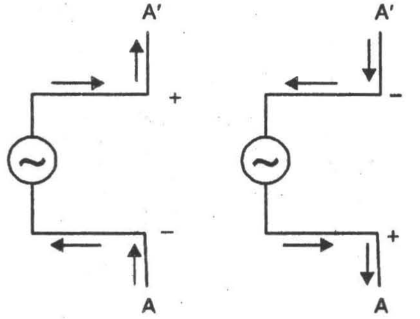


Fig. 15.2. Simple dipole antenna AA'.

A simple radio antenna that produces dipole radiation is shown in Fig. 15.2. The antenna is connected to an alternating voltage generator. The charge is accelerated in one direction and then in the other direction. Such a situation is equivalent to an oscillating electric dipole. The dipole moment at any instant is equal to either charge multiplied by the distance between them, which changes periodically. The distance is zero or the charges are together and hence the dipole moment **p** is zero at t = 0, $T/2$ and T. It is maximum at t = $T/4$ and $3T/4$. The variation of dipole moment with time is shown by the vertical lines in Fig. 15.3.

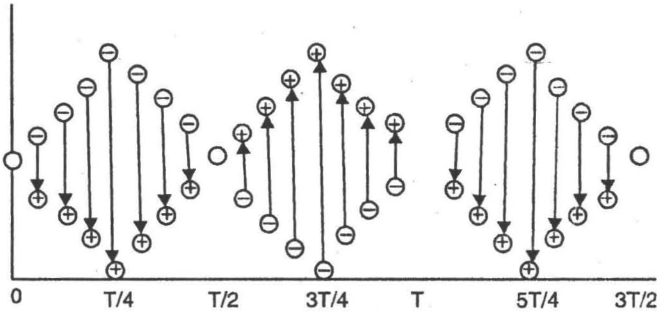


Fig. 15.3. Variation of dipole moment **p** with time t.

At time t = 0, dipole moment **p** is zero. As the two charges separate, the dipole moment **p** increases and the electric lines start from +charge and end on –charge, as shown in Fig. 15.4 (*a*). As the separation increases, dipole moment becomes maximum at the maximum separation and the field lines are as shown in Fig. 15.4(*b*). At $t > T/4$, the dipole moment starts decreasing, the electric lines are as shown in Fig. 15.4(*c*). At $t = T/2$, dipole moment is zero, the electric lines break away from the dipole as they do not have + and – charges to begin and end on. They continue to propagate. At $t > T/2$, the dipole moment **p** is reversed and the electric field is in the opposite direction **p** increases and then decreases, the lines of electric field will be similar but in reverse direction as for $0 < t < T/2$. At $t = T$, the old lines fold back to connect up to some of the new lines and form closed loops as shown in Fig. 15.4(*d*). Several loops are formed a little later. They continue to move outward.

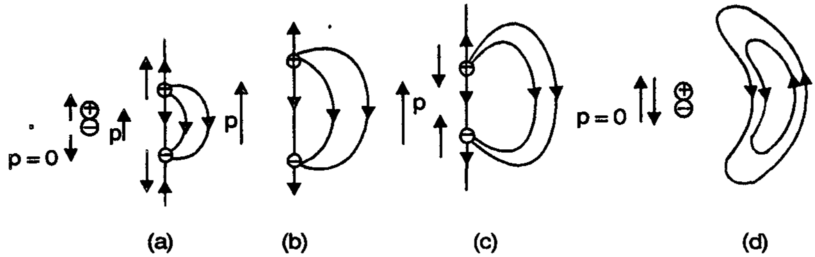


Fig. 15.4. Electric field lines due to periodic dipole.

The time varying dipole generates not only an electric field but also a magnetic field, as the moving charge constitutes a current and current gives to magnetic fields. The **E** and **B** are perpendicular to each other and to the direction of propagation.

The magnetic field at any point oscillates with the same frequency as the frequency of the oscillating current in the antenna. Fig. 15.4 shows the electric field in the right side only. The electric field lines on the left are the mirror image of those on the right. The magnetic field lines are also the closed loops in the plane perpendicular to the plane of E-lines. A complete pattern of electric and magnetic lines is shown in Fig. 15.5.

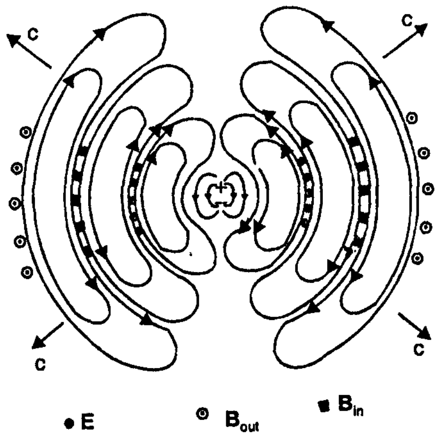


Fig. 15.5. Radiation pattern of **E** and **B** fields from an oscillating electric dipole.

15.3 RETARDED POTENTIALS

We have derived earlier the inhomogeneous wave equations for the vector and scalar potentials **A** and φ, as

$$\nabla^2\mathbf{A} - \mu\varepsilon\partial^2\mathbf{A}/\partial t^2 = -\mu\mathbf{J}_f \qquad ...(1)$$

$$\nabla^2\varphi - \mu\varepsilon\partial^2\varphi/\partial t^2 = -\rho_f/\varepsilon. \qquad ...(2)$$

These look just like the wave equation and have source terms on the right. In the static case, (1) and (2) reduce to Poisson's equations

$$\nabla^2\mathbf{A} = -\mu\mathbf{J}_f \text{ and } \nabla^2\varphi = -\rho_f/\varepsilon.$$

With the familiar solutions

$$\mathbf{A}(\mathbf{r}) = \frac{\mu}{4\pi}\int\frac{J_f(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau' \;,\; \varphi(\mathbf{r}) = \frac{1}{4\pi\varepsilon}\int\frac{\rho_f(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d\tau', \qquad ...(3)$$

where $|\mathbf{r} - \mathbf{r}'|$ is the distance from the source to the point P at which the potentials are evaluated. Because of the second time derivative terms in Eqns. (1) and (2), it is wrong to use the static solution by inserting time in the instantaneous solution (sources). The electromagnetic wave travels at the speed of light, one observes at a distance r and corresponding time t is not caused by the parameters of the source at that same time. The em-wave takes a time r/c to reach the observer. Thus, what the observer measures at time t is correlated to the sources at what is called the retarded time $t_r = t - r/c$.

Hence for non-static sources, the potentials are given as

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \text{ and } \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|}\,d\tau', \qquad ...(4)$$

where $t_r = t - |\mathbf{r} - \mathbf{r}'|/c$ is the retarded time for a given element of the distribution seated at $\mathbf{r}'$.

These are evaluated at the *retarded time* and are thus called *retarded potentials.* Here we see that the retarded potentials reduce properly to potentials in static case for which ρ and **J** are independent of time.

The retarded potentials satisfy wave equations (1) and (2). It will be wrong if we define the fields as

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|^3}|\mathbf{r}-\mathbf{r}'|\,d\tau'. \qquad ...(5)$$

and

$$\mathbf{B}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_r)}{|\mathbf{r}-\mathbf{r}'|^3}|\mathbf{r}-\mathbf{r}'|\,d\tau'. \qquad ...(6)$$

The electric and magnetic fields should be determined by using equations

$$\mathbf{E} = -\nabla\,\varphi\,(\mathbf{r}, t) - \partial\mathbf{A}\,(\mathbf{r}, t)/\partial t \qquad ...(7)$$

and

$$\mathbf{B} = \nabla \times \mathbf{A}\,(\mathbf{r}, t) \qquad ...(8)$$

respectively.

If the charge and current densities are evaluated at the advanced time $t_a = t + r/c$, the potentials are known as *advanced potentials* and defined as

$$\varphi_a(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\mathbf{r}', t_a)}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' \text{ and } \mathbf{A}_a(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t_a)}{|\mathbf{r}-\mathbf{r}'|}\,d\tau'. \qquad ...(9)$$

Advanced potentials are used along with retarded potentials for solving boundary value problems. They suggest that the potentials now depended on what the charge and current distribution will be at some time in the future, *i.e.*, the potential leads charges and currents. In other words *the effect precedes the cause.* Thus the advanced potentials have *no clear physical meaning,* they are of some theoretical interest.

15.4 RADIATION FROM AN OSCILLATING DIPOLE

A dipole consists of two tiny metal spheres separated by a short distance l and connected by a fine wire of negligible capacitance (Fig. 15.6). By short we mean that the length l is much smaller than the wave length of radiation emitted ($l << \lambda$). The system as a whole is assumed to be electrically neutral, so if at time t the charge on the upper sphere is $q(t)$, then the charge on the lower sphere is $-q(t)$. If the charge moves back and forth through the wire from one end to the other at a frequency ω, we may define

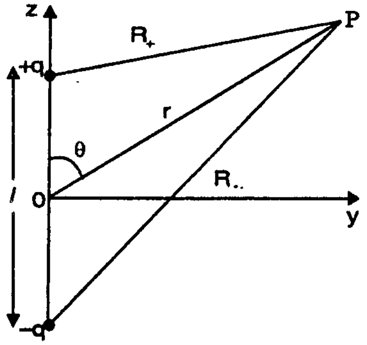


Fig. 15.6. Oscillating electric dipole.

$$q(t) = q_0 \cos \omega t. \qquad ...(10)$$

$$\therefore \quad \text{Dipole moment } \mathbf{p}(t) = q(t)\mathbf{l} = q_0 l \cos \omega t\,\mathbf{k} = p_0 \cos \omega t\,\mathbf{k} \qquad ...(11)$$

and

$$\text{current } I(t) = \frac{dq}{dt} = -q_0\,\omega \sin \omega t = -I_0 \sin \omega t, \qquad ...(12)$$

where $p_0 = q_0 l$ and $I_0 = q_0\,\omega$, the maximum values of the dipole moment and current respectively.

The retarded *scalar potential* at point P at an instant t is given by

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{q_0 \cos\omega(t - R_+/c)}{R_+} - \frac{q_0 \cos\omega(t - R_-/c)}{R_-}\right] \qquad ...(13)$$

From the geometry of Fig. 15.6. for $r >> l$, we get

$$R_\pm = \sqrt{r^2 \mp rl\cos\theta + (l/2)^2} = r\left(1 \mp \frac{1}{2}\frac{l}{r}\cos\theta\right)$$

or

$$\frac{1}{R_\pm} = \frac{1}{r}\left(1 \pm \frac{1}{2}\frac{l\cos\theta}{r}\right) \qquad ...(14)$$

$\therefore \cos\omega\,(t - R_\pm/c) = \cos\,[\omega(t - r/c) \pm \omega l \cos\theta/2c]$

$= \cos\omega\,(t - r/c)\cos\,(\omega l\cos\theta/2c) \mp \sin\omega\,(t - r/c)\sin\,(\omega l\cos\theta/2c)$

Since $l << \lambda$ or $l << c/\omega$, we get

$\cos\omega(t - R_\pm/c) = \cos\omega\,(t - r/c) \mp (\omega l\cos\theta/2c)\sin\omega(t - r/c).$ ...(15)

Putting values of $\cos\omega\,(t - R_\pm/c)$ and $1/R_\pm$ from Eqs. (15) and (14) in Eq. (13), we get

$$\varphi(\mathbf{r}, t) = \frac{p_0\cos\theta}{4\pi\varepsilon_0 r}\left[-\frac{\omega}{c}\sin\omega\left(t - \frac{r}{c}\right) + \frac{1}{r}\cos\omega\left(t - \frac{r}{c}\right)\right]$$

In the static limit $\omega = 0$ and $\varphi = p_0\cos\theta/4\pi\varepsilon_0 r^2$. It is same as proved earlier. Actually we are interested in the fields at large distances $r >> c/\omega$. In this region, known as radiation zone, the potential reduces to

$$\varphi(\mathbf{r}, t) = -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left(\frac{\cos\theta}{r}\right)\sin\omega\left(t - \frac{r}{c}\right). \qquad ...(16)$$

The retarded vector potential is determined by the current flowing in the wire

$$I(t) = dq/dt = -q_0\omega\sin\omega t.$$

Using the filamentary approximation $\mathbf{J}\,d\tau = Idz\,\mathbf{k}$, we get

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\mathbf{k}\int_{-l/2}^{l/2} -\frac{q_0\omega\sin\omega\,(t - r'/c)}{r'}\,dz', \qquad ...(17)$$

where dz' is the current element at a distance z' from the centre of the dipole and r' is the distance of point P from this element.

$$r' = (r^2 - 2z'r\cos\theta + z'^2)^{1/2} = r[1 - 2(z'/r)\cos\theta + (z'/r)^2]^{1/2}$$

Since we are interested in determining $\mathbf{A}$ at points for which $r >> l$, hence one can take $r' = r$ in the denominator.

Since $z'\cos\theta \leq l/2$ and $l << \lambda\ (= 2\pi c/\omega = cT)$, thus $cT >> z'\cos\theta$ or $\omega T >> (\omega z'/c)\cos\theta$.

With this argument, one can neglect the variation along the wire in the numerator.

$$\therefore \qquad \mathbf{A}(\mathbf{r}, t) = -\frac{\mu_0}{4\pi}\mathbf{k}\,\frac{q_0\omega\sin\omega\,(t - r/c)}{r}\int_{-l/2}^{l/2} dz' = -\frac{\mu_0 p_0\,\omega\sin\omega\,(t - r/c)}{4\pi r}\mathbf{k}. \qquad ...(18)$$

In spherical co-ordinates, we may write $\mathbf{k} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$. ...(19)

From Eq. (16), we get

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\hat{\theta}$$

$$= -\frac{p_0\omega}{4\pi\varepsilon_0 c}\left[\cos\theta\left\{-\frac{1}{r^2}\sin\omega\left(t-\frac{r}{c}\right)-\frac{\omega}{rc}\cos\omega\left(t-\frac{r}{c}\right)\right\}\hat{r}-\frac{\sin\theta}{r^2}\sin\omega\left(t-\frac{r}{c}\right)\hat{\theta}\right]$$

$$= \frac{p_0\omega^2}{4\pi\varepsilon_0 c^2}\frac{\cos\theta}{r}\cos\omega\left(t-\frac{r}{c}\right)\hat{r}. \qquad ...(20)$$

Here we have dropped the first and last terms for the fields at large distances ($r >> c/\omega$). Similarly from Eqs. (18) and (19), we get

$$\frac{\partial \mathbf{A}}{\partial t} = -\frac{\mu_0 p_0\omega^2}{4\pi r}\cos\omega\left(t-\frac{r}{c}\right)(\cos\theta\,\hat{r}-\sin\theta\,\hat{\theta}). \qquad ...(21)$$

$$\therefore \qquad \mathbf{E} = -\nabla\varphi - \frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0\, p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\theta} \qquad ...(22)$$

Since

$$\nabla\times\mathbf{A} = \frac{1}{r}\left[\frac{\partial}{\partial r}(r\,A_\theta)-\frac{\partial A_r}{\partial\theta}\right]\hat{\phi}$$

$$= -\frac{\mu_0\, p_0\omega}{4\pi r}\left[\frac{\omega}{c}\sin\theta\cos\omega\left(t-\frac{r}{c}\right)+\frac{\sin\theta}{r}\sin\omega\left(t-\frac{r}{c}\right)\right]\hat{\phi}.$$

The second term is again neglected at $r >> c/\omega$. Thus

$$\mathbf{B} = \nabla\times\mathbf{A} \simeq -\frac{\mu_0\, p_0\omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\hat{\phi}. \qquad ...(23)$$

Eqs. (22) and (23) show that **E** and **B** are in phase, mutually perpendicular and transverse. These are monochromatic waves of frequency ω travelling in the radial direction at the speed of light. The ratio of their amplitudes is given by $E_0/B_0 = c$. These are actually spherical waves and look like plane waves for large r over small regions, just as the earth surface.

The energy radiated by the oscillating electric dipole is given by Poynting vector as

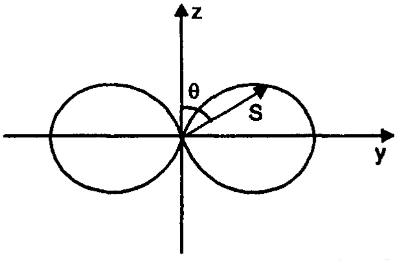


Fig. 15.7. Intensity profile for dipole radiation.

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{c}\left[\frac{p_0\omega^2}{4\pi}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t-\frac{r}{c}\right)\right]^2\hat{r} \qquad ...(24)$$

The intensity, which is the time average of **S** over a complete cycle is given by

$$<\mathbf{S}> = \frac{\mu_0 p_0^2\omega^4}{32\,\pi^2\,c}\frac{\sin^2\theta}{r^2}\hat{r}. \qquad ...(25)$$

Thus we see that there is no radiation along the axis of the dipole ($\theta = 0$), it is maximum along equatorial plane. The profile of intensity look like a donut (Fig. 15.7). The total power radiated

$$<\mathbf{P}_E> = \int <\mathbf{S}> d\tau = \int_\tau <\mathbf{S}> r^2\sin\theta\,d\theta\,d\phi = \frac{\mu_0 p_0^2\omega^4}{32\,\pi^2 c}\int_0^\pi\frac{\sin^2\theta}{r^2}r^2\sin\theta d\theta\int_0^{2\pi}d\phi$$

$$= \frac{\mu_0 p_0^2}{12\pi c}\omega^4 = \frac{p_0^2\omega^4}{(4\pi\varepsilon_0)(3c^3)}. \qquad ...(26)$$

It is independent of the radius of the sphere as expected from conservation of energy.

15.5 MAGNETIC DIPOLE RADIATION

Consider a wire loop of radius a carrying a sinusoidally varying current $I = I_0 \cos \omega t$. This acts as oscillating magnetic dipole of magnetic moment

$$\mathbf{m}(t) = \pi a^2 I(t)\,\mathbf{k} = m_0 \cos \omega t\,\mathbf{k}, \qquad \text{...(27)}$$

where $m_0 = \pi a^2 I_0 = m_{\max}$.

The scalar potential is zero because the loop is uncharged.

The retarded vector potential at point P is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{I_0 \cos \omega (t - R/c)}{R}\, dl \qquad \text{...(28)}$$

For the point P directly above the x-axis (Fig. 15.8), **A** aims in the *y-direction*, x-components will cancel on account of symmetry. Thus

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} I_0 a \mathbf{j} \int_0^{2\pi} \frac{\cos \omega (t - R/c)}{R} \cos \phi \, d\phi. \qquad \text{...(29)}$$

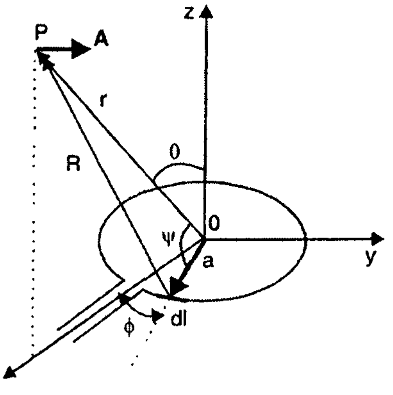


Fig. 15.8. Oscillating magnetic dipole.

From the geometry of Fig. 15.8 we get

$R^2 = r^2 + a^2 - 2ra \cos \psi = r^2 + a^2 - 2ra \sin \theta \cos \phi$

$(\because \mathbf{r} = r \sin \theta \mathbf{i} + r \cos \theta \mathbf{k}$ and $\mathbf{a} = a \cos \phi \mathbf{i} + a \sin \phi \mathbf{j})$

For a perfect dipole, $a << r$, therefore

$$R = r\left(1 - \frac{a}{r} \sin \theta \cos \phi\right) \text{ or } \frac{1}{R} = \frac{1}{r}\left(1 + \frac{a}{r} \sin \theta \cos \phi\right) \qquad \text{...(30)}$$

$$\text{and } \cos \omega\left(t - \frac{R}{c}\right) = \cos\left[\omega\left(t - \frac{r}{c}\right) + \frac{\omega a}{c} \sin \theta \cos \phi\right]$$

$$= \cos \omega\left(t - \frac{r}{c}\right) \cos\left(\frac{\omega a}{c} \sin \theta \cos \phi\right) - \sin \omega\left(t - \frac{r}{c}\right) \sin\left(\frac{\omega a}{c} \sin \theta \cos \phi\right).$$

Since the radius must be much less than the wavelength generated, $a << c/\omega$, therefore we have

$$\cos \omega\left(t - \frac{R}{c}\right) = \cos \omega\left(t - \frac{r}{c}\right) - \frac{\omega a}{c} \sin \theta \cos \phi \sin \omega\left(t - \frac{r}{c}\right). \qquad \text{...(31)}$$

Putting the values of $\cos \omega (t - R/c)$ and $1/R$ from Eqs. (30) and (31) in Eq. (29), we get

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 I_0 a}{4\pi r} \mathbf{j} \int_0^{2\pi} \left[\cos \omega\left(t - \frac{r}{c}\right) + a \sin \theta \cos \phi \left\{\frac{1}{r} \cos \omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c} \sin \omega\left(t - \frac{r}{c}\right)\right\} - \frac{\omega a^2}{rc} \sin^2 \theta \cos^2 \phi \sin \omega\left(t - \frac{r}{c}\right)\right] \cos \phi \, d\phi$$

Since $\int_0^{2\pi} \cos \phi \, d\phi = 0$, $\int_0^{2\pi} \cos^2 \phi \, d\phi = \pi$ and $\int_0^{2\pi} \cos^3 \phi \, d\phi = 0$, therefore

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0 m_0}{4\pi} \left(\frac{\sin \theta}{r}\right) \left[\frac{1}{r} \cos \omega\left(t - \frac{r}{c}\right) - \frac{\omega}{c} \sin \omega\left(t - \frac{r}{c}\right)\right] \hat{\phi} \qquad \text{...(32)}$$

Here we have replaced **j** by $\hat{\phi}$, as in general **A** points in the $\hat{\phi}$-direction.

In the radiation zone ($r >> c/\omega$), we can neglect the first term and thus have

$$\mathbf{A}(\mathbf{r}, \theta, t) = -\frac{\mu_0 m_0 \omega}{4\pi c}\left(\frac{\sin\theta}{r}\right)\sin\omega\left(t - \frac{r}{c}\right)\hat{\phi}. \qquad \text{...(33)}$$

The electric field

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} = \frac{\mu_0 m_0 \omega^2}{4\pi c}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t - \frac{r}{c}\right)\hat{\phi} \qquad \text{...(34)}$$

and the magnetic field

$$\mathbf{B} = \nabla \times \mathbf{A} = -\frac{1}{r}\frac{\partial}{\partial r}(r A_\phi)\,\hat{\theta} = -\frac{\mu_0 m_0 \omega^2}{4\pi c^2}\left(\frac{\sin\theta}{r}\right)\cos\omega\left(t - \frac{r}{c}\right)\hat{\theta}. \qquad \text{...(35)}$$

In the derivation of **B**, we have used the condition ($r >> c/\omega$). The relations (34) and (35) show that **E** and **B** are in phase, mutually perpendicular and transverse to the direction of propagation **r**. The ratio of their amplitudes is c. Thus we see that the fields of magnetic dipole are similar to the fields of an *oscillating electric dipole,* except that the directions of **B** and **E** are interchanged.

The energy flux for magnetic dipole radiation

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B}) = \frac{\mu_0}{c}\left[\frac{m_0\omega^2}{4\pi c}\,\frac{\sin\theta}{r}\cos\omega\left(t - \frac{r}{c}\right)\right]^2 \hat{r} \qquad \text{...(36)}$$

and the intensity

$$<S> = \frac{\mu_0 {m_0}^2 \omega^4}{32\,\pi^2 c^3}\,\frac{\sin^2\theta}{r^2} \qquad \text{...(37)}$$

The total radiated power $= \int_0^\pi <S> r^2 \sin\theta\, d\theta \int_0^{2\pi} d\phi$

$$<P_m> = \frac{\mu_0 {m_0}^2 \omega^4}{12\pi c^3} = \frac{1}{4\pi\varepsilon_0}\,\frac{{m_0}^2\omega^4}{3c^5}. \qquad \text{...(38)}$$

$$\therefore \qquad \frac{<P_m>}{<P_E>} = \left(\frac{m_0}{p_0 c}\right)^2 = \left(\frac{\pi a^2 I_0}{q_0 l c}\right)^2 = \left(\frac{\pi a^2 q_0 \omega}{q_0 c(\pi a)}\right)^2 = \left(\frac{a\omega}{c}\right)^2 \qquad \text{...(39)}$$

(Assuming $l = \pi a$ for the sake of comparison)

Since $a\omega/c$ is assumed to be very small, the magnetic dipole radiation is negligible and thus the electric dipole radiation should dominate over magnetic dipole radiation.

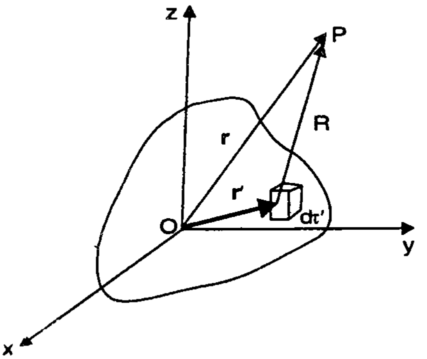


Fig. 15.9. Arbitrary distribution of charges.

15.6 RADIATION FROM AN ARBITRARY DISTRIBUTION OF CHARGES

Let us now find scalar and vector potentials due to an arbitrary distribution of charges and currents, which is localized with in some small region near the origin (Fig. 15.9).

The scalar potential at point P at any instant t due to charge on an elementary volume $d\tau'$ is given by

$$d\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{\rho(\mathbf{r}', t - R/c)}{R} d\tau'$$

Therefore due to the whole distribution of charge we have

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho(\mathbf{r}', t - R/c)}{R} d\tau' \qquad ...(40)$$

where $R = (r^2 + r'^2 - 2\mathbf{r}\cdot\mathbf{r}')^{1/2}$.

For the point of observation P far away, such that $r >> r'_{\max}$, we get

$$R = r\left(1 - \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right) \text{and} \frac{1}{R} = \frac{1}{r}\left(1 + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^2}\right), \qquad ...(41)$$

and $$\rho\left(\mathbf{r}', t - \frac{R}{c}\right) = \rho\left(\mathbf{r}', t - \frac{r}{c} + \frac{\mathbf{r}\cdot\mathbf{r}'}{rc}\right)$$

Expanding ρ as a Taylor series in the variable t, about the retarded time at the origin

$$t_0 = t - r/c. \qquad ...(42)$$

we have $\rho\left(\mathbf{r}', t - \frac{R}{c}\right) = \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\frac{\mathbf{r}\cdot\mathbf{r}'}{rc} + \frac{1}{2}\ddot{\rho}(\mathbf{r}', t_0)\left(\frac{\mathbf{r}\cdot\mathbf{r}'}{rc}\right)^2 + ...$

For an oscillating system each of ratios $\ddot{\rho}/\dot{\rho}, (\dddot{\rho}/\dot{\rho})^{1/2}, (\ddddot{\rho}/\dot{\rho})^{1/3}$ is ω, thus for $r' << c/|\ddot{\rho}/\dot{\rho}|, c/|\dddot{\rho}/\dot{\rho}|^{1/2}, c/|\ddddot{\rho}/\dot{\rho}|^{1/3}$, we may neglect third, fourth and so on terms of Taylor's expansion. We therefore have

$$\rho\left(\mathbf{r}', t - \frac{R}{c}\right) = \rho(\mathbf{r}', t_0) + \dot{\rho}(\mathbf{r}', t_0)\frac{\hat{r}\cdot\mathbf{r}'}{c}. \qquad ...(43)$$

Putting the values from Eqs. (41) and (43) in Eq. (40), and neglecting second order term, we get

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0 r}\left[\int \rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{r}}{r}.\int \mathbf{r}'\rho(\mathbf{r}', t_0)\, d\tau' + \frac{\hat{r}}{c}.\frac{d}{dt}\int \mathbf{r}'\rho(\mathbf{r}', t_0)\, d\tau'\right]$$

The first integral represents the total charge Q at time t_0, which is actually time independent. The other two integrals represent the electric dipole moment $\mathbf{p}$ (t_0) at time t_0. Thus

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\left[\frac{Q}{r} + \frac{\hat{r}\cdot\mathbf{p}(t_0)}{r^2} + \frac{\hat{r}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right] \qquad ...(44)$$

Thus we see that the arbitrary distribution of charges produces multipole expansion of the retarded potentials. The Eq. (44) represents the first three expansion terms. Higher order terms such as the quadrupole terms and so on depend on high moments of the distribution. Due to some symmetry, if these three terms vanish, the next higher order term will dominate.

The vector potential at point P is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t - R/c)}{R} d\tau'. \qquad ...(45)$$

To first order in r', it is proper to replace R by r in the integrand. The Eq. (45) may thus be written as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{J}(\mathbf{r}', t_0)}{r} d\tau'. \qquad ...(46)$$

One may prove easily that for a configuration of charges and currents confined within a volume V, we have

$$\int_{\tau} \mathbf{J} d\tau = d\mathbf{p}/dt, \text{ where } \mathbf{p} \text{ is the total dipole moment.}$$

$$\therefore \qquad \mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{\dot{\mathbf{p}}(t_0)}{r}. \qquad \text{...(47)}$$

To find the fields, let us find

$$\nabla\varphi = \frac{1}{4\pi\varepsilon_0}\left[-\frac{Q}{r^2}\hat{r} + \nabla\left(\frac{\hat{r}\cdot\mathbf{p}(t_0)}{r^2}\right) + \nabla\left(\frac{\hat{r}\cdot\dot{\mathbf{p}}(t_0)}{rc}\right)\right] \qquad \text{...(48)}$$

The first term does not contribute to the electromagnetic radiation. The radiation comes entirely from those terms in which we differentiate the argument t_0. In the radiation zone (*i.e.*, $r >> r'$), we should discard the terms proportional to $1/r^2$ in **E** and **B**. Therefore we need not require to consider the second term. Thus we are left with

$$\nabla\varphi = \frac{1}{4\pi\varepsilon_0}\nabla\left[\frac{\hat{r}}{rc}\cdot\dot{\mathbf{p}}(t_0)\right] \qquad \text{...(49)}$$

Using the vector identity for the grad of scalar product of two vector **A** and **B**

$$\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\,\mathbf{A} + \mathbf{B}\times(\nabla\times\mathbf{A}) + (\mathbf{A}\cdot\nabla)\,\mathbf{B} + \mathbf{A}\times(\nabla\times\mathbf{B})$$

and the vector property $\nabla \times \mathbf{r} = 0$, we get

$$\nabla\varphi = \frac{1}{4\pi\varepsilon_0}\left[(\dot{\mathbf{p}}(t_0)\cdot\nabla)\frac{\hat{r}}{rc} + \left(\frac{\hat{r}}{rc}\cdot\nabla\right)\dot{\mathbf{p}}(t_0) + \frac{\hat{r}}{rc}\times(\nabla\times\dot{\mathbf{p}}(t_0))\right] \qquad \text{...(50)}$$

Since **p** and $\dot{\mathbf{p}}$ are functions of $t - r/c$, the ∇ operation on them can be replaced by $\hat{r}$ $(\partial/\partial r)$ and

$$\frac{\partial}{\partial r}\dot{\mathbf{p}} = -\frac{1}{c}\ddot{\mathbf{p}}, \qquad \text{...(51)}$$

$$\therefore \qquad \nabla\varphi = \frac{1}{4\pi\varepsilon_0}\left[-\left(\dot{\mathbf{p}}(t_0)\cdot\frac{\hat{r}}{r^2c}\right)\hat{r} + \frac{\hat{r}}{rc}\cdot\ddot{\mathbf{p}}(t_0)\left(-\frac{1}{c}\right) + \frac{\hat{r}}{rc}\times\hat{r}\times\frac{\partial\dot{\mathbf{p}}(t)}{\partial r}\right]$$

$$= \frac{1}{4\pi\varepsilon_0}\left[-\left(\dot{\mathbf{p}}(t_0)\cdot\frac{\hat{r}}{r^2c}\right)\hat{r} - \frac{\hat{r}}{rc^2}\cdot\ddot{\mathbf{p}}(t_0) - \frac{\hat{r}}{rc^2}\times\hat{r}\times\ddot{\mathbf{p}}\right]$$

$$= -\frac{1}{4\pi\varepsilon_0}\left[\frac{1}{r^2c}(\dot{\mathbf{p}}(t_0)\cdot\hat{r})\,\hat{r} + \frac{\hat{r}}{rc^2}\cdot\ddot{\mathbf{p}}(t_0)\right] \qquad \text{...(52)}$$

Only the second term, having $1/r$ dependence will contribute to radiation field. It depends on the second derivative of the dipole moment and thus depends on the acceleration of the charge distribution.

On differentiating Eq. (47) with respect to t_0, we get

$$\frac{\partial\mathbf{A}}{\partial t} = \frac{\mu_0}{4\pi}\frac{\ddot{\mathbf{p}}(t_0)}{r}. \qquad \text{...(53)}$$

$$\mathbf{E}(\mathbf{r}, t) = \frac{\mu_0}{4\pi r}\left[\hat{r}\cdot\ddot{\mathbf{p}}(t_0)\,\hat{r} - \ddot{\mathbf{p}}(t_0)\right] = \frac{\mu_0}{4\pi r}\hat{r}\times\hat{r}\times\ddot{\mathbf{p}}(t_0). \qquad \text{...(54)}$$

$$\begin{aligned}\text{curl}\,\mathbf{A} &= \nabla\times\left[\frac{\mu_0}{4\pi}\cdot\frac{1}{r}\,\dot{\mathbf{p}}\,(t_0)\right] = \frac{\mu_0}{4\pi}\left[\nabla\,\frac{1}{r}\times\dot{\mathbf{p}}\,(t_0)+\frac{1}{r}\,\nabla\times\dot{\mathbf{p}}\,(t_0)\right]\\ &= \frac{\mu_0}{4\pi}\left[-\frac{1}{r^2}\,\hat{r}\times\dot{\mathbf{p}}\;(t_0)+\frac{1}{r}\,\hat{r}\times\frac{\partial}{\partial r}\,(\dot{\mathbf{p}}\;(t_0))\right]\\ &= \frac{\mu_0}{4\pi}\left[-\frac{1}{r^2}\,\hat{r}\times\dot{\mathbf{p}}\;(t_0)-\frac{1}{cr}\,\hat{r}\times\ddot{\mathbf{p}}\;\;(t_0)\right] \qquad \text{...(55)}\end{aligned}$$

On neglecting the first term, as it falls off more rapidly than r^{-1}, we get

$$\mathbf{B}(\mathbf{r},t) = \text{curl}\,\mathbf{A}(\mathbf{r},t) = -\frac{\mu_0}{4\pi cr}\left[\hat{r}\times\ddot{\mathbf{p}}\;\;(t_0)\right] \qquad \text{...(56)}$$

$$\therefore \qquad \mathbf{E}(\mathbf{r},t) = -c\hat{r}\times\mathbf{B}\,(\mathbf{r},t). \qquad \text{...(57)}$$

If we choose spherical polar coordinates with the z-axis in the direction of $\ddot{\mathbf{p}}\;\,(t_0)$, then

$$\mathbf{E}(r,\theta,t) = \frac{\mu_0}{4\pi}\,\ddot{p}\,(t_0)\left(\frac{\sin\theta}{r}\right)\hat{\theta} \qquad \text{...(58)}$$

and

$$\mathbf{B}(r,\theta,t) = \frac{\mu_0}{4\pi c}\,\ddot{p}\,(t_0)\left(\frac{\sin\theta}{r}\right)\hat{\phi}. \qquad \text{...(59)}$$

$$\begin{aligned}\therefore \qquad \mathbf{S} &= \frac{1}{\mu_0}\,(\mathbf{E}\times\mathbf{B}) = -\frac{c}{\mu_0}\,(\hat{r}\times\mathbf{B})\times\mathbf{B} = \frac{c}{\mu_0}\,\hat{r}\,|\,\mathbf{B}\,|^2\\ &= \frac{\mu_0}{16\pi^2 c}\,[\ddot{p}\,(t_0)]^2\,\frac{\sin^2\theta}{r^2}\,\hat{r} \qquad \text{...(60)}\end{aligned}$$

and the total power radiated through a sphere of radius r is

$$\begin{aligned}P &= \int <S> d\tau = \frac{\mu_0}{16\pi^2 c}\,[\ddot{p}\,(t_0)]^2\int_0^{\pi}\frac{\sin^2\theta}{r^2}\,.\,r^2\sin\theta\,d\theta\int_0^{2\pi}d\phi\\ &= \frac{\mu_0}{6\pi c}\,[\ddot{p}\,(t_0)]^2 = \frac{1}{4\pi\varepsilon_0}\cdot\frac{2}{3c^3}\,.\,[\ddot{p}\,(t_0)]^2. \qquad \text{...(61)}\end{aligned}$$

We here see that **E** and **B** are mutually perpendicular, transverse to the direction of propagation $\hat{r}$ and in the ratio $E/B = c$, as always for radiation field.

For a *single point charge q,* the dipole moment is

$$\mathbf{p}(t) = q\mathbf{z}(t), \qquad \text{...(62)}$$

where **z** is its position with respect to the origin. Thus we have

$$\ddot{\mathbf{p}}\,(t) = q\,\ddot{\mathbf{z}}\,(t) = q\,\mathbf{a}\,(t),$$

where **a** is the acceleration of the charge.

In this case the power radiated is given by

$$P = \frac{1}{4\pi\varepsilon_0}\,\frac{2}{3}\,\frac{q^2a^2}{c^3}. \qquad \text{...(64)}$$

This is known as the *Larmor formula.*

15.7 ANTENNA CHARACTERISTICS

(*a*) Antenna Patterns : An antenna pattern is a three dimensional plot of its radiation at far field. When the amplitude of **E** field is plotted, it is called field pattern or voltage pattern. When the square of the amplitude of **E** is plotted, it is called the *power pattern.*

(b) Radiation Intensity : The radiation intensity of an antenna is defined as

$$U(\theta,\phi) = r^2 P_{ave}. \qquad \text{...(65)}$$

The total average power radiated can be expressed as

$$P_{rad} = \int_S P_{ave}\, dS = \int_S P_{ave}\, r^2 \sin\theta\, d\theta\, d\phi$$

$$= \int_S U(\theta,\phi) \sin\theta\, d\theta\, d\phi = \int U(\theta,\phi)\, d\Omega.$$

$$\therefore \qquad U_{ave} = P_{rad}/4\pi. \qquad \text{...(66)}$$

(c) Directive Gain : The directive gain of an antenna is a measure of the concentration of the radiated power in a particular direction (θ, ϕ),

$$G_d(\theta,\phi) = \frac{U(\theta,\phi)}{U_{ave}} = \frac{4\pi U(\theta,\phi)}{P_{rad}}. \qquad \text{...(67)}$$

and

$$P_{ave} = (G_d/4\pi r^2)\, P_{rad}. \qquad \text{...(68)}$$

(d) Power Gain : The *radiation efficiency* of an antenna is $\epsilon_{rad} = P_{rad}/P_{in}$, where P_{in} is the average power that the antenna accepts from the feed. The product of the radiation efficiency with the directive gain is called the *power gain.*

$$G_p(\theta,\phi) = \epsilon_{rad} \cdot G_d(\theta,\phi) = \frac{4\pi U(\theta,\phi)}{P_{in}}, \qquad \text{...(69)}$$

$P_{in} = P_{rad} + P_L$, where P_L the ohmic loss of the antenna.

15.8 LIENARD-WEICHERT POTENTIALS

The retarded potentials describe the potentials of arbitrary charge and current distributions $\rho(\mathbf{r}', t_r)$ and $\mathbf{J}(\mathbf{r}', t_r)$. The fields produced by a fast moving point charge can be calculated from these potentials. If the charge is not a point charge, which is actually the case, the different portions of the charge distribution require different retarded times. Thus $\int \rho(\mathbf{r}', t_r)\, d\tau'$ does not in general represent the total charge of the system. It is because the various contributions are evaluated at different times and during the time, the charge has moved either towards or away from the field point.

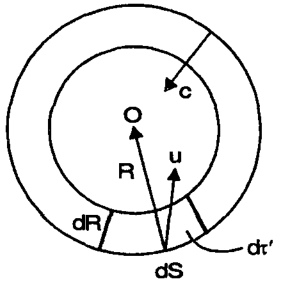


Fig. 15.10. Element dS of sphere collecting information from charge.

To understand the variation of charge, let us assume that a group of census takers converges upon the information centre with a certain speed, and counts the people every day as they travel. The correct population will differ from the observed number if the population has a net migration trend. The reported population will be greater than the true population if the migration trend is with the census takers. Similarly, the retarded potential of an approaching charge will be large than that of a receding charge at the same distance from the field point.

Let us assume that a distributed total charge q collapses towards the field point O with a velocity $\mathbf{u}$ which is comparable to c. Let $\mathbf{R}$ be the radius vector from the charge distribution to the field point O (Fig. 15.10). $\mathbf{R}$ is the retarded value of the radius. If the charge distribution is at rest, the amount of charge that the sphere will cross during the time dt as it shrinks by dR is $\rho\, dS\, dR$. If $\mathbf{u}$ is different from zero, the amount of charge that the sphere will cross will be

$$dq = \rho\, dS\, dR - \rho \frac{\mathbf{u}\cdot\mathbf{R}}{R}\, dS\, dt.$$

Since $dt = dR/c,\ dS\, dR = d\tau'$, therefore

$$dq = \rho\, d\tau' - \rho \frac{\mathbf{u}\cdot\mathbf{R}}{Rc}\, d\tau' \text{ or } \rho\, d\tau' = \frac{dq}{1 - \mathbf{u}\cdot\mathbf{R}/Rc}. \qquad \ldots(70)$$

Thus the retarded potentials become

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \int \frac{dq}{R - \mathbf{R}\cdot\mathbf{u}/c} \qquad \ldots(71)$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \int \frac{\mathbf{u}\, dq}{R - \mathbf{R}\cdot\mathbf{u}/c} \quad (\because \text{ current density } \mathbf{J} = \rho\, \mathbf{u}) \qquad \ldots(72)$$

For a point charge $\int dq = q$, therefore the potentials of a point charge are

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0} \frac{q}{R - \mathbf{R}\cdot\mathbf{u}/c}$$

and

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi} \frac{q\mathbf{u}}{R - \mathbf{R}\cdot\mathbf{u}/c} \qquad \ldots(73)$$

These relations are known as *Lienard-Wiechert potentials* in memory of the two scientists who developed these relations for a moving point charge. These are independent of the extent of the charge. Here the velocity **u** is that of the charge in its retarded position.

The field of a point charge in motion may be obtained with the relations

$$\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t \text{ and } \mathbf{B} = \nabla \times \mathbf{A}, \qquad \ldots(74)$$

where φ and **A** are given by Eqs. (73). The actual calculations are quite lengthy and complicated and therefore not solved here.

The electric field $\mathbf{E}(\mathbf{r}, t)$ is given by

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0} \frac{R}{(\mathbf{R}\cdot\mathbf{v})^3} [\mathbf{v}(c^2 - u^2) + \mathbf{R}\times(\mathbf{v}\times\mathbf{a})] \qquad \ldots(75)$$

where the quantity $\mathbf{v} = c\hat{R} - \mathbf{u}$.

Relation (75) shows that the first term falls off as the inverse square of the distance from the particle. It is independent of acceleration and is therefore known as *generalized Coulomb field*. For the velocity and acceleration both zero, this term reduces to the electrostatic value

$$\mathbf{E} = (q/4\pi\varepsilon_0 R^2)\hat{R}.$$

The second term of Eq. (75) falls off as the inverse of R and is dominant at large distances. It is responsible for electromagnetic radiation and is therefore called *radiation field* or *acceleration field* as it is proportional to the acceleration a.

The magnetic field $$\mathbf{B}(\mathbf{r}, t) = \hat{R} \times \mathbf{E}/c, \qquad \ldots(76)$$

which shows that the *magnetic field of a point charge is always perpendicular to the electric field and to the vector from the retarded point,* the direction of propagation $\hat{R}$.

EXERCISES

Example 1. *Calculate the retarded potential of an infinite straight filamentary current. Find the resulting electric and magnetic fields.*

A constant current I_0 is started at $t = 0$ in an infinite, straight filamentary conducting wire, as shown in Fig. 15.11. The wire is presumably electrically neutral, charge density $\rho = 0$ everywhere in the wire. The scalar potential produced by the current at the observation point P is therefore zero. Using the filamentary approximation $\mathbf{J}\,dv = I\,dz$ and taking $I(t) = I_0$ for $t \geq 0$ and $I(t) = 0$ for $t < 0$, the retarded vector potential at point P is given as

$$\mathbf{A}(\mathbf{r}, t) = \frac{\mu_0}{4\pi}\mathbf{k}\int_{-\infty}^{\infty}\frac{I(t - R/c)}{R}\,dz.$$

Here we have assumed that the wire lies along the z-axis.

Because of the finite speed of the disturbance, the retarded current is restricted. An element will contribute only if its distance from the observation point is less than ct. Thus

$$\begin{aligned} I(t - R/c) &= 0 && \text{where } |z|^2 \geq c^2t^2 - r^2 \\ &= I_0 && \text{when } |z|^2 < c^2t^2 - r^2 \end{aligned}$$

$$\begin{aligned} \text{Thus,}\quad \mathbf{A}(\mathbf{r}, t) &= 0 \qquad \text{for } r \geq ct \\ &= \frac{\mu_0 I_0}{4\pi}\mathbf{k}\int\frac{dz}{(z^2 + r^2)^{1/2}} \qquad \text{for } r < ct \\ &= \frac{2\mu_0 I_0}{4\pi}\mathbf{k}\,[\log_e[z + (z^2 + r^2)^{1/2}]_0^{(c^2t^2 - r^2)^{1/2}} \\ &= \frac{\mu_0 I_0}{2\pi}\mathbf{k}\log_e\left(\frac{ct + (c^2t^2 - r^2)^{1/2}}{r}\right) \end{aligned}$$

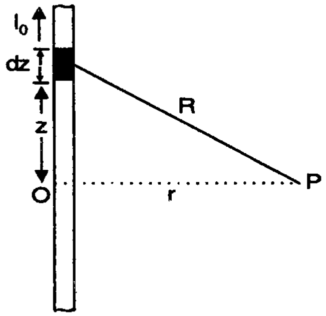


Fig. 15.11. Filamentary current.

The electric field

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\partial\mathbf{A}}{\partial t} = -\frac{\mu_0 I_0 c}{2\pi(c^2t^2 - r^2)^{1/2}}\mathbf{k}.$$

The magnetic field $\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A} = -\dfrac{\partial A_z}{\partial r}\hat{\phi} = \dfrac{\mu_0 I_0}{2\pi r}\dfrac{ct}{(c^2t^2 - r^2)^{1/2}}\hat{\phi}.$

Here the curl is evaluated in cylindrical coordinates.

At $t \to \infty$, we get $\mathbf{E} = 0$ and $\mathbf{B} = (\mu_0 I_0/2\pi r)\,\hat{\phi}$, the results of the static case.

Example 2. *A Hertzian dipole of length 5 cm is driven at a frequency $\nu = 150$ MHz. The peak current in the antenna is 30A. Determine the average radiated power and the effective resistance of the antenna.*

We know that the total power radiated by a charge q moving with uniform acceleration a is given by $P = q^2a^2/6\pi\varepsilon_0c^3$.

For a dipole of length l and driven at a frequency ν, we may substitute $I_0 l\,\omega$ for the peak value of qa. Thus the peak power is

$$P_0 = I_0^2 l^2\omega^2/6\pi\varepsilon_0 \cdot c^3.$$

The average power $\qquad P_{av} = \dfrac{1}{2}P_0 = \dfrac{I_0^2 l^2}{12\pi\varepsilon_0 c^3}\left(\dfrac{2\pi c}{\lambda}\right)^2 = \dfrac{\pi I_0^2 l^2}{3\varepsilon_0 c\lambda^2}$

In this problem $\lambda = c/\nu = 3 \times 10^8/150 \times 10^6 = 2$m, $I_0 = 30$A, $l = 5 \times 10^{-2}$m. Therefore

$$P_{av} = \frac{3.14 \times 30^2 \times (5 \times 10^{-2})^2}{3 \times (8.85 \times 10^{-12}) \times 3 \times 10^8 \times 2^2} = 221.9\ W.$$

The *effective resistance,* also called *radiation resistance,* R_{rad} of an antenna is defined by the relationship

$$P_{av} = \tfrac{1}{2} I_0^2 R_{rad}$$

$$\therefore \quad R_{rad} = \frac{2\pi(l)^2}{3\varepsilon_0 c\lambda^2} = 788\left(\frac{l}{\lambda}\right)^2 = 788\left(\frac{5\times10^{-2}}{2}\right)^2$$

$$= 0493\Omega.$$

Example 3. *Estimate the rate of energy radiation per unit area from each side of a thin uniform alternating current sheet. What effective radiation resistance in ohms is acting on a square area of this current sheet.*

Consider a current sheet in x-y plane, its width z-direction. If the current is along y-axis, the current per unit width be given as $\lambda = \lambda_0 e^{-i\omega t}\mathbf{a}_y$.

Consider a unit square area with sides parallel to x and y-axes. At large distances from the current sheet, the current in the area may be considered as a Hertzian dipole of dipole moment $\mathbf{p}$ given by

$$\dot{\mathbf{p}} = \lambda_0 e^{-i\omega t}\mathbf{a}_y.$$

Hence the power radiated averaged over one period, from unit area of the sheet is given by

$$P = \frac{1}{4\pi\varepsilon_0}\frac{\omega^2|\dot{\mathbf{p}}|^2}{3c^3} = \frac{\lambda_0^2\omega^2}{12\pi\varepsilon_0 c^3}.$$

As the thickness of the sheet is very small, the radiation is emitted mainly from the top and bottom surfaces. Therefore the power radiated per unit area from each side of the thin sheet is

$$P/2 = \lambda_0^2\omega^2/24\pi\varepsilon_0 c^3.$$

The average power is related to the amplitude of the ac current by

$$P = \tfrac{1}{2} I^2 R.$$

$$\therefore \text{ Effective radiation resistance } R = \frac{2P}{I^2} = \frac{2P}{\lambda_0^2} = \frac{\omega^2}{6\pi\varepsilon_0 c^3}.$$

Example 4. *An oscillating electric dipole $p(t)$ develops radiation field*

$$B(r,t) = -\frac{\mu_0}{4\pi rc}\mathbf{a}_r \times \frac{\partial^2}{\partial t^2} p(t - r/c) \text{ and } E(r,t) = -c\,\mathbf{a}_r \times B(r,t).$$

A point charge q at the origin is driven by a linearly polarized electromagnetic wave of angular frequency w and electric field amplitude E_0. Obtain the expressions for the radiated electromagnetic fields. Describe the state of polarization of the radiated fields.

For an oscillating charge of low speed, the influence of magnetic field of the incident radiation may be neglected. The equation of motion of the charge q, of mass m, in the field of the incident wave is

$$m\ddot{x} = q E_0 e^{-j\omega t} \quad \text{or} \quad \ddot{x} = (qE_0/m)e^{-j\omega t}.$$

The charge will oscillate with the same frequency. Hence the displacement of the charge is

$$x = -(qE_0/m\omega^2)\, e^{-j\omega t}.$$

$\therefore$ Electric dipole moment $p(t) = qx = -(q^2E_0/m\omega^2)e^{-j\omega t}$.

As

$$\frac{\partial^2}{\partial t^2} p\left(t - \frac{r}{c}\right) = \left.\frac{\partial^2 p(t')}{\partial t'^2}\right|_{t' = t - r/c} = \frac{q^2E_0}{m} e^{-j\omega(t - r/c)}$$

$$= (q^2E_0/m)\, e^{j(\kappa r - \omega t)}. \quad (\text{As } \kappa = \omega/c)$$

Thus the electric and magnetic fields are given as

$$\mathbf{B}(\mathbf{r}, t) = -\frac{\mu_0 q^2}{4\pi mrc} e^{j(\kappa r - \omega t)}\, \mathbf{a}_r \times \mathbf{E}_0$$

and

$$\mathbf{E}(\mathbf{r}, t) = -\frac{\mu_0 q^2}{4\pi mr} e^{j(\kappa r - \omega t)}\, \mathbf{a}_r \times (\mathbf{a}_r \times \mathbf{E}_0)$$

$$= \frac{\mu_0 q^2}{4\pi mr} e^{j(\kappa r - \omega t)} [(\mathbf{E}_0 \cdot \mathbf{a}_r)\, \mathbf{a}_r - \mathbf{E}_0]$$

Above relations show that **E** is in the plane of **p** and **r** and **B** is perpendicular to it. Thus the radiation emitted is linearly polarized.

Example 5. *A small loop of wire of radius a carries a current $i = i_0 \cos \omega t$. The loop is located in the xy-plane. Calculate the first non-zero multipole moment of the system. Calculate the electric and magnetic fields with the help of vector potential. Determine and discuss the angular distribution of the power radiated.*

The first non-zero multipole moment of the small current carrying loop is its magnetic dipole moment

$$\mathbf{m} = \pi a^2 i_0 \cos \omega t\, \mathbf{a}_z = \pi a^2 i_0 R_e (e^{-j\omega t})\, \mathbf{a}_z.$$

Let the center of the loop is the origin of the spherical coordinates (r, θ, ϕ). The vector potential at a point **r** for $r \to \infty$ is given by

$$\mathbf{A}(\mathbf{r}, t) = \frac{j\kappa\mu_0 e^{j\kappa r}}{4\pi r}\, \mathbf{a}_r \times \mathbf{m}, \qquad \text{where } \kappa = \omega/c.$$

$$= -\frac{j\mu_0 \omega i_0 a^2 \sin\theta}{4cr} e^{j(\kappa r - \omega t)}\, \mathbf{a}_\phi$$

$$\therefore \quad \mathbf{B} = \nabla \times \mathbf{A} = j\kappa \mathbf{a}_r \times \mathbf{A} = \frac{\mu_0 \omega^2 i_0 a^2 \sin\theta}{4c^2 r} e^{j(\kappa r - \omega t)}\, \mathbf{a}_\theta$$

and

$$\mathbf{E} = c\mathbf{B} \times \mathbf{a}_r = \frac{\mu_0 \omega^2 i_0 a^2 \sin\theta}{4cr} e^{j(\kappa r - \omega t)}\, \mathbf{a}_\phi$$

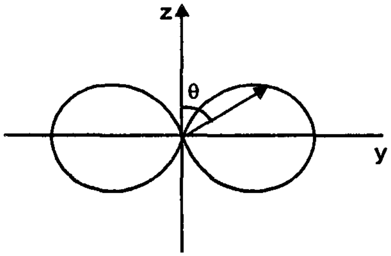


Fig. 15.12.

∴ Average poynting vector at **r** is

$$<\mathbf{S}> = \frac{1}{2} \mathrm{Re}\,(\mathbf{E} \times \mathbf{H}) = \frac{c}{2\mu_0} \mathrm{Re}\,[(\mathbf{B} \times \mathbf{a}_r) \times \mathbf{B}]$$

$$= \frac{c}{2\mu_0} |\mathbf{B}|^2\, \mathbf{a}_r = \frac{\mu_0 \omega^4 a^4 i_0^2}{32 c^3 r^2} \sin^2\theta\, \mathbf{a}_r$$

∴ Average power radiated per unit solid angle

$$\frac{dP}{d\Omega} = \frac{<S>}{r^{-2}} = \frac{\mu_0 \omega^4 a^4 i_0^2}{32 c^3} \sin^2\theta.$$

The radiated energy is distributed according to $\sin^2\theta$. The radiation is strong in the plane θ = 90° and is zero along the axis of the loop, as shown in Fig. 15.12.

Example 6. *Find the radiation resistance of the oscillating magnetic dipole and compare the radiation resistance of the electric dipole.*

We know that the average intensity of the radiations emitted from the magnetic dipole

$$\langle S\rangle = \frac{\mu_0 m_0^{\,2}\omega^4}{32\pi^2 c^3}\,\frac{\sin^2\theta}{r^2}.$$

Therefore the total radiated power

$$P_m = \int_0^{\pi} \frac{\mu_0 m_0^{\,2}\omega^4}{32\pi^2 c^3}\,\frac{\sin^2\theta}{r^2}\, r\sin\theta\, r\,d\theta \int_0^{2\pi} d\phi$$

$$= \frac{\mu_0 m_0^{\,2}\omega^4}{12\pi c^3} = \frac{1}{4\pi\varepsilon_0}\,\frac{m_0^{\,2}\omega^4}{3c^5}.$$

The magnetic moment $m_0 = \pi a^2 I_0$, $\omega = 2\pi\upsilon = 2\pi c/\lambda$.

$$\therefore \qquad P_m = \frac{1}{4\pi\varepsilon_0}\,\frac{\pi^2 a^4 I_0^{\,2}}{3c^5}\cdot\left(\frac{2\pi c}{\lambda}\right)^4 = \frac{1}{4\pi\varepsilon_0}\,\frac{16\pi^6 I_0^{\,2}}{3c}\left(\frac{a}{\lambda}\right)^4$$

The radiation resistance R_{rad} of the magnetic dipole is defined as

$$P_m = \tfrac{1}{2} I_0^{\,2}\, R_{rad}$$

$$\therefore \qquad R_{rad} = \frac{1}{4\pi\varepsilon_0}\,\frac{32\pi^6}{3c}\left(\frac{a}{\lambda}\right)^4 = \frac{9\times 10^9\times 32\times 3.14^6}{3\times(3\times 10^8)}\left(\frac{a}{\lambda}\right)^4$$

$$= 3.07\times 10^5\,(a/\lambda)^4.$$

$$\frac{\text{Radiation resistance of the magnetic dipole}}{\text{Radiation resistance of the electric dipole}} = \frac{3.07\times 10^5\,(a/\lambda)^4}{788\,(l/\lambda)^2}$$

$$= 389\, a^4/(2\pi a\lambda)^2 = 9.86\,(a/\lambda)^2.$$

Here we have assumed $l = 2\pi a$. Since $a < \lambda$, therefore the above relation shows that the radiation resistance of the magnetic dipole is negligible in comparison to the radiation resistance of the electric dipole of length of the order of circumference of the magnetic dipole.

Example 7. *Calculate the electric field of a point charge moving with constant velocity.*

For a point charge moving with constant velocity, the acceleration $a = 0$ and the electric field is given by Eq. (65) as

$$\mathbf{E}(\mathbf{r}, t) = \frac{q}{4\pi\varepsilon_0}\,\frac{R}{(\mathbf{R}\cdot\mathbf{v})^3}\,[\mathbf{v}(c^2 - u^2)] = \frac{q}{4\pi\varepsilon_0}\,\frac{c^2 - u^2}{(\mathbf{R}\cdot\mathbf{v})^3}\,R\mathbf{v}.$$

Using the relation $R\mathbf{v} = c\mathbf{R} - R\mathbf{u} = c(\mathbf{R} - \mathbf{u}\,t_r) - c\,(t - t_r)\mathbf{u}$

$$= c(\mathbf{r} - \mathbf{u}t).$$

For convenience, let the particle passes through the origin at time $t = 0$, so that

$$|\mathbf{r} - \mathbf{u}\,t_r| = c\,(t - t_r).$$

On squaring, we get

$$r^2 - 2\mathbf{r}\cdot\mathbf{u}\,t_r + u^2 t_r^{\,2} = c^2(t^2 - 2t\,t_r + t_r^{\,2})$$

or $\quad t_r^{\,2}\,(c^2 - u^2) - t_r\,(2c^2 t - 2\mathbf{r}\cdot\mathbf{u}) - r^2 + c^2 t^2 = 0,$

$$\therefore \qquad t_r = \frac{c^2 t - \mathbf{r}\cdot\mathbf{u} \pm \sqrt{(c^2 t - \mathbf{r}\cdot\mathbf{u})^2 + (c^2 - u^2)(r^2 - c^2 t^2)}}{(c^2 - u^2)}$$

To fix the sign, consider the case $u = 0$, the retarded time $t_r = t - r/c$. The vector from the retarded position to P

$$\mathbf{R} = \mathbf{r} - \mathbf{u}\,t_r$$

$$\therefore \qquad \hat{R} = (\mathbf{r} - \mathbf{u}\,t_r)/c(t - t_r).$$

We thus have

$$
\begin{aligned}
R\left(1-\frac{\mathbf{R}\cdot\mathbf{u}}{Rc}\right) &= c(t-t_r)\left[1-\frac{\mathbf{u}}{c}\cdot\frac{(\mathbf{r}-\mathbf{u}t_r)}{c\,(t-t_r)}\right] = c\,(t-t_r)-\frac{\mathbf{u}\,.\,\mathbf{r}}{c}+\frac{u^2}{c}t_r \\
&= \frac{1}{c}\left[(c^2t-\mathbf{r}\cdot\mathbf{u})-t_r\,(c^2-u^2)\right] \\
&= \frac{1}{c}\sqrt{(c^2t-\mathbf{r}\cdot\mathbf{u})^2+(c^2-u^2)\,(r^2-c^2\,t^2)}
\end{aligned}
$$

$$\therefore \quad \mathbf{R}\cdot\mathbf{v}=Rc-\mathbf{R}\cdot\mathbf{u} = \sqrt{(c^2t-\mathbf{r}\cdot\mathbf{u})^2+(c^2-u^2)\,(r^2-c^2t^2)}$$

Thus the electric field **E** is given by

$$\mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{(c^2-u^2)\,c\,(\mathbf{r}-\mathbf{u}t)}{[(c^2t-\mathbf{r}\cdot\mathbf{u})^2+(c^2-u^2)\,(r^2-c^2t^2)]^{3/2}}$$

If we define the vector from the present position of the particle to point P by $\mathbf{R}'$, then $\mathbf{R}' = \mathbf{r}-\mathbf{u}t$. If θ is the angle between $\mathbf{R}'$ and $\mathbf{u}$ (Fig. 15.12), we get

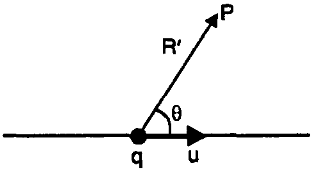


Fig. 15.12. Example 7.

$$\mathbf{R}\cdot\mathbf{v} = \mathbf{R}'c\sqrt{1-(u^2/c^2)\sin^2\theta}$$

and

$$R\mathbf{v} = c(\mathbf{r}-\mathbf{u}t) = \mathbf{R}'c.$$

$$\therefore \quad \mathbf{E}(\mathbf{r},t) = \frac{q}{4\pi\varepsilon_0}\frac{1-u^2/c^2}{(1-u^2\sin^2\theta/c^2)^{3/2}}\frac{\mathbf{R}'}{R'^3}$$

Thus **E** points along **R′** the vector from the present position of the particle, since the message came from the retarded position.

Example 8. *The transmitting and receiving antennas are separated by a distance of 300 λ and have directive gains of 25 and 18 dB, respectively. If 4 mW of power is to be received. Calculate the minimum transmitted power.*

Given that $G_{dt}(\text{dB}) = 25\text{ dB} = 10\log_{10}G_{dt}$

$\therefore \quad G_{dt} = 10^{2.5} = 316.23$

Similarly $G_{dr}(dB) = 18\,dB$ or $G_{dr} = 10^{1.8} = 63.1.$

We know that $P_r = G_{dr}G_{dt}\,[\lambda/4\pi r]^2\,P_t$

or

$$P_t = P_r\left[\frac{4\pi r}{\lambda}\right]^2\frac{1}{G_{dr}G_{dt}} = 4\times10^{-3}\left[\frac{4\pi\times300\lambda}{\lambda}\right]^2\frac{1}{63.1\times316.23}$$

$$= 2.849\text{ W.}$$

ORAL QUESTIONS

1. **Who demonstrated experimentally the production of electromagnetic waves ?**
2. **What do you mean by electromagnetic spectrum ?**
3. **A charge radiates whenever (*a*) it is moving in any manner, (*b*) it is being accelerated; (*c*) it is bound in an atom. (*b*)**
4. **A measuring device is disturbed by the (*a*) high frequency electric field, (*b*) low frequency electric field, (*c*) high frequency magnetic field. How would you separately protect the device from each one ?**

   [(*a*), (*c*) device is enclosed in a grounded shell made of a good conductor; (*b*) shell must have thickness at least a few times the depth of penetration].

5. How much time does the radiowaves take to reach 12000 km from the radiating antenna. (40 ms)
6. A quarter wave monopole antenna operating in air at 1MHz frequency must have an overall length of ..... (75 m)
7. At a distance of 6 km from a differential antenna, the field strength is 16 μV/m. The field strength at a location of 16 km from the antenna is ...... (6 μV/m)
8. A receiving antenna in an airport has a maximum dimension of 3m and operates at 100 MHz. An aircraft approaching the airport is 0.5 km from the antenna. Is it true that the aircraft is in the far field region of the antenna. (yes)
9. Radiation emitted by an antenna has angular distribution characteristic of dipole of dipole radiation when (*a*) the $\lambda > a$ the size of antenna, (*b*) $\lambda < a$ and (*c*) The antenna has the appropriate shape. (*a*)
10. Show the complete pattern of electric and magnetic lines from an oscillating electric dipole.
11. What do you mean by retarded and advanced potentials ?
12. Show the intensity profile for dipole radiation.
13. Compare the radiation powers from electric and magnetic dipoles.
14. What is Larmor formula ?
15. Define Lienard-Weichert potentials.

PROBLEMS

1. Confirm that the retarded potentials satisfy the Lorentz gauge condition.
2. Radio station FM Delhi radiates a power of 100 kW at about 90 MHz from its antenna. Estimate the strength of its electric field at Ghaziabad which is about 20 km from station. (1.2×10^{-3} V/m)
3. An atom with an atomic polarizability $\alpha(\omega)$ is subjected to an electromagnetic field $\mathbf{E} = \mathbf{E}_0 e^{j(\kappa x - \omega t)}\, \mathbf{a}_z$. The atom is assumed as being located it the origin. Final expressions for **E** and **B** fields radiated by the atom and calculate the energy radiated per unit solid angle

$$\left[-\frac{\alpha E_0 \omega^2}{4\pi\varepsilon_0 c^2 r}\sin\theta\, e^{-j\omega t}\,\mathbf{a}_\theta, \frac{-\alpha E_0\omega^2}{4\pi\varepsilon_0 c^3 r}\sin\theta e^{-j\omega t}\,\mathbf{a}_\phi, \frac{\alpha^2 E_0{}^2\omega^2\sin^2\theta}{32\pi^2\varepsilon_0 c^3}\right]$$

4. Two equal point charges $+q$ oscillate along the z-axis with their positions given by

$$z_1 = z_0 \sin\omega t, z_2 = -z_0\sin\omega t, x_i = y_i = 0 \qquad (i = 1, 2).$$

The radiation field is observed at a point $P(r, 0)$. Assume that $|r| >> \lambda >> z_0$, where λ is the wavelength of emitted radiation. Find **E** and **B** and thus calculate the power radiated per unit solid angle in the direction of r. Estimate the total power radiated.

$$\frac{i\mu_0}{4\pi}\frac{\omega^3 z_0{}^2 q}{rc}\sin\theta\cos\theta\, e^{j(kr-\omega t)}\mathbf{a}_\theta;\ \frac{j\mu_0}{4\pi}\frac{\omega^3 z_0{}^2 q}{rc^2}\sin\theta\cos\theta\, e^{j(kr-\omega t)}\,\mathbf{a}_\phi;$$

$$\frac{\mu_0}{32\pi^2}\frac{\omega^6 z_0{}^4 q^2}{c^3}\sin^2\cos^2 0, \frac{\mu_0}{60\pi}\frac{\omega^6 z_0{}^4 q^2}{c^3}.$$

5. Show that the potential of a point charge q moving with constant velocity u can be expressed as

$$\varphi(\mathbf{r}, t) = \frac{1}{4\pi\varepsilon_0}\frac{q}{R\sqrt{1-(u^2/c^2)\sin^2\theta}}$$

where **R** is the vector from the present position of the particle to point P and θ is the angle between **R** and **u**.

6. A current $I(t)$ flows around the circular ring. Derive the general expression for the power radiated. ($2\, m^2/(4\pi\varepsilon_0)\,(3c^5)$)
7. Calculate the effective radiation resistance of the half wave antenna, an antenna of length $\lambda/2$. (73.1 Ω).
8. Determine the electric field intensity at a distance of 10 km from an antenna having directive gain of 5 dB and radiating a total power of 20 kW. (0.195 V/m)

16

RELATIVITY AND ELECTRODYNAMICS

16.1 GALILEAN TRANSFORMATION

The principle of relativity (Galilean relativity) was first introduced by Newton. According to Newton, the motion of objects in a given region of space are the same among themselves, whether the region is at rest or moves uniformly linearly. We define *inertial frame of reference* or simply *inertial frames* as those systems in which the law of inertia or Newton's law holds good. A physical happening or phenomenon in an inertial system is called an *event*.

Let us consider two inertial frames S and S' with parallel axes and origins coinciding originally at $t = 0$. The frame S' is moving with velocity u relative to S along the x-direction. Let us consider an event taking place at some point P and described by coordinates (x, y, z, t) and (x', y', z', t') corresponding to two observers in frames S and S' respectively. Using Fig. 16.1, we get

$$x' = x - ut,\ y' = y,\ z' = z,\ t' = t. \qquad ...(1)$$

or

$$x = x' + ut,\ y = y',\ z = z',\ t = t' \qquad ...(2)$$

These relations are called *Gililean transformation* from one inertial frame to another.

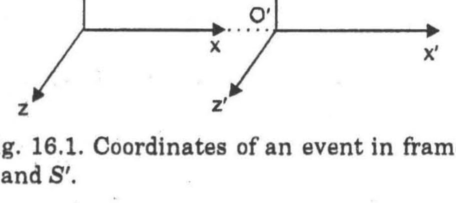


Fig. 16.1. Coordinates of an event in frames S and S'.

For a small displacement of P along x-axis, we may write

$$\Delta x' = \Delta x - u\,\Delta t \text{ and } \Delta t' = \Delta t.$$

$$\therefore\quad \Delta x'/\Delta t' = \Delta x/\Delta t - u \text{ or } v'_x = v_x - u. \qquad ...(3)$$

Similarly $v_y' = v_y$ and $v'_z = v_z$. ...(4)

If the velocity of point P changes by a small amount, we have

$$\Delta v_x'/\Delta t' = \Delta v_x/\Delta t - 0 = \Delta v_x/\Delta t \text{ or } a'_x = a_x$$

or in general $\mathbf{a}' = \mathbf{a}$. ...(5)

Thus we see that the acceleration is invariant under *Galilean transformation*. If the particle of mass m at point P is acted upon by a force $\mathbf{F}$ in an inertial frame S, we may write

$$\mathbf{F} = m\mathbf{a}. \quad ...(6)$$

Since $\mathbf{a}$ is invariant, therefore $\mathbf{F}' = m\mathbf{a}'$ ...(7)

in the frame S'.

Thus we see that Newton's law remains unchanged or invariant under Galilean transformation. Or in general *the laws of mechanics are the same in all inertial frames of reference.*

16.2 SPECIAL THEORY OF RELATIVITY

Maxwell presented his theory of electromagnetism in the form of Maxwell's equations by summarizing all the laws of electricity and magnetism and by introducing the displacement current. His theory also predicted the existence of electromagnetic waves and showed that light is just an electromagnetic waves. All known waves were transmitted by a medium-surface waves on water by a liquid, sound waves by air or an elastic solid. Therefore it was believed that electromagnetic waves need a medium to support them. The physicists named this medium *luminiferous ether* or just *ether* and assumed it to be transparent to light, of negligible density but as rigid as steel. It was assumed that Maxwell's equations were valid in any reference frame that was at rest with respect to the ether, known as *ether frame* or *rest frame* or *absolute frame.* There were two possible options : Either the earth, as it traveled in its orbit about the sun, was moving through the ether or the earth dragged the ether along its motion. The second possibility had been eliminated by James Bradley, when he explained stellar aberration.

Michelson and Morley Experiment : Michelson and Morley carried out a high precision experiment to measure the absolute velocity of earth with respect to the stationary ether by using Michelson interferometer. They assumed that the ether was at rest and was named inertial frame S, while the earth moving with velocity u along the x-direction was the inertial system S'. The apparatus consists of a light source S, lens L, mirrors M_1 and M_2, partially silvered glass plate P, compensative plate CP and telescope T, all fixed with respect to each other as shown in Fig. 16.2 (*a*). The origins of two inertial frames S and S' coincide at the time the beam of light is split into two beams *A and B* by a partially silvered plate P. At this time two coherent beams are produced by the plate P, one transmitted move along x-direction and other reflected move along y-direction. These beams each travel a distance L and are then reflected back from mirrors M_1 and M_2. On returning back by reflection, the beam A is transmitted and beam B is partially reflected by plate P. They interfere and the interference fringes can be seen through the telescope T. Assuming the propagation of light through the ether similar to the propagation of sound through air, the Galilean transformations give the velocity of light along PM_1 as c-u and along M_1P as $c + u$. Thus the time taken by the beam B to travel from plate P to mirror M_1 and back is

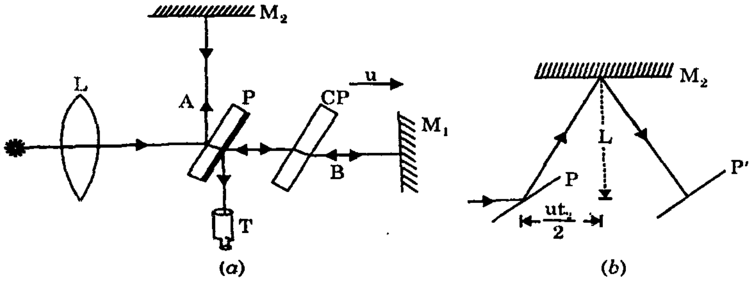


Fig. 16.2 (a) Experimental arrangement, (b) Path of beam A.

$$t_1 = \frac{L}{c-u} + \frac{L}{c+u}$$

$$= \frac{2Lc}{c^2-u^2} = \frac{2L}{c}\left(1-\frac{u^2}{c^2}\right)^{-1}$$

$$= \frac{2L}{c}\left(1+\frac{u^2}{c^2}\right) \qquad \text{...(8)}$$

Fig. 16.2 (b) shows that the beam A travels a distance given by hypotenuse of a triangle of sides L and $ut_2/2$ in time $t_2/2$ with the speed of light. Therefore,

$$\sqrt{L^2+\left(\frac{ut_2}{2}\right)^2} = \frac{ct_2}{2} \quad \text{or} \quad t_2 = \frac{2L/c}{\sqrt{1-u^2/c^2}}. \qquad \text{...(9)}$$

The time difference between the two light beams to arrive back at plate P is

$$\Delta t = t_1 - t_2 = \frac{2L}{c}\left(1+\frac{u^2}{c^2}\right) - \frac{2L}{c}\left(1+\frac{u^2}{2c^2}\right) = \frac{2L}{c}\cdot\frac{u^2}{2c^2}. \qquad \text{...(10)}$$

To remove any effects due to the failure of the two path lengths (PM_1 and PM_2) to be exactly equal, in each measurement apparatus was rotated through 90° to interchange the path lengths. The total path length difference for the two orientations is

$$\Delta L = \frac{Lu^2}{c^2} - \left(-\frac{Lu^2}{c^2}\right) = \frac{2Lu^2}{c^2}. \qquad \text{...(11)}$$

Since the earth moves around the sun with an orbital speed of about 30 km/s and the direction of motion changes during the year. Therefore for $L = 30\text{m}$, $u^2/c^2 \simeq 10^{-8}$ and $\Delta L = 2 \times 30 \times 10^{-8} = 6 \times 10^{-7}\text{m} \simeq 0.5$ fringe. Michelson and Morley with their improved apparatus, which could detect a fringe shift of 0.04 fringe, *observed no shift.* If there were an effect due to ether, the fringe pattern would shift accordingly when the apparatus is rotated through 90°. *This negative result shows that there is no experimental evidence for the existence of an ether wind.*

The Michelson-Morley experiment was repeated many times by several workers in several variations. In every case a null result was obtained. It showed that : (*i*) We could not locate the absolute rest frame, (ii) the Galilean transformations hold good for classical mechanics but not for electromagnetism, and (iii) the speed of light is invariant.

Einstein Postulates : In 1905, Albert Einstein developed the *special theory of relativity.* The theory is special because it deals with the *special case of non accelerated reference frames.* His theory deals with the physics of objects moving with high speeds and answers some of the difficulties observed in electrodynamics. The theory is based on two basic postulates :

(1) The laws of physics are the same in all inertial reference frames (Principle of equivalence).

(2) The speed of light in vacuum is always a constant equal to c, and is independent of the relative motion of the inertial systems, the source and the observer (Principle of constancy of speed of light).

The first postulate is the generalization of the fact that all physical laws, not just the Newtonian laws of mechanics, are invariant under all transformations. It denies the concepts of *absolute rest and absolute motion* or the existence of any *absolute inertial frame.* The second postulate explains the null result of the Michelson-Morley experiment. Thus the Einstein's postulates demolished the ether theory.

16.3 LORENTZ TRANSFORMATIONS

An observer in any frame S will describe an event by its location in space and time. An observer in a second frame S' will describe the same event by its space-time coordinates in frame S'. If we assume that out of two inertial frames S and S', S' is moving with velocity u with respect to S along the x-axis. The origins and axes of frames S and S' are assumed to coincide at time $t = t' = 0$. The origins of these frames at a later time t are given by $x = ut$ and $x' = 0$. A flash of light emitted at an instant $t = t' = 0$ travels as a spherical electromagnetic wave. The position P is described by (x, y, z, t) and (x', y', z', t') in S and S' frames respectively. If r and r' be the distances from the origins O and O' of these frames, then we have

$$r = ct \quad \text{or} \quad x^2 + y^2 + z^2 = c^2t^2$$

and

$$r' = ct' \quad \text{or} \quad x'^2 + y'^2 + z'^2 = c^2t'^2 \qquad ...(12)$$

Under Galilean transformations $t' = t$ and $x' = x - ut$. ...(13)

The Galilean transformation laws do not satisfy the condition that the speed of light is the same in all inertial frames. H.A. Lorentz in 1890 developed correct transformations, known as the *Lorentz transformations.* They are represented by the equations

$$x' = \gamma(x - ut), y' = y, z' = z \text{ and } t' = \gamma(t - ux/c^2), \qquad ...(14)$$

where

$$\gamma = 1/\sqrt{1 - u^2/c^2}. \qquad ...(15)$$

The inverse relations may be obtained by assuming that the frame S is moving with respect to frame S' with a velocity u along the negative x-axis. Thus we have

$$x = \gamma(x' + ut'), y = y', z = z' \text{ and } t = \gamma(t' + ux'/c^2) \qquad ...(16)$$

For $u \to 0$, the relations (13) and (14) coincide. Thus we see that Lorentz transformations reduce to Galilean transformations at $u \to 0$. If $u > c$, the space and time coordinates will be imaginary. It shows that *no object can move with a velocity greater than the velocity of light.*

Velocity Additions : Consider a particle moving in space along the x-direction and having velocity v_x as measured by an observer in inertial frame S. If this particle has velocity v'_x as measured by an observer in the frame S' moving with velocity u along the x-axis with respect to S frame, then Galilean velocity transformation gives

$$v'_x = v_x - u, \quad \text{or} \quad v_x = v'_x + u \qquad ...(17)$$

The correct relation known as Lorentz velocity transformation is given by

$$v'_x = \frac{v_x - u}{1 - u\,v_x/c^2}. \qquad ...(18)$$

For the reverse transformation, we replace u by $-u$ and interchange the primed and unprimed quantities.

$$v_x = \frac{v'_x + u}{1 + u\,v'_x/c^2}. \qquad ...(19)$$

Time Dilation : Consider two inertial frames S and S', where S' is moving with velocity u along the x-axis. Both the systems are equipped with clocks to register time intervals between two different events in space. If the time interval is $\Delta t = t_2 - t_1$ observed by an observer in S-frame and is $\Delta t' = t'_2 - t'_1$ observed by an observer in S' – frame. From Eq. (16), we get

$$\Delta t = t_2 - t_1 = \gamma\left(t'_2 + \frac{ux'_2}{c^2}\right) - \gamma\left(t'_1 + \frac{ux'_1}{c^2}\right)$$

$$= \gamma\left[(t'_2 - t'_1) + \frac{u}{c^2}(x'_2 - x'_1)\right]$$

Since the clock of S' must be at rest during the time the observer in S frame is making the observation, *i.e.* $x'_2 = x'_1$. Thus we have

$$\Delta t = \gamma(t'_2 - t'_1) = \gamma\Delta t'. \qquad ...(20)$$

Thus to *the observer in frame S, the clock of S'-frame seems to be running slow as compared to his own clock, i.e.*, moving clock always appears to run slow than an identical clock at rest with respect to the observer. This effect is called *time dilation.*

Length Contraction : Consider a rod of length L at rest in frame S. The length determined in a reference frame in which the object is at rest is called the *proper length* of the object. If the rod is lying along x-axis, with $x_1 = 0$ and $x_2 = L$. For the observer in frame S', moving with velocity u with respect to frame S, the observed length

$$L_0 = x'_2 - x'_1 = \gamma(x_2 - ut_2) - \gamma(x_1 - ut_1)$$

$$= \gamma(x_2 - x_1) - \gamma u\,(t_2 - t_1) = \gamma[L - u(t_2 - t_1)].$$

Since $t_2 = t_1$ as the observer in the S-frame measures the two ends of the rod, simultaneously. Thus the length of the rod lying in the S' frame and measured by the observer in the S-frame is

$$L = \frac{L_0}{\gamma} = L_0\sqrt{1 - \frac{u^2}{c^2}}. \qquad ...(21)$$

Thus the length of an object is maximum when measured by an observer at rest relative to the object. The length will appear contracted by a factor $\sqrt{1 - u^2/c^2}$ to an observer who is in motion relative to the object. This length is called *improper length*. This effect is called *Lorentz-Fitzgerald length contraction*. If the length of the object is oriented perpendicular to the line of motion, the proper length is again observed. Thus we see that *only the dimension along the line of motion is changed.*

16.4 RELATIVISTIC MECHANICS

Let us now see the effect of very high speeds on mass, momentum, force and energy in order for Newtonian mechanics to be invariant under Lorentz transformations.

Mass, Momentum and Force : According to Newtonian (classical) mechanics, the inertial mass of a body is constant, independent of velocity and is denoted by m_0, called the *rest mass* or *proper mass*. In *relativistic* mechanics, the *observed mass* also known as *relativistic mass* is a function of velocity and is given by

$$m = \gamma m_0 = m_0/\sqrt{1 - u^2/c^2}. \qquad ...(22)$$

It shows that $m \to m_0$ for $u << c$ and $m \to \infty$ for $u \to c$.

The momentum of this moving mass is given by

$$\mathbf{p} = m\mathbf{u} = \gamma m_0\,\mathbf{u} = m_0\mathbf{u}/\sqrt{1 - u^2/c^2}. \qquad ...(23)$$

If a particle experiences a force **F** in a fixed inertial reference frame S. If this particle is at rest in S-frame. The transformations rules give the value of this force in the another reference frame S', which is moving with velocity u relative to the frame S, as

$$F'_{\perp} = F_{\perp}\sqrt{1-u^2/c^2} \text{ and } F'_{\parallel} = F_{\parallel}. \qquad ...(24)$$

i.e., the component of **F** parallel to the motion of S' is unchanged, whereas components perpendicular are reduced by a factor $\sqrt{1-u^2/c^2}$.

Kinetic and Total Energy : If a particle with a rest mass m_0 is at rest initially in an inertial reference frame S. A force **F** accelerates it through a distance l. At the end of the motion, the particle acquires a velocity **u**. Thus we have

$$\begin{aligned} F &= \frac{dp}{dt} = m_0\frac{d}{dt}(\gamma u) = m_0\frac{d}{dt}\left(\frac{u}{\sqrt{1-u^2/c^2}}\right) \\ &= \frac{m_0}{(1-u^2/c^2)^{3/2}}\frac{du}{dt}. \end{aligned}$$

Work done in bringing the particle from rest to a speed u is the kinetic energy of the particle. Thus

$$\begin{aligned} K &= W = \int_0^l F\,dx = \int_0^t F\frac{dx}{dt}\,dt = \int_0^t F\,u\,dt \\ &= \int_0^u \frac{m_0}{(1-u^2/c^2)^{3/2}}\,u\,du = \frac{m_0c^2}{(1-u^2/c^2)^{1/2}} - m_0c^2 \\ &= mc^2 - m_0c^2. \end{aligned} \qquad ...(25)$$

Thus we see that K is not $\frac{1}{2}mu^2$. The correct value of K is $(m - m_0)\,c^2$. The quantity m_0c^2 does not depend on the velocity u and is called the *rest-mass energy* or *rest energy* (R.E.) of the particle. The total energy of the particle

$$E = mc^2 = K + m_0c^2 = K\cdot E + R\cdot E. \qquad ...(26)$$

Momentum and Total Energy : From relations (23) and (26) we get

$$p^2 = \frac{m_0^2u^2}{1-u^2/c^2} \text{ and } E^2 = m^2c^4 = \frac{m_0^2}{1-u^2/c^2}\cdot c^4$$

or

$$E^2 - p^2c^2 = \frac{m_0^2c^4}{1-u^2/c^2} - \frac{m_0^2u^2c^2}{1-u^2/c^2} = \frac{m_0^2c^4(1-u^2/c^2)}{1-u^2/c^2} = m_0^2c^4.$$

or

$$E = \sqrt{p^2c^2 + m_0^2c^4}. \qquad ...(27)$$

For a massless particle such as photon or neutrino, etc. we have

$$E = pc. \qquad ...(28)$$

Relation (27) shows that the quantity $E^2 - p^2c^2$ is *invariant*.

Invariant and Conserved Quantities : A quantity is *invariant* if it has the same value in all inertial frames. It is conserved if it has the same value before as after some process. Mass is invariant, but not conserved; energy is conserved, but not invariant; velocity is neither conserved nor invariant.

16.5 EXPERIMENTAL TESTS OF TIME-DILATION

(Decay of Unstable Particle in Motion)

The time-dilation effect is real. It can be experimentally verified with measurements of half lives of radioactive nuclei or unstable particles in motion. μ-mesons (muons) are created in the upper layers of our atmosphere by cosmic rays. These decay into electron and two neutrinos. The half life of muons is 1.5×10^{-6} s. If the muons are assumed to travel with the velocity of light, they would cover about $1.5 \times 10^{-6} \times 3 \times 10^{8} = 450$m before about half of them decayed. On account of decay, we should not expect any muons at the see level. The presence of muons in abundance was experimentally observed by Rossi and Hall in 1941. They measured the muon flux at an altitude of about 2000m and again at see level and found that $\mu_{2000}/\mu_0 = 1.4$.

The exponential time dependence character of decay shows that the ratio of these fluxes must be

$$\mu_{2000}/\mu_0 = 2^{2000/450} = 2^{4.44} = 21.7. \qquad ..(29)$$

The apparent discrepancy was resolved with the help of the theory of special relativity. Since $1.4 = 2^{1/2}$, therefore according to the clock of the muons only about half life (*i.e.,* 0.75μs) will clapse during the motion of 2000 m as measured by an observer on the earth. A particle moving with the speed of light would require $2000/3 \times 10^{8} = 6.7$ μs to cover that distance. We know that the moving clock is slowed down relative to the fixed frame. Thus the time dilation in this case is $6.7/0.75 = 8.93$.

The speed of muon corresponding to this time dilation is given by time dilation factor

$$= 8.93 = 1/\sqrt{1 - u^2/c^2} \text{ or } u = 0.994c. \qquad ...(30)$$

The time dilation effect was also checked in 1972 by Hafele and Keating. Four very accurate cesium beam atomic clocks were flown on regularly scheduled commercial jet flights around the world twice, once eastward and once westward. On comparing the time difference between the fying clocks and the reference clocks at the U. S. Naval observatory with that obtained theoretically, the time dilation was confirmed to an accuracy of about 10 percent.

16.6 INVARIANCE OF CHARGE

The electric charge is relativistically *invariant.* The experimental test of the neutrality of the hydrogen molecule and helium atom proves that the electron and proton carry charges equal in magnitude. Their motion is different in the hydrogen molecule and helium atom. If motion had any effect on the amount of charge, we would not have exact cancellation of electronic charge and nuclear charge in both the hydrogen molecule and helium atom.

The optical spectra of isotopes of the same element confirm the invariance of charge. Although there is a marked difference in the motion of the protons within the nucleus, but the comparison of spectral lines shows no discrepancy that could be attributed to even a slight difference in the nuclear charge.

We know that all charged elementary particles carry charges of precisely the same magnitude. The quantization of charge is deep and universal law of nature. We now observe that this equality of charge holds not only for two particles at rest with respect to one another, but for any state of relative motion. Thus the invariance of charge can explain the charge quantization.

The experiments show that the Gauss's law surface integral $\int \mathbf{E}\cdot d\mathbf{A}$ depends only on the number and variety of charged particles inside the surface A and not on how they are moving. This must be true for some other reference frame. Therefore we must have

$$\int_A \mathbf{E}\cdot d\mathbf{A} = \int_{A'} \mathbf{E}'\cdot d\mathbf{A}', \qquad \text{...(31)}$$

where **E** and **E**′ are the electric fields in the S and S' reference frames. We know that the events that are simultaneous in S–frame need not be simultaneous in S'–frame. If the charges lie on the boundary of surfaces A and A', the charges within A at t must be the same as those with in A' at t'. The Eq. (31) is a formal statement of the relativistic invariance of charge.

Charge is conserved and is a relativistic invariant. It is a scalar quantity.

Transformation of Surface Charge Density σ : Let us consider a sheet of charge density σ in a frame S having one edge parallel to x-axis. An observer in a second frame S' moving along x-axis with velocity u with respect to frame S will observe the charge density σ′. If Δx and Δy be the sides of the plane sheet in S-frame and $\Delta x'$ and $\Delta y'$ that in S'-frame. Then from the concept of relativistic charge invariance, we have

$$\Delta x'\Delta y'\sigma' = \Delta x\Delta y\sigma.$$

Since Δx will be contracted to $\Delta x\sqrt{1-u^2/c^2}$ and Δy will remain same in the S'–frame. Therefore

$$\Delta x\sqrt{1-u^2/c^2}\ \Delta y\sigma' = \Delta x\ \Delta y\ \sigma \text{ or } \sigma' = \sigma/\sqrt{1-u^2/c^2}. \qquad \text{...(32)}$$

If the sheet of charge is placed in y-z plane, the observed charge density in S'–frame moving in the normal direction will remain unchanged, *i.e.*, $\sigma' = \sigma$.

Transformation of Volume Charge Density : If Δx, Δy, Δz be the sides of an elementary volume of charge in S–frame. In a frame S' moving along x-axis with velocity u with respect to frame S, the volume

$$\Delta\tau' = \Delta x'\ \Delta y'\ \Delta z' = \Delta x\sqrt{1-u^2/c^2}\ \Delta y\ \Delta z = \Delta\tau\sqrt{1-u^2/c^2}.$$

From the invariance of charge, we thus have

$$\Delta q' = \rho'\,\Delta\tau' = \rho'\Delta\tau\sqrt{1-u^2/c^2} = \rho\Delta\tau$$

or

$$\rho' = \rho/\sqrt{1-u^2/c^2}. \qquad \text{...(33)}$$

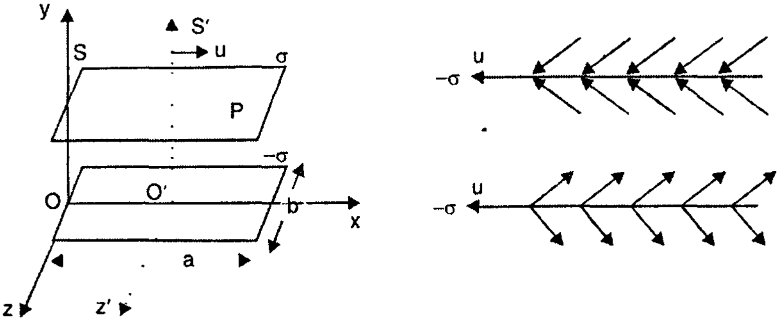
Thus we see that σ and ρ are not invariant relativistically, however the charge q is always invariant.

16.7 ELECTRIC FIELD IN DIFFERENT FRAMES OF REFERENCES

Consider a parallel plate capacitor formed by two large plates with uniform charge density σ and –σ. The plates are in x-z plane with sides a and b along x and z axes respectively, as shown in Fig. 16.3(a). If the plates are stationary in a frame S and the separation is small in comparison to the plate size, the electric field between the plates will be uniform and will be given by Gauss's law, as

$$\mathbf{E} = (\sigma/\varepsilon_0)\,\mathbf{j}. \qquad \text{...(34)}$$

Let us now consider a case, when the observer is in a S' frame, which is moving with velocity u along the x–axis with reference to the frame S. The observer will observe the length a contracted to $a\sqrt{1-u^2/c^2}$ and the surface charge density increased to $\sigma/\sqrt{1-u^2/c^2}$. The observer moving along + x – direction will find the capacitor moving along x– direction with velocity u. The moving charge plate in S' frame may not produce its field like that of a stationary plate. If the field of the moving plate tilts (actually it does not), say, in the direction shown in Fig. 16.3(*b*), the field between the plates, being the superposition of the $+\sigma$ field and the $-\sigma$ field, would still be perpendicular to the plates. The vector sum kills off the parallel components. For the uniform field, we may again apply Gauss's law and get



(*a*) Parallel plate capacitor.

(*b*) Appearance of the field of moving charged plates.

Fig. 16.3.

$$\mathbf{E}' = \left(\frac{\sigma'}{\varepsilon_0}\right)\mathbf{j} = \frac{\sigma}{\varepsilon_0\sqrt{1-u^2/c^2}}\,\mathbf{j} = \frac{1}{\sqrt{1-u^2/c^2}}\,\mathbf{E}. \qquad ...(35)$$

We often put γ for $1/\sqrt{1-u^2/c^2}$. Thus we have $E'_y = \gamma E_y$.

If the plates are rotated through an angle 90°, so that the side *b* of the plates is along the *x*– axis, we will find *b* contracted and have the same relation $E'_y = \gamma E_y$, therefore we have

$$E'_\perp = \gamma E_\perp. \qquad ...(36)$$

The subscript $\perp$ means the field perpendicular to the direction of motion of S'.

If the plates are lined up with the *y*-*z* plane, the plane separation *d* is contracted in the S' *frame*, where as *a* and *b* are the same in both he frames. Since the field of a capacitor is independent of *d*, we therefore have

$$E'_\parallel = E_\parallel. \qquad ...(37)$$

16.8 ELECTRIC FIELD OF POINT CHARGE IN UNIFORM MOTION

A point charge *q* is at rest at the origin in frame *S*. In this frame the electric field *E* at any point *P* at a distance *r* from the origin is given by

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^3}\mathbf{r} \qquad ...(38)$$

Its components are

$$E_x = \frac{1}{4\pi\varepsilon_0}\frac{qx}{(x^2+y^2+z^2)^{3/2}}, \quad E_y = \frac{1}{4\pi\varepsilon_0}\frac{qy}{(x^2+y^2+z^2)^{3/2}}$$

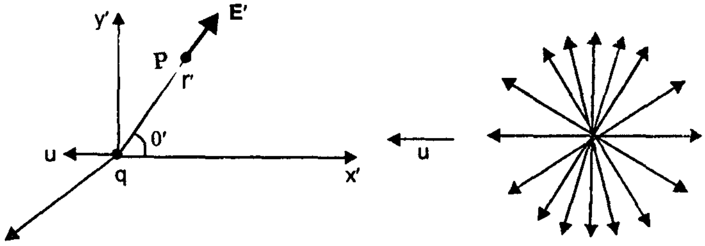
$$E_z = \frac{1}{4\pi\varepsilon_0}\frac{qz}{(x^2+y^2+z^2)^{3/2}}.$$

Let the frame S' move along *x*–axis with a velocity *u with reference to frame S*. From the property above discussed, we have,

$$E'_x = E_x = \frac{qx}{4\pi\varepsilon_0(x^2+y^2+z^2)^{3/2}}, \quad E'_y = \gamma E_y = \frac{\gamma qy}{4\pi\varepsilon_0(x^2+y^2+z^2)^{3/2}}$$

and $$E'_z = \gamma E_z = \frac{\gamma q z}{4\pi\varepsilon_0 (x^2 + y^2 + z^2)^{3/2}}.$$

These are expressed in terms of S–frame coordinates (x, y, z). To express in terms of S'–frame coordinates (x', y', z'), we require inverse transformations.



(a) (b)

Fig. 16.4. Electric field due to a point charge.

$$x = \gamma(x' + ut'),\ y = y',\ z = z'$$

and $$t = \gamma(t' + ux'/c^2).$$

For the instant $t' = 0$, when q passes the origin in frame S'. we have

$$x = \gamma x' = \gamma r_x',\ y = y' = r'_y,$$
$$z = z' = r'_z,$$

where $\mathbf{r}'$ is the vector from q to point P in the S' frame.

$$\therefore \quad E_x' = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q r'_x}{\left[(\gamma r'_x)^2 + (r'_y)^2 + (r'_z)^2\right]^{3/2}}$$

$$E'_y = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q r'_y}{[(\gamma r'_x)^2 + (r'_y)^2 + (r'_z)^2]^{3/2}}$$

and $$E_z = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q r_z'}{[(\gamma r'_x)^2 + (r'_y)^2 + (r'_z)^2]^{3/2}}$$

If the charge and the point P are in the $x'y'$ – plane, then we have

$$r'_x = r' \cos\theta',\ r'_y = r' \sin\theta',\ r'_z = 0.$$

$$\therefore \quad \mathbf{E}' = \frac{1}{4\pi\varepsilon_0} \frac{\gamma q \mathbf{r}'}{(\gamma^2 r'^2 \cos^2\theta' + r'^2 \sin^2\theta')^{3/2}} = \frac{1}{4\pi\varepsilon_0} \frac{q(1 - u^2/c^2)}{[1 - (u^2/c^2)\sin^2\theta']^{3/2}} \frac{\mathbf{r}'}{r'^3}. \quad \text{...(39)}$$

This is the expression for electric field in the reference inertial frame in which the charge moves with constant velocity u along $-x-$ direction. The relation will be same if we replace u by $-u$, *i.e.*, if the charge moves along $+x$-direction.

From above relations, we see that $E'_y/E'_x = y'/x'$. This tells that the vector $\mathbf{E}'$ makes the same angle with the x'–axis as does the radius vector $\mathbf{r}'$. Hence $\mathbf{E}'$ points radially outward along $\mathbf{r}'$. For low speed $u^2/c^2 \to 0$ and $E' = q/4\pi\varepsilon_0 r'^2$, the field due to stationary charge. For u^2/c^2 not negligible, the field is stronger at right angles to the motion of the charge than in the direction of motion, at the same distance from the charge. The density of $\mathbf{E}'$ field lines is shown in Fig. 16.4(*b*).

16.9 INTERACTION BETWEEN A MOVING CHARGE AND OTHER MOVING CHARGES

In this section, we will find an expression for the interaction between a current carrying wire and a moving charge with the help of electrostatics and relativity without using laws of magnetism. It will be the same as the Lorentz force on a charge moving in a magnetic field. Thus the *magnetism is found to be a relativistic phenomenon.*

Consider a string of positive charges moving along the right at speed v. These charges are close enough together so that we may regard them as a continuous line charge of linear charge density λ. Superimposed on it, there is a string of negative charges moving toward the left with the same speed. It is equivalent to a continuous line charge of charge density $-\lambda$. Thus the net current flows to the right, of magnitude.

$$I = 2\lambda v. \qquad \text{...(40)}$$

The net density of charge on the line is zero and the electric field **E** is also zero. Thus the electric force on a stationary point charge q some distance r from these line charges will also be zero.

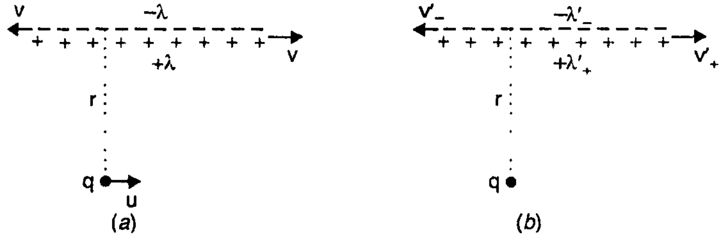


Fig. 16.5 (*a*). A line of positive and negative moving charges in a frame *S* and (*b*) in a frame *S'* moving with velocity *u*.

Suppose the charge q is set in motion to the right with speed u in the inertial reference frame S, as shown in Fig. 16.5(*a*). The net electric force on the moving charge is again zero in the frame S.

Let us now look at the same situation in the frame S' moving with velocity u to the right relative to S –frame. Thus the charge q will be at rest in frame S' as shown in Fig. 16.5 (*b*). By the relativistic velocity addition formula, the speeds of positive and negative charges in the S'- frame are

$$v_+' = \frac{v-u}{1-vu/c^2} \text{ and } v_-' = \frac{v+u}{1+vu/c^2}. \qquad \text{...(41)}$$

The two line charges will suffer different amounts of Lorentz contraction. Since v_-' is greater than v_+', the length contraction of the spacing between negative charges will be more than that between positive charges. Thus the linear charge density is increased in frame-S' The linear density of positive charge in the S'– frame is given by

$$\lambda_+' = \lambda_0 / \sqrt{1-(v_+'/c)^2} \qquad \text{...(42)}$$

Similarly the linear density of negative charge in S'frame is

$$\lambda_-' = \lambda_0 / \sqrt{1-(v_-'/c)^2}\,. \qquad \text{...(43)}$$

Here λ_0 is the linear density of charge of the positive line in its own rest system, which is related with λ moving with velocity v as

$$\lambda = \lambda_0 / \sqrt{1-v^2/c^2} \text{ or } \lambda_0 = \lambda\sqrt{1-v^2/c^2}$$

Similarly for the negative line charge $\lambda_0 = \lambda\sqrt{1-v^2/c^2}$. Thus the relations (42) and (43) may be written as

$$\lambda_+' = \frac{\lambda\sqrt{1-v^2/c^2}}{\sqrt{1-(v-u)^2(1-vu/c^2)^{-2}/c^2}} = \frac{\lambda\sqrt{1-v^2/c^2}\,(c^2-vu)}{\sqrt{(c^2-vu)^2-c^2(v-u)^2}}\,.$$

$$= \frac{\lambda\sqrt{1-v^2/c^2}\,(c^2-vu)}{\sqrt{(c^2-v^2)(c^2-u^2)}} = \frac{\lambda(1-vu/c^2)}{\sqrt{1-u^2/c^2)}} \qquad \text{...(44)}$$

Similarly $\lambda_-' = -\lambda\,(1 + vu/c^2)/\sqrt{1-u^2/c^2}$. ...(45)

Thus the net line charge density in frame S'

$$\lambda_T' = \lambda_+' + \lambda_-' = \frac{\lambda}{\sqrt{1-u^2/c^2}}\left[\left(1-\frac{vu}{c^2}\right)-\left(1+\frac{vu}{c^2}\right)\right]$$

$$= \frac{-2\lambda vu}{c^2\sqrt{1-u^2/c^2}}. \qquad ...(46)$$

Thus we see that a current carrying wire which is electrically neutral in one system will be charged in another as a result of unequal Lorentz contraction of the positive and negative charge lines.

The line charge λ_T sets up an electric field $E' = \lambda_T'/2\pi\varepsilon_0 r$. Thus the electric force experienced by charge q in frame S'

$$F' = qE' = -\frac{2\lambda vu.}{c^2\sqrt{1-u^2/c^2}}.\frac{q}{2\pi\varepsilon_0 r} = -\frac{\lambda v}{\pi\varepsilon_0 c^2 r}.\frac{qu}{\sqrt{1-u^2/c^2}}. \qquad ...(47)$$

We know that if there is a force on q in S' – frame, there must be a force in S – frame also, which we can calculate by using the transformation rules for force. Since q is at rest in frame S' and F' is perpendicular to u, the force in S – frame is given by Eq. (24), as

$$F = F'\sqrt{1-u^2/c^2} = -\frac{\lambda v}{\pi\varepsilon_0 c^2}\frac{qu}{r}. \qquad ...(48)$$

The negative sign shows that the force is attractive. Thus the charge q is attracted toward the wire by a force that is purely electrical in frame S', where the charge is at rest and the wire is charged. But in frame S, where the wire is neutral, the force F will be *nonelectrical*. If we use relations $c^2 = 1/\mu_0\varepsilon_0$ and λv = current I, then the above expression has the form

$$F = -(\mu_0 I/2\pi r)\,qu. \qquad ...(49)$$

The term $\mu_0 I/2\pi r$ is the magnetic field of a long, straight current carrying wire. Thus the above expression for force is precisely what we would have obtained by using Lorentz force equation in frame S. We, therefore conclude that *the magnetic field is produced by electrostatics and relativity taken together.*

16.10 FOUR DIMENSIONAL SPACE - MINKOWSKI SPACE

The Lorentz transformation treats x and t as equivalent variables. Minkowski suggested that *ict* may be introduced as a fourth coordinate. Thus a point in a space can be defined by four variables, three spatial coordinates x, y and z and the fourth coordinate based on time *ict*. The ordinary three dimensional space is known as *Euclidean space.* The four dimensional space, represented by four coordinates (x, y, z, ict) is known as *Minkowski space.*

The length of the position vector **R** of a point whose coordinates are (x, y, z, ict) in a frame S is given by

$$R^2 = x^2 + y^2 + z^2 + (ict)^2 = x^2 + y^2 + z^2 - c^2 t^2.$$

The length R' be the length of this position vector in a frame S' moving with velocity u along the x-axis with reference to frame S. From Lorentz transformation, we find that

$$x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2, \qquad ...(50)$$

i.e. the length of the vector in four dimensional space remains unchanged (or invariant) under a Lorentz transformation. We use notation x_1, x_2, x_3 and x_4 for the coordinates *x, y, z* and *ict* respectively. Thus

$$R^2 = x_1^2 + x_2^2 + x_3^2 + x_4^4 = x_1'^2 + x_2'^2 + x_3'^2 + x_4'^2. \qquad ...(51)$$

Thus the Lorentz transformations in terms of Minkowski notations may thus be written as

$$x_1' = \gamma(x_1 + i\beta x_4),\ x_2' = x_2,\ x_3' = x_3$$

and
$$x_4' = \gamma(x_4 - i\beta x_1).\ \text{where } \beta = u/c. \qquad ...(52)$$

These relations may be written in the form

$$x_\mu' = \sum_{\nu=1}^{4} a_{\mu\nu}\, x_\nu, \qquad ...(53)$$

where μ and ν are constants, run from 1 through 4. The coefficients $a_{\mu\nu}$ may be expressed in the form of matrix as

$$a_{\mu\nu} = \begin{bmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \qquad ...(54)$$

Four Dimensional vectors (Four vectors) – The coordinates in four space x_1, x_2, x_3, x_4 form a four dimensional vector. Thus a *four dimensional vector* $\mathbf{A}_\mu$ is a set of four quantities A_1, A_2, A_3, and A_4 that transform as the components x_1, x_2, x_3 and x_4 under a Lorentz transformation. Four dimensional vectors are also called *four vectors or world vectors.* Thus

$$A_\mu' = \sum_{\nu=1}^{4} a_{\mu\nu}\, A_\nu. \qquad ...(55)$$

The law of physics, the conservation of total relativistic energy and momentum show that $p_1, p_2, p_3, E/c$ form the *momentum-energy four vectors.*

Four Tensors : Lorentz four vectors are also called tensors of the first rank in a four dimensional space. A second rank tensor $T_{\mu\nu}$ is a set of 16 quantities, it results from the differentiation of a four vector A_μ with respect to x_ν, where μ and ν both run from1 through 4. It is labeled by two indices and transforms as

$$T_{\mu\nu}' = \sum_{\lambda,\sigma=1}^{4} a_{\mu\lambda}\, a_{\nu\sigma}\, T_{\lambda\sigma}. \qquad ...(56)$$

16.11 RELATIVISTIC ELECTRODYNAMICS

Let us now express the equations of electromagnetism using the four dimensional formalism.

(B) Continuity Equation : The continuity equation relates the charge and current density as

$$\nabla.\mathbf{J} + \partial\rho/\partial t = 0, \qquad ...(57)$$

or
$$\frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} + \frac{\partial(ic\rho)}{\partial(ict)} = 0.$$

In four dimensional space, it may be written as

$$\frac{\partial J_1}{\partial x_1} + \frac{\partial J_2}{\partial x_2} + \frac{\partial J_3}{\partial x_3} + \frac{\partial J_4}{\partial x_4} = 0, \qquad ...(58)$$

where $J_\mu = (J_1, J_2, J_3, ic\rho) = (\mathbf{J}, ic\rho)$ is the *four vector current density.* the quantity $\partial J_\mu / \partial x_\mu$ is the four *dimensional divergence of* $\mathbf{J}_\mu$. Thus the *continuity equation states that the current density* is divergenceless. One can write the continuity equation in the covariant form as

$$\Box \cdot J_\mu = \sum_{\mu=1}^{4} \frac{\partial J_\mu}{\partial x_\mu} = 0. \qquad \text{...(59)}$$

The symbol $\Box$ is thus the four dimentional analog of ∇.

(B) Lorentz Condition : The Lorentz Gauss condition relating vector and scalar potential is

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0, \qquad \text{...(60)}$$

or

$$\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{\partial (i\Phi/c)}{\partial (ict)} = 0.$$

In four dimensional space, it may be written as

$$\frac{\partial A_1}{\partial x_1} + \frac{\partial A_2}{\partial x_2} + \frac{\partial A_3}{\partial x_3} + \frac{\partial (i\Phi/c)}{\partial x_4} = 0. \qquad \text{...(61)}$$

Thus we see that A and Φ will combine to form a *four vector potential* or *electromagnetic four potential* of the form

$$A_\mu = (A_1, A_2, A_3, i\Phi/c) = (\mathbf{A}, i\,\Phi/c). \qquad \text{...(62)}$$

Equation (60) takes a similar covariant form to the covariant form of the continuity equation

$$\Box \cdot A_\mu = \sum_{\mu=1}^{4} \frac{\partial A_\mu}{\partial x_\mu} = 0. \qquad \text{...(63)}$$

(C) Wave Equation for A and Φ : The wave equations for the vector and scalar potentials are given as

$$\nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu_0\, \mathbf{J} \text{ and } \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} = -\frac{\rho}{\varepsilon_0}. \qquad \text{...(64)}$$

In four dimensional form

$$\begin{aligned} \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial (ict)^2} \\ &= \frac{\partial^2}{\partial {x_1}^2} + \frac{\partial^2}{\partial {x_2}^2} + \frac{\partial^2}{\partial {x_3}^2} + \frac{\partial^2}{\partial {x_4}^2} = \Box^2. \end{aligned}$$

where $\Box^2$ is the four dimensional analog of Laplacian operator and is known as *d'* Alembertian operator. Lorentz transformation shows that $\Box^2 = \Box'^2$.

Since J and ρ are the space and time components of the four vectors $\mathbf{J}_\mu$, therefore Eqs. (64) show that **A** and Φ will have to combine to form a four vector of the form $A_\mu = (\mathbf{A}, i\Phi/c) = (A_1, A_2, A_3, i\Phi/c)$ as obtained by Lorentz condition.

Thus the Eqs. (64) may be written in a single covariant form as

$$\Box^2 A_\mu = \sum_{v=1}^{4} \frac{\partial^2 A_\mu}{\partial {x_v}^2} = -\mu_0 \mathbf{J}_\mu. \qquad \text{...(65)}$$

(D) Maxwell's Equations : We can not write Maxwell's equations in the covariant form as was the case earlier discussed. This indicates that **E** – and **B** – fields may not combine to form four vectors. They may combine to form higher rank four tensors.

Maxwell's equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ give the following equations relating fields to the vector and scalar potentials **A** and Φ as

$$\mathbf{B} = \nabla \times \mathbf{A} \text{ and } \mathbf{E} = -\nabla \Phi - \partial \mathbf{A}/\partial t. \qquad \text{...(66)}$$

Using Eq. (62) one can show that the components of **B** and **E** may be expressed as

$$B_1 = \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} = \frac{\partial A_3}{\partial x_2} - \frac{\partial A_2}{\partial x_3};$$

$$B_2 = \frac{\partial A_1}{\partial x_3} - \frac{\partial A_3}{\partial x_1}; B_3 = \frac{\partial A_2}{\partial x_1} - \frac{\partial A_1}{\partial x_2} \qquad \text{...(67)}$$

and

$$E_1 = -\frac{\partial \Phi}{\partial x} - \frac{\partial A_1}{\partial t} \text{ or } \frac{iE_1}{c} = \frac{\partial A_1}{\partial x_4} - \frac{\partial A_4}{\partial x_1};$$

$$\frac{iE_2}{c} = \frac{\partial A_2}{\partial x_4} - \frac{\partial A_4}{\partial x_2}; \frac{iE_3}{c} = \frac{\partial A_3}{\partial x_4} - \frac{\partial A_4}{\partial x_3}. \qquad \text{...(68)}$$

These relations show that, the electric and magnetic fields can be expressed by a single equation as

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu} \qquad \text{...(69)}$$

Since the components $F_{\mu\nu}$ are the derivatives of four vector potentials, we therefore see that electric and magnetic fields are the elements of the second rank field strength tensor $F_{\mu\nu}$. It is clear from equation (69) that $F_{\mu\nu}$ is an antisymmetric tensor. Under Lorentz transformation, $F_{\mu\nu}$ transforms according to equation (56) as

$$F'_{\mu\nu} = \sum_{\lambda,\sigma}^{4} a_{\mu\lambda}\, a_{\nu\sigma}\, F_{\lambda\sigma}. \qquad \text{...(70)}$$

The field strength tensor $F_{\mu\nu}$ is thus given as

$$F_{\mu\nu} = \begin{array}{c|cccc} & \nu \rightarrow & & & \\ \hline \mu \downarrow & 0 & B_3 & -B_2 & -iE_1/c \\ & -B_3 & 0 & B_1 & -iE_2/c \\ & B_2 & -B_1 & 0 & -iE_3/c \\ & iE_1/c & iE_2/c & iE_3/c & 0 \end{array} \qquad \text{...(71)}$$

Covariant forms of Maxwell's equations – Let us consider the equations

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \qquad \text{...(72)}$$

and

$$\nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \mu_0 \mathbf{J}. \qquad \text{...(73)}$$

On expanding Eq. (72) and using the matrix of Eq. (71), we get

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = \frac{J_4}{ic\varepsilon_0} \text{ or } -ic\frac{\partial F_{41}}{\partial x_1} - ic\frac{\partial F_{42}}{\partial x_2} - ic\frac{\partial F_{43}}{\partial x_3} = \frac{J_4}{ic\varepsilon_0}$$

On adding $-ic\ \partial F_{44}/\partial x_4$ to the left, we get

$$\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \ (\mu = 4) \qquad \text{...(74)}$$

This is the covariant form of Eq. (72). A similar procedure for Eq. (73) gives

$$\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \ (\mu = 1, 2, 3) \qquad \text{...(75)}$$

This is again a covariant form of Eq. (73). The combination of Eqs. (74) and (75) gives

$$\sum_{\nu=1}^{4} \frac{\partial F_{\mu\nu}}{\partial x_\nu} = \mu_0 J_\mu \ (\mu = 1, 2, 3, 4) \qquad \text{...(76)}$$

It represents the covariant form of Eqs. (72) and (73).

Similarly, the covariant form of the homogeneous Maxwell's equations

$$\nabla \cdot \mathbf{B} = 0 \text{ and } \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \qquad \text{...(77)}$$

can be shown by the equation

$$\frac{\partial F_{\mu\nu}}{\partial x_\lambda} + \frac{\partial F_{\nu\lambda}}{\partial x_\mu} + \frac{\partial F_{\lambda\mu}}{\partial x_\nu} = 0 \ (\mu \neq \nu \neq \lambda = 1, 2, 3, 4) \qquad \text{...(78)}$$

Since $F_{\mu\nu}$ is the tensor of second rank, therefore its differentiation with respect to x_λ will be a tensor of third rank.

(E) Transformation Laws of Electric and Magnetic Fields : Let is now calculate the fields $\mathbf{E}'$ and $\mathbf{B}'$ as observed by an observer moving with velocity u along the x – axis. Since $B_1 = F_{23}$, therefore by the tensor transformation relation (56), we get

$$B_1' = F_{23}' = \sum_{\lambda,\sigma}^{4} a_{2\lambda}\, a_{3\sigma}\, F_{\lambda\sigma}. \qquad \text{...(79)}$$

From Eqs. (54) and (71), we see that many of the terms of Eq. (79) vanish. We are left with

$$B_1' = a_{22}\, a_{33}\, F_{23} = B_1.$$

Similarly

$$\begin{aligned} B_2' &= F_{31}' = \sum_{\lambda,\sigma}^{4} a_{3\lambda}\, a_{1\sigma}\, F_{\lambda\sigma} = a_{33}\, a_{11}\, F_{31} + a_{33}\, a_{14}\, F_{34} \\ &= \gamma B_2 + i\beta\gamma\,(-i\, E_3/c) = \gamma\,[B_2 + (u/c^2) E_3]. \qquad \text{...(80)} \end{aligned}$$

and

$$\begin{aligned} B_3' &= F_{12}' = \sum_{\lambda,\sigma}^{4} a_{1\lambda}\, a_{2\sigma}\, F_{\lambda\sigma} = a_{11}\, a_{22}\, F_{12} + a_{14}\, a_{22}\, F_{42} \\ &= \gamma B_3 + i\beta\gamma (iE_2/c) = \gamma [B_3 - (u/c^2) E_2]. \qquad \text{...(81)} \end{aligned}$$

$$\begin{aligned} E_1' &= ic F_{14}' = ic \sum_{\lambda,\sigma}^{4} a_{1\lambda}\, a_{4\sigma}\, F_{\lambda\sigma} = ic(a_{11}\, a_{44}\, F_{14} + a_{14}\, a_{41}\, F_{41}) \\ &= ic\,[\gamma^2(-iE_1/c) + i\beta\gamma(-i\beta\gamma)\, iE_1/c] = E_1. \qquad \text{...(82)} \\ E_2' &= ic F_{24}' = ic \sum_{\lambda,\sigma}^{4} a_{2\lambda}\, a_{4\sigma}\, F_{\lambda\sigma} = ic(a_{22}\, a_{44}\, F_{24} + a_{22}\, a_{41}\, F_{21}) \\ &= ic\,[\gamma\,(-iE_2/c) + (-i\beta\gamma)\,(-B_3)] = \gamma\,(E_2 - uB_3). \qquad \text{...(83)} \\ E_3' &= ic F_{34}' = ic \sum_{\lambda,\sigma}^{4} a_{3\lambda}\, a_{4\sigma}\, F_{\lambda\sigma} = ic(a_{33}\, a_{44}\, F_{34} + a_{33}\, a_{41}\, F_{31}) \\ &= ic\,[\gamma\,(-iE_3/c) + (-i\beta\gamma) B_2] = \gamma\,(E_3 + uB_2). \qquad \text{...(84)} \end{aligned}$$

These equations can be generalized as

$$\begin{aligned} \mathbf{E}'_{\parallel} &= \mathbf{E}_{\parallel} & \mathbf{B}'_{\parallel} &= \mathbf{B}_{\parallel} \\ \mathbf{E}'_{\perp} &= \gamma[\mathbf{E}_{\perp} - \boldsymbol{u} \times \mathbf{B}_{\perp}] & \mathbf{B}'_{\perp} &= \gamma\left[\mathbf{B}_{\perp} + \frac{\boldsymbol{u} \times \mathbf{E}_{\perp}}{c^2}\right] \qquad \text{...(85)} \end{aligned}$$

EXERCISES

Example 1. *In a Michelson – Morley experiment, the path length was 11m and light of wavelength 6000 A^0 was used. Assuming the ratio of the velocity of the earth to the velocity of light in stationary ether to be 10^{-4}, find the expected fringe shift.*

In a Michelson – Morley experiment, the total path length difference for two orientations is given by

$$\Delta L = 2L(u^2/c^2) = 2 \times 11 \times (10^{-4})^2 = 2.2 \times 10^{-7}\ m.$$

If Δn be the number of fringes moving through the cross wire of the telescope, then we have $\Delta L = \Delta n\ \lambda$, or

$$\Delta n = \Delta L/\lambda = 2.2 \times 10^{-7}/6000 \times 10^{-10} = 0.37.$$

Exaple 2. *A spaceship A of proper length L is travelling east at speed v_1 and spaceship B of proper length 2L is travelling west at speed v_2, as observed by an observer on the earth. The pilot of spaceship A sets his clock to zero when the spaceship B passes him. Calculate the time at which the pilot of A observes that the tail of B passes him.*

To solve the problem, let us assume two frames S and S' *with their origins coinciding at t* = 0. The frame S is moving. The spaceship A is at rest in frame S and the spaceship B is at rest in frame S'. If the pilots are at the origins of their respective frames. The relative speed u is given by the relation

$$u = \frac{v_1 + v_2}{1 + v_1 v_2 / c^2}.$$

If the spaceships are aligned along the x – and x'–axes, the front of spaceship A(in frame S) is at $x = 0$ and its back is at $x = -L$. Similarly, the front of spaceship B(in frame S') is at $x' = 0$ and its back is at $x' = 2L$. The time at which the fronts of two spaceships just pass each other may be taken as $t = t' = 0$. From Lorentz transformation

$$x' = \gamma(x + ut).$$

At the event the back end of spaceship B just passes the front end of spaceship A, we have $x' = 2L$ and $x = 0$. Thus

$$2L = \gamma(0 + ut) \text{ or } t = 2L/u\gamma.$$

This is the time recorded by pilot A. We can also find the same result by length contraction. The pilot A sees the length of spaceship B contracted from $2L$ to $2L/\gamma$. The spaceship B moves with a relative speed of u, the time it takes to pass is $(2L/\gamma)/u = 2L/u\gamma$.

Example 3. *In the initial frame of the fixed stars, a spaceship travels along the x-axis. Let the spaceship starts with initial velocity v = 0 when x = 0 and has constant acceleration g. Find the distance x travelled when it has achieved a velocity v (~ c).*

Consider two frames S and S' with the latter moving with a constant velocity v along the x-direction, relative to the frame S. Let the velocity and acceleration of an object moving in x-direction be $u, du/dt$ and $u', du'/dt'$ in these frames. Lorentz transformation gives

$$x = \gamma(x' + \beta ct'),\ ct = \gamma(ct' + \beta x'), \text{ where } \beta = v/c.$$

and $\gamma = (1 - \beta^2)^{-1/2}$. The velocity of the object is transformed according to the relation

$$u = (u' + v)/(1 + vu'/c^2). \qquad \text{...}(i)$$

On differentiating, we get

$$dt = \gamma\left(dt' + \frac{v}{c^2}dx'\right) = \gamma\, dt'\left(1 + \frac{vu'}{c^2}\right).$$

and $$du = du'/\gamma^2\,(1 + vu'/c^2)^2.$$

The ratio will give the relation for transformation of acceleration as

$$a = a'/\gamma^3\,(1 + vu'/c^2)^3. \qquad ...(ii)$$

If S is the inertial frame attached to the fixed stars and S' is the inertial frame in which the spaceship is momentarily at rest. Then in S' frame

$$u' = 0 \text{ and } a' = g. \qquad ...(iii)$$

$$\therefore \qquad u = v \text{ and } a = g/\gamma^3. \qquad ...(iv)$$

As the velocity of the spaceship is increased from 0 to v is the frame S. The distance travelled is

$$x = \int u\,dt = \int_0^v \frac{u\,du}{a} = \frac{1}{g}\int_0^v \frac{u\,du}{(1-\beta^2)^{3/2}} = \frac{c^2}{g}[\gamma - 1].$$

Example 4. *At exactly noon, a distress call is received on earth from a space station, which is 4.5 × 10^9 km away from the earth. The request is for some highly unstable antibiotics, which lose their potency with in 5 hrs of removal from culture. The same is sent at 1 p.m through a rocket having a speed 0.7c. At what time, the distress call was sent. Will the antibiotics still be potent on arrival at the space station.*

Since the call from the space station must be transmitted by radio, the time required for the radio signal to travel from the station to the earth

$$t = 4.5 \times 10^{12}/3 \times 10^8 = 1.5 \times 10^4 s = 4hr\ 10\ min.$$

Thus the distress call must have been sent from the space station at 8.50 A.M.

The rocket will take $4.5 \times 10^{12}/0.7 \times 3 \times 10^8$ = 21429 sec = 5 hr 57 min. 9 sec as observed by the observer on earth. The rocket will therefore arrive on the space station at 6 : 57 : 09 PM. One may therefore conclude that the medicine will loss its potency before its arrival on the destination.

The time dilation will help. According to the clock on the rocket ship, the time elapsed during this journey was less then 5 hr. 57 min 9 sec. The time recorded by this clock will be

$$t' = 21429\sqrt{1-0.7^2} = 15303 \text{ sec} = 4 \text{ hr } 15 \text{ min } 3 \text{ sec}.$$

Hence the medicine will still be potent for about next 45 m.

Example 5. *The muons are found to have life time 2.2 × 10^{-6} sec, as observed in the laboratory. Fast moving muons form a part of cosmic rays. Find the apparent life time of the muons moving with speed of 0.98 c relative to the earth. Calculate the distance travelled by them during this time.*

The mesons have lifetime of 2.2×10^{-6} sec in the laboratory frame (S– frame say) where they are at rest. In cosmic rays, the mesons move with a speed $u = 0.98c$. Thus we may assume frame of cosmic ray mesons as frame S' which has a speed of 0.98c with respect to frame S. Thus the apparent lifetime T' of the mesons in S'– frame as observed by an observer in S will be

$$T' = \frac{T}{\sqrt{1-u^2/c^2}} = \frac{2.2\times10^{-6}}{\sqrt{1-(0.98)^2}} = 11.06\times10^{-6} \text{ sec.}$$

The apparent distance travelled by the mesons in this time

$$L' = uT' = 0.98 \times 3 \times 10^8 \times 11.06 \times 10^{-6} = 3252 \text{ m}.$$

This is the distance observed by the observer in frame S. The distance observed by an observer in frame S' will be

$$L = uT = 0.98 \times 3 \times 10^8 \times 2.2 \times 10^{-6} = 646.8\text{m}.$$

Example 6. *Find the mass, momentum and total energy of an electron moving with a speed of 0.8 c.*

The rest mass of an electron m_0 is related with its moving mass relativistically as

$$m = \frac{m_0}{\sqrt{1-u^2/c^2}} = \frac{9.1095\times10^{-31}}{\sqrt{1-0.8^2}} = 1.51825\times10^{-30} \text{ kg}.$$

The relativistic momentum of the electron

$$p = \frac{m_0 u}{\sqrt{1-u^2/c^2}} = \frac{9.1095\times10^{-31}\times0.8\times3\times10^8}{\sqrt{1-0.8^2}}$$
$$= 3.6438 \times 10^{-22} \text{ kg m/sec}.$$

The total energy of the moving electron

$$E = \sqrt{p^2c^2 + m_0^2c^4} = mc^2 = (1.51825 \times 10^{-30})\,(3 \times 10^8)^2$$
$$= 1.366425 \times 10^{-13} \;\; J = 0.854 \text{ MeV}.$$

Example 7. *Two large parallel non-conducting plates separated by a distance d (along z-axis) move together along x-axis with velocity v. The upper and lower plates have uniform surface charge densities $+\sigma$ and $-\sigma$ respectively in the rest frame of the plates. Neglecting edge effects and using relativistic relations, find **E** and **B** between the plates.*

Let the electromagnetic fields be **E**′, **B**′ in frame S', where the plates are at rest and be **E**, **B** in the frame S. the field vectors transform according to

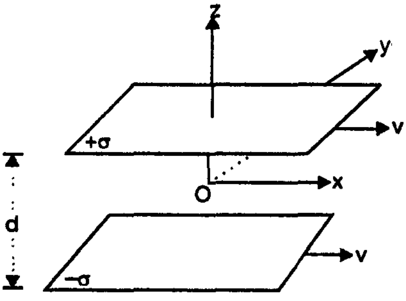


Fig. 16.6. Example 7.

$$E_x = E_x', B_x = B_x'$$
$$E_y = \gamma(E_y' - \beta c B_z'), B_y = \gamma\left(B_y' - \frac{\beta}{c}E_z'\right)$$
$$E_z = \gamma(E_z' - \beta c B_y'), B_z = \gamma\left(B_z' - \frac{\beta}{c}E_y'\right)$$

(where $\beta = v/c$ and $\gamma = (1 - v^2/c^2)^{-1/2}$

In the rest frame S',

$$B_x' = B_y' = B_z' = 0.$$
$$E_x' = E_y' = 0, E_z' = -\sigma/\varepsilon_0.$$

$\therefore E_x = 0, B_x = 0, E_y = 0, B_y = (\gamma\beta/\varepsilon_0 c)\,\sigma, E_z = -\gamma\sigma/\varepsilon_0, B_z = 0.$

Hence in the frame S (Lab-frame), the electric intensity is in the $-z$-direction with magnitude $\gamma\sigma/\varepsilon_0$ and the magnetic induction in the y-direction with magnitude $(\gamma v/\varepsilon_0 c^2)\,\sigma$.

Example 8. *Prove that* **E** · **B** *and $E^2 - c^2B^2$ are Lorentz scalars.*

To show that **E** · **B** is a Lorentz scalar, we will prove that the quantity **E.B** is invariant under Lorentz transformation, *i.e.*

$$\mathbf{E'}\cdot\mathbf{B'} = E_1'B_1' + E_2'B_2' + E_3'B_3'$$
$$= E_1B_1 + \gamma^2(E_2 - u\,B_3)\,(B_2 + uE_3/c^2) + \gamma^2(E_3 + uB_2)\,(B_3 - uE_2/c^2)$$

$$\begin{aligned}\mathbf{E}'\cdot\mathbf{B}' &= E_1B_1 + E_2B_2\gamma^2(1-u^2/c^2) + E_3B_3\gamma^2(1-u^2/c^2)\\ &= E_1B_1 + E_2B_2 + E_3B_3 = \mathbf{E}\cdot\mathbf{B}\cdot\end{aligned}$$

Similarly, we have

$$\begin{aligned}E'^2 - c^2B'^2 &= E_1'^2 + E_2'^2 + E_3'^2 - c^2(B_1'^2 + B_2'^2 + B_3'^2)\\ &= E_1^2 + \gamma^2(E_2 - uB_3)^2 + \gamma^2(E_3 + uB_2)^2 - c^2[B_1^2 + \gamma^2(B_2 + uE_3/c^2)^2 + \gamma^2(B_3 - uE_2/c^2)^2]\\ &= E_1^2 + \gamma^2[E_2^2 + u^2B_3^2 - 2uE_2B_3 + E_3^2 + u^2B_2^2 + 2uE_2B_2 - c^2B_2^2 - u^2E_3^2/c^2\\ &\qquad - 2uB_2E_3 - c^2B_3^2 - u^2E_2^2/c^2 + 2uB_3E_2] - c^2B_1^2.\\ &= E_1^2 + \gamma^2E_2^2(1-u^2/c^2) + \gamma^2E_3^2(1-u^2/c^2) - c^2B_1^2 - c^2\gamma^2B_2^2(1-u^2/c^2) - c^2\gamma^2B_3^2(1-u^2/c^2).\\ &= E_1^2 + E_2^2 + E_3^2 - c^2(B_1^2 + B_2^2 + B_3^2) = E^2 - c^2B^2.\end{aligned}$$

Exaple 9. *An infinitely long perfectly conducting straight wire of radius r carries a constant current i and charge density zero as shown by a fixed observer A. The current is due to an electron stream of uniform density moving with relativistic velocity u. A second observer B travels parallel to the wire with relativistic velocity v. Find the electromagnetic field as seen by the observer B.*

Let S and S' be the rest frames of the observers A and B respectively. The common axis say the x-axis being along the axis of conducting wire, which is fixed in S. In this frame $\rho = 0, \mathbf{j} = (i/\pi r_0^2)\,\mathbf{a}_x$, so the electric and magnetic fields in S are $\mathbf{E} = 0$.

$$\begin{aligned}\mathbf{B}(r) &= (\mu_0 ir/2\pi r_0^2)\,\mathbf{a}_\phi && (r < r_0 \text{ inside})\\ &= (\mu_0 i/2\pi r)\,\mathbf{a}_\phi && (r > r_0 \text{ outside})\end{aligned}$$

Lorentz transformation gives the electromagnetic fields in S' frame as

$$\begin{aligned}\mathbf{E}' &= \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v}\times\mathbf{B}_\perp) = -\gamma vB\,\mathbf{a}_r = -(\mu_0\gamma\, i\, v\, r/2\pi r_0^2)\,\mathbf{a}_r && (r < r_0)\\ &= -(\mu_0\gamma\, i\, v/2\pi r)\,\mathbf{a}_r && (r > r_0)\\ \mathbf{B}' &= \mathbf{B}'_\perp = \gamma\left(\boldsymbol{B}_\perp - \frac{\mathbf{v}\times\boldsymbol{E}_\perp}{c^2}\right) = \gamma\,\mathbf{B}\,\mathbf{a}_\Phi = (\mu_0 i\gamma r/2\pi r_0^2)\,\mathbf{a}_\phi && (r < r_0)\\ &= (\mu_0 i\gamma/2\pi r)\,\mathbf{a}_\phi && (r > r_0)\end{aligned}$$

The lengths r *and* r_0 are not changed by the transformation.

Example 10. *(a) Derive an expression for the force on an electron at a distance r < a from the axis of a cylindrical column of electrons of uniform charge density of* ρ_o *and radius a.*

(b) An observer in the laboratory sees a beam of circular cross section and density ρ *moving at velocity v. What force does he see on an electron of the beam at distance r < a from the axis ?*

(c) What will happen if v is near the velocity of light and an observer is moving with the beam ?

Using cylindrical coordinates with the z-axis along the axis of the cylinder. By Gauss's law and the symmetry, the electric field at a distance $r < a$ from the axis

$$\mathbf{E}(r) = (\rho_0 r/2\varepsilon_0)\,\mathbf{a}_r \qquad (r < a)$$

$\therefore$ Force on the electron at this point $\mathbf{F} = -e\mathbf{E} = -(e\rho_0 r/2\varepsilon_0)\,\mathbf{a}_r$.

This is a repulsive force as ρ_0 is negative.

(b) Let S and S' be the laboratory frame and the rest frame respectively of the column of electrons. The S' is moving with velocity v relative to S along the z-axis. The charge density may be transformed using four vector relation as $\rho = \gamma\rho_0$. In S' frame, the electric and magnetic fields are given by

$$\mathbf{E}' = (\rho_0 r'/2\varepsilon_0)\,\mathbf{a}_r,\ \mathbf{B}' = 0$$

In S– frame, one has

$$\mathbf{E}_\perp = \gamma[\mathbf{E}'_\perp - \mathbf{v}\times\mathbf{B}'_\perp) = \gamma\mathbf{E}',\ \mathbf{E}_\parallel = \mathbf{E}'_\parallel = 0.$$

$$\mathbf{B}_\perp = \gamma\left[\mathbf{B}'_\perp + \frac{\mathbf{v}}{c^2}\times\mathbf{E}'_\perp\right] = \gamma\frac{\mathbf{v}}{c^2}\times\mathbf{E}', \mathbf{B}_\parallel = \mathbf{B}'_\parallel = 0.$$

Thus the force on an electron of the beam at $r < a$ is given by

$$\mathbf{F} = -e\mathbf{E} - e\mathbf{v}\times\mathbf{B} = -e\gamma\,\mathbf{E}' - e\mathbf{v}\times\left(\gamma\frac{\mathbf{v}}{c^2}\times\mathbf{E}'\right)$$

$$= -e\gamma\mathbf{E}' + e\gamma(v^2/c^2)\,\mathbf{E}' \qquad (\because \mathbf{v} \text{ is } \perp \text{ to } \mathbf{E}')$$

As there is no transverse Lorentz contraction, $r = r'$,

$$\therefore \qquad \mathbf{F} = -e\mathbf{E}'/\gamma = -\,(e\gamma r/2\varepsilon_0 r^2)\,\mathbf{a}_r$$

(*c*) In S'– frame, the force on the electron

$$\mathbf{F}' = -e\mathbf{E}' = -\left(\frac{e\rho_0 r}{2\varepsilon_0}\right)\mathbf{a}_r = -\frac{e\rho r}{2\varepsilon_0\gamma}\mathbf{a}_r.$$

As $\gamma > 1$, $\mathbf{F}' > \mathbf{F}$. In the rest frame S' only the electric field exerts a force on the electron, while in lab frame S, a magnetic force also acts opposite to the electric force which is larger. Thus the net force on the electron is smaller as seen in S.

ORAL QUESTIONS

1. Name the important experiment which gave the negative result.
2. If the principle of equivalence were not true, what additional information would be needed for scientists to communicate with each other ?
3. If $v > c$, the space and time coordinates will be imaginary. What will be the consequences of such results ?
4. If the length of a cube contracts only on one side, what will happen to its density ?
5. Whether the twin paradox has been resolved ?
6. What do you mean by improper length ?
7. Is the kinetc energy of a particle always less than its rest energy ?
8. Is it correct to write $u = c\sqrt{1-(E_0/E)^2}$, where E is the total energy of a particle with rest energy E_0 and moving with a velocity u.
9. Name any observation which supports the theory of special relativity.
10. A cube is moving with a velocity u in the direction parallel to one of its edges. Write an expression for its volume and density.
11. Whether do σ and ρ invariant relativistically ?
12. What will happen to the electric field of a moving parallel plate capacitor.
13. Can you apply Galilean transformation for electromagnetism ?
14. Can you co-relate magnetism with relativity ?
15. What do you mean by four dimensional space and four dimensional vectors ?

PROBLEMS

1. As observed from a spaceship moving toward the earth with a speed of 0.8*c*, an athlete runs a race for 10 sec. How long does the runner think the race lasted ? (6 sec.)
2. An electron moving with a speed of 0.9*c* passes a proton moving with speed of 0.6*c*. Calculate the relative velocities of electron and proton. (0.652*c*, – 0.652 *c*)

3. A particle in frame S has velocity $v_x \mathbf{i} + v_y \mathbf{j}$. What is the velocity seen by an observer at rest in frame S' which is moving with a velocity $u\mathbf{i}$ relative to frame S.
$(v_x - u)/[1 - uv_x/c^2]\,\mathbf{i} + v_y/\gamma[1 - uv_x/c^2]\mathbf{j}$.

4. Calculate the mass and momentum of a 1GeV (a) electron and (b) proton.
(1.78×10^{-27} kg, $1 GeV/c$)

5. A small bomb is made so that it will explode exactly 5 sec after the rocket in which it is fitted is launched. If the rocket attains a speed of 0.99c with 1 sec from lift off, where will the explosion occur.
(8.49×10^9 m from earth)

6. The pions have half life of 2.6×10^{-8} sec at rest. An accelerated pions generated by an accelerator are observed to travel 40 m before half have decayed. What is the speed of the pions as measured in the laboratory ?
(0.98/c)

7. Two spaceships travel in exactly opposite directions. While passing each other, passenger in spaceship A measures the speed of spaceship B as 0.3c. The passenger in spaceship B measures that spaceship B is 30m long and spaceship A is 40m. What are the lengths of these two spaceships as measured by a passenger in A ?
(L_A = 41.9m. L_B = 28.6m)

8. At what rate must mass be converted to energy to produce a power of 20 MW ? (0.8mg/hr)

9. A proton has a speed of 0.9c. Find its (a) momentum, (b) kinetic energy and (c) total energy.

10. By how much does the mass of the ice increase because of the energy added to melt 1 kg of ice at 0°C.
(3.7×10^{-12} kg)

11. Prove that the continuity equation follows from Maxwell's equations.

12. Electrons and positrons of energy 50 GeV travel in opposite directions around a storage ring. Find the speed of each particle in the rest frame of the other ?

13. A classical electromagnetic wave satisfies the relations $\mathbf{E} \cdot \mathbf{B} = 0$ and $E^2 = c^2 B^2$, between the electric and magnetic fields. Show that these relations are valid in all frames.

14. In the exercise 9, calculate the charge density in the wire implied by the field. With what velocities do the electron and ion streams move ?
($-iv\gamma/\pi r_0^2 c^2$)

15. The uniformly distributed charge per unit length in an infinite ion beam of constant circular cross section is q. Find an expression for the force on a single beam ion that is located at radius r, assuming $R > r$ and that all the ions have the same velocity v.
($Qqr/2\pi\varepsilon_0 R^2\gamma^2$) $\mathbf{a}_\gamma$

16. Uniform beam of charged particles q/l (charge/length) moving with velocity v is distributed uniformly within a circular cylinder of radius R. Find expressions for (a) energy density and (b) momentum density.

$$\left(1+\frac{v^2}{c^2}\right)\frac{q^2 r^2}{8\pi^2\varepsilon_0 R^4 l^2},(r<R);\left(1+\frac{v^2}{c^2}\right)\frac{q^2}{8\pi^2\varepsilon_0 \pi^2 l^2},(r>R);$$
$$vq^2 r^2/4\pi^2\varepsilon_0 c^2 R^4 l^2\,\mathbf{a}_z, r<R; (vq^2/4\pi^2\varepsilon_0 c^2 r^2 l^2)\,\mathbf{a}_z, (r>R)$$

17 ALTERNATING CURRENT

17.1 ALTERNATING CURRENT

As we have discussed in article 12.8, an induced emf in the coil of turns N, area A, rotating with an angular velocity ω in a uniform field B, at any instant t is given by

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}[BNA\cos\omega t] = BNA\,\omega\sin\omega t. \qquad ..(1)$$

Thus when $\omega t = 0$, $\sin\omega t = 0$ and $\mathcal{E} = 0$

$\omega t = \pi/2$, $\sin\omega t = 1$ and $\mathcal{E} = BNA\,\omega$

$\omega t = \pi$, $\sin\omega t = 0$ and $\mathcal{E} = 0$

$\omega t = 3\pi/2$, $\sin\omega t = -1$ and $\mathcal{E} = -BNA\,\omega$

and $\omega t = 2\pi$, $\sin\omega t = 0$ and $\mathcal{E} = 0$.

Here we see that induced emf goes through a cycle, each cycle corresponds to one revolution of the coil. Instantaneous value of emf is generally represented by e and the maximum value $BNA\,\omega$ as $\mathcal{E}_0$. Thus relation (1) reduces to

$$e = \mathcal{E}_0\sin\omega t. \qquad ..(2)$$

It represents a sinusoidal alternating emf at an instant t. This will develop an instantaneous voltage v, given by

$$v = V_0\sin\omega t. \qquad ..(3)$$

The corresponding current i through the circuit is given by

$$i = I_0\sin\omega t. \qquad ..(4)$$

The time of one cycle is known as *time period* T, the number of cycles per second the *frequency* f (= $1/T$), the peak value of current or voltage the *amplitude,* the angular displacement per sec the angular velocity ω.

If an alternating current is passed through an ordinary ammeter or voltmeter, it will record the mean value for the complete cycle, as the quantity to be measured varies with time. The average value of current

$$\bar{I} = \frac{\int_0^T i\,dt}{\int_0^T dt} = \frac{\int_0^{2\pi/\omega} I_0 \sin\omega t\,dt}{\int_0^{2\pi/\omega} dt} = 0. \qquad \text{...(5)}$$

Similarly the average value of the emf for the whole cycle $V = 0$. Since these averages for the whole cycle are zero, hence *dc instruments will indicate zero deflection.* In each complete cycle the average value of current (or voltage) in first half cycle is equal and opposite to that in second half cycle.

In *ac*, the average value of current is defined as its average taken over half the period. Hence

$$I_{av} = \frac{\int_0^{T/2} i\,dt}{\int_0^{T/2} dt} = \frac{\int_0^{\pi/\omega} I_0 \sin\omega t\,dt}{\int_0^{\pi/\omega} dt} = \frac{2}{\pi} I_0$$

$$= (2/\pi) \times \textit{Peak value of current.} \qquad \text{...(6)}$$

Similarly $V_{av} = (2/\pi) \times$ *Peak value of voltage.*

A *dc* meter can be used in an *ac* circuit, if it is connected in the full wave rectifier circuit. The average value of the rectified *current is the same as the average current in any half cycle, i.e., $2/\pi$ times the maximum current I_0.* The meter can be calibrated accordingly.

Most *ac* meters are based upon the measurement of root mean square value. The quadrant electrometer used idiostatically gives a deflection proportional to the square of the applied voltage and the hot wire ammeter used for the measurement of current depends on the square of the current as the heat developed by a current is proportional to (current)^2. Thus the alternating current is measured by the strength of the steady current which would produce the same heating effect. The square root of the mean square value is called the *virtual value* and is the value given by *ac* instruments. As

$$\bar{I}^2 = \frac{\int_0^T i^2\,dt}{\int_0^T dt} = \frac{\int_0^{2\pi/\omega} I_0^2 \sin^2\omega t\,dt}{\int_0^{2\pi/\omega} dt} = \frac{I_0^2}{2}.$$

$\therefore$ Virtual current = $(I^2)^{1/2} = I_{rms} = I_{eff}$

$$= I_0/\sqrt{2} = 0.707 I_0. \qquad \text{..(7)}$$

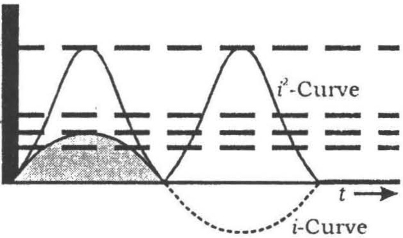


Fig. 17.1. Current Curves.

Similarly we get $V_{rms} = V_{eff} = V_0/\sqrt{2} = 0.707 V_0$.

Thus when we speak of our household power supply as 200 volts *ac*, this means that the rms voltage is 200 volts and its voltage amplitude is

$$V_0 = \sqrt{2} V_{rms} = 283 \text{ volts}.$$

The *ac* instruments are calibrated in virtual volts or virtual amperes. The ratio

$$\frac{\text{rms value}}{\text{Average value}} = \frac{V_0/\sqrt{2}}{2V_0/\pi} = \frac{\pi}{2\sqrt{2}} = 1.11 \qquad \text{...(8)}$$

is known as *form factor*. The variations of i and i^2 with time t is shown in Fig. 17.1.

17.2 ACTUAL ELEMENTS

Resistors, capacitors and inductors are named as circuit elements. Practically these elements are not ideal or pure. A resistor has capacitance between its various parts, which can be represented as a small capacitance between its ends and a small series inductance, which is extremely small when a non-inductive winding is used. An inductance coil is similar but has a large quality ratio. A capacitor has in general a small leakage conductance, representable by a large resistance in parallel, and dielectric losses, the effect of which can be represented by a small series resistance as these losses are associated with the same current that charges the capacitor. The equivalent circuits of R, L and C are shown in Fig. 17.2.

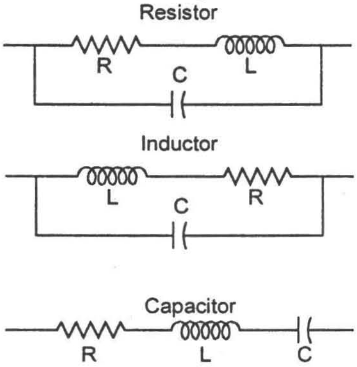


Fig. 17.2. Equivalent Circuits.

Load is a device or element which will draw current from a voltage source. The connection of a load reduces the voltage. A *short circuit between* two points is a direct connection. To short out a component a thick wire is connected in parallel with it. Load is infinite for open circuit.

17.3 CURRENT AND POTENTIAL RELATIONS

Before proceeding to the circuit analysis let us know the relations between current and potential when the sinusoidal voltage is applied to the three circuit elements, resistor, inductor and capacitor separately.

(*a*)Pure Resistor R : Suppose that an alternating potential difference $v = V_0 \sin \omega t$ is applied to a non-inductive resistor of resistance R. Then the instantaneous value of the current i passing through this resistor is given by the relation

$$v = V_0 \sin \omega t = iR, \quad \text{or} \quad i = (V_0/R) \sin \omega t = I_0 \sin \omega t. \qquad \text{...(9)}$$

The quantity V_0/R, the maximum current or the current amplitude I_0, is same as obtained with direct current. Current and voltage both vary with sin ωt, hence are in *same phase*.

(*b*)Pure Inductor : Suppose that an alternating potential difference $v = V_0 \sin \omega t$ is applied across a pure inductor having self inductance L. If i is the instantaneous current at any instant t, then

$$v = V_0 \sin \omega t = L\frac{di}{dt} \text{ or } di = \frac{V_0}{L} \sin \omega t \, dt.$$

Its integration gives

$$i = -(V_0/\omega L) \cos \omega t = I_0 \sin (\omega t - \pi/2). \qquad \text{...(10)}$$

The quantity $V_0/\omega L$ is equal to the maximum current I_0. This relation shows that the effective *ac* resistance, *i.e., the inductive reactance* of inductor, $X_L = \omega L$ and the maximum current $I_0 = V_0/X_L$. The unit of inductive reactance is also ohm. The reactance is greater for the greater inductance and for the higher frequency f or the angular velocity ω. It is also clear from equation (10) that the current in the inductor lags voltage by 90° *or the voltage across the inductor leads the current passing through it by 90°.*

(c)Pure Capacitor C : If a capacitor of capacitance C is connected across the alternating source, the instantaneous charge on the capacitor

$$q = Cv = CV_0 \sin \omega t,$$

and the instantaneous current i passing through it is given by

$$i = \frac{dq}{dt} = CV_0 \,\omega \cos \omega t = \frac{V_0}{1/\omega C} \sin(\omega t + \pi/2)$$

$$= I_0 \sin(\omega t + \pi/2), \qquad \text{....(11)}$$

This relation shows that the quantity $1/\omega C$ is the effective ac resistance or the capacitive reactance of the capacitor and is represented as X_C. It has unit as ohm.

It is also clear that the *current leads the voltage by 90° or the potential drop across the capacitor lags the current passing through it by 90°.* The relations between i and v through R, L or C and their phasor representations are shown in Fig. 17.3.

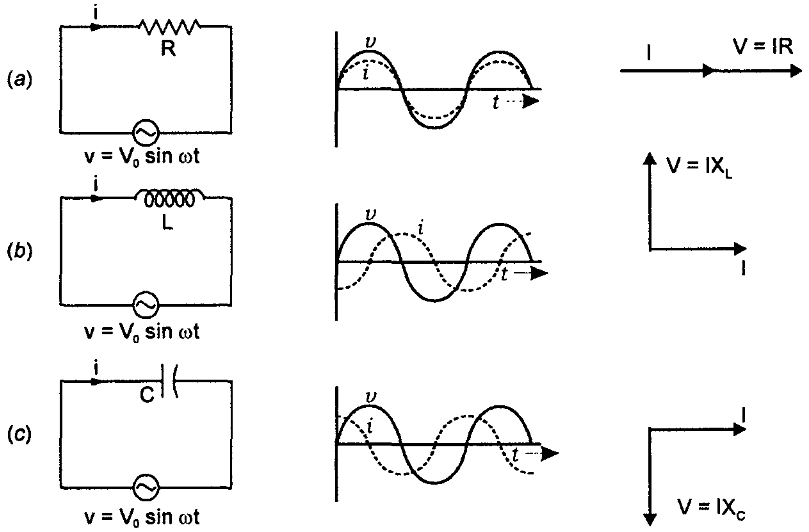


Fig. 17.3. Currents in the circuit and the potential differences across R, L or C.

17.4 PHASOR ALGEBRA

We know that the complex quantities, normally employed in *ac* circuit analysis, can be added and subtracted like coplaner vectors. Such coplaner vectors which represent sinusoidally time varying quantities are now known as *phasors.*

If x – axis and y – axis projections of a phasor **A** are known as x_A and y_A respectively, then its magnitude A is given as

$$A = \left(x_A^{\,2} + y_A^{\,2}\right)^{1/2}. \qquad \text{...(12)}$$

The angle between the direction of phasor **A** and the direction of the positive x-axis is

$$\theta = \tan^{-1}(y_A/x_A). \qquad \text{...(13)}$$

To distinguish between the x-axis projection and y-axis projection a convenient operator is used which will, when applied to a phasor, rotate it 90° counterclock wise without changing its magnitude. Let j be the such operator. When a given phasor **A**, the direction of which is along the x-axis, is multiplied by the operator j, a new phasor j **A** is obtained which will be 90° counter clockwise from **A**, *i.e.*, along y-axis. If the operator j is applied now to the phasor j **A**, a new phasor j^2**A** is obtained which will be along $-x$-axis and having same magnitude as of **A**. Thus $j^2\,\mathbf{A} = -\mathbf{A}$.

$$\therefore \qquad j^2 = -1 \text{ or } j = \sqrt{(-1)}. \qquad \text{...(14)}$$

Thus in cartesian form, a phasor **A** can be written as

$$\mathbf{A} = a + jb,$$

where a is the x-component and b the y-component of phasor **A**.

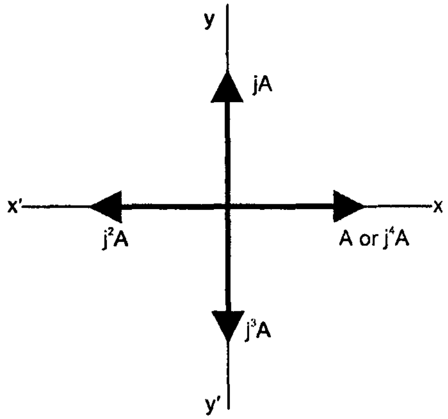


Fig. 17.4. Operator j.

In **polar form** the phasor can be written as

$$\mathbf{A} = A\cos\theta + jA\sin\theta = A(\cos\theta + j\sin\theta). \qquad \text{...(15)}$$

Here $A\cos\theta$ is the x-component of the phasor and $A\sin\theta$ the y-component. If the angle is measured in the clockwise direction from the x-axis, the phasor **A** can be written as

$$\mathbf{A} = A(\cos\theta - j\sin\theta). \qquad \text{...(16)}$$

Thus we see that $\cos\theta + j\sin\theta$ is an operator which when applied with a phasor **A**, along the x-direction, rotates it through an angle θ from its initial position. Similarly the operator $\cos\theta - j\sin\theta$ rotates the original phasor through an angle $-\theta$. It can also be proved that this operator rotates the phasor irrespective of the initial position of the phasor.

In **exponential form** the phasor **A** can thus be written as

$$\mathbf{A} = Ae^{\pm j\theta} = A(\cos\theta \pm j\sin\theta) = A\underline{|\pm\theta}. \qquad \text{...(17)}$$

$$\text{Therefore} \qquad e^{\pm j\theta} = \cos\theta \pm j\sin\theta = \underline{|\pm\theta}$$

Addition of phasors : The sum of two phasors is a phasor which is defined in magnitude and phase position by the diagonal of parallelogram formed by two phasors.

$$\mathbf{A} + \mathbf{B} = \mathbf{C} = \mathbf{B} + \mathbf{A}. \qquad \text{...(18)}$$

Substraction of phasors : In phasor algebra, phasor to be subtracted is rotated through 180° and the addition is done by completing the parallelogram.

Multiplication of phasors : The product of two phasors is a third phasor having magnitude equal to the product of the magnitudes and the phase position (phasor angle) with respect to the reference axis equal to the algebraic sum of the individual phasor angles of these phasors with respect to the same reference axis.

$$\mathbf{A} = Ae^{j\theta_A} = A\underline{|\theta_A} \text{ and } \mathbf{B} = Be^{j\theta_B} = B\underline{|\theta_B}$$

$$\therefore \qquad \mathbf{AB} = ABe^{j(\theta_A + \theta_B)} = AB\underline{|\theta_A + \theta_B}. \qquad \text{(19)}$$

Division of Phasors : When one phasor **A** is divided by another phasor **B**, it results in a phasor having magnitude A/B and the phase position with respect to the reference axis is the algebraic difference between the individual phase angles of these phasors with respect to the same reference axis. Thus

$$\mathbf{A} \div \mathbf{B} = \mathbf{C} = \frac{Ae^{j\theta_A}}{Be^{j\theta_B}} = \frac{A}{B}e^{j(\theta_A - \theta_B)} = \frac{A}{B}\underline{|\theta_A - \theta_B}. \qquad ...(20)$$

Raising a Power : A phasor may be raised to a given power n, an given integer, by multiplying the phasor by itself n times.

$$(\mathbf{A})^n = A^n \underline{|n\theta_A}. \qquad ...(21)$$

The potential difference across the inductor or the capacitor in the form of phasor can be written as under :

(*i*) Since the voltage across the inductor leads the current passing through it by 90°, hence the inductive reactance ωL can be written as $j\omega L$.

(*ii*) Since the voltage across the capacitor lags the current passing through it by 90°, hence the capacitive reactance $1/\omega C$ can be written as $-j/\omega C = 1/j\omega C$.

17.5 KIRCHHOFF'S LAWS

For the analysis of complicated a.c. circuits, Kirchhoff's laws are used. These laws are applied as in d.c. circuits (discussed in chapter 8), but are defined in complex terms as :

(1) As there is no accumulation of electric charge in a system of conductors, the sum of complex currents entering any junction equals the sum of the complex currents leaving that junction.

(2) The sum of the complex emf's in any mesh equals the sum of the complex potential drops across all elements of that mesh. Thus we can add series impedances as

$$\mathbf{Z} = \mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + ..\mathbf{Z}_n. \qquad ...(22)$$

The corresponding formula for a number of impedances in parallel is found by writing the total current as the sum of the currents through the impedance elements. Thus we can write the net impedance as given by

$$\frac{1}{\mathbf{Z}} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} + \frac{1}{\mathbf{Z}_3} + \ldots \frac{1}{\mathbf{Z}_n}. \qquad ...(23)$$

The reciprocal of the impedance is called the **admittance Y**, thus for the combination of parallel impedances we can write

$$\mathbf{Y} = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \ldots \mathbf{Y}_n. \qquad ...(24)$$

In general **Y** is a complex quantity. Thus

$$\mathbf{Y} = G + jS = \frac{1}{\mathbf{Z}} = \frac{1}{R + jX} = \frac{R - jX}{R^2 + X^2}.$$

$$\therefore \quad G = R/(R^2 + X^2) \text{ and } S = -X/(R^2 + X^2).$$

The G is known as *conductance* and S the *susceptance.*

17.6 IMPERFECT INDUCTOR (L & R IN SERIES)

If an alternating potential $v = V_0 \sin \omega t$ is applied to the circuit containing inductor and resistor in series (Fig. 17.5). This problem can be considered in two ways.

(*a*) To write down emf equation for the circuit and obtain a solution : If i is the instantaneous current in the circuit containing resistance R and inductance L, *then at any instant t,*

$$v - L\,di/dt = iR$$

or $$L\,di/dt + iR = v = V_0 \sin \omega t. \qquad ..(25)$$

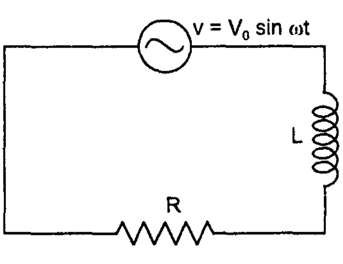


Fig. 17.5. *LR* – Circuit.

The solution of this equation is the sum of two parts : (*a*) The complimentary function, which is the solution of the equation $L\,di/dt + iR = 0$ and (*b*) the *particular integral*, any solution, which satisfies the equation (25) and contains two integration constants.

As discussed in chapter 12, the solution of the first part is given by the relation $i = i_0 e^{-Rt/L}$. It becomes zero after a considerable time. For the latter, we may assume that the periodicity of the current will be same as that of the applied emf, but can forecast nothing about its amplitude and phase. Let the solution be

$$i = A \sin(\omega t + \theta), \qquad ...(26)$$

where A and θ are constants.

Substituting for i and di/dt from Eq. (26) in Eq. (25), we get

$$L\omega A \cos(\omega t + \theta) + RA \sin(\omega t + \theta) = V_0 \sin \omega t = V_0 \sin[(\omega t + \theta) - \theta]$$

$$= V_0[\sin(\omega t + \theta)\cos\theta - \cos(\omega t + \theta)\sin\theta].$$

This is true for all values of t. Hence equating coefficients of $\sin(\omega t + \theta)$ and $\cos(\omega t + \theta)$ separately, we get

$$\omega L A = -V_0 \sin\theta \text{ and } RA = V_0 \cos\theta.$$

$$\therefore \quad \tan\theta = -\omega L/R \text{ and } A = V_0/(R^2 + \omega^2 L^2)^{1/2}$$

Hence equation for the current at time t is given as

$$i = \frac{V_0}{(R^2 + \omega^2 L^2)^{1/2}} \sin(\omega t + \theta). \qquad ...(27)$$

The angle θ is obviously negative, so that the best way of writing the expression is

$$i = \frac{V_0}{(R^2 + \omega^2 L^2)^{1/2}} \sin(\omega t - \phi), \text{ where } \tan\phi = \omega L/R. \qquad ...(28)$$

The same relation will be obtained if we assume Eq. (26) as $i = A \sin(\omega t - \phi)$.

The amplitude of the current is $I_0 = V_0/\sqrt{(R^2 + \omega^2 L^2)}$. The quantity $\sqrt{(R^2 + \omega^2 L^2)}$ plays the same roll in *ac* circuits as the ohmic resistance does in *dc* circuits. It is termed as Z and known as the *impedance* of the circuit. For very small frequencies, this reduces to R, *the ordinary resistance and for very large values of frequencies it becomes* ωL and is termed as the *reactance* of the circuit. Relation (28) indicates that the current is not in phase but lags behind the applied voltage by an angle ϕ or a fraction $\phi/2\pi$ of a period.

(*b*) **By use of the operator j :** The use of the operator **j** is of very great importance in *ac* calculations.

In cartesian coordinates a unit vector is taken along +*ve* *x*-axis. As the multiplication by the imaginary quantity **j** denotes the rotation of the positive unit vector counter clockwise through 90°, thus **j** is along *y*-axis.

If the voltages across the resistor and inductor are designated $\mathbf{V}_R$ and $\mathbf{V}_L$ and the source voltage is designated **V**. The circuit current designated **I** is the same everywhere in a series

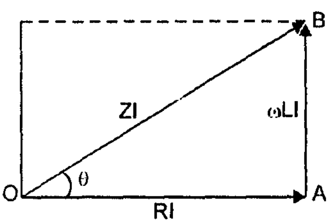


Fig. 17.6. Phasor diagram for *LR*–circuit.

circuit. We know that $\mathbf{V}_R$ and **I** are exactly in phase, and $\mathbf{V}_L$ and **I** are exactly 90° out of phase, with $\mathbf{V}_L$ leading **I**. The phase relationship among the **I**, $\mathbf{V}_R$ and $\mathbf{V}_L$ phasors is shown in Fig. 17.6. In this diagram the voltage across R is represented by the phasor **OA** and that across L by phasor **AB**. The resultant potential is represented by the phasor **OB.** Using Kirchhoff's voltage law in terms of phasor, we have

$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L = R\mathbf{I} + j\omega L\mathbf{I},$$

$$\therefore \quad \mathbf{I} = \frac{\mathbf{V}}{R + j\omega L} = \frac{\mathbf{V}}{\mathbf{Z}}, \qquad \ldots(29)$$

where $\mathbf{Z} = R + j\omega L$ is known as the complex impedance. The real part is often called the resistive or *inphase component* and the imaginary part as the reactive or *quadrature component.* The circuit impedance is given by the modulus of the complex impedance operator and the phase angle by its argument, as

$$\mathbf{Z} = R + j\omega L = Ze^{j\phi} = Z(\cos\phi + j\sin\phi).$$

$$\therefore \quad Z = \sqrt{(R^2 + \omega^2 L^2)} \text{ and } \tan\phi = \omega L / R.$$

Thus the current $$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_0 e^{j\omega t}}{Ze^{j\phi}} = \frac{V_0}{Z} e^{j(\omega t - \phi)}. \qquad \ldots(30)$$

17.7 IMPERFECT CAPACITOR (*i.e.,* R & C IN SERIES)

The capacitor is said to be linear, if its characteristic $(q - v)$ curve is a straight line passing through zero at all times. If the characteristic does not change with time, the capacitor is called time invariant capacitor. On account of absorption loss dissipated in dielectric, the real or an imperfect capacitor may be regarded as a capacitance C in series with a resistance R. If an instantaneous emf is applied to this circuit then at any instant t, when charge on the capacitor C is q, we have

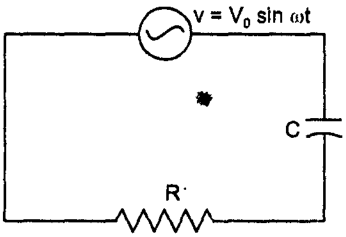


Fig. 17.7. *RC*–Series circuit.

$$iR + q/C = v = V_0 \sin \omega t.$$

As $i = dq/dt$, therefore we get

$$R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t. \qquad \ldots(31)$$

This equation is identical in form with that given by Eq. (25). The solution will be sum of two parts, one known as complimentary function which is the solution of Eq. $R dq/dt + q/C = 0$ and other known as particular integral. The solution of the first part is $q = q_0 e^{-t/RC}$, which decreases with time and becomes negligible after some time. As done in earlier case, let us assume the second part of solution as

$$q = A \sin(\omega t - \theta). \qquad \ldots(32)$$

Substituting for q and dq/dt in Eq. (31), we have

$$RA\omega\cos(\omega t - \theta) + (A/C)\sin(\omega t - \theta) = V_0 \sin[\omega t - \theta) + \theta]$$

Equating coefficients of $\cos(\omega t - \theta)$ and $\sin(\omega t - \theta)$, we have

$$RA\omega = V_0 \sin\theta \text{ and } A/C = V_0 \cos\theta.$$

From these relations, we have

$$A = \frac{V_0}{(\omega^2R^2+1/C^2)^{1/2}} = \frac{V_0C}{(1+\omega^2R^2C^2)^{1/2}} \quad \text{and} \quad \tan\theta = \omega RC.$$

$$\therefore \qquad q = \frac{V_0C}{(1+\omega^2R^2C^2)^{1/2}}\sin(\omega t-\theta), \qquad ...(33)$$

$$\text{or} \qquad i = \frac{dq}{dt} = \frac{V_0C\omega}{(1+\omega^2R^2C^2)^{1/2}}\cos(\omega t-\theta) = \frac{V_0}{(R^2+1/\omega^2C^2)^{1/2}}\sin(\omega t+\phi), \qquad ...(34)$$

where $\phi = \pi/2-\theta.$

$$\therefore \qquad \tan\phi = \tan(\pi/2-\theta) = \cot\theta = 1/\omega RC. \qquad ...(35)$$

The current is thus in advance of the applied emf by a phase angle of ϕ or $\phi/2\pi$ of a period. The amplitude of the current $I_0 = V_0/(R^2 + 1/\omega^2C^2)^{1/2}$ and the impedance is thus given as

$$Z = (R^2 + 1/\omega^2C^2)^{1/2}. \qquad ...(36)$$

Since i increases as ω increases, hence at high frequencies the circuit transmits considerable current. The impedance is infinite for zero frequency, *i.e.*, for *dc* source.

Problem can also be solved by **using operator** j. The applied *pd* is across the resistance and capacitance in series. Therefore the source voltage $\mathbf{V}$ is the sum of voltage phasors across the resistance $\mathbf{V}_R$ and the capacitor $\mathbf{V}_C$. If $\mathbf{I}$ represents the current phasor in the series circuit, then we have

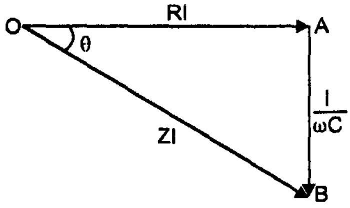


Fig. 17.8. Phasor diagram for *RC*-circuit.

$$\mathbf{V} = R\mathbf{I} - (j/\omega C)\,\mathbf{I}$$

or

$$\mathbf{I} = \frac{\mathbf{V}}{R-j/\omega C} = \frac{\mathbf{V}}{\mathbf{Z}}.$$

Hence the impedance operator

$$\mathbf{Z} = Ze^{j\theta} = Z(\cos\theta + j\sin\theta) = R - j/\omega C$$

$$\therefore \qquad Z = (R^2 + 1/\omega^2C^2)^{1/2} \text{ and } \theta = \tan^{-1}(-1/\omega RC).$$

Thus current $\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_0e^{j\omega t}}{Ze^{j\theta}} = \frac{V_0}{Z}e^{j(\omega t-\theta)} = \frac{V_0}{Z}e^{j(\omega t+\phi)}$

where $\phi = -\theta = \tan^{-1}(1/\omega RC), \qquad ...(37)$

Thus we see that the current leads the voltage or the voltage lags the current by an angle ϕ, given by the relation $\tan\phi = 1/\omega RC$.

17.8 CIRCUIT CONTAINING INDUCTOR AND CAPACITOR

Let a capacitor of capacitance C, and an inductor of inductance L of negligible resistance are connected in series. When the emf $V_0 \sin\omega t$ is applied, we have at any time t

$$v - L\,di/dt = q/C$$

$$\text{or} \qquad L\,di/dt + q/C = v = V_0\sin\omega t.$$

Since current is defined as given by $i = dq/dt$, hence this equation becomes

$$L\frac{d^2q}{dt^2} + \frac{q}{C} = V_0\sin\omega t$$

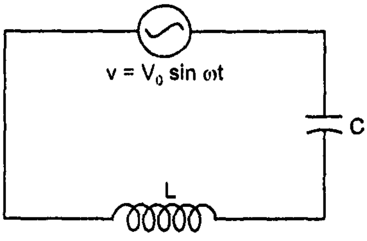


Fig. 17.9. *LC*-Series Circuit.

or
$$\frac{d^2q}{dt^2}+\frac{q}{LC} = \frac{V_0}{L}\sin\omega t. \qquad ...(38)$$

This is a linear differential equation of second order, the solution of such type of equation consists of two parts : The first part, known as complementary function, the solution of Eq. $d^2q/dt^2 + q/LC = 0$, which comes out to be $q = q_0 \sin[t/\sqrt{LC} + \pi/2]$. Thus the motion is simple harmonic and the frequency $f = 1/2\pi\sqrt{(LC)}$, known as the natural frequency of the LC circuit. The second part, known as particular integral, can the found by trial method, assuming the frequency as that of applied emf. Let us assume that the second part is given by

$$q = A\sin(\omega t + \theta). \qquad ...(39)$$

Substituting values of q and its derivatives in Eq. (38), we get

$$-\omega^2 A\sin(\omega t+\theta)+\frac{A}{LC}\sin(\omega t+\theta) = \frac{V_0}{L}\sin\omega t=\frac{V_0}{L}\sin[(\omega t+\theta)-\theta]$$
$$= V_0/L\ [\sin(\omega t+\theta)\cos\theta-\cos(\omega t+\theta)\sin\theta]$$

Equating coefficients of sin $(\omega t + \theta)$ and cos $(\omega t + \theta)$ separately, we get

$$-\omega^2 A + A/LC = (V_0/L)\cos\theta \text{ and } 0 = -(V_0/L)\sin\theta.$$

Thus
$$\theta = 0 \text{ and } A = \frac{V_0/L}{(1/LC-\omega^2)} \qquad \therefore q = \frac{V_0/L}{1/LC-\omega^2}\sin\omega t.$$

or
$$i = \frac{V_0\omega/L}{1/LC-\omega^2}\cos\omega t=\frac{V_0}{1/\omega C-\omega L}\sin(\omega t+\pi/2) \qquad ...(40)$$
$$= [V_0/(\omega L - 1/\omega C)]\sin(\omega t - \pi/2). \qquad ...(41)$$

Thus the current is exactly $\pi/2$ out of phase with the applied emf. The amplitude of current becomes infinite when $\omega = 1/\sqrt{(LC)}$, *i.e.*, when the frequency of the applied emf is exactly equal to the natural frequency of the circuit. This condition is called *resonance.*

The same result may be obtained by *phasor method.* We know that the current phasor **I** through the inductor or the capacitance is same; the voltage phasor $\mathbf{V}_L$ across the inductor leads the current by 90° and the inductive reactance is ωL, when the frequency of the applied emf is $\omega/2\pi$ the voltage phasor $\mathbf{V}_C$ across the capacitor lags behind the current by 90° and the capacitative reactance is $1/\omega C$. Thus we can represent potential differences across the inductor and capacitor by phasors as shown in Fig. 17.10. It shows that the resultant potential difference will be $|\mathbf{V}_L - \mathbf{V}_C|$. It will lead if $V_L > V_C$ and lag if $\mathbf{V}_C > \mathbf{V}_L$ the current by an angle $\pi/2$. Thus the combination is equivalent to a single element, either an inductor, or a capacitor, depending on whether the quantity $(\omega L - 1/\omega C)$ is +ve or –ve.

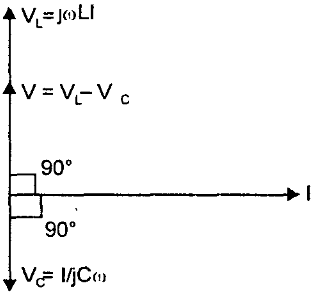


Fig. 17.10. Phasor Diagram.

Perfect Choke : When an alternating emf $v = V_0 \sin \omega t$ is applied across inductor and capacitor in parallel. If **I** be the current flowing at any time t at a junction, it will divide in two parts one being along the inductor and other along the capacitor. Using phasor algebra, we get

total current $\quad \mathbf{I} = \mathbf{I}_L + \mathbf{I}_C.$

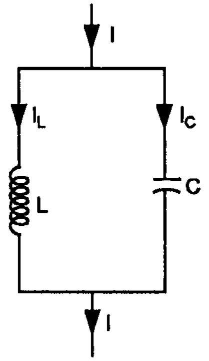


Fig. 17.11. Parallel Circuit.

$$\therefore \qquad \frac{\mathbf{V}}{\mathbf{Z}} = \frac{\mathbf{V}}{j\omega L} + \frac{\mathbf{V}}{1/j\omega C} \text{ or } \frac{1}{\mathbf{Z}} = \frac{1}{j\omega L} + j\omega C.$$

Total current **I** will be zero for all values of t, when

$$1/j\omega L + j\omega C = 0$$

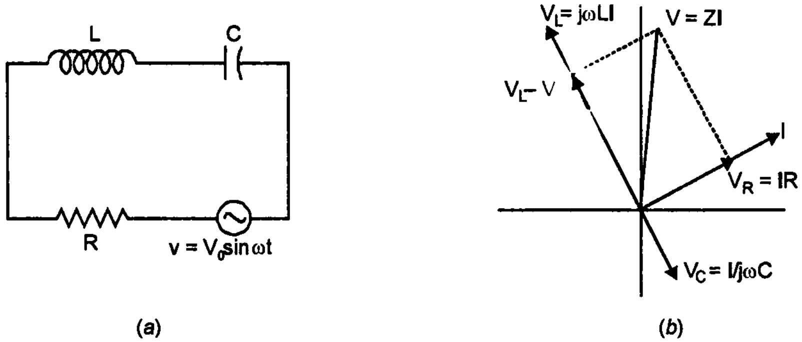
$$\text{or} \qquad \omega = 1/(LC)^{1/2}. \qquad \text{...(42)}$$

Thus the circuit acts as a perfect choke for oscillations of its own natural frequency, *i.e., the oscillations of frequency ω are prevented from passing through the circuit consisting inductor and capacitor in parallel, when this frequency equals the natural frequency of this circuit.*

Hence a filter circuit can be designed to transmit or suppress any required frequency by connecting L and C, of required values, in series for the first case and in parallel for the other.

17.9 CIRCUIT CONTAINING INDUCTOR, CAPACITOR AND RESISTOR

(*a*) Series Circuit (Acceptor Circuit) : Let an alternating voltage $V_0 \sin \omega t$ be applied to an inductor, a resistor and a capacitor, in series. If i is the instantaneous current flowing at any instant, the emf sets up in the inductance L is $-L(di/dt)$, and the voltage across the capacitor is q/C, where q is the charge on the capacitor at that instant. Thus we have



(*a*) (*b*)

Fig. 17.12. *LCR*-Series Circuit, (*b*) phasor diagram.

$$V_0 \sin \omega t - L(di/dt) - q/C = Ri,$$

$$\text{or} \qquad L(di/dt) + Ri + q/C = V_0 \sin \omega t. \qquad \text{...(43)}$$

Since $i = dq/dt$, hence this equation becomes

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin \omega t. \qquad \text{...(44)}$$

This is a differential equation whose solution consists of two parts. The first of these, known as the *complementary function*, is found by solving the equation obtained by setting the right hand side term equal to zero. If q_0 is the initial charge, this solution is obtained as in article 12.24 and is given by

$$q = \frac{1}{2} q_0 \, e^{(-R/2L)t} \left[(1 + R/\sqrt{R^2 - 4L/C}) e^{\sqrt{(R^2/4L^2 - 1/LC)}\,t} + (1 - R/\sqrt{R^2 - 4L/C}) e^{-\sqrt{(R^2/4L^2 - 1/LC)}\,t} \right] \qquad \text{...(45)}$$

Three conditions thus arise :

(*i*) If $R^2/4L^2 > 1/LC$, the square roots are real and the discharge is *non oscillatory* or *heavily damped.*

(*ii*) If $R^2/4L^2 = 1/LC$, bracket terms become 1 and the discharge is critically damped, given by relation $q = q_0 e^{-(R/2L)t}$.

(*iii*) If $R^2/4L^2 < 1/LC$, the square roots are imaginary and the discharge is oscillating due to the terms within the brackets and damped, as the amplitude is $q_0 e^{-(R/2L)t}$.

Thus we see that this part of the solution becomes negligible within a few seconds. As the alternating emf is applied continuously, we may assume that the period of alternation of the current or the discharge will be that of the applied emf. As we can not say about the amplitude and phase, let us thus assume the second part of the solution, known as *particular integral*, as

$$q = A \sin(\omega t + \theta). \qquad \text{...(46)}$$

Substituting values of q and its derivatives in Eq. (44), we get

$$-L\omega^2 A \sin(\omega t + \theta) + R\omega A \cos(\omega t + \theta) + (A/C)\sin(\omega t + \theta) = V_0 \sin \omega t = V_0 \sin(\omega t + \theta - \theta)$$

$$= V_0 [\sin(\omega t + \theta)\cos\theta - \cos(\omega t + \theta)\sin\theta]$$

Equating coefficients of $\sin(\omega t + \theta)$ and $\cos(\omega t + \theta)$ on both the sides, we get

$$-L\omega^2 A + A/C = V_0 \cos\theta \text{ and } R\omega A = -V_0 \sin\theta.$$

Solving these equations, we get

$$A = \frac{V_0}{[R^2\omega^2 + (1/C - L\omega^2)^2]^{1/2}} \text{ and } \tan\theta = -\frac{R\omega}{1/C - L\omega^2} = \frac{R}{\omega L - 1/\omega C}$$

$$\therefore \qquad q = \frac{V_0}{[R^2\omega^2 + (1/C - L\omega^2)^2]^{1/2}} \sin(\omega t + \theta). \qquad \text{...(47)}$$

$$\text{or} \qquad i = \frac{dq}{dt} = \frac{V_0}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \cos(\omega t + \theta) = \frac{V_0}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \sin(\omega t + \theta + \pi/2).$$

$$\therefore \qquad i = \frac{V_0}{[R^2 + (\omega L - 1/\omega C)^2]^{1/2}} \sin(\omega t - \phi) = I_0 \sin(\omega t - \phi) \qquad \text{...(48)}$$

Since $\tan(\pi/2 + \theta) = -\cot\theta = -(\omega L - 1/\omega C)/R$, therefore

$$\phi = \tan^{-1}(\omega L - 1/\omega C)/R. \qquad \text{...(49)}$$

Eq. (48) can also be obtained by **using phasor** or by constructing the phasor diagram for the *LCR* series circuit. If the frequency is not too high, the instantaneous current is the same in all parts of the circuit. Even in the capacitor we see that the growth of positive charge on one plate requires an equal growth of opposite charge on the other plate,which is equivalent to equal currents on both sides. Thus a single phasor **I**, of length proportional to the current amplitude $\mathbf{I}_0$ is sufficient to represent the current in each circuit elements. since the sum of sinusoidal voltages of the same frequency is always another sinusoidal voltage, hence we can say that the phasor sum of the individual generating voltages *across all three elements* is equal to the applied voltage.

As we know that operator j proceedes the *pd* $\omega L\,\mathbf{I}$ across the inductance because this *pd* leads the current by 90°, where as $-j$ precedes the *pd* $\mathbf{I}/\omega C$ across the capacitance because this *pd* lags on the current by 90°. Thus the phasor

$$\mathbf{V}_L = j\omega L\mathbf{I} \text{ and } \mathbf{V}_C = (-j/\omega C)\,\mathbf{I}.$$

therefore
$$\mathbf{V} = \mathbf{V}_R + \mathbf{V}_L + \mathbf{V}_C = R\mathbf{I} + j\omega L\,\mathbf{I} - (j/\omega C)\,\mathbf{I}$$
$$= [R + j(\omega L - 1/\omega C)]\mathbf{I}.$$

∴ Impedance
$$\mathbf{Z} = \mathbf{V}/\mathbf{I} = R + j(\omega L - 1/\omega C). \qquad ...(50)$$

Let us represent this complex quantity **Z** as
$$\mathbf{Z} = Ze^{j\phi} = Z(\cos\phi + j\sin\phi). \qquad ...(51)$$

From equations (50) and (51), we get
$$Z\cos\phi = R \text{ and } Z\sin\phi = \omega L - 1/\omega C. \text{ Therefore}$$
$$Z = \left[R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2\right]^{1/2} \text{ and } \tan\phi = \frac{\omega L - 1/\omega C}{R}. \qquad ...(52)$$

Thus the current phasor
$$\mathbf{I} = \frac{\mathbf{V}}{\mathbf{Z}} = \frac{V_0 e^{j\omega t}}{Ze^{j\phi}} = \frac{V_0}{Z}e^{j(\omega t - \phi)} \qquad ...(53)$$

or current
$$i = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}\sin(\omega t - \phi). \qquad ...(48)$$

Three cases will thus arise :

(*i*) If $\omega L > 1/\omega C$, tan ϕ becomes positive and the applied voltage thus leads the current by the phase angle ϕ.

(*ii*) If $\omega L < 1/\omega C$, tan ϕ becomes negative and the applied voltage thus lags on the current by the phase angle ϕ.

(*iii*) If $\omega L = 1/\omega C$, tan ϕ becomes zero and the applied voltage is in the same phase as that of the current. The circuit is thus purely resistive in its effect. Such a condition is known as *resonance* and frequency, known as *resonance frequency*, is given by

$$\omega L = 1/\omega C \text{ or } \omega = 1/\sqrt{(LC)}$$

∴
$$f = \omega/2\pi = 1/2\pi\sqrt{(LC)}. \qquad ...(54)$$

The current in such a case is maximum and is given by $I = V/R$, because the alternating pds across the inductor and the capacitor are then equal and 180° out of phase and therefore cancel out.

Voltage Magnification in Series Resonance Circuit. At resonance the potential differences across the inductor and the capacitor are equal and 180° out of phase and therefore cancel out hence the applied emf is merely to overcome the resistance opposition only. If an inductor or capacitor is of very large impedance the potential across it increases to a very high value. The ratio

$$\frac{\textit{pd across the inductor (or capacitor)}}{\textit{Applied emf}} = \frac{\omega L I_{rms}}{R I_{rms}} = \frac{\omega L}{R}. \qquad ...(55)$$

known as *voltage magnification,* is greater than unity. Thus for high frequencies $\omega L/R$ can be designed to magnify the voltage to any desired value.

Mechanical Analogy. Mechanical analogies can be constructed for any of the circuits described so far. For this purpose, let us consider a mass M, suspended at the free end of spiral spring. Let a force F be applied, producing an acceleration a in the direction of original extension. If v is the velocity at an instant, when displacement is x, then the equation of restoring force is

$$F - \mu x - kv = Ma,$$

or
$$M\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = F. \qquad ...(56)$$

Comparing this with the Eq. (44), we get

(i) the inductance L in the electrical problem corresponds to mass M in the mechanical one.

(ii) the resistance R in the electrical problem corresponds to k, the viscous damping force per unit velocity.

(iii) the reciprocal of capacitance (1/C) in the electrical problem corresponds to μ, the restoring force per unit displacement.

(iv) the mechanical counterpart of the charge itself is displacement and that of applied emf is mechanical force.

In both cases the nature of the solution depends on the resistance term. If k or $R = 0$, the motion is simple harmonic and undamped, while increasing k or R produces first damped harmonic motion, then critical damped and finally heavy damped for progressively larger values of this term.

Response Curves of Series Circuit. The impedance of an *LCR* series circuit depends on the frequency. The dependence is shown in Fig. 17.13, where the frequency is taken in logarithmic form because of its wide range.

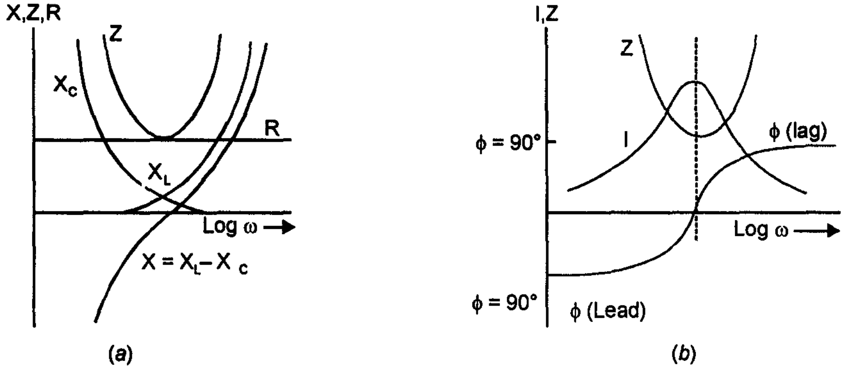


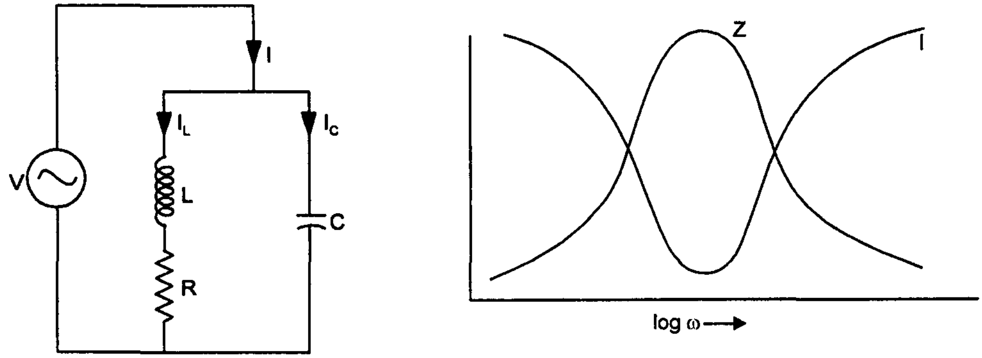
Fig. 17.13. Response curves of series circuit.

Fig. 17.13(*a*) shows that there is one particular frequency at which X_L and X_C are numerically equal. At this frequency $X_L - X_C$ is zero and the impedance Z is equal to R, and hence minimum. Fig. 17.13(*b*) shows that the current is maximum at this frequency and is doubled when the resistance is reduced to its half value.

The phase difference ϕ is also shown in the Fig. 17.13(*b*). It is clear from the curve that at low frequencies, where capacitive reactance X_C predominates, the current leads the voltage; at high frequencies, where inductive reactance X_L, predominates, the current lags the voltage. The current and voltage are in phase, at the frequency where $X_L = X_C$.

Acceptor Circuit. If the frequency of the *ac* supply is varied, or if the *ac* supply contains number of frequency components, (*i.e., radio or television signal*), then a series *LCR* circuit across the supply will give a maximum response, *i.e.*, pass a maximum current and having a maximum potential difference across its inductance for that component of frequency for which $f = 1/2\pi\sqrt{(LC)}$. This is the procedure by which a radio or television receiving set may be tuned to receive the signal from a desired station sending signals at a particular frequency. The circuit is thus known as an *acceptor circuit*, especially in radio communications.

(*b*) Parallel Circuit (Rejector Circuit) : Let us consider an alternating source connected across an inductor of inductance L in parallel with a capacitor of capacitance C [Fig. 17.14(*a*)]. If the series resistance of the inductor is R and of the capacitor is zero (as the power factors of good quality capacitors are very small). Let the instantaneous value of emf applied be $\mathbf{V}$ and the corresponding current be $\mathbf{I}$, the current through the inductor be $\mathbf{I}_L$ and through the capacitor $\mathbf{I}_C$. These latter currents ($\mathbf{I}_L$ and $\mathbf{I}_C$) will be almost in antiphase if R is very small. The total current $\mathbf{I} = \mathbf{I}_L + \mathbf{I}_C$. Therefore



(*a*) (*b*)

Fig. 17.14*a*. Parallel resonance circuit, (*b*) Response Curves.

$$\frac{\mathbf{V}}{\mathbf{Z}} = \frac{\mathbf{V}}{R+j\omega L}+\frac{\mathbf{V}}{1/j\omega C} \quad \text{or} \quad \frac{1}{\mathbf{Z}}=\frac{1}{R+j\omega L}+j\omega C \qquad \text{...(57)}$$

$$\therefore \qquad \mathbf{Y} = \frac{1}{\mathbf{Z}}=\frac{R-j\omega L}{R^2+\omega^2L^2}+j\omega C=\frac{R+j(\omega CR^2+\omega^3L^2C-\omega L)}{R^2+\omega^2L^2}.$$

The magnitude of the admittance

$$Y = \frac{1}{Z}=\frac{[R^2+(\omega CR^2+\omega^3L^2C-\omega L)^2]^{1/2}}{R^2+\omega^2L^2}. \qquad \text{...(58)}$$

The admittance will be minimum, when

$$\omega CR^2 + \omega^3L^2C - \omega L = 0.$$

or

$$\omega = \omega_0 = (1/LC - R^2/L^2)^{1/2}. \qquad \text{...(59)}$$

It gives the condition of resonance and the corresponding frequency

$$f = \frac{\omega_0}{2\pi}=\frac{1}{2\pi}\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)^{1/2} \qquad \text{...(60)}$$

is known as *resonance frequency*.

For the values of L, C, R and ω satisfying Eq. (60), the reactive component of Y vanishes or Y is real. As such a minimum admittance, *i.e.*, maximum impedance, the circuit current is minimum. Thus the parallel circuit does not allow this frequency from the source to pass in the circuit. Due to this reason the circuit is known as *rejector circuit* for such a frequency.

The reciprocal of the admittance at resonance is called the *parallel resistance* or the *dynamic resistance*. At the time of resonance the current and voltage are in phase. The dynamical resistance is thus reciprocal of the real part of the admittance.

$$\textit{Dynamic resistance} = \frac{R^2+\omega^2L^2}{R}=\frac{R^2+L^2(1/LC-R^2/L^2)}{R}=\frac{L}{CR}.$$

At resonance, the peak current from the supply, known as *make up current* is therefore $= V_0/(L/CR) = CRV_0/L$.

The *forced oscillatory current* is the current passing through capacitor and inductor circuit or simply through capacitor. The peak current through capacitor is

$$I_C = V_0/(1/\omega C) = \omega C V_0. \qquad ...(61)$$

This current will lead the applied voltage by $\pi/2$.

At resonance

$$\frac{\textit{Amplitude of oscillatory current}}{\textit{Amplitude of make up current from supply}} = \frac{V_0 \omega C}{V_0 CR/L} = \omega L/R. \qquad ...(62)$$

This ratio is known as Q-factor, which is the measure of the current magnification. Thus the *rejector circuit at resonance exhibits current magnification of $\omega L/R$ similar to the voltage magnification of the same ratio exhibited by the series acceptor circuit at resonance.*

Sharpness of Resonance in Series Circuit : Fig. 17.13(*b*) shows that the current is maximum at one particular frequency, called *resonance frequency*. It increases with the decrease of R. The sharpness of the curve thus increases with the decrement of R. The sharpness of such a curve is measured by its half width, which is the difference of two frequencies at which current amplitude is reduced to half of its maximum value. The *sharpness of resonance is large if this width is smaller. It is numerically defined as the reciprocal of the half width.* The variation of current with frequency for various values of R is shown in Fig. 17.15.

To find the sharpness two values of ω for which R/Z is $1/\sqrt{2}$ times its maximum value and their difference is determined. The value of Z will be $\sqrt{2}R$, when $X^2 = R^2$, or $X = \pm R$. Hence we can write two values of ω as

$$X = \omega_1 L - 1/\omega_1 C = R \text{ and } X = \omega_2 L - 1/\omega_2 C = -R.$$

$$\therefore \qquad (\omega_1 + \omega_2)L - \frac{\omega_1 + \omega_2}{\omega_1 \omega_2 C} = 0 \text{ or } \omega_1 \omega_2 = \frac{1}{LC} = \omega_0^2.$$

and

$$(\omega_1 - \omega_2)L + \frac{\omega_1 - \omega_2}{\omega_1 \omega_2 C} = 2R \text{ or } \omega_1 - \omega_2 = \frac{R}{L}.$$

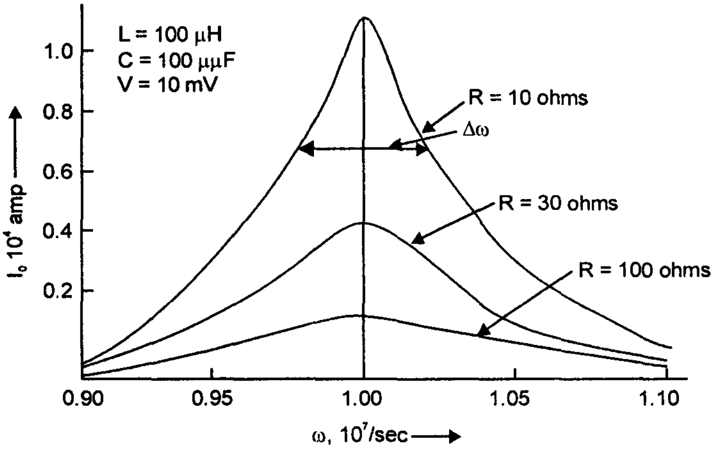


Fig. 17.15. Sharpness of resonance curve.

Hence we can write $$\frac{\omega_1-\omega_2}{\omega_0} = \frac{R}{\omega_0 L}=\frac{1}{Q_0}. \qquad ...(63)$$

Thus for the demand of sharp resonance curves (sharp tuning) a large quality factor Q_0 is desirable.

The difference between two half power frequencies $(f_1 - f_2)$ is known as the *band width of the resonant curve.* Thus

$$B\cdot W = f_1-f_2=f_0/Q_0=R/2\pi L. \qquad ...(64)$$

It might be thought that the Q value of a coil could be made very large at high frequencies, but it is not the case. The time constant L/R can be made as large as second by the use of magnetic core materials. But effective permeability decreases as frequency increases. At high frequency the skin effect increases R and the effects of distributed capacitance between the windings make the coil act like a smaller effective inductance.

17.10 POWER IN AN AC CIRCUIT

The rate of doing work or the activity of a current in the case of a steady current, is given by VI. In an alternating current both V and I vary harmonically and differ in phase. If $v = V_0 \sin \omega t$ is the applied emf across the circuit at any instant t and the corresponding current is given by $i = I_0 \sin (\omega t - \phi)$, where ϕ is the phase difference between the emf and the current. If emf and current are in volts and amperes respectively, then the instantaneous power in watts is given by

$$\begin{aligned} vi &= V_0 I_0 \sin \omega t \sin (\omega t - \phi) \\ &= V_0 I_0 [\sin^2 \omega t \cos\phi - \tfrac{1}{2}\sin 2\omega t \sin\phi]. \end{aligned}$$

The mean rate of doing work or the average power

$$\begin{aligned} P &= \frac{1}{T}\int_0^T vi\,dt=\tfrac{1}{2}V_0 I_0 \cos\phi=\frac{V_0}{\sqrt{2}}\cdot\frac{I_0}{\sqrt{2}}\cos\phi \\ &= \textit{Virtual volts} \times \textit{Virtual amperes} \times \cos \phi, \qquad ...(65) \end{aligned}$$

where the term $\cos \phi$ is known as *power factor*. It is said to be leading if current leads voltage, lagging if current lags voltage. Thus a power factor of 0.8 lagging means that current lags the voltage by $\cos^{-1} 0.8 = 36.8°$.

If the rms voltage V_{rms} and the rms current I_{rms}, also known as virtual values, are measured separately in an *ac* circuit, then the product $V_{rms} \times I_{rms}$ does not give the true power dissipated in the circuit as does the product VI for a *dc* circuit and thus gives *apparent power*. The *true power* or the *actual power* dissipated is given by

True power = Apparent power × Power factor. ...(66)

If the curves $v = V_0 \sin \omega t$ and $i = I_0 \sin (\omega t - \phi)$ are drawn on the same graph, the power curve can be obtained by plotting ordinates corresponding to the product vi at all times during the cycle. For the parts of the cycle CE and GI, the voltage and current are in the same direction and the *source of supply is doing work on the circuit.* For the parts AC, and EG of the cycle, the voltage and current are in opposite directions, the work is

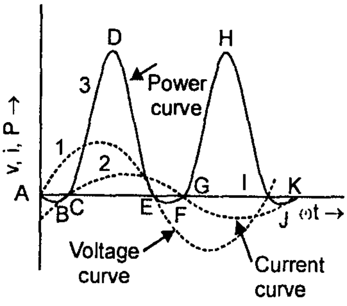


Fig. 17.16. Power curve.

negative, *i.e.*, the *circuit is doing work on the source supply*. The difference between the positive and negative areas during the time T gives the work done in the cycle, which is equal to $V_{rms} I_{rms} \cos \phi$. For $\phi = 0°$, the current and voltage are in phase and the negative portions of the graph of power (vi) against time vanish. The power is thus maximum ($V_{rms} I_{rms}$). If $\phi = 90°$, the positive areas of the power curve will equal to the negative areas. The work done is zero. The current is then stated as *wattless*. Such a case will arise when the circuit is purely inductive or capacitative. The case is similar to that of a frictionless pendulum, where the total work done by gravity upon the pendulum in a cycle is zero.

17.11 CHOKING COIL

Let us consider a coil of large inductance L and of low resistance R. The power factor for such a coil is given by

$$\cos \phi = \frac{R}{\sqrt{(R^2 + L^2 \omega^2)}} = \frac{R}{L\omega} \qquad (\text{As } R << L\omega) \qquad ...(67)$$

As $R \leq L\omega$, $\cos \phi$ is very small and thus ϕ is nearly equal to $\pi/2$. Thus the power absorbed by the coil $V_{rms} I_{rms} \cos \phi$ is very small. On account of its larger impedance $(R^2 + \omega^2 L^2)^{1/2}$, the current passing through the coil is very small. Such a coil is usually known as *choking coil and is used largely in alternating current circuits for the purpose of adjusting current to any required value without waste of energy*. The only waste of energy is due to the hysteresis loss in the iron core, which is much less than the waste of energy in the resistance that can also reduce the current if placed instead of the choking coil.

Choking coils are very much used in electronic circuits, mercury lamps and sodium vapour lamps.

17.12 QUALITY FACTOR OF RESONANT CIRCUIT

The *quality factor Q* of circuit is a quality or figure of merit used to measure the ability of circuit to discriminate between different frequencies. It is defined as 2π times the ratio of the maximum energy stored per cycle to the average energy dissipated per cycle, *i.e.*,

$$Q = \frac{2\pi(\text{maximum energy stored per cycle})}{\text{Energy dissipated per cycle}}. \qquad ...(68)$$

In general, the quality factor is a measure of the ratio of the reactance at resonance to the series resistance, the higher the value of Q, the sharper is the resonance effect.

Let us now calculate 'Q' for series and parallel resonance circuits separately.

[A] *Q of Series resonance circuit.* In *LCR* circuit at resonance the maximum energy stored occurs in the inductor when the current through it attains its maximum value and in the capacitor when the voltage across it attains its maximum value. The total energy stored remains same (the capacitor absorbs energy when the inductor releases and vice-versa).

$$\therefore \qquad \text{Maximum energy stored} = \tfrac{1}{2} L I_0^2.$$

The energy dissipated per cycle at resonance equals the heat power losses in the resistor times the time period of the wave at resonance.

$$\therefore \qquad \text{Energy dissipated per cycle} = I_{rms}^2 R T_0.$$

$$\therefore \qquad Q_s = 2\pi\left(\tfrac{1}{2} L I_0^2\right) / I_{rms}^2 R T_0.$$

Since $T_0 = 1/f_0$ = Inverse of resonance frequency, and $I_0 = \sqrt{2}I_{rms}$.

$$Q_s = \frac{2\pi \frac{1}{2} L\left[\sqrt{2}I_{rms}\right]^2}{I_{rms}{}^2 R(1/f_0)} = \frac{2\pi f_0 L I_{rms}{}^2}{I_{rms}{}^2 R} = \frac{\text{Reactive power}}{\text{Active power}}$$

$$= \frac{\omega_0 L}{R} = \frac{\text{Inductive reactance}}{\text{Series resistance}} = \frac{X_L}{R}. \qquad \text{...(69)}$$

Since at resonance, $X_L = X_C$ or $\omega_0 L = 1/\omega_0 C$, therefore

$$Q_s = X_C/R = 1/\omega_0 CR. \qquad \text{...(70)}$$

Since $\omega_0 = 1/\sqrt{LC}$, therefore the Q-factor may be written as

$$Q_s = \frac{1}{\sqrt{LC}}\frac{L}{R} = \frac{1}{R}\sqrt{\frac{L}{C}}. \qquad \text{...(71)}$$

In general the resistance of the series circuit at resonance is same as that of the coil. In this case, Q of the coil and Q of the series resonance circuit are the same. If some extra resistance is added in the circuit, Q of the circuit will be less then the Q of the coil.

At resonance $X_L = X_C$ or $\omega L = 1/\omega C$. Thus X_L (or X_C) may be increased by increasing L or decreasing C. Therefore L/C ratio must be higher at resonance for higher values of X_L or X_C and thus to obtain higher Q.

At resonance, voltage of the source $V_{in} = R \times$ maximum current $= RI_{rms}$ and the voltage across either coil or capacitor $= V_L = V_C = X_L I_{rms} = X_C I_{rms}$.

$$\therefore \quad \text{Voltage magnification} = \frac{\text{Potential difference across } L (\text{or } C)}{\text{Applied emf}}$$

$$= \frac{\omega L I_{rms}}{R I_{rms}} = \frac{\omega L}{R} = Q_s. \qquad \text{...(72)}$$

Thus the value of Q_s may be obtained by measuring experimentally the voltage across either L or C and comparing this with the applied emf in the series resonance circuit.

[*B*] *Q of the parallel resonant circuit* : The maximum energy stores in a parallel resonant circuit occurs in the capacitor when the potential difference across it is maximum, and in the inductor when the current through it is maximum. When the capacitor releases the energy, the inductor absorbs it and vice versa.

The maximum energy stored $= \frac{1}{2}CV_0{}^2$.

The energy dissipated in one cycle = Heat power losses in the resistor connected in parallel with the tank (parallel) circuit times the period of the voltage source at resonance frequency $= (V_{rms}{}^2/R)T_0$.

$$\therefore \quad Q_p = \frac{2\pi\left(\frac{1}{2}CV_0{}^2\right)}{\left(V_{rms}{}^2/R\right)T_0} = 2\pi f_0 CR = \omega_0 CR = \frac{R}{X_0} \qquad \text{...(73)}$$

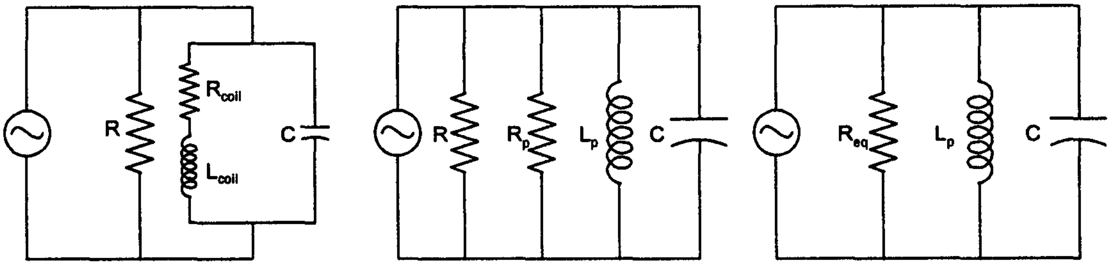
(As $T_0 = 1/f_0$ and $V_0 = \sqrt{2}V_{rms}$).

For the circuit with ideal elements, resistance of the coil is nearly zero and $Q_{coil} \geq 10$. Thus at resonance $\omega_0 L = 1/\omega_0 C$ or $\omega_0 = 1/\sqrt{LC}$. At resonance the Quality factor of the parallel circuit

$$Q_p = \frac{R}{\omega_0 L} = \frac{R}{X_L} = \frac{CR}{\sqrt{LC}} = R\sqrt{\frac{C}{L}}. \qquad \text{...(74)}$$

If the circuit is non-ideal, the series resistance R_{coil} of the coil may be converted to the equivalent parallel elements R_p. The series to parallel conversion formula gives

$$R_p = \frac{R_{coil}^2 + X_{coil}^2}{R_{coil}} \text{ and } X_p = \frac{R_{coil}^2 + X_{coil}^2}{X_{coil}}$$



(a) (b) (c)

Fig. 17.17. Conversion of non-ideal coil to equivalent ideal form.

or

$$R_p = R_{coil} + \frac{X_{coil}^2}{R_{coil}} = R_{coil} + \left(\frac{X_{coil}}{R_{coil}}\right)^2 R_{coil}$$

$$= R_{coil}\,(1 + 1/Q_{coil}^2).$$

Similarly,

$$X_p = X_{coil} + \frac{R_{coil}^2}{X_{coil}} = X_{coil} + X_{coil}\left(\frac{R_{coil}}{X_{coil}}\right)^2$$

$$= X_{coil}\,(1 + 1/Q_{coil}^2).$$

The parallel resistor [Fig. 17.17(*b*)] may be converted into a single equivalent resistor and the results in the equivalent ideal circuit [Fig. 17.17(*c*)] may be written as

$$Q_p = \omega_0 C R_{eq} = \frac{R_{eq}}{\omega_0 L_p} = R_{eq}\sqrt{\frac{C}{L_p}}, \qquad \text{...(75)}$$

where $R_{eq} = RR_p/(R + R_p)$.

17.13 BAND WIDTH OF RESONANT CIRCUIT

The series resonance curve shows that the frequencies just below and above the resonance frequency f_0 produce increased current, but a little less than the value at resonance. Similarly for parallel resonance, the impedance is high at frequencies close to f_0, but less than the maximum. The range of frequencies within which the response does not drop below 70.7 percent of its maximum value is called the band width or pass band (Fig. 17.18). Thus the

$$\text{Band width } (BW) = f_2 - f_1 = \Delta f.$$

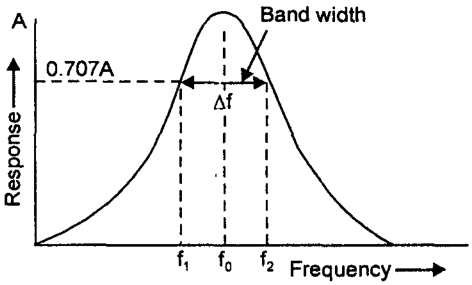


Fig. 17.18. Frequency response curve.

The frequency extremes of the band width (f_1 and f_2) are called the *cut off or corner frequencies.* These are also called the *half power frequencies* (or half power points) because the power drawn at these frequencies is half the power drawn at resonance frequency f_0. If the resonance curve is symmetrical, then we have

$$f_1 = f_0 - \tfrac{1}{2}\Delta f \text{ and } f_2 = f_0 + \tfrac{1}{2}\Delta f. \qquad ...(76)$$

Sharp resonance with high Q has a narrow band width. For the same resonance frequency, if Q decreases, the resonant response becomes broader and the band width increases. The response may be for series resonance or for parallel resonance.

For a *series resonance circuit,* the quality factor is defined as $Q_s = 2\pi f_0 L/R = f_0/(R/2\pi L)$. To find the value of Δf, the use is made of relation for current of LCR series circuit as,

$$I_{\text{half power}} = 0.717\, I_0 = 0.717\, V_0/R.$$

$\therefore$ Impedance offered by the circuit $= Z_{\text{half power}} = R/0.717 = \sqrt{2}R$.

The circuit impedance at any frequency f is given as

$$Z^2 = R^2 + (2\pi fL - 1/2\pi fC)^2.$$

For $f = f_1$ and f_2, $Z = \sqrt{2}R$, and $X_C > X_L$ at $f = f_1$ and $X_L > X_C$ at $f = f_2$, therefore

$$2R^2 = R^2 + (1/\omega_1 C - \omega_1 L)^2 \text{ or } \omega_1^2 LC + RC\,\omega_1 - 1 = 0 \qquad ...(i)$$

$$2R^2 = R^2 + (\omega_2 L - 1/\omega_2 C)^2 \text{ or } \omega_2^2 LC - RC\omega_2 - 1 = 0. \qquad ...(ii)$$

Above equations give

$$\omega_1 = \frac{-RC \pm \sqrt{(RC)^2 + 4LC}}{2LC} \text{ and } \omega_2 = \frac{RC \pm \sqrt{(RC)^2 + 4LC}}{2LC}$$

$$\therefore \quad \omega_1\omega_2 = 1/LC \text{ or } \sqrt{f_1 f_2} = 1/2\pi\sqrt{LC} = f_0 \qquad ...(77)$$

and $\omega_1 - \omega_2 = R/L$ or $2\pi(f_2 - f_1) = R/L$.

$$\therefore \quad \text{Band width } (\Delta f) = R/2\pi L = f_0/Q_s. \qquad ...(78)$$

Table 17.1. *Comparison of series and parallel resonance circuits*

Series Resonance	*Parallel resonance*
Frequency $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$	Frequency $f_0 = \dfrac{1}{2\pi}\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{L^2}}$
Current I is maximum.	Current I is minimum.
Impedance Z is minimum.	Independence Z is maximum.
Circuit acts as acceptor circuit.	Circuit acts as rejector circuit.
Quality factor $Q_s = X_L/R = \omega_0 L/R$.	Quality factor $Q_p = R/X_C = \omega_0 CR$.
Band width $= f_2 - f_1 = f_0/Q_s$.	Band width $= f_0/Q_p$.
Capacitive below f_0 but inductive above f_0.	Inductive below f_0 but capacitive above f_0.
Needs low resistance source for sharp tuning.	Needs high resistance source for sharp tuning.
Source is inside LC-circuit	Source is outside LC circuit
Voltage magnification $= \omega L/R$.	Current magnification $= \omega L/R$.

Thus we see that a circuit with a smaller band width has a higher Q, a sharper resonance and greater selectivity.

For a *parallel resonance circuit,* the resonant response increases the impedance of the circuit. The bandwidth for such a circuit is the range of frequencies within which the impedance does not drop below 70.7 percent of its maximum value.

Similar to that obtained for a series resonant circuit, the bandwidth (BW) is related with quality factor of parallel resonance circuit as

$$B{\cdot}W{\cdot} = f_0/Q_p. \qquad ...(79)$$

This relation is applicable only for the parallel resonant circuit with high Q_p, as the resonance curve becomes unsymmetrical for low Q_p values.

Summary of the series and parallel resonance circuit is given in table. 17.1.

17.14 COUPLED CIRCUIT

Let us consider an alternating emf, represented by $v = V_0 e^{j\omega t}$, applied to the primary of mutual inductance M, the primary of which has a self inductance L_1 and a resistance R_1 and the secondary is a closed circuit of inductance L_2 and a resistance R_2. If I_1 and I_2 be the currents in primary and secondary respectively, then

$$\begin{aligned} \mathbf{V} &= (R_1 + j\,\omega\,L_1)\,\mathbf{I}_1 + j\,\omega\,M\,\mathbf{I}_2 \\ &= \mathbf{Z}_1\,\mathbf{I}_1 + j\,\omega\,M\,\mathbf{I}_2. \qquad ..(80) \\ 0 &= (R_2 + j\,\omega\,L_2)\,\mathbf{I}_2 + j\,\omega\,M\,\mathbf{I}_1 \\ &= \mathbf{Z}_2\,\mathbf{I}_2 + j\,\omega\,M\,\mathbf{I}_1. \qquad ..(81) \end{aligned}$$

On solving these equations, we get

$$\mathbf{I}_1 = \frac{\mathbf{V}}{\mathbf{Z}_1 + \omega^2 M^2/\mathbf{Z}_2} = \frac{\mathbf{V}}{\mathbf{Z}_1'} \qquad ..(82)$$

and
$$\mathbf{I}_2 = -\frac{j\omega M\mathbf{V}}{\mathbf{Z}_1\mathbf{Z}_2 + \omega^2 M^2} = \frac{\mathbf{V}}{\mathbf{Z}_2'}, \qquad ..(83)$$

where $\mathbf{Z}_1'$ and $\mathbf{Z}_2'$ are two effective impedance operators for the primary and secondary circuits.

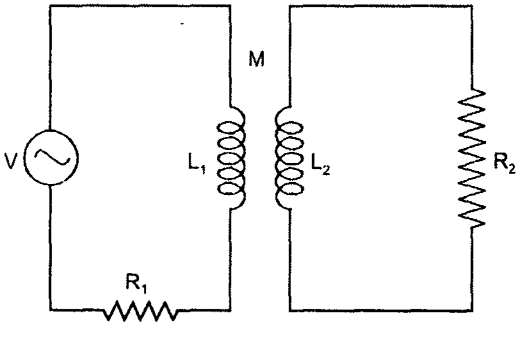


Fig. 17.19. Coupled circuit.

$$\begin{aligned} \mathbf{Z}_1' &= R_1 + j\omega L_1 + \frac{\omega^2 M^2}{R_2 + j\omega L_2} \cdot \frac{R_2 - j\omega L_2}{R_2 - j\omega L_2}. \\ &= \left[R_1 + \frac{\omega^2 M^2}{R_2^2 + \omega^2 L_2^2} R_2\right] + j\omega\left[L_1 - \frac{\omega^2 M^2}{R_2^2 + \omega^2 L_2^2} L_2\right] \\ &= R_1' + j\,\omega\,L_1'. \qquad ...(84) \end{aligned}$$

where R_1' is the effective primary resistance which is greater than R_1 by $\omega^2 M^2 R_2/(R_2^2 + \omega^2 L_2^2)$ due to the so called reflected resistance from the secondary and L_1' is the effective primary inductance which is less than L_1 by $\omega^2 M^2 L_2/(R_2^2 + \omega^2 L_2^2)$ due to *reflected inductance* from the secondary. The reflected resistance is large and reaches a maximum value for the value of $R_2 = \omega L_2$. It is also clear for $R_2 \to 0$, $L_1' \to (L_1 - M^2/L_2)$ or a positive quantity as $M^2 < L_1 L_2$ always. Thus the primary current

$$\mathbf{I}_1 = \mathbf{V}/(R_1' + j\,\omega\,L_1')$$

$$\mathbf{I}_1 = \frac{V_0 e^{j\omega t}}{\left[R_1'^2 + \omega^2 L_1'^2\right]^{1/2} e^{j\theta}} = \frac{V_0 e^{j(\omega t - \theta)}}{\left[R_1'^2 + \omega^2 L_1'^2\right]^{1/2}} = \frac{V_0}{Z_1'} e^{j(\omega t - \theta)}, \qquad \text{...(85)}$$

where $\tan\theta = \omega L_1'/R_1'$.

Thus we see that $\mathbf{I}_1$ always lags behind $\mathbf{V}$. The phase lag of current on voltage for the primary alone is $\tan^{-1} \omega L_1/R_1$. On coupling with the circuit, $L_1$and R_1 are replaced by L_1' and R_1' respectively. As $R_1' > R_1$ and $L_1' < L_1$, the phase lag reduces when such coupling takes place. When $R_2 \to 0$, this lag decreases and the coupling becomes more tight. The cosine of the phase angle becomes larger as phase angle becomes smaller and thus the *power factor and so the power taken from an ac source increases on coupling the secondary to the primary.*

The current in the secondary

$$\mathbf{I}_2 = -\frac{j\omega M}{Z_2}\mathbf{I}_1 = \frac{-j\omega M}{R_2 + j\omega L_2}\mathbf{I}_1 = \frac{\omega M e^{-j\pi/2}}{(R_2^2 + \omega^2 L_2^2)^{1/2} e^{j\phi}}\mathbf{I}_1 = \frac{\omega M}{(R_2^2 + \omega^2 L_2^2)^{1/2}} e^{-j(\pi/2+\phi)}\mathbf{I}_1, \qquad \text{...(86)}$$

where $\tan\phi = \omega L_2/R_2$.

Thus, we see that the secondary current lags the primary current by a phase angle of $(\pi/2 + \phi)$.

17.15 ALTERNATORS (AC Generators)

In a simple alternator (*alternating current generator*) coil rotates in a magnetic field. The ends of the coil are attached to two insulated brass rings (say S_1 and S_2) called slip rings. Two pieces of metal called brushes touch slip rings, one rests on each. Alternating emf is observed between two terminals of the brushes T_1 and T_2. This alternator is of no use for practical purposes, since the speed revolution of the armature should be great.

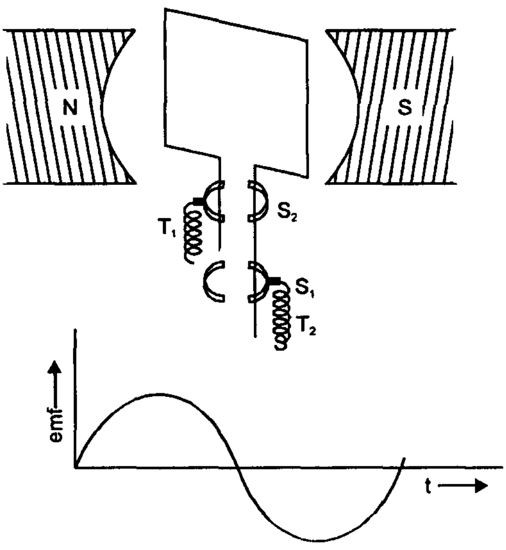


Fig. 17.20. Alternator.

An arrangement, where the armature coils are on the outside and stationary and the poles rotate inside them, is very much used for generating high voltage alternating current. The stator is of iron to decrease the reluctance of the magnetic circuit and is laminated to reduce eddy currents set up by the rotating poles. The poles are excited by coils on the rotor carrying direct current through slip rings. In large generators an electromagnet is used as a field magnet.

The *alternators differ from d.c. generators in construction as :*

(1) The former require no commutator. However slip-rings are used instead of it.

(2) In the former the magnets usually rotates and are often called the *rotor*, while the armature is called the *stator*.

(3) The former are nearly always *separately excited.*

(4) The alternator windings are nearly always of the *open coils* type, *i.e.*, the armature windings come to terminals and the circuit is not complete until the external circuit is closed.

Emf in a Generator : When the armature rotates under the poles of an external field system, or the field is rotating keeping the armature stationary, the magnetic flux linked with the armature changes and thus emf is induced. If the direction of induced current, as given by Fleming's right hand rule, is positive during the first half turn of the armature, it will be negative during the next half turn. In this way alternating current is produced by the generator (Fig. 17.20).

If N be the number of turns and A the area of the armature, the magnetic flux Φ linked with the armature if the normal makes an angle θ with the direction of magnetic field B is given by

$$\Phi = NAB \cos \theta.$$

If the coil rotates with an angular velocity ω, then the induced emf

$$\begin{aligned} e &= -\frac{d\Phi}{dt} = -\frac{d}{dt}(NAB\cos\theta) = -\frac{d}{dt}(NAB\ \cos\omega t) \\ &= NAB\ \omega \sin \omega t = \mathcal{E}_0 \sin \omega t. \qquad \text{...(87)} \end{aligned}$$

If the magnetic field B is in horizontal direction, the induced emf is zero when the armature is vertical $\omega t = 0$, it is maximum $(e = \mathcal{E}_0)$ when the armature is horizontal $\omega t = \pi/2$, it is again zero when $\omega t = \pi$, maximum $(e = -\mathcal{E}_0)$ when $\omega t = 3\pi/2$ and is again zero when the armature returns to its original position $\omega t = 2\pi$.

The frequency of ac supply is 50 cycles/sec in India and Britain and is 60 cycles/sec in U.S.A.

In the Fig. 17.21, there are eight poles and the frequency of alternation is four times the frequency of revolution. The number of magnetic poles is considerable. The number of complete alternations of emf per revolution of the rotor is half the number of poles in the rotor. It is because the turns in the coils are in series and oppositely wound. If a coil is passing a N pole, at the same time the adjacent coil is passing a S-pole. As the poles are excited through slip rings, hence the rotor circuit is broken and the current in the external circuit is broken accordingly.

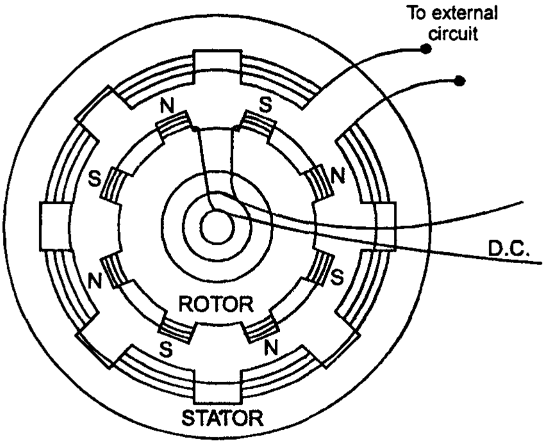


Fig. 17.21. Single phase generator.

The above type of generator is called *single phase generator*. Two phase and three phase alternators are generally designed. *In a two phase alternator*, there are two coils placed at right angles to each other so that emf in one is maximum when it is zero in the other. Thus phase difference of 90° is maintained between the two. The number of turns and area of cross section of each coil are the same, therefore the amplitude and frequency of the generated emf's will be the same. The emf in the two coils are given by

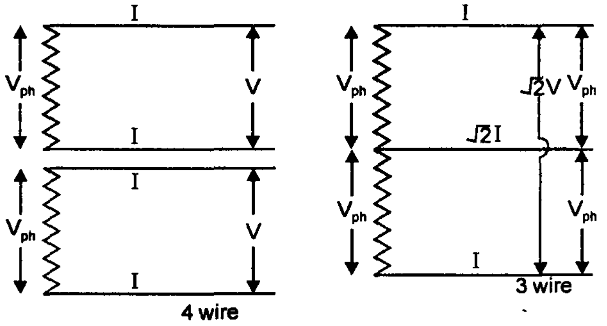


Fig. 17.22. Two phase working.

$$v = V_0 \sin \omega t,\ v = V_0 \sin(\omega t + \pi/2). \quad ...(88)$$

The alternator can be connected to the receiving apparatus by either four or three conductors. The circuit with four conductors being the same as two independent single phase circuits.

If V and I refer to line voltage and current and V_{ph} and I_{ph} refer to phase voltage and current. Then in the four wire case $V = V_{ph}$ and $I = I_{ph}$.

In the three wire case the outers carry current $I = I_{ph}$ and the voltage between outers and middle wire are $V = V_{ph}$. The middle wire carrives the resultant current $\sqrt{2}\,I$, as the two outer currents are I and 90° apart. Similarly the voltage between the two outers is $\sqrt{2}\,V$.

$$\therefore \qquad \text{Power } W = \sqrt{2}V \cdot \sqrt{2}I \cos\phi = 2VI\cos\phi.$$

where ϕ is the phase of I with respect to E.

In a *three phase generator*, there are three coils inclined at 120° to one another and are rotated at constant speed in a uniform magnetic field. The emf's generated will all be sinusoidal and of same frequency and of equal maximum values. The emf of coil B will lag behind that of A by $\frac{1}{3}$ of a period and that of coil C will lag by a further $\frac{1}{3}$ of a period $\left(\frac{2}{3}\text{in all}\right)$. Hence emf's are given by

$$v = V_0 \sin \omega t,\ v = V_0 \sin(\omega t + 2\pi/3),\ v = V_0 \sin(\omega t + 4\pi/3). \quad ...(89)$$

As one complete cycle of emf is generated by the passage of one pair of poles, hence for $2n$ poles (or n pairs) on the rotor, n cycles per revolution will be generated. If m is the number of revolutions per sec, emf of frequency nm cycles/sec will be generated.

The polyphase system has the following advantages.

(1) In polyphase alternators the total power does not fluctuate, while in a single phase generator the current fluctuates.

(2) The output power of a polyphase alternator is always greater than that of a single phase generator of the same size.

(3) The polyphase alternators are used for power generation as they are superior for transmission and distribution purposes.

Distribution of three phase alternating current : The number of terminals to carry current of 3-phases, is reduced to four terminals. The starting point of the three windings is made common and is called the *neutral*. In this way power can be handled by three heavy transmission wires and one light one, instead of six heavy wires would be required in three separate single-phase systems. There are following two ways of connecting these wires :

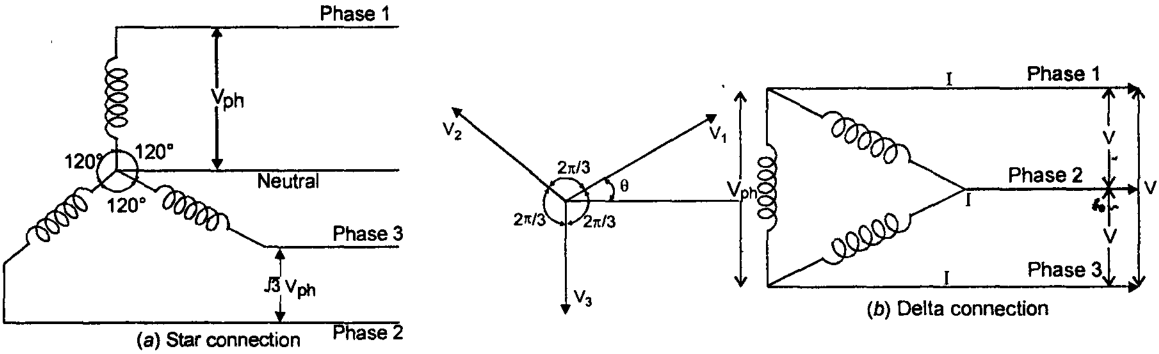
(*i*) *Star connection :* The common point of the three wires is the star point and the connecting wires are as the prongs of a star. The total current at star point is zero. A wire, known as *neutral wire* is connected with this point and three phase wires with the ends of the prongs. Single phase current can thus be obtained by connecting a load between any of the three phase wires and the neutral. If V_{ph} is the amplitude of the phase voltage, then the voltage difference between two phase wires.

$$\mathbf{V} = \mathbf{V}_2 - \mathbf{V}_1 \, (= \mathbf{V}_3 - \mathbf{V}_2 = \mathbf{V}_3 - \mathbf{V}_1)$$

or

$$V = V_{ph} \sin(\omega t + 2\pi/3) - V_{ph} \sin \omega t = 2V_{ph} \cos(\omega t + \pi/3) \sin(\pi/3)$$

$$= \sqrt{3}\, V_{ph} \cos(\omega t + \pi/3)\,. \qquad \ldots(90)$$



(*a*) Star connection

(*b*) Delta connection

Fig. 17.23. Three phase working.

Thus the voltage amplitude is $\sqrt{3}V_{ph}$ and the phase difference is 30°. Thus for a house supply of 230 volts, one line wire and a neutral wire is connected to the supply. For power supply in factories 400 volts, three line wires and a neutral wires are connected to factory. Since no current passes through the neutral wire, hence the line current is same as that of phase current. In the star connection, the line currents are equal to the phase currents. The voltage V between any pair of lines is the vector difference of voltages in the two phases supplying that pair. The phase difference between the voltage in one phase and the reverse voltage in the next phase is 60°. Thus $V = \sqrt{3}\, V_{phase}$. Hence power

$$W = 3V_{phase} I_{phase} \cos \phi = \sqrt{3}\, V I \cos \phi.$$

(*ii*) *Delta connection* : In this type of connections, three windings are joined in series. Hence the *line voltage, i.e.*, the difference of voltage between two phase wires is same as that across each winding which is known as *phase voltage*. The line current

$$\mathbf{I} = \mathbf{I}_2 + \mathbf{I}_1$$

$$I = I_{ph} \sin(\omega t + 2\pi/3) - I_{ph} \sin \omega t = 2I_{ph} \cos(\omega t + \pi/3) \sin(\pi/3)$$

$$= \sqrt{3} I_{ph} \cos(\omega t + \pi/3) \qquad \ldots(91)$$

Thus the line current is equal to $\sqrt{3}$ times the phase current.

In this way we see that the power consumption in both the cases is same.

17.16 TRANSFORMERS

The transformer consists essentially of a magnetic field circuit linking with two distinct windings. When one winding is connected to an a.c. supply an alternating flux will be set up in the core and this flux linking with the another will induce an alternating emf in it. The former winding is known as *primary* and another as *secondary*.

Transformers are used for converting an *ac* of high voltage and low current into a current of lower voltage and high current or *vice versa* with small energy loss without using moving parts. The former is known as *step-down* and the latter *step up* transformer.

Types of Transformers : There are three main types of transformers depending on the structure (*i*) *Core type* with single magnetic circuit, (2) *Shell type* with double magnetic circuit and (3) *Berry type* with distributed magnetic circuit.

The *cores* are built up of sheet iron or alloyed steel. The small transformer cores have rectangular cross sections, but the large sizes have circular cross sections. The *windings are* either of the *cylindrical* or sandwich type. The former type is mainly used on core transformers and the latter type on the shell type.

There are three types of transformers used mainly, depending on the frequency of the source in the circuit.

(*a*) *Mains frequency transformers :* These employ an iron core, laminated to avoid eddy current loss and occur in a very wide range of sizes from small laboratory components to the big transformers used at power stations.

(*b*) *Audio-frequency transformers :* These are used chiefly in radio communication and electronic circuits and contain a laminated iron core to increase coupling efficiency. They operate over the frequency range from 50 to 20,000 c/s.

(*c*) *Radio-frequency transformers :* These can not have iron cores because of large eddy current losses. Transformers without core, known as aircored also suffer from the disadvantages that the coupling coefficient and efficiency are low. More efficient device employs a dust core, consisting of a fine powder of iron mixed with an insulating binder and compressed into a core of the desired shape. These transformers are used at frequencies from 20 kc/s to 30 Mc/s.

Transformer losses : In practice the power output in the secondary is always less than the power input in the primary because of the losses due to imperfection of design. These losses are :

(*i*) *Copper losses :* The losses are due to the heating effects of inevitable resistances of the primary and secondary windings.

(*ii*) *Core losses :* The hysteresis and eddy current heating losses in the core constitute core losses. The former are minimized by the use of iron having a narrow hysteresis loop or small expenditure of energy and the latter losses are minimized by using the core of laminated sheets of soft iron which are mutually insulated.

(*iii*) *Magnetic leakage :* These losses are due to the leakage of magnetic flux between primary and secondary. In other words all the lines of magnetic flux produced by the primary current do not cut the secondary winding and the flux produced by the secondary current do not cut the primary winding. The leakage is reduced to a minimum if the two coils are interwound. For the good insulation between primary and secondary, winding should be done side by side.

(*iv*) *Self capacity of winding :* The capacitances exist (*i*) between each winding of a transformer, due to the inter turns and inter layers and (*ii*) between every pair of winding. These capacitances have a large effect on the audio frequency transformers and have practically no effect at low frequencies (power frequencies).

Efficiency. On account of the losses, the power taken from the secondary of a transformer cannot exceed that taken by the primary from the source. The efficiency of a given transformer is thus defined as

$$\eta = \frac{\text{Power output from the secondary}}{\text{Power delivered by the source}} \times 100\%$$

$$= \frac{\text{Power output}}{\text{Power input + Power lost}} \times 100\%. \qquad \text{...(92)}$$

The ideal transformer, with no losses, has efficiency one (100%).

From a constructional point of view, a *transformer* in its simplest form is a mutual inductance. Using the results of the article 17.14 for the coupled circuit, let us consider following two cases :

(*a*) **Open circuit secondary.** In an open circuit secondary, there is no secondary current, *i.e.*, $\mathbf{I}_2 = 0$, thus equation (80) gives

$$\mathbf{V} = \mathbf{V}_1 = (R_1 + j\omega L_1)\,\mathbf{I}_1,$$

$$\therefore \quad I_1 = V_1/(R_1^2 + \omega^2 L_1^2)^{1/2}. \qquad \text{...(93)}$$

The open circuit secondary emf is therefore given by

$$V_2 = \omega M I_1 = \omega M V_1/(\mathrm{R}_1^2 + \omega^2 L_1^2)^{1/2}. \qquad \text{...(94)}$$

Since $R_1^2 << \omega^2 L_1^2$ and coupling coefficient $k = M/\sqrt{(L_1 L_2)}$, hence we have

$$\frac{V_2}{V_1} = \frac{\omega M}{(R_1^2 + \omega^2 L_1^2)^{1/2}} = \frac{M}{L_1} = k\left(\frac{L_2}{L_1}\right)^{1/2}.$$

If N_1 and N_2 be the numbers of primary and secondary turns respectively, then $L_1 \propto N_1^2$ and $L_2 \propto N_2^2$ and hence

$$V_2/V_1 = k(L_2/L_1)^{1/2} = kN_2/N_1. \qquad \text{...(95)}$$

For a perfect coupling $k = 1$. It means that the secondary coils enclose the whole of the magnetic flux produced by the primary coil. Iron cored transformer is used in practice to produce so high value of k, with nearly 1 and with large values of L_1 and L_2. Really the value of k reaches to 0.99 only. Thus for a *perfect coupling and an open circuited secondary the transformation ratio* $V_2/V_1 = N_2/N_1$, *the ratio of the number of turns in the secondary coil to that in the primary.* If the resistance R_1 of the primary coil is small in comparison to the reactance ωL_1 of its inductance, the current is wattless and the power loss is zero. Thus a transformer with open circuited secondary and a low resistance primary dissipates negligible energy.

(*b*) **Closed circuited secondary.** When the secondary of the transformer is connected to a load resistance R_2, the problem becomes that of coupled circuit (article 17.14) and the effective values of the resistance and inductance of the primary circuit are given by

$$R_1' = R_1 + \frac{\omega^2 M^2 R_2}{R_2^2 + \omega^2 L_2^2} \text{ and } L_1' = L_1 - \frac{\omega^2 M^2 L_2}{R_2^2 + \omega^2 L_2^2}$$

If the resistance R_2 is small compared with ωL_2, then for the iron cored transformer for which the coupling coefficient $k = 1$ and $M = \sqrt{(L_1 L_2)}$, we get

$$R_1' = R_1 + \frac{L_1 L_2}{L_2^2} R_2 = R_1 + \frac{L_1}{L_2} R_2,$$

and $$L_1' = L_1 - \frac{L_1 L_2}{L_2} = 0. \qquad \text{...(96)}$$

Thus we see that with a load of low resistance across a secondary of transformer, the effective impedance of primary is purely resistive in nature.

Now from Eqs. (82) and (83), the ratio of the secondary to the primary current is given as

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{j\omega M}{R_2 + j\omega L_2}.$$

For a transformer with a perfect coupling $M = \sqrt{(L_1 L_2)}$ and the resistances of the two coils of the transformer are nearly zero. In addition the reactances of the primary and secondary coils are so large in comparison with the generator and load impedances. Therefore

$$\frac{\mathbf{I}_2}{\mathbf{I}_1} = \frac{j\omega M}{j\omega L_2} = \frac{M}{L_2} = \sqrt{\frac{L_1}{L_2}} = \frac{N_1}{N_2}. \qquad \text{...(97)}$$

Further the ratio of voltage drops across the primary and secondary coils are

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{R_2 \mathbf{I}_2}{\mathbf{V} - \mathbf{I}_1 R_1} = \frac{j\omega M R_2 \mathbf{V}}{[(R_1 + j\omega L_1)(R_2 + j\omega L_2) + \omega^2 M^2][\mathbf{V} - \mathbf{I}_1 R_1]}$$

Since $R_1 < j\omega L_1$, hence

$$\frac{\mathbf{V}_2}{\mathbf{V}_1} = \frac{j\omega M R_2}{j\omega L_1 R_2 - \omega^2 L_1 L_2 + \omega^2 M^2} = \frac{j\omega M R_2}{j\omega L_1 R_2} = \frac{M}{L_1}$$

$$= \sqrt{(L_2/L_1)} = N_2/N_1. \qquad \text{...(98)}$$

Thus we see that the transformation ratio (V_2/V_1) is equal to the ratio of the turns in the secondary to that in the primary. For $N_2 > N_1$, we get $V_2 > V_1$ and $I_1 > I_2$. Thus, the step-up transformer increases the voltage and decreases the current whereas a step-down transformer reduces the voltage and increases the current.

The equations (97) and (98) give $I_1 V_1 = I_2 V_2$,

i.e., *Input power = Output power.*

Thus an ideal transformer introduces no loss and has 100% efficiency. In practice only an efficiency of about 95% can be achieved in *power transformers.*

In the article (17.14), the problem was simplified by making the *L's* the total inductances, *i.e.,* including both leakage flux and mutual flux. Let us now discuss the problem in which the various fluxes are regarded as separate. The various components of the primary applied voltage are as follows :

(*i*) The voltage $j\omega L_1 \mathbf{I}_1$ to overcome the self induced emf due to the true leakage inductance L_1.

(*ii*) The voltage $j\omega(M/\rho)\,\mathbf{I}_1$ to overcome the induced emf due to the changes in current in the primary, where ρ is the turns ratio N_2/N_1.

(*iii*) The voltage $j\omega M \mathbf{I}_2$ to overcome the induced emf due to the change in current in the secondary.

Hence the emf in the primary is given as

$$\mathbf{V}_1 = R_1 \mathbf{I}_1 + j\omega L_1 \mathbf{I}_1 + j\omega(M/\rho)\mathbf{I}_1 + j\omega M \mathbf{I}_2. \qquad \text{...(99)}$$

The emf induced in the secondary $-j\omega M\mathbf{I}_1$ is used to overcome (*i*) the self inductance of the secondary $j\omega\rho M\mathbf{I}_2$ and (*ii*) the total impedance drop of both secondary and load. Therefore, we have

$$\mathbf{V}_2 = -j\omega M\mathbf{I}_1 = R_2\mathbf{I}_2 + j\omega L_2\mathbf{I}_2 + j\omega\rho M\mathbf{I}_2.$$

$$\therefore \quad \mathbf{I}_2 = -j\omega M\mathbf{I}_1/(R_2 + j\omega L_2 + j\omega\rho M)$$

$$\text{or} \quad \frac{\mathbf{I}_1}{\mathbf{I}_2} = -\left(\rho + \frac{R_2 + j\omega L_2}{j\omega M}\right).$$

Substituting the value of $\mathbf{I}_2$ in equation (99), we have

$$\begin{aligned}
\mathbf{V}_1 &= \left[R_1 + j\omega L_1 + \frac{j\omega M}{\rho} + \frac{\omega^2 M^2}{R_2 + j\omega L_2 + j\omega\rho M}\right]\mathbf{I}_1 \\
&= \left[R_1 + j\omega\left(L_1 + \frac{M}{\rho}\right) + \frac{\omega^2 M^2[R_2 - j\omega(L_2 + \rho M)]}{R_2^2 + \omega^2(L_2 + \rho M)^2}\right]\mathbf{I}_1 \\
&= \mathbf{I}_1\left\{R_1 + \frac{\omega^2 M^2 R_2}{R_2^2 + \omega^2(L_2 + \rho M)^2} + j\omega\left[L_1 + \frac{M}{\rho} - \frac{\omega^2 M^2(L_2 + \rho M)}{R_2^2 + \omega^2(L_2 + \rho M)^2}\right]\right\} \\
&= \mathbf{I}_1(R' + j\omega L'). \qquad \text{...(100)}
\end{aligned}$$

This shows that R' and L' are the equivalent resistance of the whole circuit and the equivalent inductance of the whole circuit respectively.

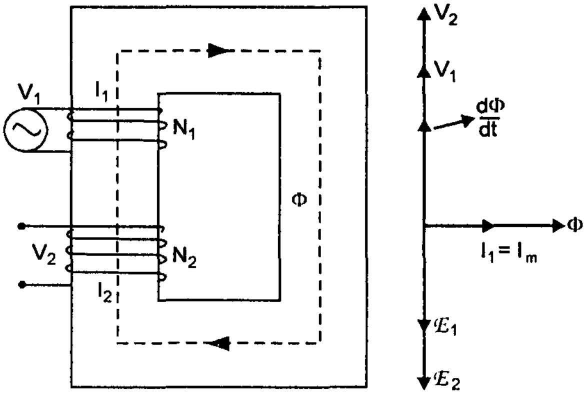


Fig. 17.24. Phasor diagram (Open circuit).

The ideal transformer, having no losses and in which all of the flux is confined to the iron core is shown schematically in Fig. 17.24. A primary winding of N_1 turns and a secondary winding of N_2 turns both encircle the core in the same sense. As the secondary is open, there is no secondary current. In the phasor diagram, phasor $\mathbf{V}_1$ represents the primary voltage. As the resistance in primary is assumed to be zero, the primary current known as the magnetizing current I_m lags the primary voltage by 90° having magnitude V_1/X_1, where X_1 is the reactance of the primary. This current sets up a flux of a amplitude Φ in the core, in phase with the current. The rate of change of flux $d\Phi/dt$ will lead Φ by 90°. By Faraday's law, we know that the induced emf $\mathcal{E} \propto -d\Phi/dt$. This *–ve* sign indicates that this emf is out of phase with $d\Phi/dt$. In the diagram, $\mathcal{E}_1$ and $\mathcal{E}_2$ represent the induced voltages at the primary and secondary terminals respectively.

If a resistance R is connected across the secondary. In this case as the primary voltage V_1 is fixed, the rate of change of flux $d\Phi/dt$ must be the same, as shown in Fig. 17.25. Hence the core flux Φ and the secondary voltage V_2 must also be the same. The core flux is now set up by the primary current I_1 and by the secondary current I_2. The secondary current I_2 is in phase with V_2 and of magnitude V_2/R. The flux Φ_2, set up by the secondary current is opposite in phase to this current. The flux Φ_1 due to the primary current is the phasor sum of Φ and $-\Phi_2$. The primary current I_1 is in phase with Φ_1.

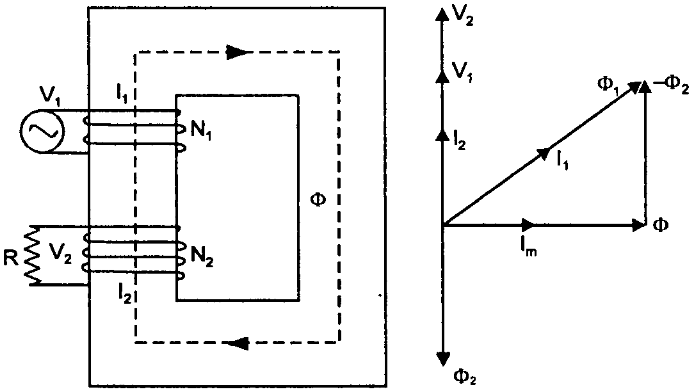


Fig. 17.25. Phasor Diagram (closed circuit).

Uses of Transformers :

(1) The *step-up and step-down transformers* are used in a.c. electrical power distribution for the domestic and industrial purposes.

(2) The *audio-frequency transformers* are used in radio-receivers, radio-telephony, radio-telegraphy and in televisions.

(3) The *radio frequency transformers* are used in radio communications at frequencies of the order of mega-cycles.

(4) The *impedance transformers* are used for impedance matching, which is necessary to get maximum power from a source of two independent circuits.

(5) The *current transformers* are used for measuring large alternating currents. The current to be measured is passed through the primary and the secondary being connected to the ammeter.

(6) The *pressure transformers* are used for measuring very high pressures. The pressures to be measured is stepped down by means of such a transformer, the low tension secondary of it is connected to the voltmeter.

(7) For measuring power on a high pressure system, both current and pressure transformers are used.

(8) The *welding transformers* are used for welding as well as in induction furnace. The ends of one or two turns of thick copper wire, forming the secondary of the transformer are connected to a pair of rods to be welded. When the current is on, the tips of the rods get red hot and started welding.

(9) *Long distance power transmission.* From the point of view of both efficiency and economy, the power must be transmitted at high voltage [suppose a given power say, 44,000 watts is transmitted from a generating station to a long distant city, it can be transmitted at a voltage 220 volts and a current of 200 amp or at a voltage of 22000 volts and a current of 2 amp. In the former case the large amount of energy (i^2Rt) will be lost as heat during transmission, a voltage drop (iR) along the line will be greater, and a very thick line wire will be required to carry the high current]. Hence the step-up transformer should be used at the generating station before transmission and the step-down transformer should be used at receiving station before its supply to the city.

17.17 ROTATING MAGNETIC FIELD AND INDUCTOR MOTOR

A rotating magnetic field is one in which the flux rotates round a fixed axis. Its magnitude remains constant while the direction changes at a constant rate. It can be produced by (*i*) merely rotating a magnet or a current carrying coil or (*ii*) a system of stationary coils supplied with single, two, three or polyphase currents.

Let us consider two coils C_1 and C_2 carrying alternating current and placed at right angles to each other. If the currents in these coils are respectively I_1 and I_2 which are equal in frequency ω, but differing in phase by θ, then we have

$$I_1 = I_{01} \sin(\omega t + \theta)$$

and $$I_2 = I_{02} \sin \omega t. \qquad \ldots(101)$$

The magnetic fields due to these currents carrying coils will be in phase with the currents and may be represented as

$$B_1 = B_{01} \sin(\omega t + \theta)$$

and $$B_2 = B_{02} \sin \omega t \qquad \ldots(102)$$

Since the coils are at right angles to each other, the resultant magnetic field is given by

$$B = \sqrt{B_1^2 + B_2^2}. \qquad \ldots(103)$$

It makes an angle ψ with B_1, such that $\tan \psi = B_2/B_1$.

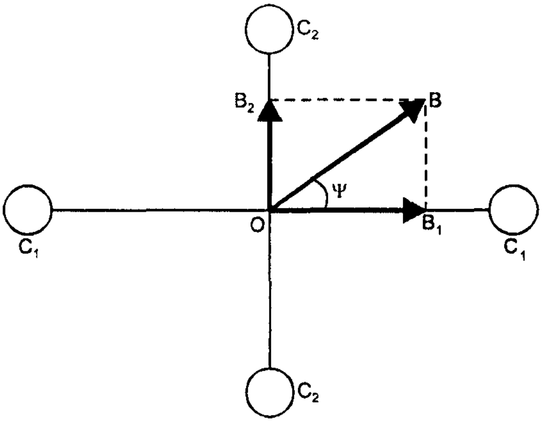


Fig. 17.26. Production of Rotating magnetic field.

The resultant field vary periodically as :

At

$t = 0$ $\quad B_2 = 0 \quad B = B_1 = B_{01} \sin \theta$

$t = -\theta/\omega,$ $\quad B_1 = 0, \quad B = B_2 = B_{02} \sin(-\theta) = -B_{02} \sin \theta$

$t = \pi/\omega$ $\quad B_2 = 0, \quad B = B_1 = -B_{01} \sin \theta$

$t = (\pi - \theta)/\omega,$ $\quad B_1 = 0, \quad B = B_{02} \sin(\pi - \theta) = B_{02} \sin \theta$

Thus *the resultant magnetic field **B** rotates (in an anti clockwise direction) and at the same time varies periodically in magnitude.*

The most important instant is when the magnetic fields have equal amplitudes and are in quadrature, *i.e.*, $B_{01} = B_{02} = B_0$ (say) and $\theta = \pi/2$. Under such conditions

$$B = B_0 \sqrt{\sin^2(\omega t + \pi/2) + \sin^2 \omega t} = B_0$$

and $$\tan \psi = \sin \omega t / \sin(\omega t + \pi/2) = \tan \omega t,$$

or $$\psi = \omega t. \qquad \ldots(104)$$

Thus the resultant magnetic field B is constant in magnitude and rotates with constant angular velocity ω, in anti-clockwise direction if the fields B_1 and B_2 are of same amplitude and frequency and B_1 is $\pi/2$ ahead of B_2 in phase.

If, on the other hand B_1 lags behind B_2 by $\pi/2$ in phase, *i.e.*, $\theta = -\pi/2$, we have

$$B_1 = B_0 \sin(\omega t - \pi/2) = -B_0 \cos \omega t$$

and $$B_2 = B_0 \sin \omega t.$$

$$\therefore \quad B = \sqrt{B_1^2 + B_2^2} = B_0$$

and $\tan\psi = -\sin\omega t/\cos\omega t = -\tan\omega t$

or $\psi = -\omega t$ ...(105)

and the rotation is thus clock-wise.

Couple acting on a coil placed in a rotating magnetic field : It has been shown above that a rotating magnetic field may be produced in a region surrounding a fixed point O by passing currents, equal in amplitude and frequency, but differing in phase by $\pi/2$, in two coils placed at right angles to each other. If a small coil of N turns and effective area A is placed in the rotating magnetic field such that the centre of the coil coincides the point O, the induced current will be produced in coil. In accordance with Lenz's law the magnetic effects of such induced currents and the original field will interact in such a manner as to reduce the relative motion between the coil and the field. Thus a couple acts on a coil, tending it to rotate in the direction of the resultant field.

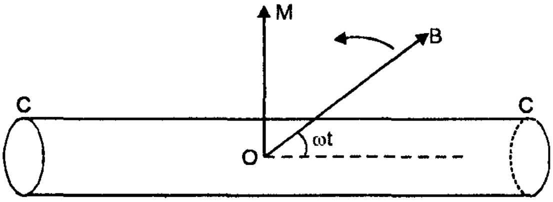


Fig. 17.27. Coil in a rotating magnetic field.

Let us calculate the couple experienced by the coil when placed in a rotating magnetic field. The total magnetic flux Φ linked with the coil at any instant t is given by

$$\Phi = NAB\sin\omega t,$$

where ωt is the angle between the direction of magnetic field B and the plane of the coil.

Due to the change of flux, the induced emf in the coil will be

$$\mathcal{E} = -\frac{d\Phi}{dt} = -\frac{d}{dt}(NAB\sin\omega t) = NAB\omega\cos(\omega t - \pi).$$

If L is the self inductance and R the resistance of the coil, then the induced current in the coil will be

$$i = \frac{NAB\omega}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t - \pi - \alpha), \qquad \text{...(106)}$$

where $\tan\alpha = \omega L/R$.

This current gives rise to a magnetic field and the coil will behave as a magnetic shell of magnetic moment NAi in the direction perpendicular to its plane. The couple acting on the coil will turn it into the direction of **B**. The moment of the couple is given by

$$\tau = \mathbf{M}\times\mathbf{B}$$

or $\tau = MB\sin(\omega t - \pi/2) = -BNAi\cos\omega t$

$$= -\frac{N^2A^2B^2\omega}{\sqrt{R^2+\omega^2L^2}}\cos(\omega t-\pi-\alpha)\cos\omega t = \frac{N^2A^2B^2\omega}{\sqrt{R^2+\omega^2L^2}}[\cos^2\omega t\cos\alpha + \sin\omega t\cos\omega t\sin\alpha]$$

Since the average value of $\cos^2\omega t$ over one cycle is $\frac{1}{2}$ and that of $\sin\omega t\cos\omega t$ is zero. Therefore the mean couple acting on the coil is

$$\tau_{av} = \frac{N^2A^2B^2\omega}{2\sqrt{R^2+\omega^2L^2}}\cos\alpha = \frac{N^2A^2B^2\omega R}{2(R^2+\omega^2L^2)}. \qquad \text{...(107)}$$

The direction of the mean couple will be the direction of rotation of the field.

The above relation shows that the mean couple depends on the value of the relative angular velocity of the field with respect to the coil. It is zero if $\omega = 0$ or ∞. If a coil is free to rotate, its angular velocity in the direction of the field rotation will increase till the rate of doing work against the friction of all kinds is equal to the rate of work done by the rotating field. When the coil is released its angular velocity increases at first. This will decrease the relative angular velocity ω of the field with respect to the coil. Thus the couple experienced by the coil will increase, which will further speed up the coil and decrease ω. However, the angular velocity of the coil will never be equal to that of the field, as in that case $\omega = 0$ and $\tau = 0$. The average couple is maximum for same value of ω between zero and infinity, can be obtained by differentiating τ_{av} w.r.t. ω and putting it as zero, *i.e.*

$$\frac{d\tau_{av}}{d\omega} = \frac{N^2A^2B^2R}{2}\frac{d}{d\omega}\left(\frac{\omega}{R^2+\omega^2L^2}\right)=0$$

or $$\frac{N^2A^2B^2R}{2}\cdot\frac{R^2+\omega^2L^2-\omega(2\omega L^2)}{(R^2+\omega^2L^2)^2}=0$$

or $$L^2\omega^2 = R^2 \text{ or } \omega = R/L. \qquad \text{...(108)}$$

As the double differentiation of τ_{av} w.r.t. ω is negative for $\omega = R/L$, hence τ_{av} is maximum for $\omega = R/L$.

Again $\tan\alpha = \omega L/R = 1$ or $\alpha = 45°$. Thus the value of α for maximum couple is 45° and the value of maximum couple is

$$\tau_{av,\,max} = \frac{N^2A^2B^2(R/L)R}{2(R^2+R^2)} = \frac{N^2A^2B^2}{4L}. \qquad \text{...(109)}$$

Induction Motor : The motion of a conductor in a rotating magnetic field has been used in the construction of induction motors besides many other partial applications. In an induction motor two pairs of electromagnets *AB* and *CD* carry alternating currents, differing in phase; this results in a rotating magnetic field. One of the coils *AB* is called the main *magnetising coil* and the other *CD* is called the *shunt coil*, the two being at right angles to each other. As the coils are supplied ac by the same source, a phase difference can be introduced by putting an extra inductance *L* into circuit *CD* or a capacity *C* into the circuit *AB* or both. If the electromagnets are similar, then the magnetic fields will have same amplitudes and the resultant magnetic field will be a rotating field.

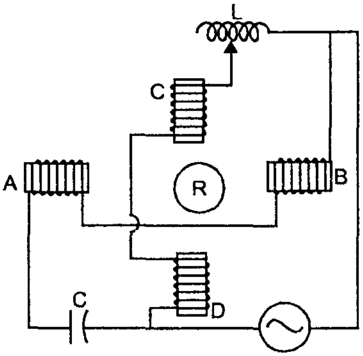


Fig. 17.28. Induction Motor.

A mass of metal or a system of closed coils, known as rotor, is mounted upon an axle in the rotating field. It will experience a deriving couple as discussed above. There are two types of rotor winding, the *squirrel cage* and the *phase wound*. The former consists of a number of copper bars threaded through slots in a laminated rotor core, with their ends connected to strong copper end rings, and is used only for comparatively small motors. The latter is provided with a distributed winding. The separate phases are generally connected in star and the three free ends connected each to one of three insulated slip rings mounted on the motor shaft.

In polyphase induction motor, the current in its rotor is not drawn from the supply, but is induced by relative motion of the rotor and the magnetic field produced by the stator currents. The stator is built exactly like an alternating stator, carries a polyphase winding similar to an alternating winding. An ac induction motor in which its speed is strictly proportional to the frequency of the current is called a *synchronous motor*. In such a motor, the coils of the rotor are not short circuited but connected through slip rings to a dc source and the stator is supplied an ac. The rotor is thus driven at exactly the same speed as the rotating field.

ALTERNATING CURRENT BRIDGES

17.18 IMPEDANCE BRIDGE

The theory of the Wheatstone bridge for the comparison of resistances, using dc source, has already been discussed in chapter 8. The theory of ac bridges is very important as the capacitance and inductance measurements are now made by the aid of these bridges. A simple bridge consisting of four impedances, arranged as in the Wheatstone bridge is known as *impedance bridge*. The cell and galvanometer of the familiar dc bridge are replaced by an ac generator and a suitable detector D respectively. The detector is generally a pair of headphones, giving a minimum sound at balance with an ac source of about 1 kc/sec. A vibration galvanometer is also used as a detector using the mains with a step-down transformer as the source of supply.

To deduce the condition for balance, let us assume that $\mathbf{I}_1$, $\mathbf{I}_2$ and $\mathbf{I}_0$ be the cyclic currents in the loops, as indicated in Fig. 17.29. Applying Kirchhoff's laws for the mesh $ABDA$, we get

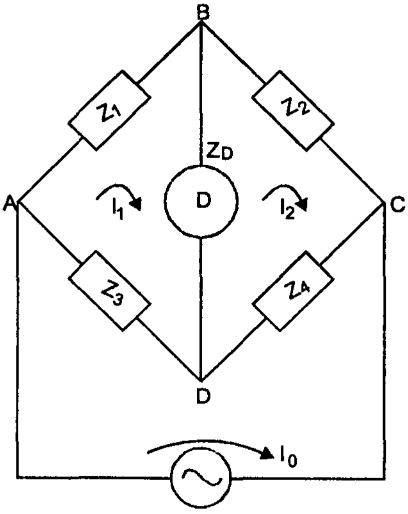


Fig. 17.29. Impedance Bridge.

$$\mathbf{I}_1(\mathbf{Z}_1 + \mathbf{Z}_D + \mathbf{Z}_3) - \mathbf{I}_2\mathbf{Z}_D - \mathbf{I}_0\mathbf{Z}_3 = 0$$

and for the mesh $BCDB$, we get

$$\mathbf{I}_2(\mathbf{Z}_2 + \mathbf{Z}_4 + \mathbf{Z}_D) - \mathbf{I}_1\mathbf{Z}_D - \mathbf{I}_0\mathbf{Z}_4 = 0.$$

Solving these equations the current through the detector is given by

$$\mathbf{I} = \mathbf{I}_1 - \mathbf{I}_2 = \frac{\mathbf{I}_0(\mathbf{Z}_1\mathbf{Z}_4 - \mathbf{Z}_2\mathbf{Z}_3)}{\mathbf{Z}_D(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4) + (\mathbf{Z}_1 + \mathbf{Z}_3)(\mathbf{Z}_2 + \mathbf{Z}_4)}.$$

If the detector and generator are interchanged, the current through the detector

$$\mathbf{I} = \frac{\mathbf{I}_0(\mathbf{Z}_2\mathbf{Z}_3 - \mathbf{Z}_1\mathbf{Z}_4)}{\mathbf{Z}_D(\mathbf{Z}_1 + \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4) + (\mathbf{Z}_1 + \mathbf{Z}_2)(\mathbf{Z}_3 + \mathbf{Z}_4)}$$

The bridge is balanced when the current $\mathbf{I}$ is zero, *i.e.*,

$$\mathbf{Z}_1\mathbf{Z}_4 = \mathbf{Z}_2\mathbf{Z}_3 \text{ or } \mathbf{Z}_1/\mathbf{Z}_2 = \mathbf{Z}_3/\mathbf{Z}_4. \qquad ...(110)$$

The result remains identical when detector and generator are interchanged as the denominator has no effect at the time of balance.

All these Z's are complex impedances and are comprise of, in general, resistances, inductances and capacitances. Writting each Z in the form $R + jX$,

we have $(R_1 + jX_1)(R_4 + jX_4) = (R_2 + jX_2)(R_3 + jX_3)$

$$R_1R_4 - X_1X_4 + j(X_1R_4 + X_4R_1) = R_2R_3 - X_2X_3 + j(X_2R_3 + X_3R_2).$$

Equating reals and unreals on each side, we get

$$R_1R_4 - X_1X_4 = R_2R_3 - X_2X_3 \text{ or } R_1R_4 = R_2R_3 \text{ and } X_1X_4 = X_2X_3, \quad ...(111)$$

and
$$X_4R_1 + X_1R_4 = X_3R_2 + X_2R_3 \text{ or } \frac{R_1}{X_1} + \frac{R_4}{X_4} = \frac{R_2}{X_2} + \frac{R_3}{X_3}$$

or
$$\phi_1 + \phi_4 = \phi_2 + \phi_3. \quad ...(112)$$

These are two conditions to be satisfied in order to obtain a continuous balance. Except in special circumstances, all ac bridges involve a double balance. The former is usually a balance of the resistive components of the bridge and could be made with a dc source and moving coil galvanometer if the bridge arms contained no capacitors. If the balance conditions obtained are independent of the frequency of the supply, then the use of a supply with nonsinusoidal wave form does not introduce serious errors as the harmonic components will be balanced out as well as the fundamental. However a single frequency is desirable because the value of the electrical components involved may be frequency conscious.

17.19 MEASUREMENT OF INDUCTANCE

(*a*) **Maxwell's Impedance Bridge :** It is a simple type of ac bridge for the comparison of an inductance with a variable standard inductance. The ratio arms R_3 and R_4 are non-inductive resistances. The variable standard self inductance of constant resistance R_2 is L_2. L_1 is the unknown inductance with resistance R_1 and R is an adjustable noninductive resistance which may be placed in series with this inductance, as shown in Fig. 17.30, if required, by changing the contact lead from A to a. Between terminals AC and BD are the source (ac or dc) and the detector (head-phone or galvanometer) respectively.

At the time of balance, using Eq. (110), we have

$$\frac{R_1 + j\omega L_1}{R_2 + j\omega L_2} = \frac{R_3}{R_4}$$

or
$$R_1R_4 + j\omega L_1R_4 = R_2R_3 + j\omega L_2R_3. \quad ...(113)$$

This equation is complex and is really equivalent to two equations. Thus the bridge may be balanced when the sum of all the horizontal components, *i.e.*, the real terms is zero and the sum of all the vertical components, *i.e.*, the imaginary terms is zero. Hence

$$R_1R_4 - R_2R_3 = 0 \text{ or } R_1/R_2 = R_3/R_4, \quad ...(114)$$

and
$$j\omega(L_1R_4 - L_2R_3) = 0 \text{ or } L_1 = L_2R_3/R_4. \quad ...(115)$$

Therefore a perfect balance implies two things – *that the ohmic resistances are balanced and that the inductances are in the ratio of the resistances in corresponding bridge arms.* The first balance is obtained as in the ordinary Wheatstone bridge by connecting a small variable resistance R is series with the inductance of large value.

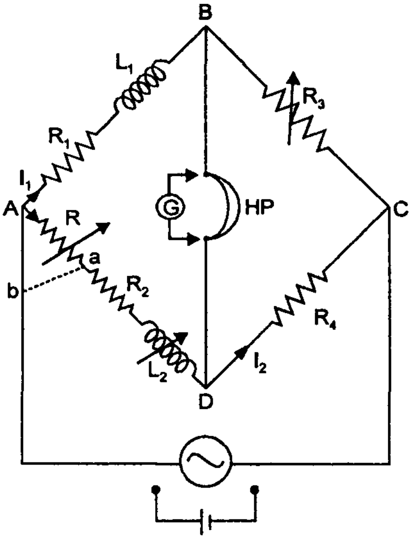


Fig. 17.30. Maxwell's *L*-bridge.

The bridge suffers from the *disadvantage* that the balance conditions cannot be satisfied independently of one another. If L_2 is constant, the two conditions of balance are semi-independent as a change in R_3 affects a change in R. If L_2 is variable and $R_3 \approx R_4$, the conditions are again semi-independent as a change in L_2 affects a change in R. Conditions can be made independent of each other only if R is kept constant and L_2 and R_3 are varying.

L_1 and L_2 are usually kept away to avoid mutual inductance M. Let us examine the effect of M between coils L_1 and L_2. If $\mathbf{I}_1$ and $\mathbf{I}_2$ are the current phasors for the arms ABC and ADC of the balanced bridge, then we have

$$(R_1 + j\omega L_1 + j\omega M)\,\mathbf{I}_1 = (R_2 + j\omega L_2 + j\omega M)\,\mathbf{I}_2.$$

and $$R_3\mathbf{I}_1 = R_4\mathbf{I}_2.$$

Eliminating $\mathbf{I}_1$ and $\mathbf{I}_2$, we have

$$R_1R_4 + j\omega R_4\,(L_1 + M) = R_2R_3 + j\omega R_3\,(L_2 + M).$$

Then the required conditions are :

$$R_1R_4 = R_2R_3 \text{ or } R_1/R_2 = R_3/R_4. \qquad \text{...(116)}$$

and $$R_4(L_1 + M) = R_3(L_2 + M) \text{ or } (L_1 + M)/(L_2 + M) = R_3/R_4. \qquad \text{...(117)}$$

Combining these equations, we get

$$\frac{L_1 + M}{L_2 + M} = \frac{R_3}{R_4} = \frac{R_1}{R_2}. \qquad \text{...(118)}$$

If $R_3 = R_4$ and L_2 is varied to L_1, then the balance point is unaffected by mutual inductance M.

In a properly constructed bridge, the resistance coils should be free from inductance and capacitance. In practice in all *ac* bridges residual reactances in R_3 or R_4 are always present. To avoid this effect we adjust $L_1/L_2 = R_1/R_2 = R_3/R_4 = 1$.

If the order of magnitude of the inductance is entirely unknown, one may begin with $R_3 = R_4$, and obtain balance by varying R. The resistance R may be transferred to the other side of the bridge, if required. Now *dc* source is replaced by *ac* source and moving coil galvanometer is replaced by the headphone or vibration galvanometer. The balance position is obtained by adjusting variable inductance L_2. On account of the limited range of the inductance L_2, the chances are of not obtaining of exact balance. The bridge is then rebalanced for direct and then for alternating currents. A good balance may be obtained after two or three trials. If for $R_3 = R_4$, the sound is apparently the same, a large change in the ratio is required, say for example $R_3/R_4 = 10$. If it does not give results, $R_3/R_4 = 1/10$ may be tried. A final attempt to obtain a perfect balance may be made by altering R and L_2 slightly.

Eqs. (114) and (115) show that the conditions are free from ω. However the balance conditions are affected by variation of ω due to *eddy current and skin effects. Self capacitances of* R_1 *and* R_2 *are effective specially at higher* ω. *The earth capacitance, effective at high frequencies, is avoided by shielding all the branches from one another.*

(*b*) Maxwell's L/C bridge : It is one of the earliest forms of bridge for the measurement of self inductance. With its aid a self inductance is measured in terms of a capacitance and two resistances.

The arm DC contains the coil under test in series with a variable non-inductive resistance box R_4. The arms AB, BC and AD are non-inductive resistances, or more exactly resistances of very low residual inductance. A standard variable capacitor C is placed in parallel with the resistance R_1, opposite to the arm containing L, as shown in Fig. 17.31.

The bridge is balanced, when

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_3}{\mathbf{Z}_4} \text{ or } \frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{R_3}{R_4 + j\omega L}.$$

Here R_4 includes the resistance of the inductance L.

$$\frac{1}{\mathbf{Z}_1} = \frac{1}{R_1} + j\omega C \text{ or } \mathbf{Z}_1 = \frac{R_1}{1 + j\omega CR_1}.$$

$$\therefore \quad \frac{R_1}{R_2(1 + j\omega CR_1)} = \frac{R_3}{R_4 + j\omega L}$$

or $\quad R_1R_4 + j\omega LR_1 = R_2R_3 + j\omega CR_1R_2R_3.$

Equating real and imaginary parts, we have

$$R_1R_4 = R_2R_3 \text{ or } R_1/R_2 = R_3/R_4, \quad \text{...(119)}$$

and $\quad j\omega LR_1 = j\omega\, CR_1R_2R_3 \quad \text{or} \quad L = CR_2R_3.$ ...(120)

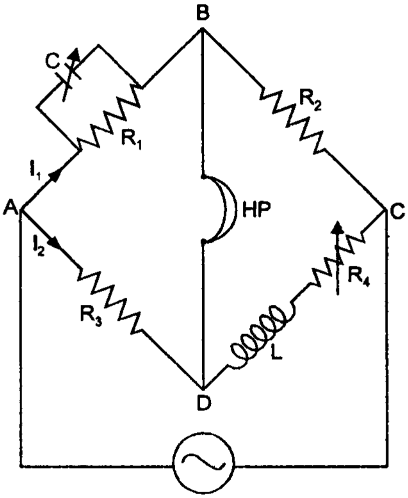


Fig. 17.31. Maxwell's L/C Bridge.

Thus we see that balance in this bridge is independent of the frequency. This bridge does not permit the two conditions of balance to be attained independently if the two variable quantities are resistances. These two conditions of balance are completely independent if a variable capacitor is used. Good mica capacitors arranged in decade dials over the range 0 – 1. 11 μF, to which a variable air capacitor of capacitance 0.001 μF may be added in parallel, are of practical importance for the continuous variation of capacitance.

In practice R_1/R_2 is best kept as a simple ratio, whilst R_4 is varied first to obtain in phase minimum, then C is varied to obtain a quadrature minimum. Two or three alternate variations of R_4 and C should give a good balance. The variable resistance in series with L should include decade subdivided ohm for the fine adjustment, preferably a small carbon plate rheostat may be used as the value of R_4 is not required for the calculation of L. *The coil L will always have some self capacitance so it will not be possible to obtain a perfect balance.*

In practice C is always imperfect, it is accompanied by either a small resistance r in series or a high resistance R in parallel. Assuming r in series, the balance conditions are given by

$$\mathbf{Z}_1/R_2 = R_3/(R_4 + j\omega L).$$

Here $\mathbf{Z}_1$ is given by

$$\frac{1}{\mathbf{Z}_1} = \frac{1}{R_1} + \frac{j\omega C}{1 + j\omega Cr} = \frac{1 + j\omega Cr + j\omega CR_1}{R_1(1 + j\omega Cr)}$$

$$\therefore \quad \frac{R_1(1 + j\omega Cr)/[1 + j\omega C(R_1 + r)]}{R_2} = \frac{R_3}{R_4 + j\omega L}$$

or $\quad R_1R_4 + j\omega LR_1 + j\omega CrR_1R_4 - \omega^2 CrR_1 L = R_2R_3 + j\omega CR_2R_3(R_1 + r).$

Equating real and imaginary parts, we get

$$R_1R_4 - \omega^2 CrR_1L = R_2R_3$$

and $\quad LR_1 + CrR_1R_4 = CR_2R_3(R_1 + r)$

$$\therefore \quad \frac{R_4 - \omega^2 rLC}{R_3} = \frac{R_2}{R_1} \text{ and } \frac{L}{R_3} + \frac{rCR_4}{R_3} = CR_2 + \frac{rCR_2}{R_1}$$

Eliminating R_2/R_1, we get

$$L = CR_2R_3/(1 + \omega^2C^2r^2). \qquad ...(121)$$

This expression shows that the correction term involves the power factor of the capacitor (ωCr). For a high class mica capacitor the power factor is less than 10^{-4}, thus the correction is less (1 in 10^8) and may be taken to be zero.

A perfect balance is not possible due to the presence of harmonics in the source, since even the formulae at balance are independent of frequency. Where the highest accuracy is required, the effects of stray capacitance must be eliminated by the use of the Wagner earth, to be discussed latter.

(c) Owen's Bridge : This bridge is used to measure self inductance in terms of a standard capacitance C_2 and two resistances R_2 and R_3. The bridge is arranged as shown in Fig. 17.32. Assuming that the capacitors are without losses and that all the resistances are perfectly non-reactive, the balance conditions are given by

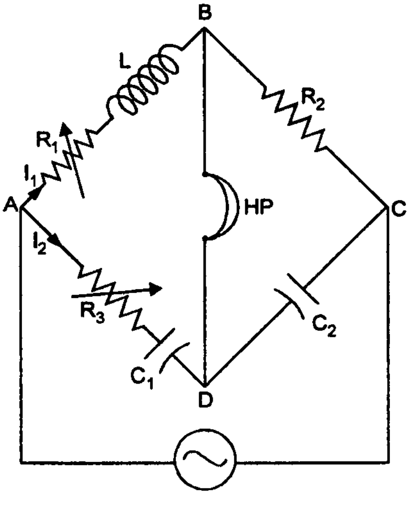


Fig. 17.32. Owen's Bridge.

$$\frac{R_1 + j\omega L}{R_2} = \frac{R_3 + 1/j\omega C_1}{1/j\omega C_2},$$

or

$$R_2R_3 + R_2/j\omega C_1 = L/C_2 + R_1/j\omega C_2.$$

where R_1 includes coil resistance R.

Equating real and imaginary parts, we get

$$L = C_2R_2R_3, \qquad ...(122)$$

and

$$C_1/C_2 = R_2/R_1 \qquad ...(123)$$

Both the conditions of balance are independent of each other and of the frequency. It can be shown that before the actual balance, one component of the current through the detector depends upon ω (varies inversely as ω) and the other does not. It is true also for all bridges in which final balance is independent of frequency.

Phasor diagram of the bridge : The voltage **V** across the bridge is represented by phasor **EF** (Fig. 17.33). Let $\mathbf{I}_1$ and $\mathbf{I}_2$ be the current phasors for the arms *ABC* and *ADC* respectively. When the bridge is balanced $\mathbf{I}_1$ will lag behind **V** through ϕ_1 (say) and $\mathbf{I}_2$ will lead through ϕ_2 (say). These ϕ_1 and ϕ_2 are given by the relations

$$\tan\phi_1 = \frac{\omega L}{R_1 + R_2} \text{ and } \tan\phi_2 = \frac{1}{\omega C R_3},$$

where

$$1/C = 1/C_1 + 1/C_2.$$

$\therefore$

$$\tan\phi_1 \tan\phi_2 = \frac{L}{(R_1 + R_2)R_3C} = \frac{L}{R_3(R_1 + R_2)}\left[\frac{1}{C_1} + \frac{1}{C_2}\right]$$

$$= \frac{L}{R_3(R_1 + R_2)}\left[\frac{R_1}{R_2C_2} + \frac{1}{C_2}\right] = \frac{L}{R_2R_3C_2} = 1$$

or

$$\phi_1 + \phi_2 = \pi/2. \qquad ...(124)$$

If $\mathbf{EG} = R_3\mathbf{I}_2$ and **GH** is normal to **EG** and of length $\mathbf{I}_2/j\omega C_1$, then **E H** gives the potential drop across arm *AD* of the bridge. This must be equal to the potential drop across the arm *AB* of the bridge, *i.e.*, **EI** + **IH**, where **EI** = $R_1\mathbf{I}_1$ and **IH** = $j\omega L\mathbf{I}_1$. **IH** is normal to **EI**. We have also proved that $\phi_1 + \phi_2 = \pi/2$, **EI** is normal to **EG** or the currents $\mathbf{I}_1$ and $\mathbf{I}_2$, are in quadrature. Now **HF** is $R_2\mathbf{I}_1$ and since it is also $\mathbf{I}_2/j\omega C_2$, it must be normal to $\mathbf{I}_2$ Thus *GHF* is a straight line.

Condition (122) may be secured by varying R_3 and condition (123) either by varying R_1 or by varying C_1, the former is preferable. This bridge is extremely convenient and possesses the great practical advantage as the balance is obtained by adjusting resistances only. As it consists only one inductance, under test, all mutual reactance between inductances are avoided.

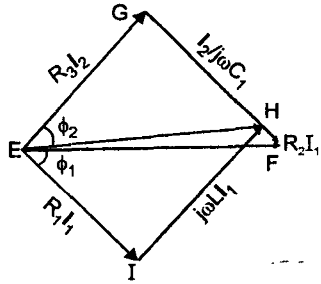


Fig. 17.33. Phasor Diagram.

Owen has shown that even with limited apparatus the range of this bridge is very wide. Thus with $C_1 = C_2 = 0.3\ \mu F$ and values of R_2 from 1 to 200Ω, self inductances from $2\mu H$ to $0.5H$ can be measured accurately. The residuals due to the inductance of leads and terminals may be easily eliminated by obtaining a balance first with the inductance L and then with a short circuit across L. If R_3 and R_3' are the respective values of the R_3 for these two balances, then we have to a close approximation

$$L = C_2R_2\,(R_3 - R_3'). \qquad \text{...(125)}$$

This second balance is necessary for large values of L and can be avoided for low values of L.

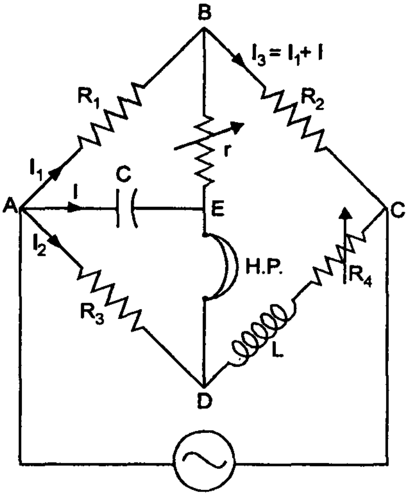


Fig. 17.34. Anderson's Bridge.

(*d*) Anderson's Bridge : This is the modified form of Maxwell's bridge. The double balance is obtained by adjusting resistances only, the standard capacitor being of a fixed value. The modification consists in placing a non-inductive resistance r in series with the capacitor, the combination being in parallel with R_1, the other components and the detector are placed as shown in Fig. 17.34.

At the time of balance the potential at E is equal to the potential at D. Applying Kirchhoff's laws we have for mesh *ABEA*

$$R_1\mathbf{I}_1 - (r + 1/j\omega C)\,\mathbf{I} = 0, \qquad \text{...(126)}$$

for mesh *AEDA*

$$(1/j\omega C)\,\mathbf{I} - R_3\mathbf{I}_2 = 0, \qquad \text{...(127)}$$

and for mesh *BCDB*

$$R_2\,(\mathbf{I}_1 + \mathbf{I}) - (j\omega L + R_4)\,\mathbf{I}_2 + r\mathbf{I} = 0 \qquad \text{...(128)}$$

Substituting values of $\mathbf{I}_1$ and $\mathbf{I}_2$ from Eqs. (126) and (127) in Eq. (128), we have

$$\frac{R_2}{R_1}\left(r+\frac{1}{j\omega C}\right)\mathbf{I}+R_2\mathbf{I}-\frac{(j\omega L+R_4)}{j\omega CR_3}\mathbf{I}+r\mathbf{I} = 0. \qquad \text{...(129)}$$

Equating real and imaginary parts, we get

$$\frac{R_2}{R_1}r+R_2-\frac{L}{CR_3}+r = 0$$

or

$$L = CR_3\,[r\,(1 + R_2/R_1) + R_2] \qquad \text{...(130)}$$

and $$\frac{R_2}{R_1\omega C}-\frac{R_4}{R_3\omega C} = 0, \text{ or } \frac{R_1}{R_2}=\frac{R_3}{R_4}. \qquad ...(131)$$

These conditions are independent of the frequency of the source and can be satisfied independently of each other by adjusting R_4 (the variable resistance in series with the inductance coil L) in the latter case and r in the former case. Eq. (130) shows that the ac balance is possible only when $L > CR_2R_3$, otherwise, r will be negative. Since there is no other inductance, all mutual reaction between neighbouring inductances is avoided.

It is best to use $R_1 : R_2 = 1 : 1$. The formulae then reduce to

$$R_3 = R_4 \text{ and } L = CR_3\,[2r + R_2]. \qquad ...(132)$$

Phasor Diagram. The Phasor diagram of Anderson's bridge is shown in Fig. 17.35. In it **EF** is **V**, the potential phasor for the applied source and is also equal to the potential difference between terminals A and C. If $\mathbf{I}_2$ represents the current in ADC, then $\mathbf{EG} = R_3\mathbf{I}_2$, $\mathbf{GH} = j\omega L\,\mathbf{I}_2$ and $\mathbf{HF} = R_4\mathbf{I}_2$, so that **HF** is || to **EG** and **GH** is ⊥ to **EG** or **HF**. Since the potential of point E is the same as that of D. hence $\mathbf{V}_{AE} = \mathbf{V}_{AD}$ or $(1/j\omega C)\,\mathbf{I} = R_3\mathbf{I}_2$. This relation shows that **I** is in quadrature with $\mathbf{I}_2$. If we draw $\mathbf{GI} = r\mathbf{I}$ then the vector **EI** is $R_1\mathbf{I}_1$. The vector **IF** is $R_2\mathbf{I}_3 = R_2\,(\mathbf{I}_1 + \mathbf{I})$. The directions of $\mathbf{I}_1$ and $\mathbf{I}_3$ are thus determined from this diagram.

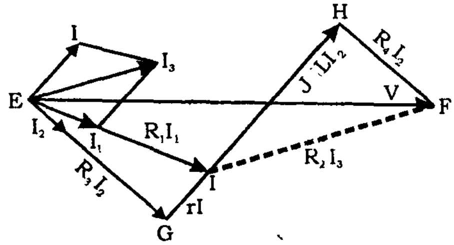


Fig. 17.35. Phasor Diagram.

First of all the circuit is arranged as a simple Wheatsone's bridge connecting the inductance in the unknown arm CD. The resistances R_1 and R_2 are fixed to 10 ohm each and R_3 is varied until the galvanometer deflection changes direction. The expected value is calculated with the formula (131).

The experiment is repeated with $R_1/R_2 = 10$ and 100 and the actual resistance in the inductance arm R_L is calculated. Now R_1 and R_2, are re-adjusted to the ratio 10 : 10 and a fractional resistance R' is placed in the series with the inductance. Now R_3 and R' are varied for no deflection in galvanometer. In this case $R_L + R' = R_4 = R_3$. To increase the sensitivity resistances are so arranged that $R_1 = R_2 = \frac{1}{2}R_3$.

Now galvanometer and dc source are replaced by headphones and ac source (oscillator) respectively and capacitance C and resistor r (0 to 100 $k\Omega$) are inserted as shown in Fig. 17.34. R_1, R_2, R_3 and R_4 are kept as such and the bridge is balanced for minimum sound with the help of resistance r. The experiment may be repeated by taking different capacitances. Substituting these values in Eq. (132), self inductance of a given coil can be calculated.

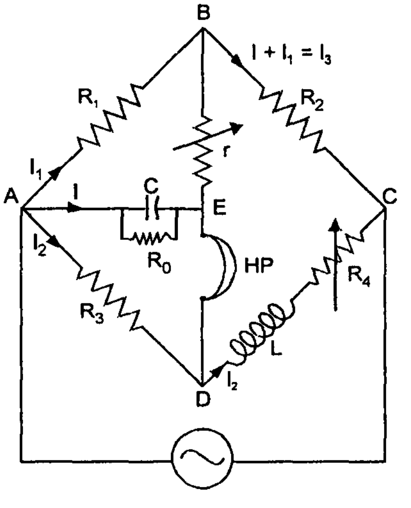


Fig. 17.36. Anderson's bridge with imperfect capacitor.

Relative Mertis of Anderson's and Owen's bridges when capacitor is slightly imperfect. To examine the effect of an imperfect capacitor, let us assume that such a defect may be represented by a high resistance R_0 in parallel with capacitor C. When the bridge is balanced let the current vectors be as indicated in Fig. 17.36. Applying Kirchhoff's laws, for the meshes *ABEA, AEDA* and *BCDB*, we have

$$R_1\mathbf{I}_1-\left[r+\frac{R_0}{1+j\omega CR_0}\right]\mathbf{I} = 0, \qquad ...(133)$$

$$\frac{R_0}{1+j\omega CR_0}\mathbf{I}-R_3\,\mathbf{I}_2 = 0 \qquad ...(134)$$

and $$R_2\,(\mathbf{I}+\mathbf{I}_1)-j\omega L\mathbf{I}_2-R_4\mathbf{I}_2+r\mathbf{I} = 0. \qquad ...(135)$$

Substituting values of $\mathbf{I}_1$ and $\mathbf{I}_2$ from Eq. (133) and (134) respectively in Eq. (135) and equating the real and imaginary parts, we get

$$R_1R_3+rR_3+\frac{R_3R_1}{R_2}r+R_0R_3 = \frac{R_0R_1R_4}{R_2} \qquad ...(136)$$

and $$L = CR_3\,[r\,(1+R_2/R_1)+R_2]\ . \qquad ...(137)$$

Thus we see that imperfection of capacitor has no effect on the measured value of the self inductance L and has a small effect on the measured value of R_4.

To examine the effect of an imperfect capacitor in Owen's bridge, let us assume that the effect may be represented by a small series resistance R_4. The conditions for balance are given by

$$\frac{R_1+j\omega L}{R_2} = \frac{R_3+1/j\omega C_1}{R_4+1/j\omega C_2}.$$

Solving this relation and equating real and imaginary parts, we have

$$R_1R_4+L/C_2 = R_2R_3 \qquad ...(138)$$

and $$\omega LR_4-R_1/\omega C_2+R_2/\omega C_1 = 0 \qquad ...(139)$$

Thus both conditions for balance are effected by the imperfectness. The second condition involves ω, showing that the source must be free from harmonics. This shows that, *Omen's bridge is inferior to the Anderson's bridge.*

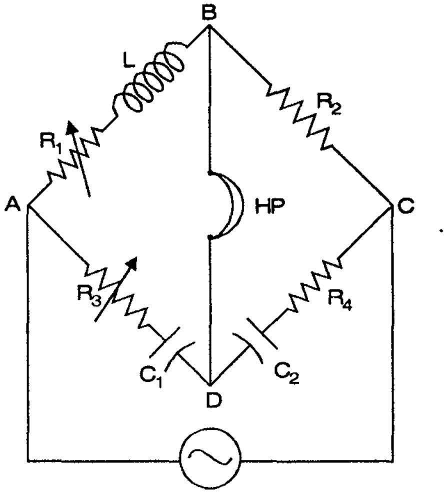


Fig. 17.37. Owen's bridge with imperfect capacitor.

17.20 MEASUREMENT OF MUTUAL INDUCTANCE

(*a*) Modified Carey Foster Method : The Carey Foster bridge was devised to determine the mutual inductance between two coils in terms of a standard capacitance and two known resistances. A variable resistance is included in series with the standard capacitor and one arm is short circuited in Heydweiller's modification of the original Carey Foster bridge. It is undesirable to classify this circuit as a bridge, but it is the usual practice.

When R_2 is zero the resistance R_1 is obviously connected directly across the supply (neglecting the primary of M). The primary of the mutual inductance is connected so that the voltage induced by it in arm CD neutralizes the drop due to the current $\mathbf{I}_2$ in this branch. When $R_2 = 0$, the drop across the arm CD must be zero for the balance position. The balance is obtained by varying R_3 and R_4.

As it is not of the Wheatstone bridge type, hence the general relation $\mathbf{Z}_1/\mathbf{Z}_2 = \mathbf{Z}_3/\mathbf{Z}_4$ can not be used. Using Kirchhoff's laws we thus have

$$\mathbf{I} = \mathbf{I}_1 + \mathbf{I}_2, \text{(first law)} \qquad ...(140)$$

$$R_1\mathbf{I}_1 - (R_3 + 1/j\omega C)\,\mathbf{I}_2 = 0 \qquad \text{(for mesh } ABDA) \qquad ...(141)$$

and $$(R_4 + j\omega L)\,\mathbf{I}_2 + j\omega M\,(\mathbf{I}_1 + \mathbf{I}_2) = 0. \qquad \text{(for mesh } BCDB) \qquad ...(142)$$

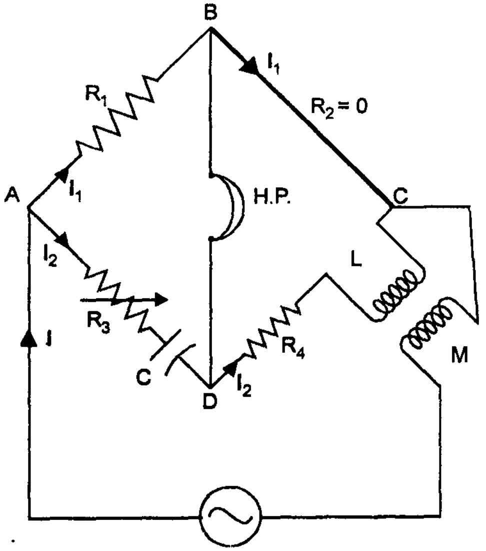


Fig. 17.38. Modified Carey Foster's bridge.

Substituting value of $\mathbf{I}_1$ from Eq. (141) in Eq. (142), we get

$$R_4\mathbf{I}_2 + j\omega L\,\mathbf{I}_2 + j\omega\frac{M}{R_1}\left(R_3 + \frac{1}{j\omega C}\right)\mathbf{I}_2 + j\omega M\,\mathbf{I}_2 = 0.$$

Equating real and imaginary parts we have

$$R_4 + M/CR_1 = 0 \text{ or } M = -\,CR_1R_4 \qquad ...(143)$$

and $$L + MR_3/R_1 + M = 0 \text{ or } L = -M(R_3/R_1 + 1). \qquad ...(144)$$

These equations represent the conditions of balance. These conditions require that M shall be negative. Thus we have

$$M = CR_1R_4 \text{ and } L = M\,(R_3/R_1 + 1). \qquad ...(145)$$

These balance conditions are independent of one another and frequency of the source can be obtained by varying R_4 and R_3 and keeping R_1 constant. Second condition shows that $L > M$. The balance cannot be obtained if the leads of the secondary or the primary are wrapped in the wrong way.

If the capacitor is imperfect, $R_3 = R + r$, where R is the known resistance and r is the contribution of imperfectness, then we have

$$L = M\,[1 + (R + r)/R_1] \text{ or } R = R_1\,[L/M - 1] - r. \qquad ...(146)$$

The power factor for the capacitor ωCr can be calculated by plotting R against R_1. Since R_4 includes the resistance of the coil (secondary of the mutual inductance), we can write $R_4 = R' + r'$, where R' is the known resistance. Then

$$M = CR_1\,(R' + r') \text{ or } R' = M/CR_1 - r'. \qquad ...(147)$$

A graph between R' and $1/R_1$ gives us the value of M, if C is known. The value of r' can be calculated by the intersection.

In the derivation of balance conditions we have assumed $R_2 = 0$. If the arm BC has a small but finite resistance ΔR, then the balance conditions, obtained by using Kirchhoff's laws, are

$$R_3\,\Delta R + M/C = R_1R_4 \qquad ...(148)$$

and $$L = M\,[1 + R_3/R_1] - \Delta R/\omega^2 CR_1. \qquad ...(149)$$

These equations show that the balance is independent of frequency only if $\Delta R = 0$.

(b) Maxwell's Method : This method is used for the comparison of two unequal mutual inductances. One M_2 is a standard variable and other is measured in terms of it. L_1 and L_2 are the self inductances of their secondary windings. R_1 and R_2 are the total resistances of two branches. To obtain exact balance a variable self inductance L is inserted in either branch so that it is in series with L_1 or L_2.

At the time of a balance, if $\mathbf{I}$ is the current in the source circuit and $\mathbf{I}'$ in the loop $ABCDA$, and inductance L is in series with L_1, then

$$R_1\mathbf{I}' + j\omega(L_1 + L)\mathbf{I}' + j\omega M_1\mathbf{I} = 0. \quad \text{(for mesh } ABDA\text{)} \quad ...(150)$$

and $R_2\mathbf{I}' + j\omega L_2\mathbf{I}' + j\omega M_2\mathbf{I} = 0.$ (for mesh $BCDB$) ...(151)

Substituting value of $\mathbf{I}$ from Eq. (150) to Eq. (151),we get

$$(R_2 + j\omega L_2)\mathbf{I}' - \frac{M_2}{M_1}[R_1 + j\omega(L_1 + L)]\mathbf{I}' = 0. \quad ...(152)$$

Equating real and imaginary parts we have

$$R_2 - M_2R_1/M_1 = 0 \text{ or } M_1/M_2 = R_1/R_2$$

and $L_1 + L - M_1L_2/M_2 = 0$ or $M_1/M_2 = (L_1 + L)/L_2$.

$$\therefore \quad M_1/M_2 = R_1/R_2 = (L_1 + L)/L_2. \quad ...(153)$$

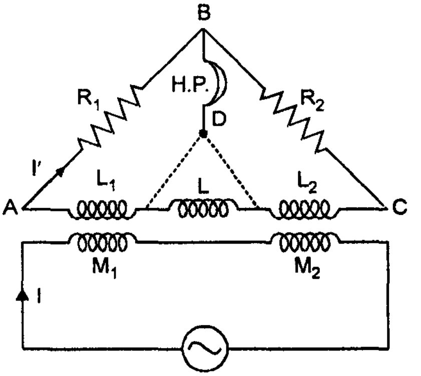


Fig. 17.39. Maxwell's Method for M_1/M_2.

15.21 MEASUREMENT OF CAPACITANCE

The methods for determination of self and mutual inductances in terms of standard capacitance and resistance may equally well be applied conversely to the measurement of capacitance if the standards of inductance are available. Other few methods are :

(a) De Sauty's Bridge : This method is the simplest way of comparing two capacitances. In this bridge two capacitors whose capacitances are to be compared (or one unknown and other of known capacitance) are connected as shown in Fig. 17.40(a). Audio frequency oscillator and headphone are connected in the gaps AC and BD respectively.

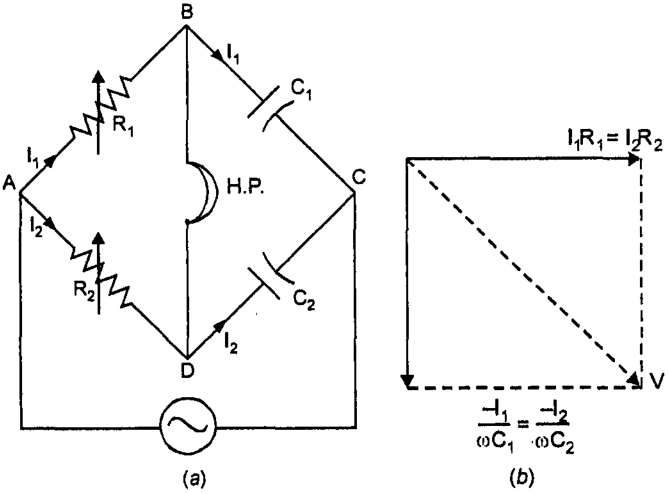
If the power factors of the capacitors are neglected, then at the balance, we have

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_3}{\mathbf{Z}_4},$$

$$\therefore \quad \frac{R_1}{R_2} = \frac{1/j\omega C_1}{1/j\omega C_2}.$$

Thus $R_1/R_2 = C_2/C_1$

or $\quad C_1 = C_2R_2/R_1. \quad ...(154)$



(a) (b)

Fig. 17.40. De Sauty's bridge.

A comparison of capacitances is therefore made in terms of variable, *non inductive resistances* R_1 and R_2. The balance is also independent of frequency. For maximum sensitivity C_2 should be equal to C_1. The perfect balance can be obtained only in the case of air capacitor.

In practice the comparison of ordinary capacitor is not so simple, as the capacitors are not both free from dielectric losses. As the behaviour of a capacitor with an imperfect dielectric depends on the frequency, it is important that the correct periodicity be employed. Imperfect capacitor may be regarded as made up of a capacitance associated with a resistance, either a small resistance in a series or a large resistance in parallel. To obtain an accurate balance, it is necessary to balance out the associated resistance.

(*b*) **Wien's Bridge :** It is a direct modification of De Sauty's bridge. In it C_1 and C_2, the capacitors to be compared, are assumed as perfect capacitors in series with small hypothetical resistances r_1 and r_2 respectively, where r_1 and r_2 are the loss components of the capacitors. R_3 and R_4 are the resistances required to obtain exact balance. All the resistances in the bridge are supposed to be non-inductive. At the time of balance.

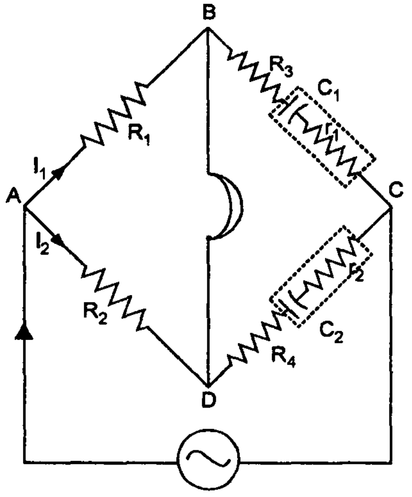


Fig. 17.41. Wien's Bridge.

$$\mathbf{Z}_1/\mathbf{Z}_2 = \mathbf{Z}_3/\mathbf{Z}_4.$$

$$\frac{R_1}{R_2} = \frac{R_3+r_1+1/j\omega C_1}{R_4+r_2+1/j\omega C_2},$$

or $$R_1(R_4+r_2+1/j\omega C_2) = R_2(R_3+r_1+1/j\omega C_1).$$

Equating real and imaginary parts, we have

$$R_1/R_2 = (R_3+r_1)/(R_4+r_2) \qquad ...(155)$$

and $$R_1/R_2 = C_2/C_1. \qquad ...(156)$$

These balance conditions may be satisfied, though not independently of another. The balance arms R_1 and R_2 are variable and R_3 and R_4 are the adjustable resistances used to bring the potential differences between *BC* and *CD* into phase. Two resistances (R_3 and R_4) are included because it may not be known at the start which of the capacitors has the lower power factor. Only one of the resistances can be used with R_3 and R_4 both zero, the radio R_1/R_2 is adjusted until the indication of the detector is a minimum. The balance is then improved by adjusting R_3 or R_4 as the case may be and still further improved by adjusting R_1/R_2 and so on. Thus a perfect balance is obtained by successive adjustments.

(*c*) **Schering Bridge :** Schering bridge is the most accurate bridge method for the determination of (small) capacitances of standard capacitors. It is also used for the measurement of permittivities, *i.e.* dielectric constants. *It is now most widely used for measuring dielectric loss and power factor and in the testing of cables and insulators at high voltages.*

In its simplest form the bridge is shown in Fig. 17.42. The capacitor under test C_1 is placed in the arm *AB* of the bridge. *R* is an imaginary resistance representing its dielectric loss component. A standard capacitor C_2 is placed in the arm *AD*. The remaining arms contain non-inductive resistances R_2 and R_3. In parallel with R_3 a continuously variable calibrated air capacitor C_3 is connected. Such an air capacitor will have negligible power factor. The impedances of arms *AB* and *AD* are usually very high and that of arms *BC* and *CD* are usually small compared with those of the former. Due to this reason the detector (vibration galvanometer) and the resistances are at potential only a few volts even when a high voltage supply (~ 10^5 volts) is used.

At the time of balance, we have

$$\mathbf{Z}_1/\mathbf{Z}_2 = \mathbf{Z}_3/\mathbf{Z}_4.$$

Here $$\frac{1}{\mathbf{Z}_1} = \frac{1}{R}+\frac{1}{1/j\omega C_1} \text{ or } \mathbf{Z}_1 = \frac{R}{1+j\omega C_1 R}$$

$$\mathbf{Z}_2 = R_2,\ \mathbf{Z}_3 = 1/j\omega C_2 \text{ and } \mathbf{Z}_4 = R_3/(1+j\omega C_3 R_3).$$

$$\therefore\quad \frac{R}{R_2(1+j\omega C_1 R)} = \frac{1/j\omega C_2}{R_3/(1+j\omega C_3 R_3)}$$

$$\text{or}\quad \frac{R(1-j\omega C_1 R)}{R_2(1+\omega^2 C_1^2 R^2)} = -j\frac{(1+j\omega C_3 R_3)}{\omega C_2 R_3} = \frac{\omega C_3 R_3 - j}{\omega C_2 R_3}.$$

Equating real terms, we get

$$R/(1+\omega^2 C_1^2 R^2) = C_3 R_2/C_2. \qquad \text{...(157)}$$

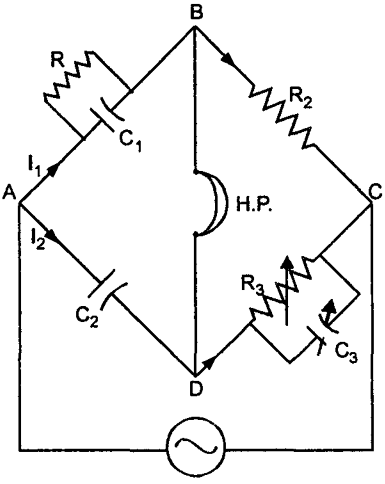


Fig. 17.42. Schering Bridge.

Fig. 17.43 represents phasor diagram, **OC** is the potential phasor applied to the bridge from the supply. **OA** = $\mathbf{V}_1$, the pd across *AB* (or *AD*) and phasor **OB** = $\mathbf{V}_2$, the pd across *BC* (or *DC*). The current phasor $\mathbf{I}_1$ flowing in arms *AB* and *BC* is represented by **OH**, while **OF** represents the current phasor $\mathbf{I}_2$ flowing in *AD* and *DC*. **OG** and **OJ** represent the component parts of current $\mathbf{I}_1$ through C_1 and R respectively. Similarly **OD** and **OE** represent the components of current $\mathbf{I}_2$ through R_3 and C_3 respectively. It is clear from figure that δ, the angle between current and potential phasors through capacitor C_1 is given by

$$\cos\delta = \frac{V_1\omega C_1}{V_1\sqrt{(1/R^2+\omega^2 C_1^2)}} = \frac{\omega C_1 R}{\sqrt{(1+\omega^2 C_1^2 R^2)}} \qquad \text{...(158)}$$

Thus Eq. (157) reduces to

$$\cos^2\delta/\omega^2 C_1^2 R = C_3 R_2/C_2$$

$$\text{or}\quad C_1 = C_2\cos^2\delta/\omega^2 C_1 C_3 R R_2.$$

From triangles *GOH* and *EFO*, we have

$$\tan\delta = \frac{V_1/R}{V_1\omega C_1} = \frac{V_2\omega C_3}{V_2/R_3},$$

hence $1/\omega C_1 R = \omega R_3 C_3$ or $R_3 = 1/\omega^2 C_1 C_3 R$.

Thus the expression for C_1 reduces to

$$C_1 = C_2 R_3\cos^2\delta/R_2 = C_2 R_3/R_2, \text{ since } \delta \text{ is small.} \qquad \text{...(159)}$$

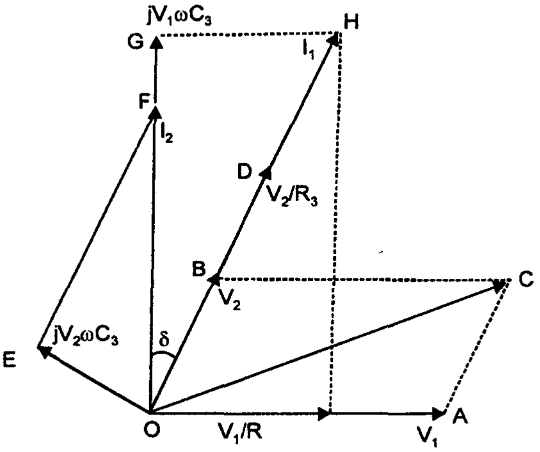


Fig. 17.43. Phasor Diagram.

To obtain balance, R_2, C_2 and C_3 are kept fixed and R_3 is adjusted for minimum sound. If the sound is not minimized C_3 is also adjusted first. By doing alternate adjustments of C_3 and R_3 a perfect balance is obtained.

The power factor of C_1 is cos δ, we may write it as

$$\cos\delta = \omega C_1 R. \qquad \text{...(160)}$$

17.22 MEASUREMENT OF FREQUENCY (LOW)

In *ac* bridge used for the measurement of inductance or capacitance the balance conditions are independent of frequency that is why the freedom of the source from harmonics is not an essential requirement. The situation is entirely different in the measurement of the source frequency as the balance is to hold only at one frequency. If harmonics are there the balance point is blurred when a telephone is used as a detector, a vibration galvanometer is also impracticable as this would have to be turned afresh to every new frequency.

Robinson's Frequency Bridge : It is the modified form of the Wein's capacitance bridge. It consists two high class standard capacitors C_1 and C_2, preferably of equal value, and four non-inductive resistors R_1, R_2, R_3 and R_4. R_1 is in parallel with C_1, R_3 in series with $C_2$and R_2 and R_4 constitute the third and fourth arms of the bridge.

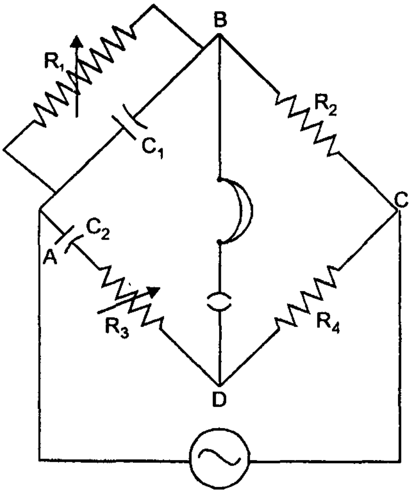


Fig. 17.44. Robinson's frequency bridge.

$$\frac{1}{\mathbf{Z}_1} = \frac{1}{R_1}+\frac{1}{1/j\omega C_1} \text{ or } \mathbf{Z}_1 = \frac{R_1}{1+j\omega C_1R_1}$$

$$\mathbf{Z}_3 = R_3+\frac{1}{j\omega C_2}=\frac{1+j\omega C_2R_3}{j\omega C_2}$$

At balance, we have

$$\frac{R_1}{(1+j\omega C_1R_1)R_2} = \frac{1+j\omega C_2R_3}{j\omega C_2R_4}$$

$$\text{or} \quad j\omega C_2R_1R_4 = (1+j\omega C_2R_3)(1+j\omega C_1R_1)R_2.$$

Equating in turn the real and imaginary parts, we get

$$R_3/R_1 + C_1/C_2 = R_4/R_2, \qquad \ldots(161)$$

$$\text{and} \quad 1-\omega^2 C_1C_2R_1R_3 = 0 \quad \text{or } \omega^2 = 1/C_1C_2R_1R_3. \qquad \ldots(162)$$

Robinson simplified these conditions for balance by making $C_1 = C_2 = C$ (*say*) and $R_1 = R_3 = R$ (*say*). Now above conditions become

$$R_4 = 2R_2 \qquad \ldots(163)$$

$$\text{and} \quad \omega = 1/CR \text{ or frequency } f = 1/2\pi CR. \qquad \ldots(164)$$

Thus the bridge can be used for the measurement of frequency by adjusting

$R_1 = R_3$, $R_4 = 2R_2$ and $C_1 = C_2$.

For C constant, $f \propto 1/R$ (or $\propto$ conductance), thus the scale of the instrument becomes linear when resistance boxes are replaced by conductance boxes. This bridge has the great merit that there is no inductive coupling in any part of the bridge.

Any harmonic present in the input voltage will appear in the detector arm at balance as the conditions for balance consist ω. A sensitive ear can find out balance position with the fundamental even in the presence of harmonics.

17.23 WAGNER EARTHING DEVICE

At high frequencies stray capacitances arise error in the results obtained with the bridges previously described. Let us suppose that $\mathbf{Z}_1$, $\mathbf{Z}_2$, $\mathbf{Z}_3$ and $\mathbf{Z}_4$ comprise the four impedances forming an *ac* bridge and one of them is on test. $\mathbf{Z}_5$ and $\mathbf{Z}_6$ are the two variable impedances of the Wagner earth branch connected across the *ac* supply, the central point of which is earthed. The detector is connected by a two-way key either to B or to B', the common terminals of impedances $\mathbf{Z}_3$ and $\mathbf{Z}_4$, and $\mathbf{Z}_5$ and $\mathbf{Z}_6$ respectively. The stray capacitances are represented by the broken lines in the figure. They will connect points *A, B, C* and *D* to earth and current will flow from the bridge to earth. The flow of current from the points *A* and *C* to earth can not affect the balance of the bridge but the currents through $\mathbf{Z}_1$ and $\mathbf{Z}_2$ (or $\mathbf{Z}_3$ and $\mathbf{Z}_4$) cannot be equal if currents flow from the points *B* and *D* to earth. If the point *B* is earthed, the currents through $\mathbf{Z}_3$ and $\mathbf{Z}_4$ are unequal even when the detector current is zero. It means that we do not have natural balance. Wagner overcome this difficulty by bringing points *B* and *D* at earth potential without connecting actually to earth. It was done by using subsidiary bridges *AB′ CD*.

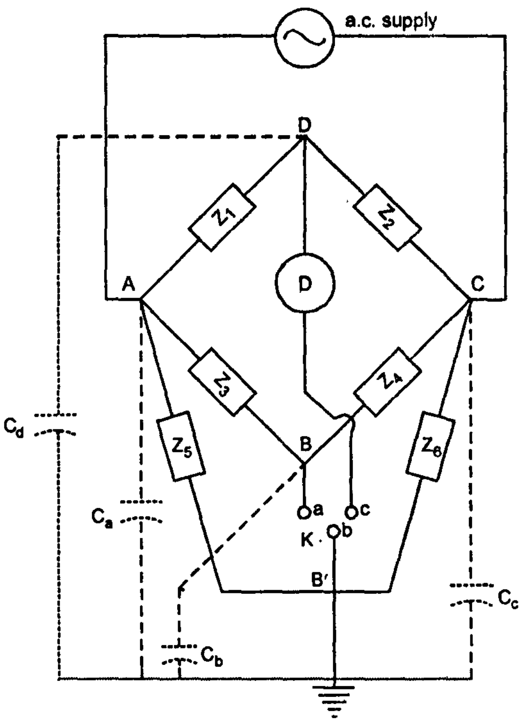


Fig. 17.45. Wagner earthing device.

In Wagner device, the main bridge *ABCD* is first balanced by adjusting $\mathbf{Z}_3$ or $\mathbf{Z}_4$ or both. As we obtain best obtainable balance, the detector is connected to earthed point *B′* and the balance is obtained by adjusting $\mathbf{Z}_5$ and $\mathbf{Z}_6$. As *B′* is at earth potential, at balance, point *D* is at earth potential but because of the current to earth at *B′* it cannot be said that $\mathbf{Z}_1/\mathbf{Z}_2 = \mathbf{Z}_5/\mathbf{Z}_6$. The main bridge is balanced again by adjusting slightly $\mathbf{Z}_3$ and $\mathbf{Z}_4$. These processes are repeated till perfect balance is obtained. At this time detector current is zero either we connect the detector to *B* or to *B′*. Thus points *D*, *B* and *B′* are at the same potential. In this way points *B* and *D* are brought at earth potential without connecting to earth actually. The stray capacitances C_b and C_d have no effect on the balance conditions and the relation $\mathbf{Z}_1/\mathbf{Z}_2 = \mathbf{Z}_3/\mathbf{Z}_4$ is satisfied.

17.24 MEASUREMENT OF DIELECTRIC CONSTANT

Wagner earth device is needed when the impedance to be found is high (*e.g.*, small capacitance). The dielectric constant can be measured by measuring capacitance of a capacitor, first with air as dielectric medium and the second with the substance under investigation as dielectric medium. The latter is naturally greater than the previous one. The ratio of these capacitances gives the dielectric constant. Since such capacitors have a high reactance, hence Wagner earth device should be used in that bridge, used for the capacitance measurement. A simple bridge used for this purpose is a de-Sauty bridge with a Wagner earth (Fig. 17.46).

C is a variable wireless receiving capacitor of maximum capacitance 500 *pF*. It is carried below a polystyrene plate supported on the upper edge of a cylindrical glass jar of ample size. On the top of the plate there are two terminals connected to the fixed and moving vanes respectively. This plate is placed in the arm *AD* of the bridge. C_1 and C_2 are two variable air capacitors of the maximum value 0.0005 μ*F* (500 *pF*). R_1, R_2 and *R* are non inductive resistance boxes. The source is an oscillator supplying about 15 volts at a frequency of 1200 *c/s*.The detector is a pair of earphones. Starting with air as medium, alternate balances for the main bridge and then for the Wagner earth circuit are found by varying R_1 and R_2. The vessel (jar) is filled with the liquid under test. The balances are again found alternately by adjusting R_1 and R_2 as done before. Let R' be the new value of R, then

$$CR = kCR' \text{ or } k = R/R'. \qquad ...(165)$$

The error is 5%, if the Wagner earth in not used.

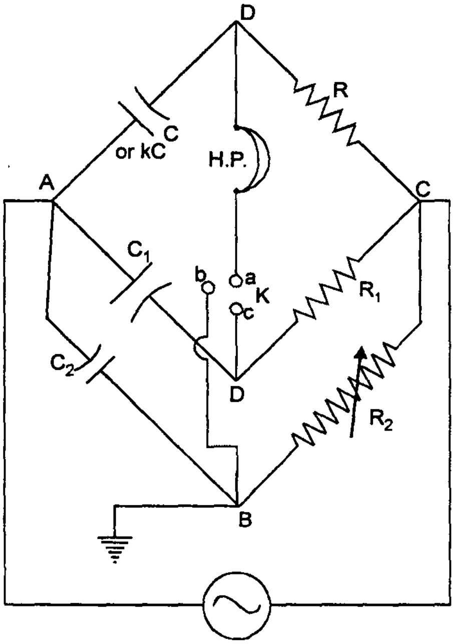


Fig. 17.46. De-Sauty's bridge with a Wagner earth.

17.25 MEASUREMENT OF SPECIFIC CONDUCTIVITY

Kohlraush's bridge : The specific conductivity of the electrolyte is determined by using Kohlraush's bridge, the modified form of the Wheatstone's bridge in which the effect of polarization is reduced by using an *ac* source. the back emf is reduced by using electrodes of large area.

The non-inductive resistances R_1 and R_2 are connected in the ratio arms. R_3 and a capacitor C are connected in parallel and thus form the resistance arm. The voltameter is connected in the unknown arm. It acts as a parallel combination of a resistance R_x and a capacitor of capacitance C_x. A supply in the range from 10 to 20 V at frequencies upto 1 kc/sec and high impedance telephones (used as a detector) are connected in the source arm AC and detector arm BD respectively (Fig. 17.47).

At the balance condition

$$\frac{\mathbf{Z}_1}{\mathbf{Z}_2} = \frac{\mathbf{Z}_3}{\mathbf{Z}_4}.$$

Here $\mathbf{Z}_1 = R_1, \mathbf{Z}_2 = R_2$

$$\mathbf{Z}_3 = \frac{R_3}{1+j\omega CR_3}, \quad \mathbf{Z}_4 = \frac{R_x}{1+j\omega C_x R_x}$$

$$\therefore \quad \frac{R_1}{R_2} = \frac{R_3(1+j\omega C_x R_x)}{R_x(1+j\omega CR_3)}$$

or $R_1 R_x + j\omega CR_1 R_3 R_x = R_2 R_3 + j\omega C_x R_x R_2 R_3$,

Equating real and imaginary parts, we get

$$R_1 R_x = R_2 R_3 \text{ or } R_x = R_2 R_3/R_1. \qquad ...(166)$$

and $\omega CR_1 R_3 R_x = \omega C_x R_x R_2 R_3$

$$\text{or} \quad C_x = CR_1/R_2. \qquad ...(167)$$

The R_3 and C are adjusted alternatively for minimum sound in the detector. With the help of above equations, the inherent capacitance C_x is measured.

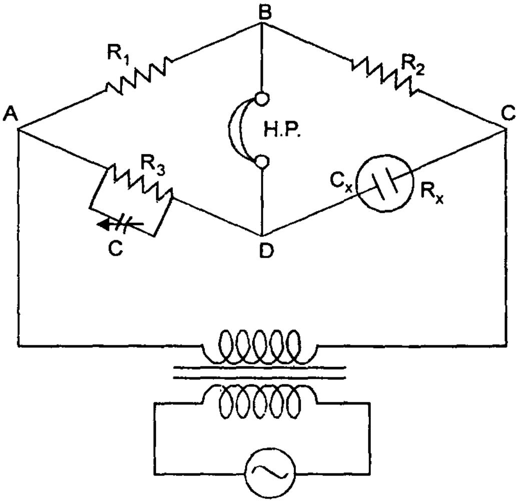


Fig. 17.47. Kohlraush's Bridge.

Kohlrausch from his experimental results suggested that the dissociation theory is only tenable for the *weak electrolytes, e.g.*, ammonia and acetic acid and most of the organic acids and salts. For *strong electrolytes, e.g.*, inorganic salts and acids, degree of dissociation for a solution α is practically unity for all concentrations. The variation of Λ, the equivalent conductivity with concentration is mainly due to variation of the ionic mobilities with the concentration. Experimental results show that the equivalent conductivity decreases at high concentrations and becomes practically constant at low concentrations. Debye and Huckel gave a theory applicable to strong electrolytes. The readers are advised to read it from **Electricity and Magnetism**, vol I, by *J.H. Fewkes* and *John Yarwood*.

EXERCISES

Example 1. *A circular coil, having 100 turns and of radius 25 cm rotates uniformly 10 times per sec about a vertical diameter in a place where the angle of dip is 60° and the resultant field strength is 30 amp / meter. Assuming self inductance of the coil as negligible, find (a) the maximum emf in volts, (b) the maximum current in amperes and (c) the rate of production of heat per second in the coil of resistance 10 Ω*

The vertical and horizontal components of magnetic field are 30 sin 60° and 30 cos 60° respectively. The vertical component is always || to the vertical coil, hence no emf is induced in the coil as the resultant flux through the coil is zero. The horizontal component gives the induced emf only. The flux associated with the coil at an angle θ is given by

$$\Phi = BAn \cos\theta = \mu_0 nAH \cos\theta.$$

$\therefore$ Induced emf $\mathcal{E} = -d\Phi/dt = \mu_0 nAH \sin\theta\ (d\theta/dt)$.

Hence $\mathcal{E}$ will have maximum value V_0, when $\sin\theta = 1$.

$$\therefore \quad V_0 = \mu_0 nAH\,(d\theta/dt) = \mu_0 nA\,(30 \cos 60°)\,(d\theta/dt)$$
$$= 4\pi \times 10^{-7} \times 100 \times \pi \times (0.25)^2 \times 30 \times (1/2) \times 20\pi$$
$$= 2.325 \times 10^{-2} \text{ volt.}$$

The maximum current $\quad I_0 = \dfrac{V_0}{R} = \dfrac{2.325\times10^{-2}}{10} = 2.325\times10^{-3}$ amp.

$\therefore$ Rate of production of heat $= I_{rms}{}^2 R = \frac{1}{2} I_0{}^2 R = 2.70 \times 10^{-5}$ joule/sec.

Example 2. *A coil of self inductance 0.7 henry is joined in series with a non-inductive resistance of 50 Ω. Calculate the wattless and power components of current when connected to a supply of 200 volts at a frequency of 50 cycles / sec.*

Impedance $\quad Z = \sqrt{(R^2 + \omega^2 L^2)}$

$$= \sqrt{[50^2 + (2\pi \times 50 \times 0.7)^2]}$$
$$= 225.5 \text{ ohms.}$$

$\therefore$ Current $\quad I = \dfrac{V}{Z} = \dfrac{200}{225.5} = 0.887$ amp.

and $\quad \tan\phi = \dfrac{\omega L}{R} = \dfrac{2\pi\times50\times0.7}{50} = 4.4$

or $\quad \phi = 77° 12'$.

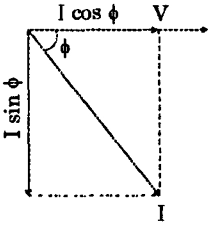


Fig. 17.48. Example 2.

The current lags behind the *ac* emf by an angle ϕ. Thus the current phasor **I** will make angle ϕ in clockwise direction with the emf phasor **V**. The component $I \cos\phi$ in phase with **V** is responsible for the power dissipation and is known as the power component of **I**. The component $I \sin \phi$, which is 90° out of phase with **V**, does not contribute to the power and is known as *wattless current.*

$\therefore$ Power component of current$= I \cos \phi = 0.887 \times \cos 77° 12' = 0.197$ amp.

and wattless component of the current $= I \sin \phi = 0.887 \times \sin 77° 12' = 0.865$ amp.

Example 3. *Show that when a coil of inductance L and resistance R is attached to two terminals at which an emf $v = V_0 \sin \omega t$ is maintained, the average rate of consumption of energy is $\frac{1}{2} V_0^2 R/(R^2 + \omega^2L^2)$.*

This problem is same as given in article 17.6. The current flowing through the circuit is given by

$$i = \frac{V_0}{\sqrt{(R^2+\omega^2L^2)}} \sin(\omega t - \phi). \quad \text{where } \phi = \tan^{-1}\frac{\omega L}{R}.$$

As the energy will only be dissipated in the circuit resistance hence the average rate of consumption of energy

$$U_{av} = \frac{\int_0^{T=2\pi/\omega} i^2 R\,dt}{\int_0^{T=2\pi/\omega} dt} = \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \frac{V_0^2 R}{R^2+\omega^2L^2} \sin^2(\omega t - \phi)\,dt$$

$$= \frac{V_0^2 R\omega}{2\pi(R^2+\omega^2L^2)} \frac{1}{2}\int_0^{2\pi/\omega} [1-\cos 2(\omega t-\phi)]\,dt.$$

$$= \frac{V_0^2 R\omega}{4\pi(R^2+\omega^2L^2)}\left[t - \frac{\sin 2(\omega t-\phi)}{2\omega}\right]_0^{2\pi/\omega} = \frac{V_0^2 R}{2(R^2+\omega^2L^2)}.$$

Example 4. *An ac arc lamp needing 150 volts is put across mains at a pd. of 200 volts and a chocking coil in series with the lamp. The lamp takes 10 amperes and the frequency is 50 per second. Find (i) the pressure produced by the chocking coil reaction and (ii) the self inductance of the chocking.*

Total emf applied $\mathbf{V}$ = *pd* $\mathbf{V}_1$ across the lamp + *pd* $\mathbf{V}_2$ across choke coil.

$$\therefore \quad V = \sqrt{(V_1^2 + V_2^2)}$$

$$\text{or} \quad V_2 = \sqrt{(V^2 - V_1^2)} = \sqrt{(200^2 - 150)^2} = 132.3 \text{ volts.}$$

Current will be same in the circuit, hence the pd across the choke of self inductance L is

$$V_2 = \omega LI = 2\pi f LI$$

$$\text{or} \quad L = \frac{V_2}{2\pi f I} = \frac{132.3}{2\times 3.14\times 50\times 10} = 0.042 \text{ henry.}$$

Example 5. *In an oscillating LC circuit, calculate the charge in terms of maximum charge on the capacitor when the whole energy is shared between electric field and magnetic field. Assuming L = 10 mH and C = 1.0 μF, find the time required for this condition.*

Let q_m be the maximum charge on the capacitor.

$\therefore$ Maximum energy associated with the capacitor $= q_m^2/2C$. If q be the charge on the capacitor, when half of the energy is converted into electric energy linked with the inductor, then

$$q^2/2C = \tfrac{1}{2}q_m{}^2/2C \text{ or } q=q_m/\sqrt{2}.$$

To find time, assuming $\phi = 0$,

$$q = q_m \cos \omega t = q_m/\sqrt{2}.$$

$$\therefore \quad \cos \omega t = 1/\sqrt{2}, \quad \omega t = 45° = \pi/4 \text{ or } t = \pi/4\omega.$$

As

$$\omega = \frac{1}{\sqrt{(LC)}} = \frac{1}{\sqrt{(10\times10^{-3}\times10^{-6})}} = 10^4 \text{ radians/sec.}$$

$$\therefore \quad t = \frac{\pi}{4\times10^4} = 7.85\times10^{-5} \text{ sec.}$$

Example 6. *A 20 volts 5 watt lamp is to be used on a.c. mains of 200 volts 50 cycles/sec. Calculate the (i) capacitor, (ii) inductor to be put in series to run the lamp. How much pure resistance should be included in place of the above devices so that the lamp can run on its rated voltage. Which of the above arrangements will be more economical and why ?*

The current required by the lamp $= \dfrac{\text{Power}}{\text{Voltage}} = \dfrac{5}{20} = 0.25$ amp.

∴ The resistance of the lamp = Voltage/current = 20/0.25 = 80 ohms.

If the lamp is to run at 200 volts, 50 cycles/sec. a.c. mains, a capacitor (or choke) must be placed in series with the lamp in order to increase its effective resistance so that the current through it is 0.25 amp only.

(*i*) Let C be the capacitance of the required capacitor, then the impedance of the circuit becomes

$$Z = \sqrt{(R^2+X_c{}^2)} = \sqrt{(R^2+1/\omega^2C^2)} = \sqrt{[80^2+1/(2\pi\times50)^2C^2]}.$$

The current in the circuit must be 0.25 amp, hence

$$I = V/Z = 200/\sqrt{(80^2+1/100^2\pi^2C^2)} = 0.25$$

$$\text{or} \quad 80^2 + 1/(100\pi)^2 C^2 = 800^2$$

$$\therefore \quad C = 4.0\ \mu F.$$

(*ii*) Let L be the inductance of the choke placed in series with the lamp to set the current at 0.25 amp. then the impedance of the circuit becomes

$$Z = \sqrt{(R^2+X_L{}^2)} = \sqrt{(R^2+\omega^2L^2)} = \sqrt{[80^2+(2\pi\times50)^2L^2]}$$

The current in the circuit is

$$I = V/Z = 200/\sqrt{(80^2+100^2\pi^2L^2)} = 0.25$$

$$\text{or} \quad 80^2 + 100^2\pi^2L^2 = 800^2$$

$$\therefore \quad L = 2.53 \text{ henry.}$$

(*iii*) Let r be the resistance placed in series with the lamp in place of a choke, then the current in the circuit

$$I = V/(R+r) = 200/(80+r) = 0.25$$

$$\text{or} \quad r = 720 \text{ ohms.}$$

(*iv*) In the last case, *i.e.*, the insertion of a resistance of 720 ohm to run the lamp, the dissipation of energy is maximum, while in the case of a choke in series the power consumption is practically zero ($R << X_L$ or X_C). Hence it will be more economical to use inductor or capacitor in series with the lamp to run it.

Example 7. *A 200-ohm resistor and a 1 μ F capacitor are connected in series across a 120 volts (r.m.s.), 60 cycles / sec. line. (a) What is the impedance ? (b) What is the r.m.s. value of the circuit. (c) What is the power dissipated in the circuit ? (d) What will be the reading of an a.c. voltmeter-connected across the resistor and across the capacitor ?*

(*a*) The capacitive reactance $X_C = 1/\omega C = 1/2\pi fC$

$\therefore \quad X_C = 1/2 \times 3.14 \times 60 \times 1 \times 10^{-6} = 2652.6$ ohm.

When a capacitor C is connected in series with a resistance R, the impedance of the circuit becomes

$$Z = \sqrt{R^2 + X_C{}^2} = \sqrt{2000^2 + 2652.6^2} = 3322.1 \text{ ohms.}$$

(*b*) The r.m.s. value of the circuit is given by

$$I_{rms} = \frac{V_{rms}}{Z} = \frac{120}{3322.1} = 0.036 \text{ amp.}$$

(*c*) In the circuit containing resistor and capacitor, the phase lag is given by ϕ and the power factor

$$\cos \phi = R/Z = 2000/3322.1 = 0.602.$$

Therefore the power dissipated in the circuit

$$P = V_{rms} I_{rms} \cos \phi = 120 \times 0.036 \times 0.602 = 2.6 \text{ watts.}$$

(*d*) The potential differences across the resistor and the capacitor are given by

$$V_R = I_{rms} \times R = 0.036 \times 2000 = 72 \text{ volts}$$

$$V_C = I_{rms} \times X_C = 0.036 \times 2652.6 = 95.5 \text{ volts.}$$

Thus the a.c. voltmeter will read 72 and 95.5 volts, when connected across the resistor and capacitor respectively.

Example 8. *A series resonant circuit has L = 1.01 H, C = 20 μF and R = 70 ohms to which an emf of 100 V. (r.m.s.) is applied. At which frequency of the applied emf will it resonate with maximum response.*

In a *LCR* series resonance circuit, the impedance of the circuit is minimum and is equal to the resistance of the circuit, *i.e.*,

$$Z = R = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$$

i.e., $\quad \omega L = 1/\omega C$ or $\omega^2 = 1/LC$.

Thus the frequency of the applied emf for the maximum response

$$f = \omega/2\pi = 1/2\pi\sqrt{LC}$$

$$= 1/2 \times 3.14\,(1.01 \times 20 \times 10^{-6})^{1/2} = 35.41\ c/sec.$$

Example 9. *If in the LCR-series circuit a.c. voltmeter reads 30 watts across the resistance, 80 volts across the inductance and 40 volts across the capacitor, what is the applied emf ?*

The voltmeter reads the r.m.s. value of the voltage. The total applied emf may be obtained by a vector method. The voltage across the inductance $\mathbf{V}_L$ leads by 90° on $\mathbf{V}_R$ and across the capacitor $\mathbf{V}_C$ lags behind by 90°. Therefore the resultant voltage is the phasor sum of $\mathbf{V}_R$ and $(\mathbf{V}_L \sim \mathbf{V}_C)$ which are mutually perpendicular to each other.

$$\therefore \quad V_{rms}{}^2 = V_R{}^2 + (V_L - V_C)^2 = 30^2 + (80 - 40)^2 = 900 + 1600$$

or $$V_{rms} = 50 \text{ volts.}$$

Another method is to use the relations

$$V_R = RI_{rms},\ V_L = \omega L I_{rms},\ V_C = I_{rms}/\omega C.$$

As
$$\begin{aligned} V_{rms} &= I_{rms}Z = I_{rms}\sqrt{R^2+(\omega L-1/\omega C)^2} = \sqrt{(I_{rms}R)^2+(I_{rms}\omega L-I_{rms}/\omega C)^2} \\ &= \sqrt{V_R{}^2+(V_L-V_C)^2} = \sqrt{30^2+(80-40)^2} \\ &= 50 \text{ volts.} \end{aligned}$$

Example 10. *A coil of resistance 60 ohm and inductance 3 henry is connected in series with a capacitor of 4 μF and an a.c. supply of 200 volts and 50 c/s. Calculate (i) the impedance in the circuit. (ii) the phase difference between current and voltage and (iii) the potential difference across the inductor and capacitor.*

(*i*)The impedance of the *LCR*-series-circuit is given by

$$Z = \sqrt{R^2+(X_L-X_C)^2},$$

Given
$$\begin{aligned} R &= 60 \text{ ohms}, X_L = \omega L = 2\pi fL = 2 \times 3.14 \times 50 \times 3 = 942 \text{ ohms} \\ X_C &= 1/\omega C = 1/2\pi fC = 1/2 \times 3.14 \times 50 \times 4 \times 10^{-6} \\ &= 796.2 \text{ ohms.} \end{aligned}$$

$$\therefore \quad Z = \sqrt{60^2+(942-796.2)^2} = 157.7 \text{ ohms.}$$

(*ii*) The phase difference between current and voltage is given by

$$\begin{aligned} \phi &= \tan^{-1}\frac{X_L-X_C}{R} = \tan^{-1}\frac{942-796.2}{60} = \tan^{-1} 2.43. \\ &= 67.6°. \end{aligned}$$

(*iii*) The r.m.s. value of current passing through the circuit

$$I_{rms} = V_{rms}/Z = 200/157.7 = 1.27 \text{ amp.}$$

∴The potential difference across the inductor $V_L = I_{rms}X_L = 1.27 \times 942 = 1194.7$ volts and across the capacitor $V_C = I_{rms}X_C = 1.27 \times 796.2 = 1009.8$ volts.

Example 11. *In an a.c. circuit an inductance of 1 H, a capacitor of 0.2 μF and a resistance of 3000 ohms are connected in series to a 10 volt a.c. source of 1000/ 2π cycles/sec. Find the peak current in the circuit and power consumed.*

In a series *LCR* circuit, the peak current is given by

$$I_0 = \frac{V_0}{\sqrt{R^2+(\omega L-1/\omega C)^2}}$$

In this problem $R = 3000\Omega$, $\omega = 2\pi f = 1000$, $L = 1\ H$, $C = 0.2 \times 10^{-6}\ F$ and $V_{rms} = 10$ volts. Therefore $V_0 = \sqrt{2}V_{rms} = 10\sqrt{2}$. Substituting these values in the above relation, we get

$$\begin{aligned} I_0 &= 10\sqrt{2}/\sqrt{3000^2+(1000\times1-1/1000\times0.2\times10^{-6})^2} \\ &= 2.83 \text{ milliamperes.} \end{aligned}$$

$$\therefore \quad I_{rms} = I_0/\sqrt{2} = 2.83\times10^{-3}/\sqrt{2} = 2\times10^{-3} \text{ amp.}$$

The power factor is given by

$$\cos\phi = R/\sqrt{R^2+(\omega L-1/\omega C)^2}$$

$$= 3000/\sqrt{3000^2+(1000-1/1000\times0.2\times10^{-6})^2} = 0.6$$

$\therefore$ Power consumed in the circuit

$$P = I_{rms}V_{rms}\cos\phi = 2\times10^{-3}\times10\times0.6 = 1.2\times10^{-2}\text{ watt.}$$

Example 12. *A 60 cycles a.c. circuit has a resistance of 2 ohms and inductance of 10 millihenries. What is the power factor ? What capacitance placed in the circuit will make the power factor unity ?*

Power factor of a LR-circuit is given by

$$\cos\phi = R/\sqrt{R^2+(\omega L)^2}$$

In this problem, given that $R = 2$ ohms, $L = 10\times10^{-3}$ henry and $\omega = 2\pi f = 2\times3.14\times60$

$$\therefore \quad \cos\phi = 2/\sqrt{2^2+(2\times3.14\times60\times10^{-2})^2} = 0.469.$$

When the capacitor is placed in the circuit in series with the resistor and inductor, the power factor

$$\cos\phi = R/\sqrt{R^2+(\omega L-1/\omega C)^2}$$

It will be unity, if

$$R = Z = \sqrt{R^2+(\omega L-1/\omega C)^2}$$

$$\omega L - 1/\omega C = 0 \quad\text{or}\quad C = 1/\omega^2 L = 1/4\pi^2 f^2 L.$$

Substitution of numerical values gives

$$C = 1/4\times(3.14)^2\times60^2\times10^{-2} = 703.6\ \mu F.$$

Example 13. *The alternating emf of rms value 100 volts and 50 cycles is applied to a circuit containing the inductor of 5 henries and the capacitor of 2 microfarads and resistor of 1000 ohms in series. What rms current flows? What would be the effect of (a) doubling and (b) halving the frequency?*

The maximum value of emf = $V_0 = V_{rms}\sqrt{2} = 100\sqrt{2}$

The current in LCR circuit is given as

$$i = \frac{V_0}{\sqrt{[R^2+(\omega L-1/\omega C)^2]}}\sin(\omega t-\phi),$$

where $\phi = \tan^{-1}(\omega L - 1/\omega C)/R,$

$\therefore$ Maximum value of current $I_0 = V_0/\sqrt{[R^2+(\omega L-1/\omega C)^2]}$

$$= \frac{100\sqrt{2}}{\sqrt{[1000^2+(2\pi\times50\times5-1/2\pi\times50\times2\times10^{-6})^2]}}$$

$$= 0.09998\sqrt{2} = 0.1414\text{ amp.}$$

or *rms* value of current $= I_{rms} = I_0/\sqrt{2} = 0.09998$ amp.

This current is nearly same as due to the resistance only as the bracket term is very small. It will lag behind the applied emf by a phase

$$\phi = \tan^{-1}\frac{\omega L - 1/\omega C}{R} = \tan^{-1}\frac{2\pi \times 50 \times 5 - 1/2\pi \times 50 \times 2 \times 10^{-6}}{1000}$$
$$= \tan^{-1}(-0.02) = -0.02 \text{ radian } (-\text{ve for lag})$$

(because the angle is very small hence tan $\phi = \phi$ in radian)

$\therefore$ $$\text{Time advance} = \frac{0.02}{2\pi}\frac{1}{50} = 6.4 \times 10^{-5} \text{ sec.}$$

Rest part of the example is left as an exercise.

Example 14. *Find the voltage across the various elements, i.e., resistor, capacitor and inductor, which are in series and having values 1000Ω, 1 μF and 2 henry respectively.*

Given emf as $v = (100/\sqrt{2}) \sin 1000\, t$ volts.

The rms value of voltage across the source = $100\sqrt{2}/\sqrt{2}$ = 100 volts and $\omega = 2\pi f = 1000 \text{ sec}^{-1}$.

$\therefore$ $$I_{rms} = V_{rms}/Z = V_{rms}/\sqrt{[R^2 + (\omega L - 1/\omega C)^2]}$$
$$= \frac{100}{\sqrt{[1000^2 + (1000 \times 2 - 1/1000 \times 1 \times 10^{-6})^2]}} = 0.0707 \text{ amp.}$$

The current will be same everywhere in the circuit, therefore

PD across resistor = $V_R = I_{rms} R = 0.0707 \times 1000 = 70.7$ volts

PD across inductor = $V_L = I_{rms}\, \omega L = 0.0707 \times 1000 \times 2 = 141.4$ volts

PD across capacitor = $V_C = I_{rms}/\omega C = 0.0707 \times (1/1000 \times 10^{-6}) = 70.7$ volts.

The rms voltages do not add directly, as

$$V_R + V_L + V_C = 282.8 \text{ volts,}$$

which is not the source voltage 100 volts. The reason is that these voltages are not in phase and can be added by phasor algebra only.

Example 15. *A circuit consists of a non inductive resistor of 50Ω, a coil of a inductance 0.3 H and resistance 2Ω, and a capacitor of 40 μF in series and is supplied with 200 volts rms at 50 cycles/sec. Find the current lag or lead and the power in the circuit.*

Total resistance in the circuit = 50 + 2 = 52 ohms.

Inductive reactance $X_L = \omega L = 2 \times 3.14 \times 50 \times 0.3 = 94.2$ ohms.

Capacitive reactance $X_C = 1/\omega C = 1/2 \times 3.14 \times 50 \times 40 \times 10^{-6} = 79.6$ ohms.

$\therefore$ $$I_{rms} = \frac{V_{rms}}{\sqrt{[R^2 + (X_L - X_C)^2]}} = \frac{200}{\sqrt{[52^2 + (94.2 - 79.6)^2]}}$$
$$= 3.7 \text{ amp.}$$

As $X_L > X_C$, hence the current lags behind the applied emf by an angle ϕ, given by

$$\tan\phi = \frac{X_L - X_C}{R} = \frac{94.2 - 79.6}{52} = 0.28$$

or $$\phi = 15°5'.$$

$$\text{Power factor } \phi = \frac{R}{\sqrt{[R^2 + (X_L - X_C)^2]}} = \frac{52}{\sqrt{[52^2 + (14.6)^2]}} = 0.96.$$

∴ Apparent power = $I_{rms}\, V_{rms} = 3.7 \times 200 = 740$ watts,

and true power = $I_{rms}\, V_{rms} \cos\phi = 740 \times 0.96 = 710.4$ watts.

Same value is also obtained by the relation

$$\text{true power} = I_{rms}^{\,2} R = (3.7)^2 \times 52 = 711.9 \text{ watts.}$$

Example 16. *A coil of inductance 2mH and resistance 15W is connected in parallel with a capacitor of magnitude 0.001 μF. Find (a) the frequency at which the current from an a.c. supply is minimum, (b) the peak value of this make up current if $V_0 = 2$ volts and (c) the peak oscillatory current under these conditions.*

We know that in this type of circuit the current from the supply is minimum at the resonant frequency of the circuit. The frequency is given by

$$f = \frac{1}{2\pi}\sqrt{\left(\frac{1}{LC}-\frac{R^2}{L^2}\right)} = \frac{1}{2\times 3.14}\sqrt{\left(\frac{1}{2\times 10^{-3}\times 10^{-9}}-\frac{15^2}{(2\times 10^{-3})^2}\right)}$$

$$= 112.5 \text{ kilo cycles/sec.}$$

Make up current peak value at resonance

$$= \frac{\text{Peak value of emf}}{\text{Dynamic resistance}} = \frac{V_0}{L/CR} = \frac{V_0 CR}{L} = \frac{2\times 10^{-9}\times 15}{2\times 10^{-3}} = 15\times 10^{-6} \text{ amp.} = 15\ \mu\text{A}.$$

The peak oscillatory current at resonance

$$= V_0/(1/\omega C) = V_0\,\omega C = 2\times 2\pi \times 112.5\times 10^3 \times 10^{-9} = 1.41 \text{ mA.}$$

Example 17. *In an a.c. circuit given that $X_C = 200\ \Omega$, $R = 100\Omega$ and the applied voltage $V_{rms} = 100$ volts. If the inductive reactance X_L is varied over a wide range by varying L, find the condition for resonance and the condition for maximum choke drop.*

We know that the current in a *LCR*-series circuit is given by

$$I_{rms} = V_{rms}/\sqrt{[R^2+(X_L-X_C)^2]}$$

Thus the potential difference across the choke is given by

$$V_L = X_L\, I_{rms} = X_L\, V_{rms}/\sqrt{R^2+(X_L-X_C)^2}$$

The condition for resonance is

$$X_L = X_C = 200\ \Omega \text{ and } Z = R = 100\Omega$$

$$\therefore \qquad \text{current } I_{rms} = V_{rms}/R = 100/100 = 1 \text{ amp,}$$

and the potential difference across the choke

$$V_L = X_L\, I_{rms} = 200 \times 1 = 200 \text{ volts.}$$

The potential drop across the choke will be maximum, when

$$dV_L/dX_L = 0,\ i.e.,$$

$$\frac{[R^2+(X_L-X_C)^2]^{1/2} V_{rms} - X_L V_{rms}\,\tfrac{1}{2}[R^2+(X_L-X_C)^2]^{-1/2}\times 2(X_L-X_C)}{R^2+(X_L-X_C)^2} = 0$$

$$\text{or} \qquad R^2+(X_L-X_C)^2 = X_L\,(X_L-X_C)$$

$$\therefore \qquad X_L = (R^2+X_C^{\,2})/X_C.$$

In the present problem, $X_C = 200\Omega$ and $R = 100\Omega$,

$$\therefore \qquad X_L = (100^2 + 200^2)/200 = 250\Omega.$$

Hence the current in the circuit

$$I_{rms} = \frac{V_{rms}}{\sqrt{[R^2 + (X_L - X_C)^2}} = \frac{100}{[100^2 + (250-200)^2]^{1/2}} = 0.89 \text{ amp.}$$

and $$V_L = X_L I_{rms} = 250 \times 0.89 = 223.3 \text{ volts.}$$

Example 18. *A circuit containing resistor R_1, inductor L_1 and capacitor C_1, connected in series gives resonance at the same frequency as the second similar combination R_2, L_2 and C_2. If the two circuits are connected in series show that the whole circuit will resonate with the same frequency f.*

In the series-resonant circuit, the frequency is given by the relation $f = 1/2\pi\sqrt{LC}$. In this problem these two sets of LCR series circuit have the same frequency, hence

$$f = \frac{1}{2\pi\sqrt{L_1C_1}} = \frac{1}{2\pi\sqrt{L_2C_2}}$$

$$\therefore \qquad L_1C_1 = L_2C_2 \text{ or } L_1 = L_2C_2/C_1.$$

When these two circuits are connected in series, the total inductance $L = L_1 + L_2$ and the total capacitance C is given by

$$\frac{1}{C} = \frac{1}{C_1} + \frac{1}{C_2} \text{ or } C = \frac{C_1C_2}{C_1 + C_2}.$$

The resonant frequency of the whole (or combined) circuit is given by

$$f = \frac{1}{2\pi\sqrt{LC}} = \frac{1}{2\pi\sqrt{(L_1 + L_2)\,C_1C_2/(C_1 + C_2)}}$$

$$= \frac{1}{2\pi\sqrt{\left(\frac{L_2C_2}{C_1} + L_2\right)\frac{C_1C_2}{C_1 + C_2}}} = \frac{1}{2\pi\sqrt{L_2\left(\frac{C_1 + C_2}{C_1}\right)\frac{C_1C_2}{C_1 + C_2}}}$$

$$= \frac{1}{2\pi\sqrt{L_2C_2}} = f \text{ (frequency of each circuit).}$$

Example 19. *A capacitor of 250 pF is connected in parallel with a coil having inductance of 0. 16 mH and effective resistance 20Ω. Calculate (i) the resonance frequency and (ii) circuit impedance at resonance.*

The resonant frequency of the LCR-parallel circuit is given by

$$f = \frac{1}{2\pi}\sqrt{\frac{1}{LC} - \frac{R^2}{L^2}}.$$

In this problem, given $L = 0.16 \times 10^{-3}$ henry, $C = 250 \times 10^{-12}$ farad, and $R = 20$ ohms. Therefore

$$f = \frac{1}{2 \times 3.14}\sqrt{\frac{1}{0.16 \times 10^{-3} \times 250 \times 10^{-12}} - \frac{20^2}{(0.16 \times 10^{-3})^2}} = 7.9 \times 10^5 \text{ Hz.}$$

(*ii*) The circuit impedance at resonance is given by

$$Z = \frac{L}{CR} = \frac{0.16 \times 10^{-3}}{250 \times 10^{-12} \times 20} = 3.2 \times 10^4 \text{ ohms.}$$

Example 20. *An A.C. circuit consists of a 220 ohms resistance and a 0.7 henry choke. Find the power absorbed from 220 volt 50 cycle mains connected to this circuit if the resistance and choke are joined (i) in series and (ii) in parallel.*

(*i*) When the resistance R and choke of inductance L are connected in series, then the impedance Z of the a.c. circuit is given as

$$Z = \sqrt{R^2 + \omega^2 L^2} = \sqrt{R^2 + (2\pi f L)^2}$$

Substituting numerical values, we get

$$Z = \sqrt{(220)^2 + (2 \times 3.14 \times 50 \times 0.7)^2} = 310.99 \text{ ohm}$$

$$\therefore \quad I_{rms} = V_{rms}/Z = 220/310.99 = 0.707 \text{ amp.}$$

Therefore the power absorbed in the circuit

$$P = I_{rms} V_{rms} \cos\phi = I_{rms} V_{rms} (R/Z)$$
$$= 0.707 \times 220 \times (220/310.99) = 110.1 \text{ watts.}$$

(*ii*) When the resistance and choke are parallel, the entire power will be absorbed in the resistance, as the choke does not absorb any power.

$$\therefore \quad \text{Power absorbed } P = I_{rms}^2 R = V_{rms}^2/R$$
$$= 220^2/220 = 220 \text{ watts.}$$

Example 21. *An emf $v = V_0 \sin \omega t$ is applied to a circuit which consists of a self inductance L of neglible resistance in series with a variable capacitor C. The capacitor is shunted by a variable resistance R. Find the value of C for which the amplitude of the current is independent of R.*

To make the problem easy, let us make use of phasor algebra. The complex impedance of the circuit (Fig. 17.49) $\mathbf{Z} = j\omega L + \mathbf{Z}'$,

where $\mathbf{Z}'$ is due to C and R in parallel, given by

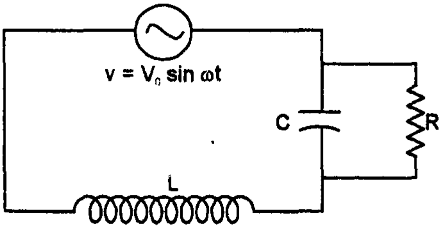


Fig. 17.49. Example 21.

$$\frac{1}{\mathbf{Z}'} = \frac{1}{R} + j\omega C = \frac{1 + j\omega CR}{R}$$

or

$$\mathbf{Z}' = \frac{R}{1 + j\omega CR} = \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$\therefore \quad \mathbf{Z} = j\omega L + \frac{R(1 - j\omega CR)}{1 + \omega^2 C^2 R^2}$$

$$= \frac{R}{1 + \omega^2 C^2 R^2} + j\left(\omega L - \frac{\omega C R^2}{1 + \omega^2 C^2 R^2}\right)$$

The magnitude of $\mathbf{Z}$ is thus given by

$$Z = \left[\frac{R^2}{(1 + \omega^2 C^2 R^2)^2} + \left(\omega L - \frac{\omega C R^2}{1 + \omega^2 C^2 R^2}\right)^2\right]^{1/2}$$

or
$$Z^2 = \frac{R^2}{(1+\omega^2C^2R^2)^2} + \omega^2L^2 + \frac{\omega^2C^2R^4}{(1+\omega^2C^2R^2)^2} - \frac{2\omega^2LCR^2}{1+\omega^2C^2R^2}$$
$$= \frac{R^2 - 2\omega^2LCR^2}{1+\omega^2C^2R^2} + \omega^2L^2.$$

The peak value of current will be independent of R, if Z or Z^2 is also independent of R. It is possible when

$$R^2 - 2\omega^2LCR^2 = 0, \text{ or } C = 1/2\omega^2L. \quad \therefore \omega = 1/\sqrt{2LC}.$$

Example 22. *An AC is applied to a LCR-circuit, in which a resistor of resistance R is in series with a parallel LC-circuit, as shown in Fig. 17.50. Calculate (a) the impedance of the circuit, (b) the power dissipated in the circuit and (c) the condition of resonance.*

The L and C are in parallel, therefore the equivalent impedance of the parallel LC circuit is given as

$$\frac{1}{\mathbf{Z}_{LC}} = \frac{1}{j\omega L} + j\omega C \text{ or } \mathbf{Z}_{LC} = \frac{j\omega L}{1-\omega^2LC}.$$

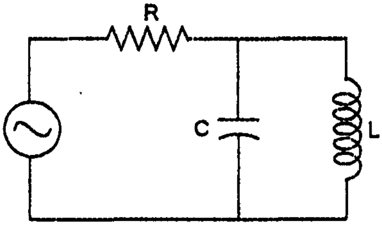


Fig. 17.50. Example 22.

Since R is in series with LC combination, therefore total impedance of the LCR circuit

$$\mathbf{Z} = R + \frac{j\omega L}{1-\omega^2LC} = \frac{R(\omega^2LC-1) - j\omega L}{\omega^2LC-1}$$

Thus the current in the circuit, if the applied source is given by the relation $v = V_0 \sin \omega t$, is given by

$$i = \frac{V_0(\omega^2LC-1)}{\sqrt{R^2(\omega^2LC-1)^2 + \omega^2L^2}} \sin(\omega t + \phi),$$

where $\phi = \tan^{-1} [\omega L/R\,(\omega^2LC - 1)]$

At resonance $\omega^2LC - 1 = 0$ or $\omega = \omega_0 = 1/\sqrt{LC}$.

At this stage no current flows through the resistor and the oscillations in LC part of the circuit keep the voltage at the LC end of the resistor the same as the voltage at the source end of the resistor.

The average power dissipated in the circuit

$$P_{av} = \frac{1}{2}V_0^2 \frac{(\omega^2LC-1)}{\sqrt{R^2(\omega^2LC-1)^2 + \omega^2L^2}} \cos\phi = \frac{V_0^2}{2R} \frac{(\omega^2 - \omega_0^2)\cos\phi}{\sqrt{(\omega^2-\omega_0^2)^2 + (\omega/RC)^2}}$$

where $\omega_0 = 1/\sqrt{LC}$, the resonance frequency. Thus at resonance the power is least and the circuit acts as rejector circuit.

Example 23. *A series LCR-circuit has inductor of 1 mH, a capacitor of 0.1 μF and a resistor of resistance 10Ω. Calculate the resonant frequency of the circuit. What is the separation between the two half power points, the power factor for the circuit and the frequency corresponding to a half power point ?*

The resonant frequency of the series LCR–circuit.

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi\sqrt{(LC)}} = \frac{1}{2\times 3.14\,(1\times 10^{-3}\times 0.1\times 10^{-6})^{1/2}}$$
$$= 1.592 \times 10^4 \text{ cycles/sec.}$$

The angular frequency separation for the two half power points

$$2\Delta\omega = R/L = 10/10^{-3} = 1 \times 10^4 \text{ radians/sec,}$$

which is quite large, showing that the resonance curve is flat.

If R decreases, $2\Delta\omega$ decreases showing the sharpness of the resonance. The resonance is very sharp for $R = 0.1$ ohm, as for this half power point separation is only 10^2 radians/sec.

The separation in frequency

$$2\Delta f = 2\Delta\omega/2\pi = 1.592 \times 10^3 \text{ cycles/sec.}$$

∴ Frequencies corresponding to half power points are

$$f + \Delta f = 1.672 \times 10^4 \text{ cycles/sec.}$$

and

$$f - \Delta f = 1.512 \times 10^4 \text{ cycles/sec.}$$

The power factor at resonance $= R/Z_{\text{Resonance}} = R/R = 1$

and at half power points $= R/Z_{hpp} = R/R\sqrt{2} = 1/\sqrt{2}$.

Example 24. *Two FM radio stations broadcast at the same strength from the same nearby distance one at a frequency of 91.1 MHz and the other at 91.3 MHz. Construct a simple series LCR-circuit with a given inductor of 1 μH inductance and variable resistance and capacitance to act as a receiver unique to one station. To limit the power from the unwanted station to 1% of the power received from the desired station, calculate the values of R and C.*

If we want to construct LCR circuit for the station broadcasting at a frequency of 91.3 *MHz*, the LCR series circuit must have resonance frequency given by

$$\omega_0 = 2\pi f = 2 \times \pi \times 91.3 \times 10^6 = 5.74 \times 10^8\, Hz.$$

The resonance frequency is defined from L and C alone. Thus with a known value of L, C is determined as

$$\omega_0 = 1/LC \text{ or } C = 1/L\,\omega_0^2$$
$$\therefore \quad C = 1/1 \times 10^{-6} \times (5.74 \times 10^8)^2 = 3.04 \times 10^{-12} F = 3.04\, pF.$$

To restrict the power of the unwanted station of frequency 91.1 MHz or $\omega = 2\pi$ (91.1 *MHz*) $= 5.72 \times 10^8\, Hz$, the sharpness is adjusted with the resistance R.

We know that the average power dissipated in LCR series circuit is

$$P_{av} = \frac{1}{2}\frac{V_0^2 R}{Z^2} = \frac{V_0^2 R}{2[(\omega L - 1/\omega C)^2 + R^2]} = \frac{V_0^2 R\omega^2}{2[(\omega^2 - \omega_0^2)^2 L^2 + \omega^2 R^2]}$$

Power dissipated at resonance $P_{res} = V_0^2/2R$.

Since two stations have same strength, thus we may consider the same value of V_0. Thus we have

$$\frac{P_{av}}{P_{res}} = \frac{R^2\omega^2}{L^2(\omega^2 - \omega_0^2)^2 + \omega^2 R^2} = \frac{1}{100}$$

$$99R^2\omega^2 = L^2(\omega^2 - \omega_0^2)^2 \text{ or } R = L\,(\omega + \omega_0)\,(\omega - \omega_0)/\sqrt{99}\omega$$

Since $\omega - \omega_0 = 2\pi(f - f_0) = 2\pi \times 0.2 \times 10^6$ and $\omega + \omega_0 = 2\omega$, therefore

$$R = 2L(2\pi)\,(f - f_0)/\sqrt{99} = 2 \times 10^{-6} \times 2\pi \times 0.2 \times 10^6/\sqrt{99}$$
$$= 2.51\Omega.$$

Example 25. *Determine the circuit parameters of an LCR series circuit, which has a resonance frequency of 10kHz a band width of 1000Hz and draws 15.3 W from a 200V generator operating at the resonance frequency of the circuit.*

For a series *LCR*-circuit operating at resonance. Therefore $V_R = V_{\text{Total}} = 200V$.

$$\text{Power } P_R = V_R^2/R \quad \text{or} \quad 15.3 = 200^2/R.$$

$$\therefore \quad R = 2.61\, k\Omega.$$

The quality factor of the circuit $Q_S = f_0/B.W. = 10 \times 10^3/1000 = 10$.

$$\text{Since} \quad Q_s = \frac{\omega_0 L}{R} = \frac{2\pi f_0 L}{R} \quad \text{or } 10 = \frac{2\pi \times 10^4 L}{2.61 \times 10^3}\,.$$

$$\therefore \quad L = 416\, mH.$$

$$\text{The resonance frequency } f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ or } 10^4 = \frac{1}{2\pi(0.416C)^{1/2}}$$

$$\therefore \quad C = 610\, pF.$$

Example 26. *A series LCR circuit has a quality factor of 5.1 at a frequency of 100kHz. If the power dissipation of the circuit is 100W when it draws a current of 1 amp. Determine (a) the circuit parameters, (b) band width and (c) the half power frequencies.*

Power dissipation of the circuit is due to the resistance *R*,

$$\therefore \quad P = P_R = I^2R \quad \text{or} \quad 100 = 1^2 \times R, \therefore R = 100\Omega.$$

The quality factor of the *LCR*-circuit is defined as

$$Q_S = \frac{\omega_0 L}{R} \text{ or } 5.1 = \frac{2\pi \times 100 \times 10^3 L}{100}$$

$$\therefore \quad L = 0.812\, mH.$$

$$\text{The resonance frequency } f_0 = \frac{1}{2\pi\sqrt{LC}} \text{ or } 10^5 = \frac{1}{2\pi\sqrt{0.812 \times 10^{-3} C}}.$$

$$\therefore \quad C = 3.12\, nF.$$

$$\text{Band width} = \frac{f_0}{Q_S} = \frac{100 \times 10^3}{5.1} = 19.6\, kHz.$$

If f_1 and f_2 be the half power frequencies, then

$$f_0 = \sqrt{f_1 f_2} \text{ or } f_1 f_2 = (100 \times 10^3)^2 = 10^{10}.$$

$$\text{Since} \quad \text{B.W.} = f_2 - f_1 = 19.6\, kHz, \text{ therefore, we get}$$

$$f_1 = 90.7\, kHz \text{ and } f_2 = 110\, kHz.$$

Example 27. *A 150 HP three phase star connected motor works on a supply whose line voltage is 4000V. The motor efficiency is 90% and power factor 90%. Calculate the line current.*

$$\text{Motor output} = 746 \times 150 \text{ Watts.}$$

$$\therefore \quad \text{Motor intake } W = \frac{\text{Output}}{\text{Efficiency}} = \frac{746 \times 150}{0.90} = 124333.3 \text{ W}$$

$$\text{Since} \quad W = \sqrt{3}\, VI \cos\phi, \text{ therefore}$$

$$I = \frac{W}{\sqrt{3} V \cos\phi} = \frac{124333.3}{\sqrt{3} \times 4000 \times 0.9} = 19.94 \text{ amp.}$$

Example 28. *A three phase transmission line consists of 1/2″ conductors placed at the corners of an equilateral triangle of side 4 ft. The line is 20 km long and supplies a load of 4000 kW at a voltage of 20,000 V and a power factor of 0.8. If the frequency is 50 c/sec, find the drop of volts in the line. Given the resistance of one conductor as 5Ω*

We know that the inductance of the loop in henrys/m = $4 \log_e (d/r) \times 10^{-7}$.

In the present problem $d = 48''$, $r = 1/4''$

$$\therefore \text{ Inductance of the line of 20 km} = 4 \times 2.3026 \log_{10} (48 \times 4) \times 20 \times 10^3 \times 10^{-7}$$
$$= 4.2 \times 10^{-2} \text{ H.}$$

Reactance of the loop $X_L = \omega L = 2\pi \times 50 \times 0.042 = 13.19\Omega$.

The current in a single phase line using two of the conductors and transmitting 4500 $kW = 4000 \times 10^3/20,000 \times 0.8 = 250$A.

Poential drop across resistance = $2 \times 5 \times 250 = 2500$V

Potential drop across reactance = $13.19 \times 250 = 3297.5$V

Voltage at generating end has x and y components given by

$$V_x = 20,000 + 2500 \times 0.8 + 3297.5 \times 0.6 = 23978.5 \text{ V}$$
$$V_y = 3297.5 \times 0.8 - 2500 \times 0.6 = 1138\text{V}$$
$$\therefore \qquad V = \sqrt{V_x^2 + V_y^2} = 24005\text{V}$$

Therefore voltage drop = 4005V.

Example 29. *A step-up transformer works on 220 volts and gives 2 ampere to an external resistor. The turn ratio between the primary and secondary coils is 2 : 25. Assuming 100% efficiency, find the secondary voltage, primary current and power delivered.*

If I_p and I_s be the currents in the primary and secondary respectively and V_p and V_s the respective voltages across them. Therefore for 100% efficiency, we have power loss zero and

$$I_p V_p = I_s V_s \quad \text{or} \quad I_p/I_s = V_s/V_p = N_s/N_p.$$

In this problem, given the turn ratio $N_s/N_p = 25/2$, $V_p = 220$ volts and $I_s = 2$ amp.

$$\therefore \qquad V_s/220 = 25/2 \text{ or } V_s = 2750 \text{ volts}$$
$$\text{and} \qquad I_p/2 = 25/2 \text{ or } I_p = 25 \text{ amp.}$$
$$\therefore \qquad \text{Power delivered} = I_s V_s = 2 \times 2750 = 5500 \text{ watts.}$$

Example 30. *Calculate the currents in the primary and secondary circuits with no-load and on load conditions, in an iron cored transformer having $R_1 = 1\Omega$, $L_1 = 0.006$ H, $R_2 = 50\Omega$, $L_2 = 0.080$H, $M = 0.5$H, turns ratio $N_s/N_p = \rho = 2$, $V_1 = 100$V at $f = 50$ c/s.*

The iron cored transformer may be considered as the mutually inductive circuits. Each winding has its own separate resistances R_1 and R_2 and total inductances L_1 and L_2. M is the mutual inductance of the circuits.

In the first case with no load, the secondary resistance R_2 should be considered as infinity and the primary current $I_{1,0}$ may be given by

$$\mathbf{V}_1 = \mathbf{I}_{1,0} (R_1 + j\omega L_1) + j\omega M\mathbf{I}_2 = \mathbf{I}_{1,0} [R_1 + j\omega (L_1 + M/\rho)]$$
$$\text{or} \qquad 100 = \mathbf{I}_{1,0} [1 + j \times 2 \times 3.14 \times 50 (0.006 + 0.5/2)] = \mathbf{I}_{1,0} [1 + j\, 80.384]$$
$$\therefore \qquad \mathbf{I}_{1,0} = \frac{100}{(1 + j80.384)} = \frac{100(1 - j80.384)}{1^2 + 80.384^2} = 0.0155 - j1.2438.$$

or $$I_{1,0} = 1.244 \text{ amp. and } \phi_0 = \tan^{-1}(-1.2438/0.0155) = 270°43'.$$

In the second case, with a load $R_2 = 50\Omega$, we have relation for V_1 and I_1 with on load condition, as given by equation (101). Substituting the numerical values, we get

$$100 = \mathbf{I}_1[11.4887 + j\,9.2452]$$

or $$\mathbf{I}_1 = 100(11.4887 - j\,9.2452)/217.464 = 5.283 - j\,4.251,$$

$$I_1 = 6.781 \text{ amp. and } \phi_1 = \tan^{-1}(-4.251/5.283) = 321°11'.$$

The current in the secondary with on-load condition is given by

$$\mathbf{I}_2 = j\omega M\mathbf{I}_1/(R_2 + j\omega L_2 + j\omega\rho M).$$

Substituting the numerical values, we get

$$\mathbf{I}_2 = \left(\frac{j157}{50 + j25.12 + j314}\right)(5.283 - j4.251) = \frac{j157(5.283 - j4.251)(50 - j339.12)}{50^2 + 339.12^2}$$

$$= 2.6776 - j\,1.5732$$

$$\therefore \quad I_2 = 3.105 \text{ amp. and } \phi_2 = 329°24'$$

ORAL QUESTIONS

1. Give the physical reasons behind the facts that impedance of an inductor increases with frequency while the impedance of a capacitor decreases with frequency.
2. How can one separate sinusoidal voltages of frequencies 10^5 and 10 *Hz* by inserting either a 2μ*F* capacitor or a 20 *mH* inductor in series with the power source ?
3. Which of the two combinations of inductor and a capacitor is required to (*a*) transmit a required frequency, (*b*) to suppress it ?
4. Point out as many similarities as you can between *LC*-resonant circuit and the vibration of a point mass at the end of a spring.
5. Why a parallel *LC*-circuit is inductive, but a series *LC* circuit is capacitive below f_0.
6. Under what conditions can current be supplied to an *ac* circuit without delivering power to it ?
7. Which will be an ideal chocking coil ?
8. What are the conditions for resonance in *LCR* circuit ?
9. Why the *LCR*-series circuit is called an acceptor circuit ?
10. What are the possible ways of reducing current in an *a.c.* circuit ? Explain which one is better ?
11. What is the effect on *Q* and band width of a parallel resonance circuit if its shunt damping resistance is decreased from 50*k*Ω to 10*k*Ω ?
12. What happened to the voltage and current if the turns in the secondary of the transformer is doubled ?
13. How emf is developed in the secondary of the transformer ? How the primary is affected by increase of load in the secondary ?
14. A transformer has n_1 primary turns and n_2 secondary turns, 60% of the magnetic flux due to current in the primary goes out as leakage. Find the ratio of primary to secondary voltage with no load on the secondary.
15. How can you explain two balance conditions in *ac* bridges ?
16. Why the Owen's bridge is inferior to Anderson's bridge ?
17. What do you expect if no *ac* balance is obtained in Anderson's bridge ?
18. What are the relative merits of the *ac* bridges used for capacitance measurement ?
19. Whether a 6 volt step down transformer may be used to perform bridge experiments with headphone as a detector ? Give reasons.
20. What is the function of Wagner earthing device ?

PROBLEMS

1. A circular coil of negligible self inductance, having 100 turns and of radius 25 cm, rotates uniformly five times per second about a vertical diameter in a plane where the magnetic dip is 60° and the resultant magnetic field strength is 38 amp/meter. Calculate (*a*) V_0, (*b*) I_0 and (*c*) the rate of production of heat in the coil, if its resistance is 10 ohms.
   (1.172×10^{-2} V. 1.172×10^{-3} amp., 6.88×10^{-6} joule/sec.)
2. Find the rms value of current in the circuit, consisting of an inductor of inductance 100 *mH* and a resistor of resistance 2000 ohms in series along with a *ac* source of emf 50*V* at a frequency of 1000 cycles/sec. Also calculate *pd* across the inductor and across the resistor. *(24 mA, 15 V, 47.6V)*
3. A 500Ω light bulb is being operated at 40 watts by an alternating sine voltage. Find both the rms and peak values for this voltage. *(141 V, 200 V)*
4. An alternating emf 200 virtual volts at 50 cycles is connected to a circuit of resistance 1 ohm and inductance 0.01 henry. What is the current and lag in phase between the emf and the current ?
   (60.3 amp., 72°)
5. An *ac* voltage of 220 rms volts at a frequency of 40 cycles per sec is applied to a circuit containing a pure inductance of 0.01 henry and a pure resistance of 6 ohms in series. Find (*a*) the current, (*b*) the *pd* across the circuit elements, (*c*) the lag in time between the voltage and the current.
   (38.83 amp, 202.98 V, 96.83 V, 0.00158 sec)
6. A 100 volts, 50 watts light bulb is to be run by 200 volts 50 cycles supply. Calculate the capacity of the capacitor to be used in series with the bulb. *(9.18 μF)*
7. A resistor and inductor are connected in series across a voltage $v = 283 \sin 314\, t$. The current is found to be $4 \sin (314\, t - \pi/4)$, find the values of the inductance and resistance.
   (0.16 henry, 50 ohms)
8. A generator having emf $282 \sin (100\, \pi t)$ and internal resistance 2*k*Ω is connected in series with an inductance (40/π*H*), capacitance (10/πμ*F*) and resistance (2*k*Ω). Calculate (*i*) the rms value of current flowing in the circuit, (*ii*) the rms value of the voltage across the inductor and (*iii*) the frequency at which the impedance of the circuit will be purely resistive.
   (0.35 amp., 141 volts, 25 hertz)
9. An arc lamp consumes 10 amp. at 40 volts. Calculate the power factor when it is connected with a suitable value of choke coil required to run the arc lamp on *ac* mains of 200 volts (rms) and 50 cycles per sec. Calculate also the energy wasted per second if it was to run with a resistance on 200 volts *d.c.* *(0.2, 1600 watts)*
10. An electric lamp which runs at 100 volts *dc* and consumes 10 amp current is connected to *ac* mains at 150 volts 50 cycles/sec with a choke coil in series. Calculate the inductance and drop of voltage across the choke. Neglect the resistance of choke. *(0.036H, 111.8 volts)*
11. A circuit operating at 360/2π *Hz* contains a 1 μ*F* capacitor and a 20 Ω resistor. How large an inductor must be added in series to make the phase angle for the circuit zero ? Calculate the current in the circuit if the applied voltage is 120 volts. *(7.7 henry, 6A)*
12. If a 0.03 henry inductor, a 10 Ω resistor and a 2μ*F capacitor* are connected in series. At what frequency will they resonate ? What will be the phase angle at resonance ? *(650 Hz, 0)*
13. A series circuit consists of a resistance of 10Ω, an inductance 5 *mH* and a capacitor whose capacitance is 800 *pF*. An emf of amplitude 5 volts and of variable frequency is connected across the above series combination. Find values for the frequency at which the amplitude of the current will be (*a*) a maximum, (*b*) one half its maximum value.
   ((a) 7.96×10^4 cycles/sec; (b) 7.99×10^4 and 7.93×10^4 cycles/sec]
14. A series *LCR* circuit has *L* = 1 *mH*, *C* = 0.1μ*F* and *R* = 10Ω. Calculate resonance frequency of the circuit. *(1.59×10^4 c/s)*
15. A series circuit contains *R* = 300 ohm, *C* = 2μ*F*, and *L* = 0.02 *H* is connected to a 9 volts *a.c.* supply. Find (*a*) the resonant frequency, (*b*) the current in the circuit at resonance and (*c*) the

power absorbed at resonance. If the applied frequency is twice that at resonance, then what will be the absorbed power ?

16. A resistance of 10 ohm is joined in series to an inductance of 0.5 henry. What capacitance should be put in series with the combination to obtain maximum current ? What will be the potential difference across the resistance, inductance and capacitance ? The current is being suplied by a 200 volt and 50 cycle *a.c.* mains. *(2μF, 200 volts, 3140 volts, 3140 volts)*
17. Two coils, one of resistance 2 Ω and self inductance 0.015 *H*, the other having a resistance of 1 Ω and a self inductance of 0.08 *H* are arranged in parallel on a 200 volt 50 cycles/sec frequency circuit. Calculate the currents flowing in each circuit and through the whole system.
18. When a potential difference of 200 volts and 50 hertzs is applied to an inductive circuit, the current in it is 10 amp and the power consumed is 1000 watts. Find the power factor, resistance and inductance of the circuit. *(0.5, 10Ω, 3.04 mH)*
19. A resistance of 40Ω is joined in parallel with a reactance of 30Ω, across an *ac* supply. Obtain values for the resistance and reactance which, when connected in series with the same supply, would take the same current and have the same power factor. *(14.4Ω, 19.2 Ω)*
20. A average power to the plant is 250 *kW* and the power factor is 0.707 lagging. The generators with a capacitor in parallel is connected across the plant. If the generator is sinusoidal with ω = 377 radians/sec. and the effective value is 2300 volts, determine the value of the capacitance such that the power factor is 1. *(125μF)*
21. A series resonance circuit has a resonance frequency of 150 *kHz*, a band width of 75 *kHz* and quality factor $Q_s = 2$. Determine the cut off frequencies. *(117.1 kHz, 192.1 kHz)*
22. A series resonance circuit has a resistance of 1000Ω and cut off frequencies of 20 *kHz* and 100 *kHz*. Determine (*a*) the band width, (*b*) the resonance frequency, (*c*) the quality frequency of circuit at the resonance frequency, (*d*) the inductance and (*e*) the capacitance. *(80 kHz, 44.72 kHZ. 0.50, 2mH, 0.3A)*
23. A tank circuit consisting of a capacitor in parallel with a coil whose inductance and resistance are 3.2 *mH* and 56Ω respectively, is driven at its resonance frequency of 22.3 *kHz* from a current source whose source resistance is 8081 Ω. If the band width is 4.05 *kHz*, calculate the capacitance of the tank circuit. Determine the amount of additional parallel resistance required to change the band width to 6 *kHz*. *(15.7 μF, 5184 Ω)*
24. A coil of inductive reactance 3Ω and resistance 4Ω is connected in parallel with a capacitor branch having a capacitive reactance 8Ω and a series resistance 6Ω. The *ac* source of rms voltage 100*V* and freqeuncy of 50 *Hz* is applied across the combination. Calculate (*a*) the impedance of the circuit, (*b*) line circuit, (*c*) power factor and (*d*) the average power supplied by the source. $(2\sqrt{5}\,\Omega, 10\sqrt{5}\ amp, 11/5\sqrt{5}, 2200W)$
25. The internal resistance of an *ac* source is 10*k*Ω. (*a*) What should be the turns ratio of a transformer to match the source to a load of resistance 10 ohms ? (*b*) If the voltage amplitude of the source is 100 *V*, what is the voltage amplitude of the secondary on open circuit ? *(0.0316, 3.16V)*
26. A current of 6 amperes at 220 volts flow in the primary of a transformer. In the secondary of the same transformer, a current of 1 ampere at 100 volts is available. Find the power loss in the transformer. *(1000 watts)*
27. In the *LCR*-series circuit, voltages across *L*, *C* and *R* come out at 30*V*, 60*V* and 40*V* respectively for one *ac*-source. If *ac* frequency is doubled, keeping voltage amplitude constants, the voltage across *R* will remain same, will be doubled or halved.
28. In the ***LCR*** circuit with signal generator of fixed out put voltage, it is found that when the frequency is increased continuously the current supplied by the generator has a minimum value *V*/*R* at a certain frequency. The possible circuit arrangement will be (*a*) ***LCR*** in series, (*b*) ***LCR*** in parallel, (*c*) *C* in parallel with *L* & *R* in series. *(b)*
29. A series *LCR* circuit is connected to an *ac* source of variable frequency. When the frequency is continuously increased, the power frequency (*a*) goes on increasing continuously, (*b*) goes on decreasing continuously, (c) becomes maximum at a particular frequency. *(c)*

18

NETWORK ANALYSIS AND NETWORK THEOREMS

18.1 NETWORK

A *network* is a circuit containing *circuit elements* or *branches*. The *circuit elements* are of two types : (*a*) *passive* and (*b*) *active*. The inductor, resistor and capacitor with two terminals are regarded as *passive elements* as they are not fundamental sources of power. The quantitative measure of these elements are inductance, resistance and capacitance respectively. The *active elements* are those which feed the power into the given circuits. These are of two types : (*a*) *voltage source* and (*b*) *current source*. An *ideal voltage source* is an active device which can maintain a constant voltage across its terminals irrespective of the current supplied by it. It is also called *constant voltage source*. It is always equal to the *open-circuit voltage*. In an actual voltage source, the voltage across its terminals decreases as the load resistance is decreased and the more current is supplied by the source. This voltage is highest when the circuit is an open circuit, *i.e.,* when the source does not supply any current. If the load resistance reduces to zero, *i.e.,* if the output terminals of the voltage source are *short-circuited*, the current will be infinite. A *constant current source* is an active device which can supply the constant current to any load resistance that is connected across its terminals. In such a case the terminal voltage is directly proportional to the load resistance.

A group of elements in series is called *branch*. The junction of two or more branches is known as a *node* or *vertex*. A set of branches forming a closed path in a network is called a *mesh*. The network is said to be a linear network if it consists of linear impedances. A linear impedance is any impedance that obey's Ohm's law. The resistors, inductors and capacitors fall into this category.

18.2 NETWORK ANALYSIS

Besides the various circuit theorems, to be discussed in the next article, that aid the analysis of circuits, the following several methods are used for solving network problems.

1. Series-Parallel Connections. When the circuit consists of series and parallel branches, the first step is to simplify each series and parallel branch separately. For this purpose the assistance of Kirchhoff's laws is needed. Let us briefly discuss following two types of networks with series-parallel connections :

(*a*) *Ladder networks :* Such networks have alternate series and parallel elements. They are called ladder networks as they look like ladders. To determine the input resistance of such a network, we start at the section of the network farthest from the source, combining elements moving towards the input terminals.

Consider a simple ladder network with each element of equal impedances $\mathbf{Z}$, as shown in Fig. 18.1. To find the total impedance, let us start at the section farthest from the source. This section CDHGC is composed of two impedances $\mathbf{Z}_7$ and $\mathbf{Z}_8$ in series and the combination is in parallel with an impedance $\mathbf{Z}_6$. The equivalent impedance of this section is

$$\mathbf{Z}_{CG} = \frac{(\mathbf{Z}_7 + \mathbf{Z}_8)\,\mathbf{Z}_6}{\mathbf{Z}_6 + \mathbf{Z}_7 + \mathbf{Z}_8} = \frac{2}{3}\mathbf{Z}.$$

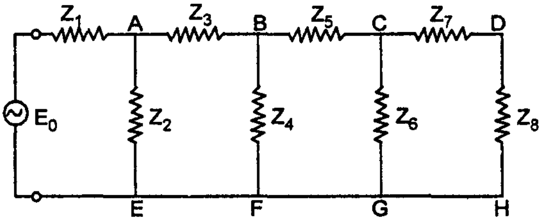


Fig. 18.1. Ladder network.

This is in series with $\mathbf{Z}_5$ and the combination is in parallel with $\mathbf{Z}_4$. Thus the equivalent impedance

$$\mathbf{Z}_{BF} = \frac{\mathbf{Z}_4\,(\mathbf{Z}_5 + \mathbf{Z}_{CG})}{\mathbf{Z}_4 + \mathbf{Z}_5 + \mathbf{Z}_{CG}} = \frac{\mathbf{Z}\left(\mathbf{Z} + \frac{2}{3}\mathbf{Z}\right)}{\mathbf{Z} + \mathbf{Z} + \frac{2}{3}\mathbf{Z}} = \frac{5}{8}\mathbf{Z}.$$

Using the same procedure

$$\mathbf{Z}_{AE} = \frac{\mathbf{Z}_2(\mathbf{Z}_3 + \mathbf{Z}_{BF})}{\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_{BF}} = \frac{\mathbf{Z}\left(\mathbf{Z} + \frac{5}{8}\mathbf{Z}\right)}{\mathbf{Z} + \mathbf{Z} + \frac{5}{8}} = \frac{13}{21}\mathbf{Z}.$$

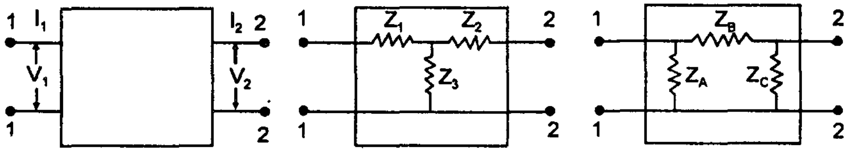
$\therefore$ Input impedance $\mathbf{Z}_{in} = \mathbf{Z}_1 + \mathbf{Z}_{AE} = \mathbf{Z} + \frac{13}{21}\mathbf{Z} = \frac{34}{21}\mathbf{Z}$

and the feeder current $\mathbf{I} = \mathbf{E}_0/\mathbf{Z}_{in}$.

(*b*) *Bridge networks :* Bridge networks are series and parallel circuits that have many applications. The basic bridge circuit, called the wheatstone's bridge is used for measuring unknown resistance, the other modified bridges are used for measuring the inductance, capacitance, internal resistance of the measuring instruments and dielectric constant of the medium. These bridges have been discussed in the previous chapter.

2. T-and Π-Networks (Star and Delta networks). The above simple principle alone is not sufficient to solve all problems. Any complicated electrical network can be represented, at a single frequency, by a *T*-or Π-network. *T*- and Π-networks are also called star and Delta or Wye and Delta networks respectively.

Convenient impedance values which may be chosen are with open circuit and short circuit. The impedance measured across the 1, 1 terminals with 2, 2 terminals open circuited are designated as $\mathbf{Z}_{1OC}$ and the measurement with 2, 2 terminals short circuited as $\mathbf{Z}_{1SC}$. Similar measurements are taken at the 2, 2 terminals and are designated as $\mathbf{Z}_{2OC}$ and $\mathbf{Z}_{2SC}$.



(*a*) (*b*) (*c*)

Fig. 18.2. *T*-and Π-network.

There are eight quantities, the magnitudes and phase angles of $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{I}_1$ and $\mathbf{I}_2$, which must be established by the equivalent network. If a load $\mathbf{Z}_R$ is connected between terminals 2, 2 then with the relation $V_2 = I_2\mathbf{Z}_R$, one magnitude and one phase angle will be determined and we left with other three magnitudes and three phase angles to be determined. Thus we require at least three adjustable impedances to set up an equivalent network. These three impedances can be arranged in two different ways as *T*-and Π-networks.

T-network would be said to be equivalent to the network at Fig. 18.2 (*b*), if $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$ are selected to have the same external currents and voltages as the original network (*a*), and hence the open and short-circuit measurements must be identical. Therefore for *T*-network

$$\mathbf{Z}_{1OC} = \mathbf{Z}_1 + \mathbf{Z}_3. \qquad ...(1)$$

$$\mathbf{Z}_{1SC} = \mathbf{Z}_1 + \mathbf{Z}_2\mathbf{Z}_3/(\mathbf{Z}_2 + \mathbf{Z}_3). \qquad ...(2)$$

$$\mathbf{Z}_{2OC} = \mathbf{Z}_2 + \mathbf{Z}_3. \qquad ...(3)$$

$$\mathbf{Z}_{2SC} = \mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3/(\mathbf{Z}_1 + \mathbf{Z}_3). \qquad ...(4)$$

From Eqs. (1) and (2) we get

$$\mathbf{Z}_{1OC} - \mathbf{Z}_{1SC} = \mathbf{Z}_3 - \mathbf{Z}_2\mathbf{Z}_3/(\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{Z}_3^2/(\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{Z}_3^2/\mathbf{Z}_{2OC}.$$

$$\mathbf{Z}_3 = \pm[\mathbf{Z}_{2OC}(\mathbf{Z}_{1OC} - \mathbf{Z}_{1SC})]^{1/2}. \qquad ...(5)$$

From Eqs. (1) and (3), we get

$$\mathbf{Z}_1 = \mathbf{Z}_{1OC} - \mathbf{Z}_3 \text{ and } \mathbf{Z}_2 = \mathbf{Z}_{2OC} - \mathbf{Z}_3 \qquad ...(6)$$

Similarly for an equivalent Π-section [Fig. 18.2(*c*)], we have

$$\mathbf{Z}_{1OC} = \mathbf{Z}_A(\mathbf{Z}_B + \mathbf{Z}_C)/(\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C). \qquad ...(7)$$

$$\mathbf{Z}_{1SC} = \mathbf{Z}_A\mathbf{Z}_B/(\mathbf{Z}_A + \mathbf{Z}_B). \qquad ...(8)$$

$$\mathbf{Z}_{2OC} = \mathbf{Z}_C(\mathbf{Z}_A + \mathbf{Z}_B)/(\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C). \qquad ...(9)$$

Solving these relations, we have

$$\mathbf{Z}_A = \frac{\mathbf{Z}_{2OC}\mathbf{Z}_{1SC}}{\mathbf{Z}_{2OC} - [\mathbf{Z}_{2OC}(\mathbf{Z}_{1OC} - \mathbf{Z}_{1SC})]^{1/2}} \qquad ...(10)$$

$$\mathbf{Z}_B = \frac{\mathbf{Z}_{2OC}\mathbf{Z}_{1SC}}{[\mathbf{Z}_{2OC}(\mathbf{Z}_{1OC} - \mathbf{Z}_{1SC})]^{1/2}} \qquad ...(11)$$

$$\mathbf{Z}_C = \frac{\mathbf{Z}_{2OC}\mathbf{Z}_{1SC}}{\mathbf{Z}_{1OC} - [\mathbf{Z}_{2OC}(\mathbf{Z}_{1OC} - \mathbf{Z}_{1SC})]^{1/2}} \qquad ...(12)$$

Conversion between T-and Π-sections. It is convenient to convert directly *T*-section to equivalent Π-section.

From relations (1–3) and (7–9), we get

$$\mathbf{Z}_1 + \mathbf{Z}_3 = \mathbf{Z}_A(\mathbf{Z}_B + \mathbf{Z}_C)/(\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C),$$

$$\mathbf{Z}_1 + \mathbf{Z}_2\mathbf{Z}_3/(\mathbf{Z}_2 + \mathbf{Z}_3) = \mathbf{Z}_A\mathbf{Z}_B/(\mathbf{Z}_A + \mathbf{Z}_B),$$

$$\mathbf{Z}_2 + \mathbf{Z}_3 = \mathbf{Z}_C(\mathbf{Z}_A + \mathbf{Z}_B)/(\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C).$$

All branches of the T-network is thus given as

$$\mathbf{Z}_1 = \frac{\mathbf{Z}_A\mathbf{Z}_B}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C}, \mathbf{Z}_2 = \frac{\mathbf{Z}_B\mathbf{Z}_C}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C}, \mathbf{Z}_3 = \frac{\mathbf{Z}_C\mathbf{Z}_A}{\mathbf{Z}_A + \mathbf{Z}_B + \mathbf{Z}_C} \qquad ...(13)$$

If the impedances of the Π-circuit are known, the above equations may be used to determine the equivalent T-network.

The inverse transformation can be obtained, if we have values of $\mathbf{Z}_A$, $\mathbf{Z}_B$ and $\mathbf{Z}_C$ in terms of $\mathbf{Z}_1$, $\mathbf{Z}_2$ and $\mathbf{Z}_3$ as

$$\mathbf{Z}_A = \square/\mathbf{Z}_2, \mathbf{Z}_B = \square/\mathbf{Z}_3, \mathbf{Z}_C = \square/\mathbf{Z}_1,$$

where $$\square = \mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1. \qquad ...(14)$$

3. Nodal Equations. The alternate method of solving complicated network problems is the *nodal method*. In this method admittances (Y) are used instead of impedance and the voltage of the node is represented by E_a, E_b, E_c, etc. From any reference point a (Fig. 18.3), currents are going to its neighbouring points, because of its own potential. Similarly owing to potentials of the neighbouring points currents are coming to this point. As there is no accumulation of charge at this point, hence

$$I_1 - I_2 - I_4 = 0. \qquad ...(15)$$

For the solution of the problem some convenient junction between elements is chosen as a reference node (say point O) and the voltages at different nodes are assumed with respect to this reference point. The figure shows that

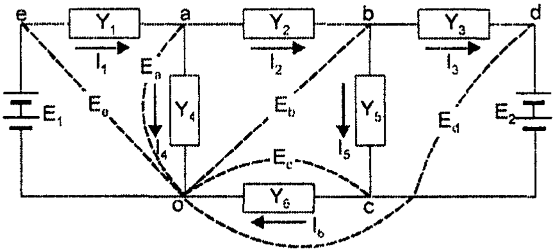


Fig. 18.3. Nodal equation method.

$$E_e = E_1 \text{ and } E_d = E_c + E_2. \qquad ...(16)$$

Current summation *at junction a* can be written as

$$(E_1 - E_a)\,Y_1 - (E_a - E_b)\,Y_2 - E_aY_4 = 0. \qquad ...(17)$$

At junction b $$I_2 - I_3 - I_5 = 0, \qquad ...(18)$$

or $$(E_a - E_b)Y_2 - (E_b - E_2 - E_c)Y_3 - (E_b - E_c)Y_5 = 0. \qquad ...(19)$$

At junction c $$I_3 + I_5 - I_6 = 0, \qquad ...(20)$$

or $$(E_b - E_2 - E_c)Y_3 + (E_b - E_c)Y_5 - E_cY_6 = 0. \qquad ...(21)$$

These equations may be written as

$$-(Y_1 + Y_2 + Y_4)\,E_a + Y_2E_b = -Y_1E_1 \qquad ...(22)$$

$$Y_2E_a - (Y_2 + Y_3 + Y_5)\,E_b + (Y_3 + Y_5)E_c = -Y_3E_2 \qquad ...(23)$$

$$(Y_3 + Y_5)\,E_b - (Y_3 + Y_5 + Y_6)E_c = Y_3E_2. \qquad ...(24)$$

From these relations we can get potential information at any junction. As at high frequencies voltmeters are more commonly used and are more accurate than ammeters, therefore nodal method is very important.

4. Maxwell's Method. In this method matrices and determinants are used. Maxwell suggested cycle currents to flow through different loops and the two following terms : (1) *Self Resistance – Sum of* resistances through which any cyclic current is passing. (2) *Mutual*

Resistance – The sum of resistances through which two current loops are passing. The latter is negative if the loop currents are passing in opposite directions and is zero when there is no common path.

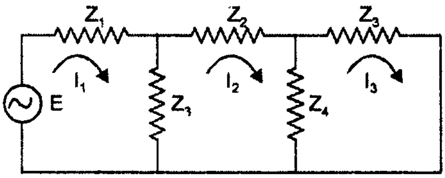


Fig. 18.4. Maxwell method.

Let us consider a general network such as Fig. 18.4. The three equations may be written as

$$E = (\mathbf{Z}_1 + \mathbf{Z}_3)I_1 - \mathbf{Z}_3 I_2. \qquad ...(25)$$

$$0 = -\mathbf{Z}_3 I_1 + (\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4)I_2 - \mathbf{Z}_4 I_3 \qquad ...(26)$$

$$0 = -\mathbf{Z}_4 I_2 + (\mathbf{Z}_4 + \mathbf{Z}_5)I_3. \qquad ...(27)$$

The coefficients of the currents in these circuit equations may be written as

$$[\mathbf{Z}] = \begin{bmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & -\mathbf{Z}_3 & 0 \\ -\mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 & -\mathbf{Z}_4 \\ 0 & -\mathbf{Z}_4 & \mathbf{Z}_4 + \mathbf{Z}_5 \end{bmatrix} \qquad ...(28)$$

This *impedance matrix* may be written as

$$[\mathbf{Z}] = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & 0 \\ \mathbf{Z}_{12} & \mathbf{Z}_{22} & \mathbf{Z}_{23} \\ 0 & \mathbf{Z}_{23} & \mathbf{Z}_{33} \end{bmatrix}, \qquad ...(29)$$

where $\mathbf{Z}_{11} = \mathbf{Z}_1 + \mathbf{Z}_3$ the impedance of first mesh, $\mathbf{Z}_{22} = \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4$ the impedance of second mesh, $\mathbf{Z}_{33} = \mathbf{Z}_4 + \mathbf{Z}_5$ the impedance of third mesh. Also $-\mathbf{Z}_3 = \mathbf{Z}_{12}$ the mutual impedance between meshes 1 and 2, $-\mathbf{Z}_4 = \mathbf{Z}_{23}$ the mutual impedance between meshes 2 and 3. The original mesh equations (25–27) may thus be written as

$$\begin{bmatrix} E \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mathbf{Z}_{11} & \mathbf{Z}_{12} & 0 \\ \mathbf{Z}_{12} & \mathbf{Z}_{22} & \mathbf{Z}_{23} \\ 0 & \mathbf{Z}_{23} & \mathbf{Z}_{33} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix}. \qquad ...(30)$$

The currents can be calculated by simple algebra. We thus get

$$I_1 = \frac{[(\mathbf{Z}_2 + \mathbf{Z}_3)(\mathbf{Z}_4 + \mathbf{Z}_5) + \mathbf{Z}_4\mathbf{Z}_5]E}{(\mathbf{Z}_4 + \mathbf{Z}_5)(\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3) + (\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{Z}_4\mathbf{Z}_5}, \qquad ...(31)$$

$$I_2 = \frac{\mathbf{Z}_3(\mathbf{Z}_4 + \mathbf{Z}_5)E}{(\mathbf{Z}_4 + \mathbf{Z}_5)(\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3) + (\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{Z}_4\mathbf{Z}_5}, \qquad ...(32)$$

$$I_3 = \frac{\mathbf{Z}_3\mathbf{Z}_4 E}{(\mathbf{Z}_4 + \mathbf{Z}_5)(\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3) + (\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{Z}_4\mathbf{Z}_5} \qquad ...(33)$$

The determinant of **Z** may be written as

$$\begin{aligned} \Delta_z &= (\mathbf{Z}_1 + \mathbf{Z}_3)\begin{vmatrix} \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 & -\mathbf{Z}_4 \\ -\mathbf{Z}_4 & \mathbf{Z}_4 + \mathbf{Z}_5 \end{vmatrix} + \mathbf{Z}_3\begin{vmatrix} -\mathbf{Z}_3 & -\mathbf{Z}_4 \\ 0 & \mathbf{Z}_4 + \mathbf{Z}_5 \end{vmatrix} \\ &= (\mathbf{Z}_1 + \mathbf{Z}_3)[(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4)(\mathbf{Z}_4 + \mathbf{Z}_5) - \mathbf{Z}_4^2] - \mathbf{Z}_3^2(\mathbf{Z}_4 + \mathbf{Z}_5) \\ &= (\mathbf{Z}_4 + \mathbf{Z}_5)(\mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_1\mathbf{Z}_3 + \mathbf{Z}_2\mathbf{Z}_3) + (\mathbf{Z}_1 + \mathbf{Z}_3)\mathbf{Z}_4\mathbf{Z}_5. \end{aligned} \qquad ...(34)$$

If the first column of the matrix be replaced by the left hand side E column matrix, then the determinant

$$\Delta_1 = \begin{vmatrix} E & -\mathbf{Z}_3 & 0 \\ 0 & \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 & -\mathbf{Z}_4 \\ 0 & -\mathbf{Z}_4 & \mathbf{Z}_4 + \mathbf{Z}_5 \end{vmatrix} = [(\mathbf{Z}_2 + \mathbf{Z}_3)(\mathbf{Z}_4 + \mathbf{Z}_5) + \mathbf{Z}_4\mathbf{Z}_5]E \qquad ...(35)$$

Similarly we have

$$\Delta_2 = \begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & E & 0 \\ -\mathbf{Z}_3 & 0 & -\mathbf{Z}_4 \\ 0 & 0 & \mathbf{Z}_4 + \mathbf{Z}_5 \end{vmatrix} = \mathbf{Z}_3\,(\mathbf{Z}_4 + \mathbf{Z}_5)E \qquad \text{...(36)}$$

and

$$\Delta_3 = \begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 & -\mathbf{Z}_3 & E \\ -\mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_4 & 0 \\ 0 & -\mathbf{Z}_4 & 0 \end{vmatrix} = \mathbf{Z}_3\,\mathbf{Z}_4 E. \qquad \text{...(37)}$$

Thus we see that the currents can be written as

$$I_1 = \frac{\Delta_1}{\Delta_z}, I_2 = \frac{\Delta_2}{\Delta_z} \text{ and } I_3 = \frac{\Delta_3}{\Delta z}. \qquad \text{...(38)}$$

Thus we see that to find expression for I_r, we have to write determinant Δ_r formed by replacing the r_{th} column with the mesh voltage column and then divide it by the matrix determinant Δ_z.

18.3 NETWORK THEOREMS

Network theorems are very powerful tools for solving advanced and complicated network problems. Some of the important theorems are :

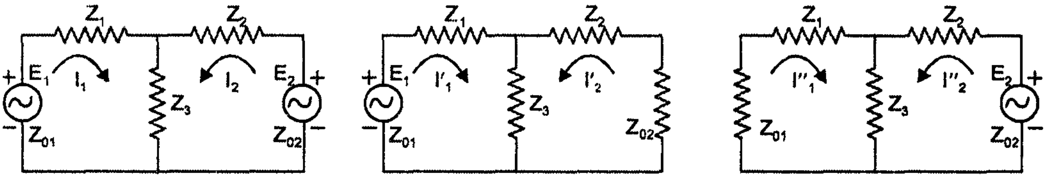
1. Superposition Theorem. *In any linear network containing linear impedances and potential sources, the current flowing in any element is the phasor sum of the currents that would flow in that element by each potential source, all other sources being replaced at that time by their internal impedances.*

Consider a simple case of network consisting two meshes with two potential sources E_1 and E_2. For this network, the mesh currents are as shown in Fig. 18.5. The mesh equations may thus by written as

$$E_1 = (\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_{01})\, I_1 + \mathbf{Z}_3 I_2 \qquad \text{...(39)}$$

and

$$E_2 = \mathbf{Z}_3 I_1 + (\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_{02})\, I_2, \qquad \text{...(40)}$$



(a) (b) (c)

Fig. 18.5. Networks for superposition theorem.

The currents may be obtained by using Maxwell's method

$$I_1 = \frac{\begin{vmatrix} E_1 & \mathbf{Z}_3 \\ E_2 & \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_{02} \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_{01} & \mathbf{Z}_3 \\ \mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_{02} \end{vmatrix}}$$

or

$$I_1 = [(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_{02})\, E_1 - \mathbf{Z}_3\, E_2]/\Delta_z, \qquad \text{...(41)}$$

and similarly

$$I_2 = [-\mathbf{Z}_3 E_1 + (\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_{01}) E_2]/\Delta_z, \qquad \text{...(42)}$$

where Δ_z is the impedance matrix.

To prove superposition theorem, let us first find I_1' and I_2' by short circuiting emf E_2 (but not its impedance) and then I_1'' and I_2'' short circuiting emf E_1 (but not its impedance). Using Fig. 18.5(*b*) we get,

$$I_1' = \frac{(\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_{02})E_1}{\Delta_z} \text{ and } I_2' = \frac{-\mathbf{Z}_3 E_1}{\Delta_z} \qquad ...(43)$$

Using Fig. 18.5(*c*), we get

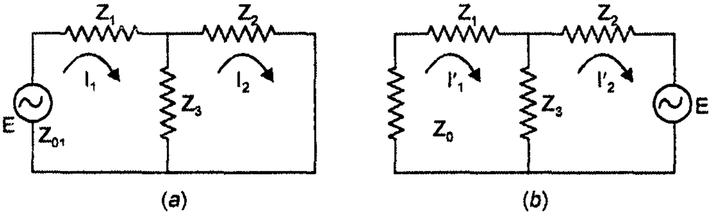
$$I_1'' = \frac{-\mathbf{Z}_3 E_2}{\Delta_z} \text{ and } I_2'' = \frac{(\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_{01})E_2}{\Delta_z} \qquad ...(44)$$

Equations (41) to (44) show that

$$I_1 = I_1' + I_1'' \quad \text{and} \quad I_2 = I_2' + I_2''. \qquad ...(45)$$

Hence we have proved superposition theorem. This theorem can also be used if voltages of different frequencies are introduced.

2. Reciprocity Theorem : *Any linear network with the linear and bilateral elements and a single voltage source or a single current source, the current in one mesh is equal to the current in the latter when the same emf is placed in the former (but not its impedance).* This may also be stated as : *If a voltage source in branch AB of a network causes current I amperes in branch CD, shifting the voltage source (but not its impedance) to branch CD will cause the same current I to appear in branch AB.* In other words *the interchange of an ideal current source and an ideal voltage source in any network will not change the voltmeter and ammeter readings.*



(*a*) (*b*)

Fig. 18.6. Networks for Reciprocity theorem.

Consider a two mesh network, as shown in Fig. 18.6.

Applying Kirchhoff's laws to the meshes, when the voltage source is in the first mesh [Fig. 18.6(*a*)], we get,

$$E = (\mathbf{Z}_1 + \mathbf{Z}_0 + \mathbf{Z}_3)\, I_1 - \mathbf{Z}_3 I_2 \qquad ...(46)$$

$$0 = -\mathbf{Z}_3 I_1 + (\mathbf{Z}_2 + \mathbf{Z}_3)\, I_2. \qquad ...(47)$$

$$\therefore \quad I_2 = \mathbf{Z}_3 E/[(\mathbf{Z}_1 + \mathbf{Z}_0)\,(\mathbf{Z}_2 + \mathbf{Z}_3) + \mathbf{Z}_2\mathbf{Z}_3] \qquad ...(48)$$

Similarly for the Fig. 18.6(*b*), we get

$$0 = (\mathbf{Z}_1 + \mathbf{Z}_0 + \mathbf{Z}_3)\, I_1' - \mathbf{Z}_3 I_2' \qquad ...(49)$$

$$E = -\mathbf{Z}_3 I_1' + (\mathbf{Z}_2 + \mathbf{Z}_3) I_2' \qquad ...(50)$$

$$\therefore \quad I_1' = \mathbf{Z}_3 E/[(\mathbf{Z}_1 + \mathbf{Z}_0)\,(\mathbf{Z}_2 + \mathbf{Z}_3) + \mathbf{Z}_2\mathbf{Z}_3]. \qquad ...(51)$$

Thus we see that $I_2 = I_1'$, as required by the *reciprocity theorem*. The ratio of emf in one mesh (or branch) to the current in another mesh (or branch) is called the transfer impedance, represented by $\mathbf{Z}_{tr}$.

In terms of transfer impedance, Eqns. (48) and (51) give

$$E/I_2 = E/I_1'$$

$$\text{or} \quad (\mathbf{Z}_{tr})_{12} = (\mathbf{Z}_{tr})_{21}. \qquad ...(52)$$

Thus we see that the transfer impedance is invariant to the change of position of voltage source.

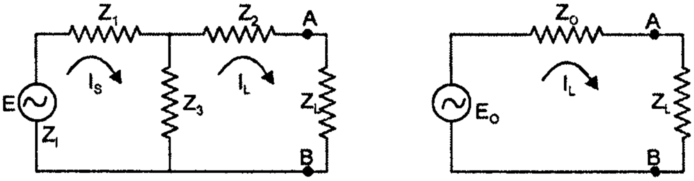
Eq. (52) may be written as

$$I_2/E = I_1'/E, \qquad ...(53)$$

i.e., the ratio of response transform to excitation transform is invariant to an interchange of the position in the network of the excitation and the response. It is the another way to state the theorem.

3. Thevenin's Theorem. *Any two terminals linear network containing generators (voltage sources) and impedances may be replaced with an equivalent circuit consisting of voltage source E_0 in series with an impedance $\mathbf{Z}_0$. The value of E_0 is the open circuit (when there is no load) voltage between the terminals of the network and $\mathbf{Z}_0$ is the impedance appearing across the terminals when all generators in the network have been replaced by their internal impedances. All circuit elements must be linear bilateral.*

Consider two terminal network containing two meshes, one active and other passive. If load impedance $\mathbf{Z}_L$ is placed between the terminals, then the mesh equations are written as



(a) (b)

Fig. 18.7. Networks for Thevenin's theorem.

$$E = (\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i) I_S - \mathbf{Z}_3 I_L \qquad \text{...(54)}$$

$$0 = -\mathbf{Z}_3 I_S + (\mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_L) I_L \qquad \text{...(55)}$$

Solving these Eqs. for the currents, we get

$$I_L = \frac{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i & E \\ -\mathbf{Z}_3 & 0 \end{vmatrix}}{\begin{vmatrix} \mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i & -\mathbf{Z}_3 \\ -\mathbf{Z}_3 & \mathbf{Z}_2 + \mathbf{Z}_3 + \mathbf{Z}_L \end{vmatrix}}$$

$$= \frac{E\mathbf{Z}_3}{(\mathbf{Z}_2 + \mathbf{Z}_L)(\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i) + \mathbf{Z}_3(\mathbf{Z}_1 + \mathbf{Z}_i)}$$

$$= \frac{E\mathbf{Z}_3/(\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i)}{\mathbf{Z}_2 + \mathbf{Z}_L + \mathbf{Z}_3 (\mathbf{Z}_1 + \mathbf{Z}_i)/(\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i)}. \qquad \text{...(56)}$$

Fig. 18.7(*a*) shows that the open circuit voltage across *AB* terminals is

$$E_0 = E\mathbf{Z}_3/(\mathbf{Z}_1 + \mathbf{Z}_3 + \mathbf{Z}_i), \qquad \text{...(57)}$$

and the total impedance between terminals when the generator is replaced by its internal impedance

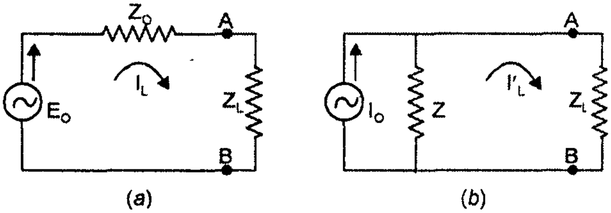
$$\mathbf{Z}_0 = \mathbf{Z}_2 + \frac{1}{1/\mathbf{Z}_3 + 1/(\mathbf{Z}_1 + \mathbf{Z}_i)} = \mathbf{Z}_2 + \frac{\mathbf{Z}_3(\mathbf{Z}_1 + \mathbf{Z}_i)}{\mathbf{Z}_1 + \mathbf{Z}_i + \mathbf{Z}_3}. \qquad \text{...(58)}$$

These quantities E_0 and $\mathbf{Z}_0$ are called Thevenin's components. Thus the Eq. (56) reduces in terms of these components, as

$$I_L = E_0/(\mathbf{Z}_0 + \mathbf{Z}_L). \qquad \text{...(59)}$$

The current is same as obtained from an equivalent circuit, known as *Thevenin's equivalent network*, consisting of a voltage source $\mathbf{E}_0$ in series with an impedance $\mathbf{Z}_0$, hence is the theorem. It may be generalized to any number of generators by using superposition theorem.

(4) Norton's Theorem : *Any two terminal active network, consisting of linear impedances and generators may be replaced by a current generator in parallel with an impedance. The current is equal to that obtained through a short circuit across the two terminals and the impedance is that viewed across the two terminals when all the generators are replaced by their internal impedances.*



(a) (b)

Fig. 18.8. Networks for Norton's theorem.

This theorem may be easily proved by considering a Thevenin's equivalent network shown in Fig. 18.8(a). Now this may be replaced by a circuit consisting current source I_0 in parallel with an impedance Z appearing between two terminals A and B. The current source is called the Norton's current and the impedance $\mathbf{Z}$ is called the Norton's impedance. The current flowing in the network (a) is given by

$$I_L = E_0/(\mathbf{Z}_0 + \mathbf{Z}_L). \quad \text{...(60)}$$

Applying the current division law in network Fig. 18.8(b), we have

$$I_L' = \mathbf{Z}I_0/(\mathbf{Z} + \mathbf{Z}_L). \quad \text{...(61)}$$

If $\mathbf{Z} = \mathbf{Z}_0$, the load current I_L can be made equal to I_L. Thus from above relations we have

$$I_0 = E_0/\mathbf{Z}_0. \quad \text{...(62)}$$

which is the short circuit current of the Thevenin's equivalent network. Thus we see that interchange of voltage and current sources with the help of Thevenin's and Norton's theorems gives a method of network analysis. The main difference between these is *that in the former case a voltage source is removed from a circuit by short circuiting its emf while in the latter case current source is removed by opening circuit.* Both these equivalent circuits can be used to analyse the complicated problems, however the former is used when the load impedance is greater than the source impedance, while the latter is used when the sources have high internal impedances.

(5) Maximum Power-Transfer Theorem : *This theorem states that the power received by one network from another connected network is maximum when the impedance of the receiving network is conjugate to the impedance of the connected network looking at its* two terminals. The process of bringing about this condition is called impedance matching.

Let us consider a two terminal active linear network connected to a two terminal passive linear network. The former can be replaced by Thevenin's equivalent active network consisting of open circuit potential and the total impedance when all the generators are replaced by their internal impedances and the latter networks is replaced by equivalent impedance $\mathbf{Z}_R$. Let the impedances $\mathbf{Z}_0$ and $\mathbf{Z}_R$ are of complex nature, *i.e.*,

$$\mathbf{Z}_0 = R_0 + jX_0 \text{ and } \mathbf{Z}_R = R_R + jX_R. \quad \text{...(63)}$$

Thus the current flowing in the network

$$I = \frac{E_0}{\mathbf{Z}_0 + \mathbf{Z}_R} = \frac{E_0}{R_0 + R_R + j(X_0 + X_R)} \quad \text{...(64)}$$

$$\therefore \quad I = E_0 / \sqrt{(R_0 + R_R)^2 + (X_0 + X_R)^2}$$

or

$$I_{rms} = E_{rms} / \sqrt{(R_0 + R_R)^2 + (X_0 + X_R)^2}$$

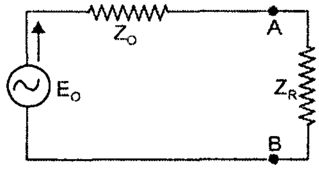


Fig. 18.9. Networks for maximum power transfer theorem.

The power delivered to the passive network or to the impedance $\mathbf{Z}_R$ is given as

$$P = I_{rms} Z_R I_{rms} \cos\phi = I_{rms}{}^2 Z_R \cos\phi$$

Since

$$Z_R = \sqrt{R_R{}^2 + X_R{}^2} \text{ and } \cos\phi = R_R / \sqrt{R_R{}^2 + X_R{}^2}$$

$$\therefore \quad P = I_{rms}{}^2 R_R = \frac{E_{rms}{}^2 R_R}{(R_0 + R_R)^2 + (X_0 + X_R)^2}. \quad \text{...(65)}$$

If the impedance of receiving network, *i.e.*, $\mathbf{Z}_R$ $(= R_R + jX_R)$ is varied then the maximum power transfer may be calculated first by considering X_R variable and then R_R only.

(*i*)If X_R is varying, then power will be maximum when

$$\frac{\partial P}{\partial X_R} = \frac{-E_{rms}{}^2 R_R (2)(X_0 + X_R)}{[(R_0 + R_R)^2 + (X_0 + X_R)^2]^2} = 0.$$

or

$$X_R + X_0 = 0 \text{ or } X_R = -X_0.$$

$$\therefore \quad P_{max} = E_{rms}{}^2 R_R/(R_R + R_0)^2. \quad \text{...(66)}$$

(*ii*)If R_R is varying, then power transfer will be maximum, when

$$\frac{\partial P}{\partial R_R} = \frac{E_{rms}{}^2 [(R_R + R_0)^2 - 2R_R (R_R + R_0)]}{(R_R + R_0)^4} = 0.$$

or

$$R_R = R_0.$$

$$\therefore \quad P_{max} = E_{rms}{}^2/4R_0. \quad \text{...(67)}$$

Hence the total impedance of the passive network

$$\mathbf{Z}_R = R_R + jX_R = R_0 - jX_0, \quad \text{...(68)}$$

which is conjugate to the impedance of the active network looking at two terminals, *i.e.*, to the equivalent impedance $\mathbf{Z}_0 = R_0 + jX_0$.

$$\text{Power generated from the generator} = \frac{E_{rms}{}^2}{\text{Total resistance}} = \frac{E_{rms}{}^2}{R_0 + jX_0 + R_0 - jX_0}$$

$$= E_{rms}{}^2/2R_0. \quad \text{...(69)}$$

Thus we see that the maximum power is transferred to the passive network when the impedance of this network is conjugate to the impedance of the active network. The amount of this power is half to the total power generated. In other words we can say that the power lost in the internal generator is equal to the power delivered to the receiving network.

(6) Millman's Theorem : It is just an extension of Norton's theorem and is very useful in simplifying electronic circuits. This theorem provides a simple method for finding the common voltage across any number of parallel branches with different voltage sources, Fig. 18.10(*a*).

This theorem states that *a set of parallel voltage sources is equivalent to a single voltage source called Millman equivalent voltage source of emf*

$$\mathbf{E}_M = \frac{(\mathbf{E}_1/\mathbf{Z}_1) + (\mathbf{E}_2/\mathbf{Z}_2) + (\mathbf{E}_3/\mathbf{Z}_3) + \ldots}{(1/\mathbf{Z}_1) + (1/\mathbf{Z}_2) + (1/\mathbf{Z}_3) + \ldots} \quad \text{...(70)}$$

and *impedance* $\mathbf{Z}_M$ given by

$$1/\mathbf{Z}_M = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3 + \ldots \quad \text{...(71)}$$

To prove it, all parallel voltage sources are converted into equivalent current sources as shown in Fig. 18.10(*b*).

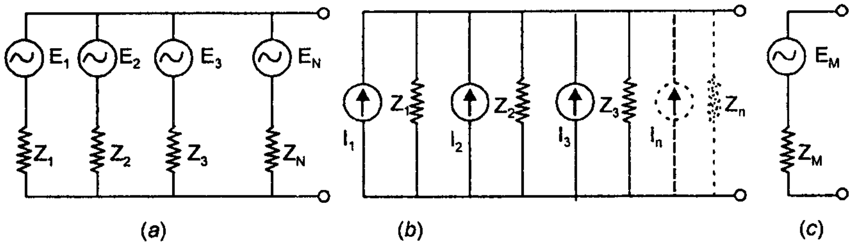


Fig. 18.10(*a*). Voltage sources in parallel, (*b*) Equivalent current sources and (*c*) Millman equivalent circuit.

The current supplied by the i^{th} source is $\mathbf{I}_i = \mathbf{E}_i/\mathbf{Z}_i$.

Therefore the phasor sum of the currents supplied by the current sources

$$\mathbf{I}_T = \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \ldots \mathbf{I}_n, \qquad \ldots(72)$$

Sum of impedances of the current sources is given as

$$1/\mathbf{Z}_T = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3 + ..1/\mathbf{Z}_n. \qquad \ldots(73)$$

In terms of admittances of the current sources

$$\mathbf{Y}_T = \mathbf{Y}_1 + \mathbf{Y}_2 + \mathbf{Y}_3 + \ldots \mathbf{Y}_n. \qquad \ldots(74)$$

Thus the circuit of equivalent current sources is equivalent to a current source with parameters

$$\mathbf{I}_C = \mathbf{I}_T \text{ and } \mathbf{Z}_C = \mathbf{Z}_T = 1/\mathbf{Y}_T. \qquad \ldots(75)$$

The current to voltage conversion relations give the parameters of Millman equivalent voltage source [Fig. 18.10(*c*), as

$$\mathbf{E}_M = \mathbf{I}_C\mathbf{Z}_C = \mathbf{I}_T\mathbf{Z}_T = \mathbf{I}_T/\mathbf{Y}_T \qquad \ldots(76)$$

and

$$\mathbf{Z}_M = \mathbf{Z}_C = 1/\mathbf{Y}_T. \qquad \ldots(77)$$

Eqns. (76) and (77) are same as that of equations (70) and (71) respectively.

7. Compensation Theorem : *This theorem states that any impedance in a network may be replaced by a generator of zero internal impedance and of emf equal to the potential difference across the replaced impedance at every instant.*

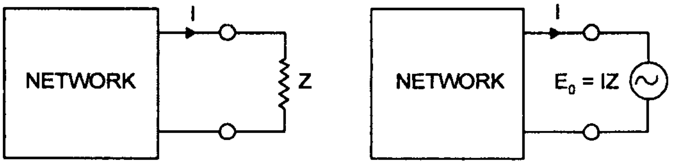


Fig. 18.11(*a*). Network with an impedance across two terminals, (*b*) network with the source equivalent to this impedance.

Fig. 18.11(*a*) represents a network of linear impedances and generators connected with an impedance **Z**, considered as load impedance. This impedance may be replaced by a generator of emf equal to the potential difference across **Z**, *i.e.*, $e_0 = \mathbf{IZ}$.

This theorem applies to the effect produced on the network currents by a change Δ **Z** in the impedance of a network branch. This effect may be found by replacing Δ **Z** with a generator of emf = **I** Δ **Z**, where **I** is the original current which flowed in the branch.

18.4 FOUR TERMINAL NETWORK

A *four terminal network*, also called *two terminal pair network*, is treated as a black box with two input terminals and two output terminals, as shown in Fig. 18.12. The black box may contain linear network (either active or passive) such as transformer, filter, transistor and amplifier. A four terminal network is also called *two port network*, if for all t and all possible external connections at these terminals, the current entering the network by terminal 1 is equal to that leaving the network by terminal 1′ and the current entering the network by terminal 2 is equal to that leaving the network by terminal 2′. Thus a four terminal network is more general than a two port network. The port at the left is usually referred to as the *input port*, whereas the port at the right is usually referred to as the *output port*.

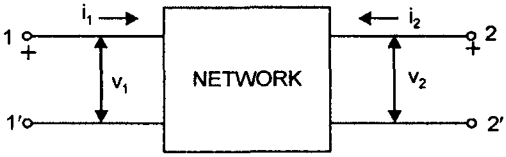


Fig. 18.12. Four terminal network.

In applying two port network theory, the actual two port is transformed into an equivalent two port model with specially defined parameters. The most commonly used such parameters are impedance parameters, admittance parameters and hybrid parameters. The first two are used when elements of the two port consists of resistors, inductors and capacitors. Hybrid parameters are generally used in transistors, amplifiers, etc. The equivalent two port parameters may be calculated from current and voltage measurements made at the input and output terminals. Let the terminal quantities are v_1, i_1, v_2 and i_2. Any pair of these variables may be arbitrarily chosen as independent, giving two equations. The solution of these equations give two other dependent variables. We may choose two independent variables out of the four in different ways. Let us discuss three important cases in details as under.

(*a*) **Impedance Parameters :** Making the choice of i_1 and i_2 as independent variables implies the general relations.

$$v_1 = f(i_1, i_2) \text{ and } v_2 = f(i_1, i_2).$$

These may be written as

$$dv_1 = (\partial v_1/\partial i_1)\, di_1 + (\partial v_1/\partial i_2)\, di_2$$
$$dv_2 = (\partial v_2/\partial i_1) di_1 + (\partial v_2/\partial i_2)\, di_2.$$

For sufficiently small a-c signals these equations may be written as

$$\mathbf{V}_1 = z_{11}\mathbf{I}_1 + z_{12}\mathbf{I}_2.$$
$$\mathbf{V}_2 = z_{21}\mathbf{I}_1 + z_{22}\mathbf{I}_2,$$

or
$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{V}_2\end{bmatrix} = \begin{bmatrix} z_{11} & z_{12}\\ z_{21} & z_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{I}_2\end{bmatrix}. \qquad ...(78)$$

Here z-parameters have been defined as the slopes of the v-i relations and are known as *open circuit impedance parameters* of the network and are useful in general network analysis but not for transistor studies.

Using $\mathbf{I}_1 = 0$ and $\mathbf{I}_2 = 0$ in equation (78), we have

$$z_{11} = \partial v_1/\partial i_1 = \text{Open circuit input impedance } (\mathbf{I}_2 = 0)$$
$$z_{12} = \partial v_1/\partial i_2 = \text{Open circuit reverse transfer impedance } (\mathbf{I}_1 = 0)$$
$$z_{21} = \partial v_2/\partial i_1 = \text{Open circuit forward transfer impedance } (\mathbf{I}_2 = 0)$$
$$z_{22} = \partial v_2/\partial i_2 = \text{Open circuit output impedance } (\mathbf{I}_1 = 0).$$

Eqation (78) may be written as

$$\mathbf{V} = \mathbf{ZI}, \text{ where the matrix}$$

$$\mathbf{Z} = \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \qquad \text{...(79)}$$

is called the *open circuit impedance matrix* of the two port network.

Z-parameters may be determined experimentally from phasor current and phasor voltage measurements made during open circuit tests.

(*b*) **Admittance Parameters :** If the choice of v_1 and v_2 as the independent variables is made, then the general relations become

$$di_1 = (\partial i_1/\partial v_1)\, dv_1 + (\partial i_1/\partial v_2)\, dv_2.$$
$$di_2 = (\partial i_2/\partial v_1)\, dv_1 + (\partial i_2/\partial v_2)\, dv_2.$$

For sufficiently small a-c signals where the partial derivatives are constant, we have

$$\mathbf{I}_1 = y_{11}\mathbf{V}_1 + y_{12}\mathbf{V}_2$$
$$\mathbf{I}_2 = y_{21}\mathbf{V}_1 + y_{22}\mathbf{V}_2.$$

or
$$\begin{bmatrix} \mathbf{I}_1 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_1 \\ \mathbf{V}_2 \end{bmatrix} \qquad \text{...(80)}$$

Here y-parameters are known as *short circuit* admittance parameters. Using $\mathbf{V}_1 = 0$ and $\mathbf{V}_2 = 0$ in equation (80) we have

$y_{11} = \partial i_1/\partial v_1$ = Short circuit input admittance $(\mathbf{V}_2 = 0)$
$y_{12} = \partial i_1/\partial v_2$ = Short circuit reverse transfer admittance $(\mathbf{V}_1 = 0)$
$y_{21} = \partial i_2/\partial v_1$ = Short circuit forward transfer admittance $(\mathbf{V}_2 = 0)$
$y_{22} = \partial i_2/\partial v_2$ = Short circuit output admittance $(\mathbf{V}_1 = 0)$

Eqn (80) may be written as $\mathbf{I} = \mathbf{YV}$, where the matrix

$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \qquad \text{...(81)}$$

is called the *short circuit admittance matrix* of the two port network.

y-parameters may be determined experimentally from phasor current and phasor voltage measurements made during short circuit tests.

It is easy to prove from Eqns (79) and (81) that

$$\mathbf{Z} = \mathbf{Y}^{-1} \text{ or } \mathbf{Y} = \mathbf{Z}^{-1} \qquad \text{...(82)}$$

$$\therefore \quad z_{11} = \frac{y_{22}}{\Delta_Y},\ z_{12} = -\frac{y_{12}}{\Delta_Y},\ z_{21} = -\frac{y_{21}}{\Delta_Y},\ z_{22} = \frac{y_{11}}{\Delta_Y}$$

$$y_{11} = \frac{z_{22}}{\Delta_Z},\ y_{12} = -\frac{z_{12}}{\Delta_Z},\ y_{21} = -\frac{z_{21}}{\Delta_Z},\ y_{22} = \frac{z_{11}}{\Delta_Z}$$

where $\Delta_Y = \det|\mathbf{Y}|$ and $\Delta_z = \det|\mathbf{Z}|$.

(*c*) **Hybrid Parameters :** The third choice of i_1 and v_2 as independent variables leads to a set of network parameters of considerable value in transistor circuit analysis. It gives

$$v_1 = f(i_1, v_2),\ i_2 = f(i_1, v_2)$$
$$dv_1 = (\partial v_1/\partial i_1)\, di_1 + (\partial v_1/\partial v_2)\, dv_2$$

and
$$di_2 = (\partial i_2/\partial i_1)\, di_1 + (\partial i_2/\partial v_2)\, dv_2.$$

For sufficiently small a-c signals where the partial derivatives become constants, we have

$$\begin{aligned}\mathbf{V}_1 &= h_{11}\,\mathbf{I}_1 + h_{12}\,\mathbf{V}_2\\ \mathbf{I}_2 &= h_{21}\,\mathbf{I}_1 + h_{22}\,\mathbf{V}_2.\end{aligned}$$

or

$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{I}_2\end{bmatrix} = \begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix}\begin{bmatrix}\mathbf{I}_1\\ \mathbf{V}_2\end{bmatrix} \qquad ...(83)$$

Here h-parameters are known as *hybrid parameters*, as they are mixed parameters compared to purely impedance and admittance parameters. These are defined under either open or short circuit conditions. We have $\mathbf{V}_2 = 0$ for a short circuit and $\mathbf{I}_1 = 0$ for an open circuit. Thus we have two sets of h-parameters, as defined below :

$$\begin{aligned}h_{11} &= \partial v_1/\partial i_1 = \text{Short circuit input impedance} && (\mathbf{V}_2 = 0)\\ h_{12} &= \partial v_1/\partial v_2 = \text{Open-circuit reverse voltage gain} && (\mathbf{I}_1 = 0)\\ h_{21} &= \partial i_2/\partial i_1 = \text{Short-circuit forward current gain} && (\mathbf{V}_2 = 0)\\ h_{22} &= \partial i_2/\partial v_2 = \text{Open-circuit output admittance.} && (\mathbf{I}_1 = 0)\end{aligned}$$

Here we see that one parameter is an impedance, one an admittance and two are dimensionless ratios. The condition for the two port to be reciprocal is that $h_{21} = -h_{12}$.

In equation (83) we may define the matrix

$$\begin{bmatrix}h_{11} & h_{12}\\ h_{21} & h_{22}\end{bmatrix} = \mathbf{H}. \qquad ...(84)$$

The matrix **H** is called a *hybrid matrix*. The hybrid matrices describe a two port when the voltage of one port and the current of the other are taken as the independent variables.

The h-parameters may be determined experimentally from phasor current and phasor voltage measurements made during output port short-circuited and input port open-circuited.

These three sets of parameters have been found useful in electronic circuit analysis. They can be measured under the specified conditions or evaluated from the characteristics. If one set of parameters are given, other two can be calculated using equations (78), (80) and (83). One can find

$$\begin{aligned}z_{11} &= h_{11} - h_{12}h_{21}/h_{22};\ z_{12} = h_{12}/h_{22}.\\ z_{21} &= -h_{21}/h_{22};\ z_{22} = 1/h_{22}.\end{aligned} \qquad ...(85)$$

Conversion chart of two port matrices is shown in Table 18.1.

Table 18.1. *Conversion chart of matrices*

	Z		**Y**		**H**	
Z	z_{11}	z_{12}	z_{22}/Δ_Y	$-y_{12}/\Delta_Y$	Δ_H/h_{22}	h_{22}/h_{22}
	z_{21}	z_{22}	$-y_{21}/\Delta_Y$	y_{11}/Δ_Y	$-h_{21}/h_{22}$	$1/h_{22}$
Y	z_{22}/Δ_z	$-z_{12}/\Delta_z$	y_{11}	y_{12}	$1/h_{11}$	$-h_{12}/h_{11}$
	$-z_{21}/\Delta_z$	z_{11}/Δ_z	y_{21}	y_{22}	h_{21}/h_{11}	Δ_H/h_{11}
H	Δ_z/z_{22}	z_{12}/z_{22}	$1/y_{11}$	$-y_{12}/y_{11}$	h_{11}	h_{12}
	$-z_{21}/z_{22}$	$1/z_{22}$	y_{21}/y_{11}	Δ_Y/y_{11}	h_{21}	h_{22}

For each set of equations represented in matrix form we may write down an equivalent circuit. The equivalent circuit that satisfies equation (78) is shown in Fig. 18.13(*a*) and has been called the z-equivalent or z-parameter equivalent circuit (model). Each of the z-parameters have the dimensions of impedance and the quantities $z_{12}\mathbf{I}_2$ and $z_{21}\mathbf{I}_1$ are dependent voltage sources. The z-parameters are not particularly only for transistor studies.

The equivalent circuit that satisfies Eq. (80) is shown in Fig. 18.13(*b*) and is known as the y-equivalent. Each of the y-parameters have the dimensions of admittance. $y_{12}\mathbf{V}_2$ and $y_{21}\mathbf{V}_1$ are the two dependent current generators. The y-parameters are primarily applicable at high frequencies and are often employed to describe a transistor at a particular frequency.

The equivalent circuit that satisfies Eq. (83) is shown in Fig. 18.13(*c*) and is known as the ***hybrid (or h) parameters equivalent.*** Since the defining equations must obey Kirchhoff's laws, hence h_{11} must be an impedance, h_{22} an admittance while h_{12} and h_{21} are dimensionless. For low-frequency analysis of junction transistors h_{11} and h_{22} will be resistive.

Besides the above cases. Three following cases are also possible :

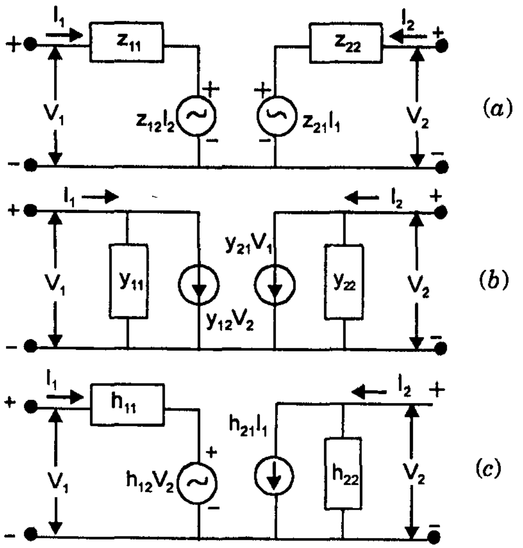


Fig. 18.13. Equivalent circuits, (*a*) z-parameter model, (*b*) y-parameter model and (*c*) h-parameter model.

(*d*) Choosing $\mathbf{V}_1$ and $\mathbf{I}_2$ as independent variables, we get

$$\mathbf{I}_1 = g_{11}\mathbf{V}_1 + g_{12}\mathbf{I}_2$$
$$\mathbf{V}_2 = g_{21}\mathbf{V}_1 + g_{22}\mathbf{I}_2$$

or
$$\begin{bmatrix}\mathbf{I}_1\\ \mathbf{V}_2\end{bmatrix} = \begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ \mathbf{I}_2\end{bmatrix}, \quad \text{...(86)}$$

where the matrix $\begin{bmatrix}g_{11} & g_{12}\\ g_{21} & g_{22}\end{bmatrix}$ is called 'G' matrix.

(*e*) Choosing $\mathbf{V}_2$ and $\mathbf{I}_2$ as independent variables, we get

$$\mathbf{V}_1 = a_{11}\mathbf{V}_2 - a_{12}\mathbf{I}_2$$
$$\mathbf{I}_1 = a_{21}\mathbf{V}_2 - a_{22}\mathbf{I}_2$$

or
$$\begin{bmatrix}\mathbf{V}_1\\ \mathbf{I}_1\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_2\\ -\mathbf{I}_2\end{bmatrix} \quad \text{...(87)}$$

where the matrix $\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$ is called 'A' matrix.

(*f*) Choosing $\mathbf{V}_1$ and $\mathbf{I}_1$ as independent variables we get

$$\mathbf{V}_2 = b_{11}\mathbf{V}_1 - b_{12}\mathbf{I}_1$$
$$\mathbf{I}_2 = b_{21}\mathbf{V}_1 - b_{22}\mathbf{I}_1$$

or
$$\begin{bmatrix}\mathbf{V}_2\\ \mathbf{I}_2\end{bmatrix} = \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}\mathbf{V}_1\\ -\mathbf{I}_1\end{bmatrix} \quad \text{...(88)}$$

where the matrix $= \begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}$ is called 'B' matrix.

Cascaded Networks : In the case of cascaded networks, we generally use 'A' matrices, as the current direction is in the forward direction (*i.e.*, outward) at the output port. Thus for the system of two networks in series, as shown in Fig. 18.14, we have

$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \mathbf{V}_2 \\ -\mathbf{I}_2 \end{bmatrix}$$

and
$$\begin{bmatrix} \mathbf{V}_2 \\ \mathbf{I}_2 \end{bmatrix} = \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \begin{bmatrix} \mathbf{V}_3 \\ -\mathbf{I}_3 \end{bmatrix}$$

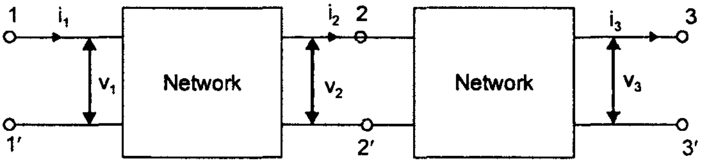


Fig. 18.14. Cascaded network with two stages.

or
$$\begin{bmatrix} \mathbf{V}_1 \\ \mathbf{I}_1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \begin{bmatrix} \mathbf{V}_3 \\ \mathbf{I}_3 \end{bmatrix} = \begin{bmatrix} a_{11}^T & a_{12}^T \\ a_{21}^T & a_{22}^T \end{bmatrix} \begin{bmatrix} \mathbf{V}_3 \\ \mathbf{I}_3 \end{bmatrix}$$

$$\therefore \quad \begin{bmatrix} a_{11}^T & a_{12}^T \\ a_{21}^T & a_{22}^T \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{bmatrix} \qquad \text{...(89)}$$

In this way, we may add large number of networks.

EXERCISES

Example 1. *A resistance of 10Ω is in series with an inductive reactance of 6.28Ω and the combination is in parallel with a circuit consisting of a resistance of 20Ω in series with a capacitance of 100 μF. The whole is in series with a resistance of 15Ω and an inductive reactance 12.56Ω. Calculate the equivalent resistance, reactance and impedance of the whole circuit.*

Since the frequency of the supply is not given, let us assume it as 50 c/s. The reactance of the capacitor

$$\mathbf{X}_2 = 1/j\omega C = -j/314 \times 10^{-4} = -j31.85.$$

For branch I, $\mathbf{Z}_1 = 10 + j\,6.28$

and for branch II, $\mathbf{Z}_2 = 20 - j\,31.85.$

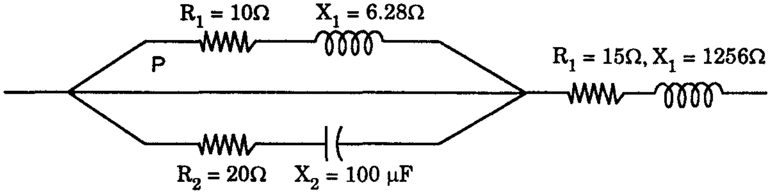


Fig. 18.15. Example 1.

Since these branches are in parallel, the total impedance of the combination in parallel is given as

$$\frac{1}{\mathbf{Z}_C} = \frac{1}{\mathbf{Z}_1} + \frac{1}{\mathbf{Z}_2} = \frac{1}{10+j6.28} + \frac{1}{20-j31.85}$$

$$\text{or} \quad \mathbf{Z}_C = \frac{(10+j6.28)(20-j31.85)}{30-j25.57} = \frac{(400-j192.9)(30+j25.57)}{30^2+25.57^2}$$

$$= 10.90 + j\,2.86$$

It is in series with an impedance $\mathbf{Z}_3 = 15 + j\,12.56$

$$\therefore \quad \mathbf{Z}_{\text{total}} = \mathbf{Z}_C + \mathbf{Z}_3 = 25.90 + j\,15.42$$

$$= R_{total} + jX_{total}$$

$\therefore$ The resistance $R = 25.90\Omega$, total reactance $X = 15.42\Omega$

and total impedance $Z = (25.90^2 + 15.42^2)^{1/2} = 30.14\Omega$.

Example 2. *Calculate open circuit voltage and impedance between two terminals E and F in the network shown in Fig. 18.16. Calculate also the load current flowing through a load impedance* $\mathbf{Z}_L = 10 - j\,7.5$.

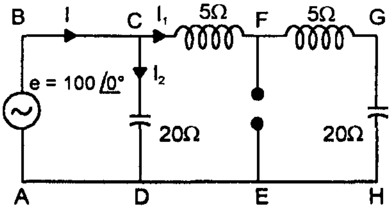


Fig. 18.16. Example 2.

For calculating the impedance $\mathbf{Z}_0$, between terminals E and F, let us remove potential source e and short circuit this branch.

Thus we have

$$\mathbf{Z}_0 = \frac{(j5-j20)(j5)}{(j5-j20)+(j5)} = j7.5.$$

The total impedance across the source e is

$$\mathbf{Z} = \frac{\mathbf{Z}_{CD}\mathbf{Z}_{CGH}}{\mathbf{Z}_{CD}+\mathbf{Z}_{CGH}} = \frac{(-j20)(j5+j5-j20)}{-j20+(j5+j5-j20)} = -j6.66.$$

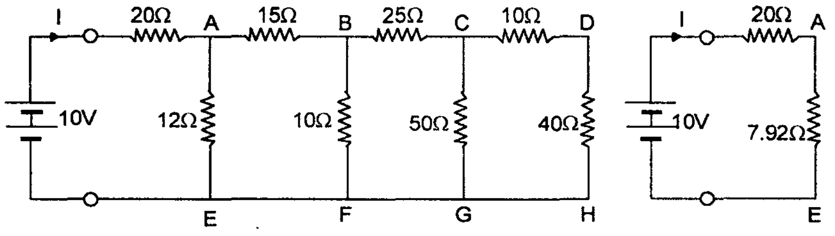
$$\therefore \text{ Current in } BC \text{ branch, } \mathbf{I} = \frac{e}{\mathbf{Z}} = \frac{100\ \angle 0^\circ}{-j6.66} = \frac{100\ \angle 0^\circ}{6.66\ \angle -90^\circ} = 15\ \angle 90^\circ = j15.$$

$$\text{Current in } CGH \text{ branch, } I_1 = \frac{-j20}{-j30} \times j15 = j10 = 10\ \angle 90^\circ$$

Hence the pd across E and F terminals $= \mathbf{Z}_{FGH} \times I_1 = -j15 \times j10 = 150\ \angle 0^\circ$

$$\therefore \quad \text{Load current } \mathbf{I}_L = \frac{150\ \angle 0^\circ}{j7.5+10-j7.5} = 15\ \angle 0^\circ.$$

Example 3. *For the circuit shown in Fig. 18.17, determine (a) the driving point impedance (input resistance), (b) feeder current and (c) the voltage drop across the 50Ω resistor.*



(*a*) (*b*)

Fig. 18.17(*a*). Given ladder circuit, (*b*) Equivalent circuit.

(*a*) Starting at the section *CDHG* farthest from the driver, in which 10Ω and 40Ω are in series, thus $R_{CDH} = 10 + 40 = 50\Omega$ and the combination is in parallel with the 50Ω resistor. The relation for the combination of two parallel resistances gives

$$R_{C.G.} = \frac{R_{CG} \times R_{CDH}}{R_{CG} + R_{CDH}} = \frac{50 \times 50}{50+50} = 25\Omega.$$

This is in series with the 25Ω resistor, thus $R_{BCG} = 25 + 25 = 50\Omega$. This combination is in parallel with the 10Ω resistor. Therefore, we get

$$R_{B.F.} = \frac{R_{BF} \times R_{BCG}}{R_{BF} + R_{BCG}} = \frac{10 \times 50}{10 + 50} = \frac{25}{3} = 8.33\Omega.$$

This is in series with the 15 Ω resistor, thus $R_{ABF} = 15 + 8.33 = 23.33\ \Omega$. This combination is in parallel with the 12 Ω resistor. Therefore, we get

$$R_{A.E.} = \frac{R_{AE} \times R_{ABF}}{R_{AE} + R_{ABF}} = \frac{12 \times 23.33}{12 + 23.33} = 7.92\Omega$$

$$\therefore \quad R_{in} = 20 + 7.92 = 27.92\ \Omega.$$

(*b*) Thus the feeder current $I = \frac{V}{R_{in}} = \frac{10}{27.92} = 0.358$ amp.

(*c*) To determine the voltage drop across the 50Ω resistor, the current through it is required. There are several methods for finding this value. Let us make use of Kirchhoff's current law at node *A*.

$$I = I_{AE} + I_{AB}.$$

Using the current divider equation,

$$I_{AB} = I\frac{R_{AE}}{R_{AE} + R_{ABF}} = \frac{0.358 \times 12}{12 + 23.33} = 0.122 \text{ amp.}$$

Applying Kirchhoff's current law to node *B*.

$$I_{AB} = I_{BF} + I_{BC}.$$

Using the current divider equation

$$I_{BC} = I_{AB}\frac{R_{BF}}{R_{BF} + R_{BCG}} = \frac{0.122 \times 10}{10 + 50} = 0.020 \text{ amp.}$$

Applying Kirchhoff's current law to node *C*,

$$I_{BC} = I_{CG} + I_{CD}.$$

Using the current divider equation

$$I_{CG} = I_{BC}\frac{R_{CDH}}{R_{CG} + R_{CDH}} = \frac{0.020 \times 50}{50 + 50} = 0.010 \text{amp.}$$

∴ Voltage across 50 Ω resistor = $I_{CG} \times R_{CG} = 0.010 \times 50 = 0.5$ Volt.

Example 4. *Design a Π network, which is equivalent to the under mentioned 'T' network.*

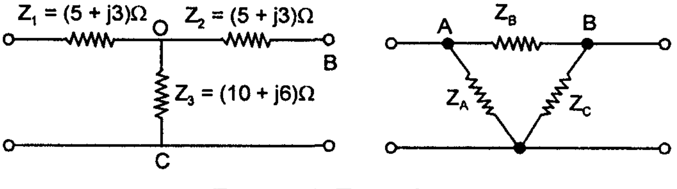


Fig. 18.18. Example 4.

We know that the impedances of the Π circuit may be determined if we have the impedances of the *T*-network with the following relations.

$$\mathbf{Z}_A = \square/\mathbf{Z}_2,\ \mathbf{Z}_B = \square/\mathbf{Z}_3 \text{ and } \mathbf{Z}_C = \square/\mathbf{Z}_1,$$

where $\square = \mathbf{Z}_1\mathbf{Z}_2 + \mathbf{Z}_2\mathbf{Z}_3 + \mathbf{Z}_3\mathbf{Z}_1.$

In this problem $\mathbf{Z}_1 = 5 + j3,\ \mathbf{Z}_2 = 5 + j3 \text{ and } \mathbf{Z}_3 = 10 + j6$

$$\therefore \quad \square = (5 + j\,3)(5 + j\,3) + (5 + j\,3)(10 + j\,6) + (10 + j\,6)(5 + j\,3)$$

$$= 16 + 30j + 32 + j\,60 + 32 + j\,60 = 80 + j\,150.$$

Therefore
$$\mathbf{Z}_A = \frac{80+j150}{5+j3} = \frac{(80+j150)(5-j3)}{(5+j3)(5-j3)} = 25+j15$$

$$\mathbf{Z}_B = \frac{80+j150}{10+j6} = \frac{(80+j150)(10-j6)}{(10+j6)(10-j6)} = 12.5+j7.5$$

and
$$\mathbf{Z}_C = \frac{80+j150}{5+j3} = \frac{(80+j150)(5-j3)}{(5+j3)(5-j3)} = 25+j15.$$

Example 5. *Find the current drawn from the battery of 20 volts connected to a circuit as shown in Fig. 18.19(a).*

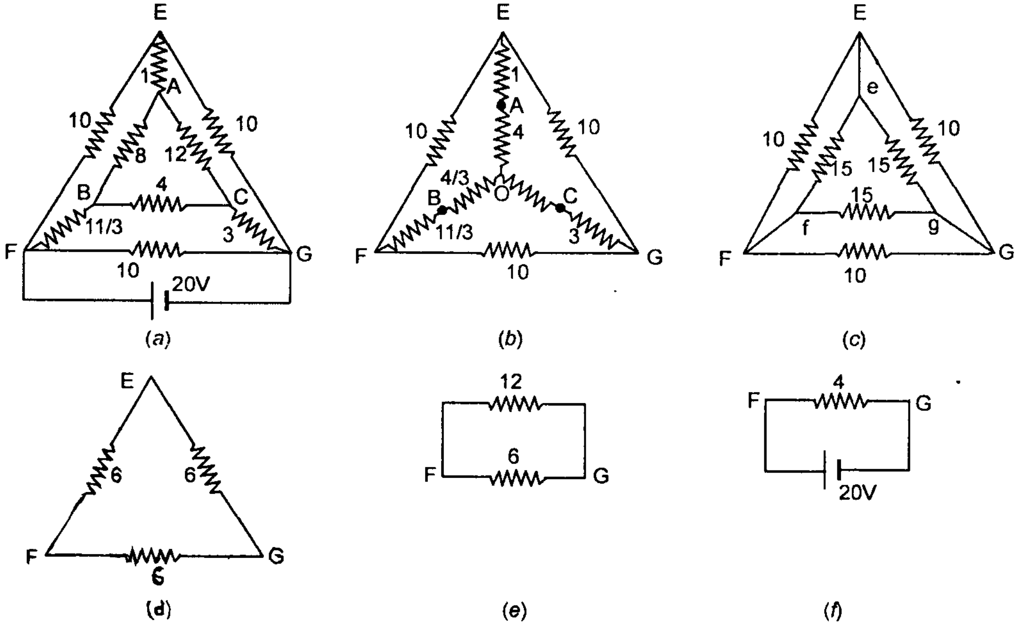
Let us first reduce the delta connection *ABC* to a star connection as shown in Fig.18.19 (*b*). If *O* be the central point of the star (or *T*) circuit, then the resistances *OA*, *OB* and *OC* may be obtained by using equations (13), as given below. Since the circuit consists of pure resistances, hence the impedances **Z**'s are to be replaced by *R*'s.

$$\therefore \qquad R_{OA} = \frac{R_{AC}\times R_{AB}}{R_{AC}+R_{AB}+R_{BC}} = \frac{12\times 8}{12+8+4} = 4\Omega,$$

$$R_{OB} = \frac{R_{AB}\times R_{BC}}{R_{AC}+R_{AB}+R_{BC}} = \frac{8\times 4}{12+8+4} = \frac{4}{3}\Omega$$

and
$$R_{OC} = \frac{R_{BC}\times R_{AC}}{R_{AC}+R_{AB}+R_{BC}} = \frac{4\times 12}{12+8+4} = 2\Omega.$$

These resistances are added with the series resistances originally present in the circuit. Now these resistances 5, 5, 5 make star network. This star network is now converted into a delta as shown in Fig. (*c*).



(*a*) (*b*) (*c*)

(*d*) (*e*) (*f*)

Fig. 18.19. Example 5.

$$R_{ef} = \frac{5\times5+5\times5+5\times5}{5}=15\,\Omega$$

$$R_{eg} = 15\,\Omega \text{ and } R_{fg} = 15\,\Omega.$$

Since these are in parallel with originally present sides of resistance of 10 Ω each, the effective value of the sides of *EFG* are 6 Ω each.

$$\left[\frac{1}{R}=\frac{1}{10}+\frac{1}{15}=\frac{1}{6}\text{ or } R=6\,\text{ohm}\right]$$

This gives two parallel resistances *FG* and (*FE* + *EG*), *i.e.*, 6 and (6 + 6). The combination 12 × 6/(12 + 6) = 4 Ω acts across the battery of voltage 20 volts. Hence the current drawn

$$I = V/R = 20/4 = 5 \text{ amp}.$$

Example 6. *Calculate all voltages and currents in the network shown in Fig. 18.20, using node-voltage analysis.*

To calculate voltages and currents, we first find the node voltage V_B from *B* to *D*. Let I_1, I_2 and I_3 be the currents through the resistors R_1, R_2 and R_3 respectively. The equation of currents at node *B* is.

$$I_1 + I_2 = I_3. \qquad \text{...}(i)$$

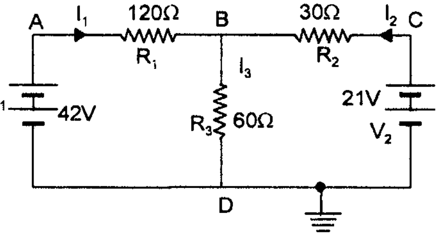


Fig. 18.20. Example 6.

If V_{R1} and V_{R2} be the potential differences across the resistors R_1 and R_2 respectively, we have

$$I_1 = \frac{V_{R_1}}{R_1}=\frac{V_{R_1}}{120}, I_2=\frac{V_{R_2}}{R_2}=\frac{V_{R_2}}{30} \text{ and } I_3=\frac{V_{R_3}}{R_3}=\frac{V_{R_3}}{60}.$$

Substituting the values of I_1, I_2 and I_3 in the equation (*i*), we get

$$\frac{V_{R_1}}{120.}+\frac{V_{R_2}}{30} = \frac{V_{R_3}}{60} \qquad \text{...}(ii)$$

Since $V_{R_3} = V_B$ the node voltage at node *B*, therefore we get

$$V_1 = V_{R_1}+V_B \quad \text{or} \quad V_{R_1} = V_1 - V_B = 42 - V_B \qquad \text{...}(iii)$$

and

$$V_2 = V_{R_2}+V_B \quad \text{or} \quad V_{R_2} = V_2 - V_B = 21 - V_B \qquad \text{...}(iv)$$

Thus Eqn. (*ii*) becomes

$$\frac{42-V_B}{120}+\frac{21-V_B}{30} = \frac{V_B}{60}, \text{ or } V_B = 18 \text{ volts}.$$

$$\therefore \quad V_{R_1} = 42-18=24 \text{ volts, and } V_{R_2}=21-18=3 \text{ volts}.$$

This gives I_1 = 24/120 = 0.2 amp, I_2 = 3/30 = 0.1 amp and I_3 = 0.3 amp.

Example 7. *In a network given below, calculate the mesh currents I_1, I_2 and I_3 by method of determinants. Hence obtain the voltages V_1 and V_2 and also the voltage across the 6Ω resistor.*

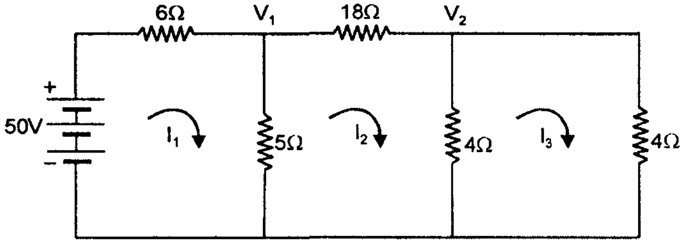


Fig. 18.21. Example 7.

The three equations for the three meshes may be written as

$$50 = (6+5)I_1 - 5I_2 \qquad ...(i)$$

$$0 = -5I_1 + (5+18+4)I_2 - 4I_3 \qquad ...(ii)$$

$$0 = -4I_2 + (4+4)I_3. \qquad ...(iii)$$

These equations may be written in the form of determinants as

$$\begin{vmatrix} 50 \\ 0 \\ 0 \end{vmatrix} = \begin{vmatrix} 11 & -5 & 0 \\ -5 & 27 & -4 \\ 0 & -4 & 8 \end{vmatrix} \begin{vmatrix} I_1 \\ I_2 \\ I_3 \end{vmatrix}$$

The determinant of resistance may be written as

$$\Delta_R = [R] = 11\begin{bmatrix} 27 & -4 \\ -4 & 8 \end{bmatrix} + 5\begin{vmatrix} -5 & -4 \\ 0 & 8 \end{vmatrix} = 11\,(27 \times 8 - 4 \times 4) + 5\,(-40) = 2000.$$

$$\Delta_1 = \begin{vmatrix} 50 & -5 & 0 \\ 0 & 27 & -4 \\ 0 & -4 & 8 \end{vmatrix} = 50\begin{vmatrix} 27 & -4 \\ -4 & 8 \end{vmatrix} = 10000.$$

$$\Delta_2 = \begin{vmatrix} 11 & 50 & 0 \\ -5 & 0 & -4 \\ 0 & 0 & 8 \end{vmatrix} = -50\begin{vmatrix} -5 & -4 \\ 0 & 8 \end{vmatrix} = 2000.$$

$$\Delta_3 = \begin{vmatrix} 11 & -5 & 50 \\ -5 & 27 & 0 \\ 0 & -4 & 0 \end{vmatrix} = 50\begin{vmatrix} -5 & 27 \\ 0 & -4 \end{vmatrix} = 1000.$$

Thus the currents are given by

$$I_1 = \frac{\Delta_1}{\Delta_R} = \frac{10000}{2000} = 5\,\text{amp}, \; I_2 = \frac{\Delta_2}{\Delta_R} = \frac{2000}{2000} = 1\text{ amp. and } I_3 = \frac{\Delta_3}{\Delta_R} = \frac{1000}{2000} = 0.5\text{ amp.}$$

The voltage $V_1 = (I_1 - I_2)\,5 = (5-1) \times 5 = 20$ volts

$V_2 = (I_2 - I_3)\,4 = (1 - 0.5) \times 4 = 2$ volts.

and voltage across $6\Omega = 6I_1 = 6 \times 5 = 30$ volts.

Example 8. *Calculate the current in 6Ω resistance in Fig. 18.22.*

Let I_1 and I_2 be the currents in the meshes *ABEDA* and *BCFEB* respectively as shown in Fig. 18.22.

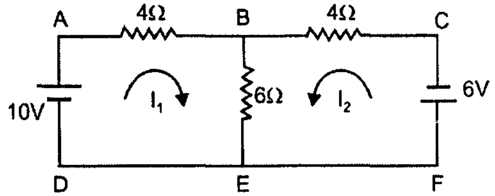


Fig. 18.22. Example 8.

There are two methods of finding current in 6Ω resistance.

(*i*) *Kirchhoff's Laws* : Applying Kirchhoff's first law at point *E*, we get current along $BE = I = I_1 + I_2$. Now applying Kirchhoff's second law to the mesh *ABEDA*

$$I_1 \times 4 + (I_1 + I_2) \times 6 = 10, \text{ or } 10I_1 + 6I_2 = 10 \qquad ...(i)$$

and to the mesh *BCFEB*

$$I_2 \times 8 + (I_1 + I_2) \times 6 = -6 \text{ or } 6I_1 + 14I_2 = -6 \qquad \ldots(ii)$$

Solving equations (*i*) and (*ii*), we get $I_1 = 22/13$ and $I_2 = -15/13$ amp.

∴ Current in the *BE* arm $= I_1 + I_2 = 7/13$ amp.

(*ii*) *Superposition Theorem :* If the second source is replaced by its internal resistance (*i.e.*, zero) and I is the current in *AB* branch, then the external resistance of the whole circuit will be $R = 4 + 6 \times 8/(6 + 8) = 52/7$.

$$\therefore \qquad \text{Current } I = \frac{V}{R} = \frac{10}{52/7} = \frac{35}{26}.$$

Its component along 6Ω resistance

$$I_1' = \frac{8}{8+6} I = \frac{8}{14} \times \frac{35}{26} = \frac{10}{13} \text{ amp.}$$

If the first source is replaced by its internal resistance zero, and the current in the *BC* arm is I, then the external resistance of the current

$$R = 8 + 4 \times 6/(4 + 6) = 52/5 \text{ amp.}$$

$$\therefore \qquad \text{Current } I = \frac{V}{R} = \frac{-6}{52/5} = \frac{-15}{26} \text{ amp.}$$

Its component along the 6Ω resistance

$$I_2' = \frac{4}{10} I = \frac{4}{10} \times \left(\frac{-15}{26}\right) = -\frac{3}{13} \text{ amp.}$$

The superposition theorem gives the total current in 6Ω resistance as

$$I' = I_1' + I_2' = 10/13 - 3/13 = 7/13 \text{ amp.}$$

Example 9. *Calculate the current in branch BC for the network shown in Fig. 18.23, by using superposition theorem.*

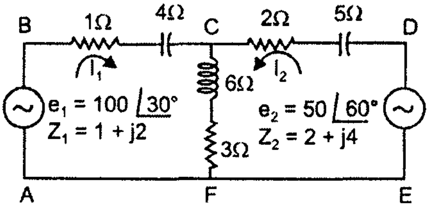


Fig. 18.23. Example 9.

We know that the impedance of capacitor $= -j/\omega C$ and of inductance $= j\omega L$. In our present problem

$$\mathbf{Z}_{11} = 1 + j\,2 + 1 - j\,4 + 3 + j\,6 = 5 + j4,$$
$$\mathbf{Z}_{22} = 2 - j5 + 2 + j4 + 3 + j6 = 7 + j5,$$
$$\text{and} \qquad \mathbf{Z}_{12} = 3 + j6.$$

If I_1' and I_2' be the currents in the left and right meshes respectively when the source from the second mesh is replaced by its impedance. Then we have

$$e_1 = \mathbf{Z}_{11}\mathbf{I}_1' + \mathbf{Z}_{12}\mathbf{I}_2'$$
$$0 = \mathbf{Z}_{12}\mathbf{I}_1' + \mathbf{Z}_{22}\mathbf{I}_2'.$$

$$\text{or} \qquad \mathbf{I}_1' = \frac{e_1 \mathbf{Z}_{22}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} = \frac{100(\cos 30° + j \sin 30°)(7 + j5)}{(5 + j4)(7 + j5) - (3 + j6)^2}$$

$$= \frac{50(\sqrt{3} + j)(7 + j5)}{(15 + j53) - (-27 + j36)} = -\frac{50[7\sqrt{3} - 5 + j(7 + 5\sqrt{3})]}{40 + j17}$$

Similarly the current $\mathbf{I}_1''$ in the left hand mesh, when the source from this mesh is replaced by its impedance, is given by

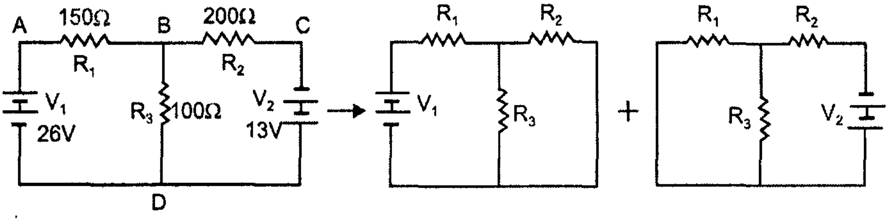
$$\mathbf{I}_1'' = \frac{-e_2\mathbf{Z}_{12}}{\mathbf{Z}_{11}\mathbf{Z}_{22}-\mathbf{Z}_{12}^2} = \frac{-50(\cos 60°+j\sin 60°)(3+j6)}{42+j17}$$

$$= -\frac{25(1+j\sqrt{3})(3+j6)}{42+j27} = -\frac{25[3-6\sqrt{3}+j(3\sqrt{3}+6)]}{42+j17}$$

By the superposition theorem, the current

$$\mathbf{I}_1 = \mathbf{I}_1' + \mathbf{I}_2'' = \frac{500\sqrt{3}-325+j(200+175\sqrt{3})}{42+j17} = 15.3+j5.8.$$

Example 10. *Determine the potential difference between B and D points of a circuit. [Fig. 18.24(a)] acting as voltage divider with two sources.*



(a) (b) (c)

Fig. 18.24. Example 10.

To determine the potential difference between B and D points, the use is made of superposition theorem. Both the voltage sources are ideal as their internal resistances are zero. Thus the current in the branch BD, when the second source V_2 is replaced by a short circuit [Fig. (*b*)] is obtained as under :

Since R_2 and R_3 are parallel and their equivalent resistance of $R_2\,R_3/(R_2 + R_3) = 100 \times 200/300 = 200/3\Omega$ is in series with the resistance $R_1 = 150\Omega$. The total resistance $R' = 150 + 66.66 = 216.66\ \Omega$. Thus the current I' due to a potential source $V_1 = 26V$ is $26/216.66\ \Omega = 0.12$ amp. This current is shared by R_2 and R_3 resistances connected in parallel. From the law of division of current in the branches of parallel circuit, we get

$$I_{BD} = \frac{R_2}{R_2+R_3}I' = \frac{200}{200+100}\times(0.12) = 0.08 \text{ amp.}$$

For the second part when the first source V_1 is replaced by a short circuit [Fig. (*c*)], we have R_1 and R_3 in parallel and their equivalent resistance $R_1R_3/(R_1 + R_3) = 150 \times 100/250 = 60\Omega$. The current I'' due to the potential source $V_2 = -13/260 = -0.05$ amp. It contributes to the resistance R_3 a current

$$I_{BD} = \frac{R_1}{R_1+R_3}I'' = \frac{150}{150+100}\times(-0.05) = -0.03 \text{ amp.}$$

Thus the current in the BD branch due to both the sources = 0.08 – 0.03 = 0.05 amp.

∴ Potential difference across the branch $BD = 100 \times 0.05 = 5V$.

Example 11. *Make use of the network shown in Fig. 18.25, calculate the current in branch EF and prove reciprocity theorem by actually calculating the current in AB when the same potential is removed from branch AB and inserted in branch EF.*

Mesh equations for currents are given as

$$e = \mathbf{Z}_{11}\mathbf{I}_1 + \mathbf{Z}_{12}\mathbf{I}_2.$$

$$0 = \mathbf{Z}_{12}\mathbf{I}_1 + \mathbf{Z}_{22}\mathbf{I}_2.$$

where $\mathbf{Z}_{11} = 2 - j2 + 2 - j8 = 4 - j10.$

$\mathbf{Z}_{22} = 2 + j4 + 1 - j4 + 2 - j8 = 5 - j8$

and $\mathbf{Z}_{12} = 2 - j8.$

The current in second mesh is given by

$$\mathbf{I}_2 = \frac{-e\mathbf{Z}_{12}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} = \frac{-100(2-j8)}{(4-j10)(5-j8)-(2-j8)^2}$$

$$= 100\,(2 - j8)/(-50j) = -16 - j4$$

Similarly the current in first mesh, when its source is introduced in the second mesh is given by

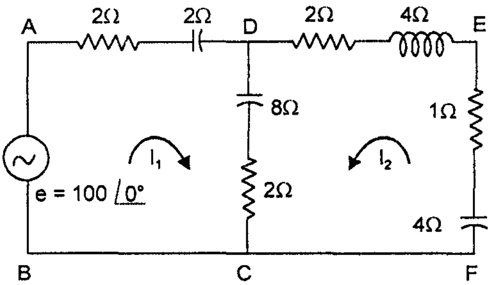


Fig. 18.25. Example 11.

$$\mathbf{I}_1 = \frac{-e\mathbf{Z}_{12}}{\mathbf{Z}_{11}\mathbf{Z}_{22} - \mathbf{Z}_{12}^2} = -16 - j4.$$

Hence the reciprocity theorem is proved.

Example 12. *Determine the current in the load resistance R_L of the ladder-network (Fig. 18.26), using Thevenin's theorem.*

This problem may be solved by a repeated application of Thevenin's theorem. As a first step, let us treat the network to the right of R_2 as the load resistance. The comparison of the left side of the network with the Fig. 18.5 gives us.

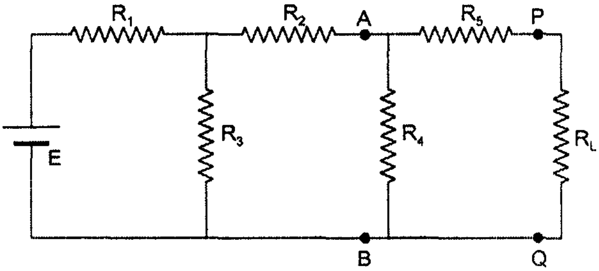
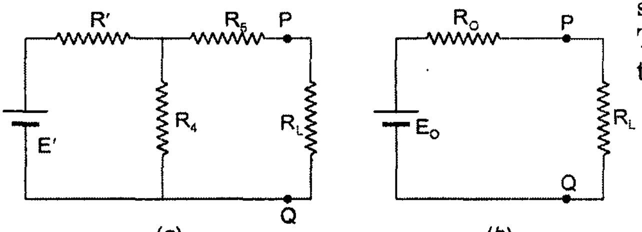


Fig. 18.26. Ladder network.

Open circuit voltage across $AB = ER_3/(R_1 + R_3) = E'$ (says) and the total resistance between terminals A and B when the generator is replaced by its internal resistance (which is zero in this

$$= R_2 + R_3 R_1/(R_1 + R_3) = R' \text{ (say)}$$



(a) (b)

Fig. 18.27. Example 12.

The use of Thevenin's theorem thus reduces the original circuit to that of Fig. 18.27(*a*). To solve the problem, let us again apply Thevenin's theorem to this circuit. For this the open circuit voltage across PQ is given by

$$E_0 = \frac{E'R_4}{R'+R_4} = \frac{E R_3 R_4/(R_1+R_3)}{(R_2+R_4)+R_1 R_3/(R_1+R_3)}$$

$$= \frac{E R_3 R_4}{(R_1+R_3)(R_2+R_4)+R_1 R_3}$$

and the total resistance between the terminals P and Q when generator is replaced by its internal resistance

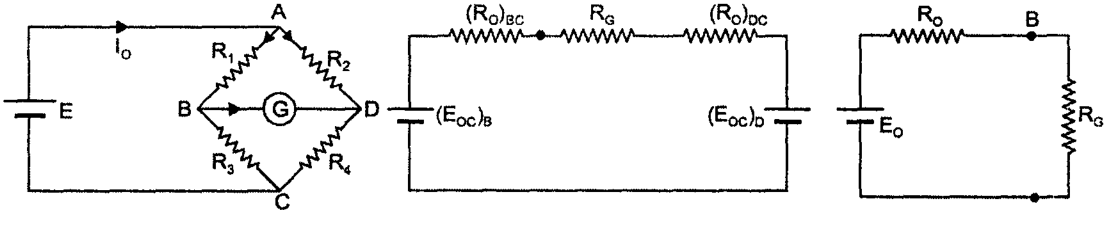
$$R_0 = R_5 + \frac{R' R_4}{R'+R_4} = R_5 + \frac{R_4[R_1 R_2 + R_1 R_3 + R_2 R_3]}{[R_1 R_2 + R_1 R_3 + R_2 R_3 + (R_1+R_3)R_4]}$$

Here E_0 and R_0 are called Thevenin's components. The equivalent circuit is as shown in Fig. 18.27(*b*). Thus the current in the load resistance

$$I_L = \frac{E_0}{R_0+R_L} = \frac{ER_3R_4}{(R_5+R_L)[(R_1+R_3)(R_2+R_4)+R_1R_3]+R_4(R_1R_2+R_1R_3+R_2R_3)}.$$

Example 13. *Find the current in the galvanometer arm of a Wheatstones bridge using the Thevenin's theorem.*

Since the Thevenin's theorem can determine the current flowing through the load resistance connected between the two output terminals of a network. Therefore we assume the galvanometer as a load of resistance $R_L = R_G$.



(*a*) (*b*) (*c*)

Fig. 18.28. Example 13.

Let us first apply Thevenin's theorem to the point B on the bridge. The open circuit voltage at point B, *i.e.*, when R_G is removed

$$(E_{OC})_B = ER_3(R_1+R_3)$$

and the resistance across B and C when the source is short circuited, is

$$(R_0)_{BC} = R_1R_3/(R_1+R_3).$$

Similarly the application of Thevenin's theorem to the point D gives

$$(E_{OC})_D = ER_4/(R_2+R_4)$$

and $$(R_0)_{DC} = R_2R_4/(R_2+R_4).$$

Thus the bridge is reduced to a circuit shown in Fig. 18.28(*b*). If the current passes in the galvanometer arm from point B to D, this circuit will be further reduced to the form shown in Fig. 18.28(*c*), where

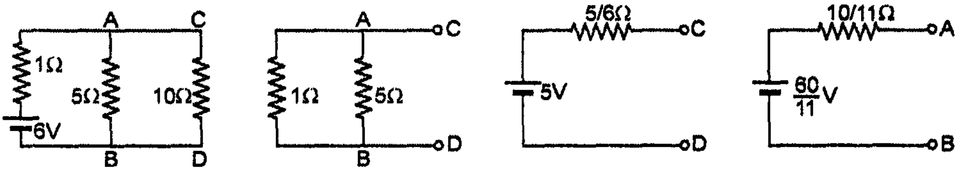
$$E_0 = (E_{OC})_B - (E_{OC})_D$$

and $$R_0 = (R_0)_{BC} + (R_0)_{DC}$$

∴ Current $$I_g = E_0/R_0.$$

If $(E_{OC})_D > (E_{OC})_B$, the current I_g will flow in opposite direction. If $(E_{OC})_D = (E_{OC})_B$ the current will be zero and the bridge will be balanced.

Example 14. *A 6 volt battery having internal resistance of 1 Ω is connected to parallel combination of 5 Ω and 10 Ω Calculate the current through these resistances, using Thevenin's theorem.*



(*a*) (*b*) (*c*) (*d*)

Fig. 18.29. Example 14.

To find the current through 10Ω resistance, let us first calculate the open circuit voltage across C and D. When CD arm is open circuited, the current flowing through 5Ω resistance $I = 6/(1 + 5) = 1$ amp. Thus the voltage across AB becomes $5 \times 1 = 5$ volts. It will be same as the required open circuit voltage, E_0.

When the battery is replaced by its internal resistance, [Fig. (*b*)], the total resistance between the terminals C and D is given by

$$1/R_0 = 1/1 + 1/5 \text{ or } R_0 = 1 \times 5/(1 + 5) = 5/6 \text{ ohm.}$$

Thus the Thevenin's equivalent circuit, as shown in Fig. (*c*), has components E_0 (= 5 volts) and R_0 (= 5/6 ohm). The current in the load resistance (10 ohms), when connected across C and D is given as

$$I_{CD} = \frac{5}{5/6+10} = \frac{6}{13} \text{ amp.}$$

Proceeding exactly as above, we can find the current through 5Ω resistance.

The open circuit voltage E_0' across A and B is obtained first by calculating current through 10Ω resistance when AB arm is open circuited. This current $I' = 6/(1 + 10) = 6/11$ amp. Therefore the open circuit voltage across A and B is

E_0' = Voltage across the 10Ω resistance = $\frac{6}{11} \times 10 = \frac{60}{11}$ volt.

When the battery is replaced by its internal resistance, the total resistance between the terminals A and B is

$$R_0' = 1 \times 10/(1 + 10) = 10/11 \text{ ohm.}$$

Thus the Thevenin's equivalent circuit, as shown in Fig. (*d*) has components $E_0' = 60/11\ V$ and $R_0' = 10/11\Omega$.

Thus the current in the AB arm $I_{AB} = \dfrac{60/11}{10/11+5} = \dfrac{12}{13}$ amp.

Example 15. *Convert linear network Fig. 18.7(a) with $Z_1 = 2$, $Z_2 = 4$, $Z_3 = 3$, $Z_L = 5\Omega$ and $E = 10$ volts into Thevenin's equivalent network and then into Norton's equivalent circuit. Show that the power delivered to the load in both the cases is same.*

For converting this network in Thevenin's equivalent, let us calculate, Thevenin's components E_0 and Z_0 as

$$E_0 = \frac{EZ_3}{Z_1+Z_3} = \frac{10\times3}{2+3} = 6 \text{ volts.}$$

and

$$Z_0 = Z_2 + \frac{Z_1 Z_3}{Z_1+Z_3} = 4 + \frac{2\times3}{2+3} = 5.2 \text{ ohms.}$$

$\therefore$Power delivered to the load $Z_L = I_L^2 Z_L = \left(\dfrac{6}{5.2+5}\right)^2 \times 5 = 1.73$ watts.

The Thevenin's equivalent network is shown in right part of the Fig. 18.7. Now applying Norton's theorem, we have

$$I_0 = E_0/Z_0 = 6/5.2 = 1.15.$$

The current through the load can be obtained by applying division law in network as

$$I_L = \frac{I_0 Z_0}{Z_0+Z_L} = \frac{6}{5.2+5} = 0.588.$$

∴ Power delivered to the load = $(6/10.2)^2 \times 5 = 1.73$ watts.

which is same as that in the Thevenin's equivalent network.

Example 16. *Calculate the load current in the following network, using Norton's theorem.*

First of all let us calculate Thevenin's components.

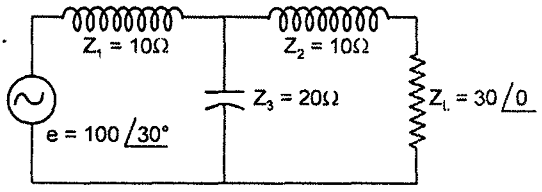


Fig. 18.30. Example 16.

$$\mathbf{e}_0 = \frac{e\mathbf{Z}_3}{\mathbf{Z}_1+\mathbf{Z}_3} = \frac{100/\underline{|30^\circ}}{j10-j20}(-j20) = 200\underline{|30^\circ}.$$

$$\text{and } \mathbf{Z}_0 = \mathbf{Z}_2 + \frac{\mathbf{Z}_1\mathbf{Z}_3}{\mathbf{Z}_1+\mathbf{Z}_3} = j10 + \frac{j10\times(-j20)}{j10-j20} = j30.$$

Using Norton's theorem, we have load current

$$\mathbf{I}_L = \frac{\mathbf{e}_0}{\mathbf{Z}_0+\mathbf{Z}_L} = \frac{200\underline{|30^\circ}}{j30+30} = \frac{200\underline{|30^\circ}}{30\sqrt{2}\underline{|45^\circ}}$$

$$= 4.71\underline{|-15^\circ}$$

Example 17. *A a.c. generator (emf = 40 volts) delivers a maximum power of 10 W to an external load resistance. Calculate the internal resistance of the source and its short-circuit current. How much power will it dissipate when its terminals are shorted ?*

From maximum power theorem, we know that the maximum power will be delivered to the load if the impedance of the network is conjugate to the impedance of the receiving network. In the present problem we have given only pure resistances, hence

Source resistance R_s = External load resistance R_L

and the maximum power delivered to the load

$$P_{L,\,\max} = \frac{E^2}{4R_L} = \frac{E^2}{4R_s} \quad \text{or } R_s = \frac{E^2}{4P_{L,\,\max}}.$$

In the present problem, $E = 40$ volts and $P_{L,\,\max} = 10$ watts.

$$\therefore \qquad R_s = 40^2/4 \times 10 = 40\Omega,$$

The short circuit current of the source $= E/R_s = 40/40 = 1$ amp.

When the source terminals are shorted, *i.e.*, $R_L = 0$, the power dissipated will be

$$P = E^2/R_s = 40^2/40 = 40 \text{ watts.}$$

Example 18. *Equivalent circuit of the operation of the valve 6J 5 is shown in Fig. 18.31. If* $Y_g = 20$, $Y_{gk} = j0.8$, $Y_{gp} = j\,0.7$, $Y_p = 140$, $Y_{pk} = j\,0.9$, $Y_L = 8$. *The admittances are in micro mhos and* $\mu = 20$. *Calculate the output potential.*

Applying the Millman theorem to node P and taking K as the reference node, we have

$$E_L = \frac{-\mu E_g Y_p + E_g Y_{gp}}{Y_p+Y_L+Y_{gp}+Y_{pk}} = \frac{[-20\times140+j\,0.7]E_g}{140+8+j\,0.7+j\,0.9}$$

$$\text{or } \frac{\mathbf{E}_L}{\mathbf{E}_g} = \frac{-2800+j\,0.7}{148+j1.6}. \qquad ...(i)$$

Now applying the theorem to node G, we get

$$\mathbf{E}_g = \frac{e_s Y_g + E_L Y_{gp}}{Y_g+Y_{gp}+Y_{gk}} = \frac{e_s 20 + E_L\, j\,0.7}{20+j\,0.7+j\,0.8} \qquad ...(ii)$$

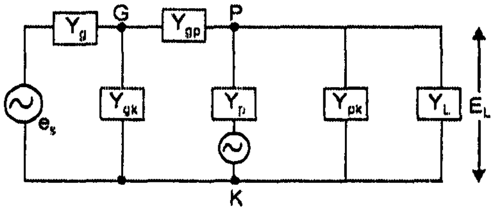


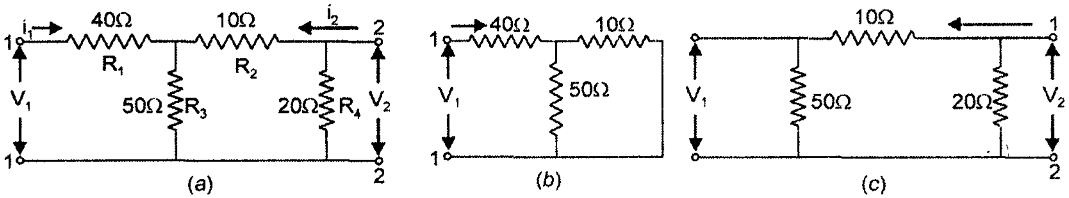
Fig. 18.31. Example 18.

Solving Eqs. (*i*) and (*ii*) we get

$$\mathbf{E}_L = \frac{-2800+j0.7}{140+j1.6}\times\frac{e_s 20+jE_L 0.7}{20+j1.5}$$

$$\therefore \quad = (-12.14+j\,9.09)\,e_s = (15.49\underline{|143^\circ 8'})e_s.$$

Example 19. *Find the values of the 'h' parameters for the circuit of Fig. 18.32.*



(a) (b) (c)

Fig. 18.32. Example 19.

We define the 'h_{11}' parameter as

$$h_{11} = \mathbf{V}_1/\mathbf{I}_1. \qquad (\mathbf{V}_2 = 0)$$

$\mathbf{V}_2 = 0$, if the terminals 2-2 are short circuited (Fig. *b*)

$$\therefore \quad h_{11} = 40+\frac{10\times 50}{10+50} = 40+\frac{500}{60} = 48.3\,\text{ohms.}$$

Since $\quad h_{12} = \mathbf{I}_2/\mathbf{I}_1 \qquad (\mathbf{V}_2 = 0),$

Therefore the ratio of current at $\mathbf{V}_2 = 0$, may be obtained from Fig. (*b*) using the current division formula.

$$-\mathbf{I}_2 = \frac{50}{10+50}\mathbf{I}_1 = \frac{50}{60}\mathbf{I}_1 \qquad \therefore h_{21} = \frac{\mathbf{I}_2}{\mathbf{I}_1} = -0.83.$$

Since $\quad h_{12} = \mathbf{V}_1/\mathbf{V}_2. \qquad (\mathbf{I}_1 = 0).$

Therefore the ratio of input and output voltages with the terminals 1-1 open circuited, may be obtained from Fig. (*c*) using the voltage division formula,

$$\mathbf{V}_1 = \frac{50}{10+50}\mathbf{V}_2 \qquad \therefore h_{12} = \frac{\mathbf{V}_1}{\mathbf{V}_2} = \frac{50}{60} = 0.83.$$

Since $\quad h_{22} = \mathbf{I}_2/\mathbf{V}_2 \qquad (\mathbf{I}_2 = 0),$

The impedance seen at terminals 2, 2 is

$$Z = \frac{20\times(50+10)}{20+(50+10)} = \frac{20\times 60}{80} = 15\Omega.$$

$$\therefore \quad h_{22} = \frac{I_2}{V_2} = \frac{1}{15} = 0.067\text{ mho.}$$

Thus the '*h*' parameters h_{11}, h_{12}, h_{21} and h_{22} are 48.3Ω, 0.83, – 0.83 and 0.067 mho, respectively.

ORAL QUESTIONS

1. What are the differences between the network, loop and mesh ?
2. What will be the current if the output terminals to the voltage source are short-circuited ?
3. Whether the algebraic sum of all voltages around any mesh or any loop is zero ?
4. What do you mean by T and Π-networks. How are they also called star and delta network ?
5. What is the nodal method for solving complicated networks ?
6. What is the uses of star and delta connections ?
7. What do you mean by self and mutual resistances in network problems ?
8. Whether the reciprocity theorem is correct for ac generators in circuits ?
9. What are the main differences between Thevenin and Norton's equivalent circuits.

10. How is a voltage source converted to a current source or vice versa ?
11. If the power lost in the internal generator is maximum what will be the power delivered to the receiving generator ?
12. For what type of circuit is Millman's theorem used ?

PROBLEMS

1. Referring to Fig. 18.1, where $\mathbf{Z}_1 = 45\Omega$, $\mathbf{Z}_2 = 12\Omega$, $\mathbf{Z}_3 = 18\Omega$, $\mathbf{Z}_4 = 15\Omega$; $\mathbf{Z}_5 = 40\Omega$, $\mathbf{Z}_6 = 60\Omega$, $\mathbf{Z}_7 = 10\Omega$, $\mathbf{Z}_8 = 50\Omega$ and $E_0 = 12$ volts, determine (*a*) feeder current and (*b*) the voltage drop across *the* 6Ω resistor. *(0.224 amp, 0.336 volt)*
2. Calculate the current in the battery and galvanometer arms, of a Wheatstone's bridge network having values $R_1 = 10\Omega$, $R_2 = 20\Omega$, $R_3 = 30\Omega$, $R_4 = 40\Omega$ and $R_g = 50\Omega$ with the delta-star transformation method. The d.c. source is a battery of emf 2 volts and internal resistance 1Ω. *(91.2 mA, 2.5 mA)*
3. Calculate the resistance between the points *A* and *B* of a triangular pyramid *ABCD*, built up of six wires, whose resistances are as follows; *AB* = *AC* = *DC* = *DB* = 1Ω and *CB* = *AD* = 2Ω. *(7 / 12Ω)*
4. Determine the ammeter reading in Fig. 18.33, using (*a*) loop current analysis and (*b*) node voltage analysis. (10.4 amp)
5. Two batteries ($E_1 = 2.05\ V$, $r_1 = 0.05\Omega$; $E_2 = 2.15V$, $r_2 = 0.04\Omega$, are connected in parallel and the combined network is again in parallel with a resistance of 1Ω. Calculate the current in first battery branch by the principle of superposition. *(0.18 amp)*
6. Determine the current through $R_3 = 15\Omega$ of the network shown in Fig. 18.20, where $V_1 = 10V$, $V_2 = 20V$, $R_1 = 10\Omega$, $R_2 = 30\Omega$, using Thevenin's Theorem, Norton's Theorem and Superposition Theorem. Compare the results. *(5 / 9 amp)*
7. Determine the currents in the various meshes of a ladder network Fig. 18.26, by the method of determinants. Given that $E = 30V$, $R_1 = R_2 = R_3 = 4\Omega$ and $R_4 = R_5 = 5\Omega$ and $R_L = 10\Omega$. *(4.52, 1.54, 0.38 amp)*
8. Calculate the currents in the branches *AB* (= 4Ω), *BC* (= 10Ω) and *BE* (= 6Ω) by using the superposition theorem, in a network in which the positive terminal of a battery of 20 volts and internal resistance of 2Ω is connected through the resistances *AB* and *BC* with the positive terminal of the another battery ($E = 1V$ and $r = 2\Omega$). The negative terminals of these batteries are joined with the end *E* of the resistance *BE*. *(1.50, 0.33, 1.83 amp)*
9. Calculate the current in branch *BC for the network shown in Fig. 18.23 if* $e_1 = 100\underline{|0^\circ}$, $Z_1 = 1 + j3$, $e_2 = 50\angle 30^\circ$ $Z_2 = 1 + j5$, in *BC* arm $R = 1\Omega$, $C = 4\Omega$, in *CF* arm $R = 20\Omega$, $C = 8\Omega$ and in *CF* arm $R = 2\Omega$ and $L = 4\Omega$. *(15.77 – j 8.92)*
10. How the Thevenin's theorem can be used to study the charging of a capacitor when a voltage divider arrangement is used and the capacitor itself is used as a load. Determine the time constant and the voltage across the capacitor ($C = 1000\ \mu F$) at the end of the period. Also calculate the currents in R_1 and R_2 at this stage. Given that $E = 25$ volts, $R_1 = 10\Omega$ and $R_2 = 40\ k\Omega$. The capacitor is charged across the resistance R_2. *(8 sec, 12.6V, 1.24 mA, 0.315 mA)*
11. Calculate the open circuit voltage and Thevenin resistance of a simple two-terminal network. In it two resistors 20 and 30Ω and a source of 100 volts are connected in series and the terminals are the ends of resistance 30Ω. *(60 V, 12Ω)*
12. A condenser is connected with a voltage divider arrangement (*i.e.*, a battery of emf $\mathcal{E}$ is connected across the two resistors R_1 and R_2 in series and a capacitor *C* is connected across R_2). Determine the time constant and the voltage across the capacitor at the end of this period. *[$CR_1R_2/(R_1 + R_2)$, $\mathcal{E}R_2/(R_1 + R_2)$]*
13. For the network shown in Fig. 18.30, calculate the current through load impedance Z_L using Norton's theorem if $e_i = 100\angle 0^\circ$ instead of $100\angle 30^\circ$. (4.71 $\angle 45^\circ$.
14. A Π network is composed of resistances, each of 100Ω. It has a 10 volt source at one pair of terminals and a variable resistance R_L at the other pair. Find the value R_L corresponding to the maximum transfer of power and the magnitude of the power. *(50Ω, 125 mW)*
15. Determine the Millman equivalent of the four parallel voltage sources ($\mathcal{E}_1 = 12V$, $R_1 = 0.2\Omega$, $\mathcal{E}_2 = 15V$, $R_2 = 0.3\Omega$, $\mathcal{E}_3 = 14V$, $R_3 = 0.7\Omega$, $\mathcal{E}_4 = 12V$ and $R_4 = 0.4\Omega$). Calculate the current to the load of 10Ω. *(9.8V, 0.082Ω; 0.97 amp)*
16. Determine the equivalent $\mathcal{Y}$ parameters for the two port network in Fig. 18.32.

19

UNITS AND THEIR ABSOLUTE MEASUREMENTS

19.1 SYSTEM OF UNITS

Some thousands of years ago when man wandered from place to place in groups, living on wild grains he gathered and the animals he hunted, there was no need for any system of measurement for trading with his follows. The beginning of civilization came and the systems of measurement developed. With the advance of science during the 17th and 18th century scientists began more and more to rail against the confusion presented by multiplicities of systems of measurement. The units generally discussed at school level are *C.G.S., F.P.S.* and *M.K.S.*

Electrostatic and Electromagnetic Systems : Unlike the other branches of Physics, the study of units in electromagnetism is confusing on account of various systems of electrical units used at various times. To understand the well known theoretical systems of units *electrostatic system of units (esu)* and *electromagnetic system of units* (emu), we start with two force laws.

Coulomb force between two charges $F = C_e(q_1 q_2/r^2)$. ...(1)

Magnetic force between two current carrying wires per unit length

$$dF/dl = C_m\,(2i_1 i_2/r). \qquad ...(2)$$

In the electrostatic system of units, the unit of length, mass and time are chosen in C.G.S. system and we take $C_e = 1$ and $C_m = 1/c^2$. Thus

$$F = \frac{q_1 q_2}{r^2} \text{ and } \frac{dF}{dl} = \frac{2i_1 i_2}{c^2 r}. \qquad ...(3)$$

Thus in the *esu* system, the fundamental unit is the unit of electric charge. It is known as *statcoulomb* and is defined as a charge such that *two such charges at 1 cm apart in air exert*

1 dyne of force upon one another. The esu unit of magnetic field **B** is defined by the equation $\mathbf{F} = q\,(\mathbf{v} \times \mathbf{B})$.

In the electromagnetic system of units, the unit of length, mass and time are also chosen in C.G.S. system, but we take $C_m = 1$ and $C_e = c^2$.

Thus $$F = c^2 \frac{q_1 q_2}{r^2} \text{ and } \frac{dF}{dl} = \frac{2i_1 i_2}{r}, \qquad \text{...(4)}$$

Thus is emu system, the fundamental unit is the unit of electric current. It is known as abampere and is defined as : *Two very long filamentary currents are one abampere each if the force of interaction per unit length at 1 cm separation is 2 dynes per cm.*

In emu-system, the unit of magnetic pole was also defined as the fundamental unit, similar to the unit of charge in *esu*-system. Since monopole does not exist, therefore the unit of current as the fundamental unit in emu-system is prefered.

As the quantities used in electricity and magnetism can not be defined in terms of the fundamental units of mass, length and time/and the fourth fundamental unit is required. In the early works, this fourth unit was introduced in two different ways. Consequently two different systems of electrical units : (*i*) electrostatic system of units (*e.s.u.*) and (*ii*) electromagnetic system of units (*e.m.u.*) were developed. The former was based upon theoretical considerations while the later on the experimental facts. The permittivity of the medium was assumed as the fourth fundamental unit in the former system of units and the permeability in the later. Let us derive other electrical quantities in terms of fundamental units in these two cases.

19.2 ELECTROSTATIC SYSTEM OF UNITS

The force between two charges q_1 and q_2 placed at a distance r apart is given by the relation

$$F = q_1 q_2 / 4\pi\varepsilon r^2, \qquad \text{...(5)}$$

where ε is the permittivity or dielectric constant of the medium in which charges are situated. In the electrostatic system of units ε is taken as the fourth dimension. ε is usually written as ε_0 for empty space. The dimensions of charge can be obtained as

(*i*) $$[q]\,[q] = [F]\,[\varepsilon_0]\,[L^2] = [MLT^{-2}]\,[\varepsilon_0 L^2]$$

or $$[q] = [\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-1}].$$

The dimensions of the other electrical units can be found by using appropriate relations as :

(*ii*) Work done W = charge q × potential difference V.

$$\therefore \quad [V] = \frac{[W]}{[q]} = \frac{[ML^2T^{-2}]}{[\varepsilon_0^{1/2} M^{1/2} L^{3./2} T^{-1}]} = [\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}].$$

The same will be the dimensions for *electromotive force.*

(*iii*) Electric intensity E = Force experienced by a unit charge = F/q.

$$\therefore \quad [E] = \frac{[F]}{[q]} = \frac{[MLT^{-2}]}{[\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-1}]} = [\varepsilon_0^{-1/2} M^{1/2} L^{-1/2} T^{-1}].$$

(*iv*) Electric induction or displacement $D = \varepsilon_0 E$.

$$\therefore \quad [D] = [\varepsilon_0]\,[E] = [\varepsilon_0^{1/2} M^{1/2} L^{-1/2} T^{-1}].$$

(*v*) Electric flux Φ = Electric field × Area

$$[\Phi] = [\varepsilon_0^{-1/2} M^{1/2} L^{-1/2} T^{-1}]\,[L^2] = [\varepsilon_0^{-1/2} M^{1/2} L^{3/2} T^{-1}].$$

(*vi*) Electric current i = Rate of flow of charge = q/t.

$\therefore \quad [i] = [q]/[T] = [\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-2}].$

(*vii*) Resistance R = Emf/current = V/i.

$$\therefore \quad [R] = \frac{[V]}{[i]} = \frac{[\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-2}]} = [\varepsilon_0^{-1} L^{-1} T].$$

(*viii*) Capacitance C = Charge q/potential difference V

$$\therefore \quad [C] = \frac{[q]}{[V]} = \frac{[\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-1}]}{[\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]} = [\varepsilon_0 L].$$

(*ix*) Inductance L = Emf V/Rate of change of current (di/dt)

$$\therefore \quad [L] = \frac{[V]}{[i]\,[T^{-1}]} = \frac{[\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-2}]\,[T^{-1}]} = [\varepsilon_0^{-1} L^{-1} T^2]$$

(*x*) Magnetic field intensity $H = idl \sin\theta/r^2$.

$\therefore \quad [H] = [i]\,[L]/[L]^2 = [\varepsilon_0^{1/2} M^{1/2} L^{1/2} T^{-2}].$

(*xi*) Magnetic induction B = Induced emf/lv

$\therefore \quad [B] = [\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]/[L][LT^{-1}] = [\varepsilon_0^{-1/2} M^{1/2} L^{-3/2}].$

(*xii*) Magnetic moment M = couple/$B \sin\theta$

$\therefore \quad [M] = [ML^2T^{-2}]/[\varepsilon^{-1/2} M^{1/2} L^{-3/2}] = [\varepsilon_0^{1/2} M^{1/2} L^{7/2} T^{-2}]$

(*xiii*)Magnetic pole strength m = Magnetic moment/length of the magnet.

$[m] = [M]/[L] = [\varepsilon_0^{1/2} M^{1/2} L^{5/2} T^{-2}].$

19.3 ELECTROMAGNETIC SYSTEM OF UNITS

The force between two magnetic poles m_1 and m_2, situated at a distance r apart is given by the relation

$$F = \mu m_1 m_2 / 4\pi r^2, \qquad \text{...(6)}$$

where μ is the magnetic permeability of the medium. For the free space $\mu = \mu_0$. As neither μ nor m can be expressed in $[M]$, $[L]$ and $[T]$, hence to represent m, one can assume μ as a fourth dimension. Thus for free space

(*i*) $\quad [F] = [\mu_0]\,[m]\,[m]/[L^2]$

$\therefore \quad [m] = [\mu_0^{-1} F L^2]^{1/2} = [\mu_0^{-1} MLT^{-2} L^2]^{1/2} = [\mu_0^{-1/2} M^{1/2} L^{3/2} T^{-1}].$

The dimensions of other electric quantities can be obtained from fundamental formula as follows :

(*ii*) Magnetic induction B = Force experienced by a unit pole F/m.

$$\therefore \quad [B] = \frac{[F]}{[m]} = \frac{[MLT^{-2}]}{[\mu_0^{-1/2} M^{1/2} L^{3/2} T^{-1}]} = [\mu_0^{1/2} M^{1/2} L^{-1/2} T^{-1}].$$

(*iii*) Magnetic field intensity $H = B/\mu_0$.

$\therefore \quad [H] = [B]/[\mu_0] = [\mu_0^{-1/2} M^{1/2} L^{-1/2} T^{-1}].$

(*iv*) Magnetic flux Φ = Magnetic induction B × Area A.

$\therefore \quad [\Phi] = [B]\,[A] = [\mu_0^{1/2} M^{1/2} L^{-1/2} T^{-1}]\,[L^2] = [\mu_0^{1/2} M^{1/2} L^{3/2} T^{-1}].$

(*v*) Magnetic moment M = Pole strength m × length of the magnet $2l$.

$$\therefore \quad [M] = [m]\,[l] = [\mu_0^{-1/2}M^{1/2}L^{3/2}T^{-1}]\,[L] = [\mu_0^{-1/2}\,M^{1/2}\,L^{5/2}\,T^{-1}].$$

(*vi*) Intensity of magnetisation I = Magnetic moment/volume.

$$\therefore \quad [I] = [M]/[\tau] = [\mu_0^{-1/2}M^{1/2}L^{5/2}T^{-1}]/[L^3] = [\mu_0^{-1/2}\,M^{1/2}L^{-1/2}T^{-1}],$$

(*vii*) Electric current is related with magnetic field as

$$H = i\,dl\,\sin\theta/r^2.$$

$$\therefore \quad [i] = [H]\,[r^2]/[l] = [\mu_0^{-1/2}M^{1/2}\,L^{-1/2}T^{-1}][L^2]/[L]$$

$$= [\mu_0^{-1/2}\,M^{1/2}L^{1/2}T^{-1}].$$

(*viii*) Electric charge q = current i × time t.

$$\therefore \quad [q] = [i]\,(t) = [\mu_0^{-1/2}M^{1/2}\,L^{1/2}\,T^{-1}]\,[T] = [\mu_0^{-1/2}\,M^{1/2}L^{1/2}].$$

(*ix*) Electric field intensity E = Force experienced by a unit charge = F/q

$$\therefore \quad [E] = \frac{[F]}{[q]} = \frac{[MLT^{-2}]}{[\mu_0^{-1/2}M^{1/2}L^{1/2}]} = [\mu_0^{1/2}M^{1/2}L^{1/2}T^{-2}].$$

(*x*) Potential difference V = Work done by a unit charge = W/q.

$$\therefore \quad [V] = \frac{[W]}{[q]} = \frac{[ML^2T^{-2}]}{[\mu^{-1/2}M^{1/2}L^{1/2}]} = [\mu_0^{1/2}M^{1/2}L^{3/2}T^{-2}]$$

(*xi*) Resistance R = Potential difference V/Current i.

$$\therefore \quad [R] = \frac{[V]}{[I]} = \frac{[\mu_0^{1/2}M^{1/2}L^{3/2}T^{-2}]}{[\mu_0^{-1/2}M^{1/2}L^{1/2}T^{-1}]} = [\mu_0 LT^{-1}].$$

(*xii*) Capacitance $C = q/V$.

$$[C] = [\mu_0^{-1/2}\,M^{1/2}\,L^{1/2}]/[\mu_0^{1/2}\,M^{1/2}\,L^{3/2}\,T^{-2}] = [\mu_0^{-1}L^{-1}T^2].$$

(*xiii*) Inductance L = emf V/Rate of change of current di/dt.

$$\therefore \quad [L] = \frac{[V]}{[i]/[t]} = \frac{[\mu_0^{1/2}M^{1/2}L^{3/2}T^{-2}]}{[\mu_0^{-1/2}M^{1/2}L^{1/2}T^{-2}]} = [\mu_0 L].$$

19.4 COMPARISON BETWEEN THE ELECTROSTATIC AND ELECTROMAGNETIC SYSTEMS OF UNITS

On comparing the dimensions of any electric quantity in both the systems of units, it is seen that they differ. It is impossible to have different dimensions for the same electric quantity. Thus one is bound to consider dimensions of μ_0 and ε_0.

If we compare dimensions of charge q obtained in both the systems of units, then

In *e.s.u.* $[\varepsilon_0^{1/2}\,M^{1/2}\,L^{3/2}\,T^{-1}]$ = In *e.m.u.* $[\mu_0^{-1/2}\,M^{1/2}\,L^{1/2}]$.

$$\therefore \quad \mu_0^{-1/2}\,\varepsilon_0^{-1/2} = [LT^{-1}], \qquad \text{...(7)}$$

which are the dimensions of velocity. The magnitude of this velocity can be found by measuring a given charge first in *e.m.u.* and then in *e.s.u.* The ratio of these values is found nearly equal to the velocity of light. Thus we see that

$$[q]_{emu}/[q]_{esu} = \text{velocity of light } c.$$

Similarly we have :

$$(i)\quad \frac{C_{emu}}{C_{esu}} = \frac{[\mu_0^{-1}L^{-1}T^2]}{[\varepsilon_0 L]} = \mu_0^{-1}\varepsilon_0^{-1}\,[L^{-2}T^2] = c^2.$$

$$(ii)\quad \frac{i_{emu}}{i_{esu}} = \frac{[\mu_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{3/2} T^{-2}]} = \mu_0^{-1/2} \varepsilon_0^{-1/2} [LT^{-1}]^{-1} = c.$$

$$(iii)\quad \frac{V_{emu}}{V_{esu}} = \frac{[\mu_0^{1/2} M^{1/2} L^{3/2} T^{-2}]}{[\varepsilon_0^{-1/2} M^{1/2} L^{1/2} T^{-1}]} = \mu_0^{1/2} \varepsilon_0^{1/2} [LT^{-1}] = c^{-1}.$$

$$(iv)\quad \frac{R_{emu}}{R_{esu}} = \frac{[\mu_0 L T^{-1}]}{[\varepsilon_0^{-1} L^{-1} T]} = \mu_0 \varepsilon_0 [LT^{-1}]^2 = c^{-2}.$$

$$(v)\quad \frac{L_{emu}}{L_{esu}} = \frac{[\mu_0 L]}{[\varepsilon_0^{-1} L^{-1} T^2]} = \mu_0 \varepsilon_0 [LT^{-1}]^2 = c^{-2}.$$

$$(vi)\quad \frac{H_{emu}}{H_{esu}} = \frac{[\mu_0^{-1/2} M^{1/2} L^{-1/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{1/2} T^{-2}]} = \mu_0^{-1/2} \varepsilon_0^{-1/2} [LT^{-1}]^{-1} = c.$$

$$(vii)\quad \frac{M_{emu}}{M_{esu}} = \frac{[\mu_0^{-1/2} M^{1/2} L^{5/2} T^{-1}]}{[\varepsilon_0^{1/2} M^{1/2} L^{7/2} T^{-2}]} = \mu_0^{-1/2} \varepsilon_0^{-1/2} [LT^{-1}]^{-1} = c$$

$$(viii)\quad \frac{E_{emu}}{E_{esu}} = \frac{[\mu_0^{1/2} M^{1/2} L^{1/2} T^{-2}]}{[\varepsilon_0^{-1/2} M^{1/2} L^{-1/2} T^{-1}]} = \mu_0^{1/2} \varepsilon_0^{1/2} [LT^{-1}] = c^{-1}.$$

Similarly we can compare the values in *emu* and *esu* of other electrical quantities.

19.5 PRACTICAL AND INTERNATIONAL UNITS

The electrostatic units of charge and current and the electromagnetic unit of potential are inconveniently small for practical purpose. The practical units are coulomb for charge, ampere for current and volt for potential. From these are derived the ohm for resistance, watt for power, farad for capacity and henry for inductance.

International conference on Electrical Standards (London, 1908) defined these practical units as given below and thus named as '*International units.*

One *international ampere* is that current which will deposit 0.001118 gm of silver in one sec, in the electrolysis of $AgNO_3$.

One international ohm is the resistance of a column of mercury of length 106.3 cm. at 0°C, of a uniform cross sectional area and 14.452 gm. in mass.

One international volt is defined as the potential difference across 1 international ohm carrying 1 international ampere. It was also defined in terms of emf of the Weston cell, assuming it to be 1.0183 volts.

Absolute measurements have now been made by different methods. Practical units are therefore based on the absolute system of units directly. The absolute units are related by powers of 10 to the electromagnetic units and differ slightly from the international units. Modern instruments are calibrated in absolute practical units. The relationship between the international units and practical units are

1 international ampere = 0.999835 amp.

1 international ohm = 1.000425 ohms.

1 international volt = 1.00033 volts.

19.6 MEASUREMENT OF ABSOLUTE QUANTITIES

1. Determination of resistance in absolute measure by Lorenz Method. Lorenz method is based on the principle of measuring the inductance and time as inductance has dimensions of length in *emu*, and the resistance has dimension $\mu_0 LT^{-1}$ in *emu*.

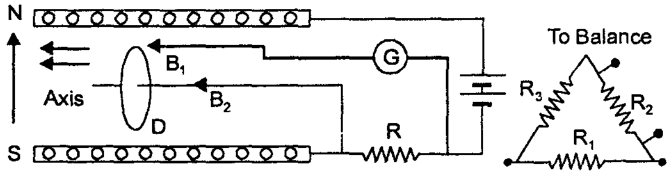


Fig. 19.1. Lorenz method.

In this method, a solenoid of n turns per unit length is placed in east-west direction. A circular disc D is mounted concentrically such that its plane is perpendicular to the magnetic lines of forces. The disc is driven with uniform speed by means of an electric motor. Thus an *emf* is set up between the central axis (axle) and the circumference of the disc. To measure this *emf* two small metal brushes B_1 and B_2 are connected with the two terminals of the experimental resistance R. A battery and a galvanometer are connected as shown in Fig. 19.1.

Let the disc of radius r be rotated through an angle θ, then the area swept out is $\frac{1}{2}\theta r^2$. If B is the flux density, then the flux cutting the disc perpendicularly is $\frac{1}{2}\theta r^2 B$.

If the disc is having m revolutions per sec, the time taken to turn through angle θ is $\theta/2\pi m$ sec.

$$\therefore \text{ emf between the centre and circumference} = \frac{d\Phi}{dt} = \frac{\frac{1}{2}\theta r^2 B}{\theta/2\pi m} = \pi r^2 mB. \qquad \text{...(8)}$$

This is in *emu*, if B is in gauss and r in centimeters.

If i *emu* current is passing through the solenoid, the magnetic flux

$$B = 4\pi\mu_0 ni. \qquad \text{...(9)}$$

The solenoid should be at right angles to the magnetic meridian to eliminate the effect of the earth's magnetic field. Hence

$$\text{Induced emf } \mathcal{E} = \pi r^2\, 4\pi m\mu_0 ni = 4\mu_0\pi^2\, r^2\, nmi \text{ emu}.$$

This *emf* is balanced against the p.d across a resistor R which is in series with the solenoid. Hence for no deflection in galvanometer

Induced *emf* $4\mu_0\pi^2 r^2\, nmi$ = p.d. across the resistance iR

$$\therefore \qquad R = 4\mu_0\pi^2 r^2\, nm. \qquad \text{...(10)}$$

This equation is dependent on n, r and m, which are measured in absolute units of length and time.

The method suffers errors due to the following causes : (*i*) the earth magnetic field, (*ii*) the significant value of the thermo-emf. (*iii*) the requirement of the measurement of disc radius whilst it is moving with a very high speed and (*iv*) the difficulty in electrical connections to the centre and circumference of the rotating disc.

Instead of a long solenoid a coil system is more useful. If M be the mutual inductance between the disc and the coil system, then

$$\begin{aligned} emf &= \text{Rate of change of } Mi = Mim \\ &= \text{p.d. across the resistance at the time of balance} \\ \therefore \qquad R &= mM, \qquad \text{...(11)} \end{aligned}$$

where M the mutual inductance is dependent on the geometrical dimensions. As the resistance in the coil circuit is small, hence Lord Rayleigh and Mrs Sidgwick used an arrangement of three resistors. R_2 was the smallest, whilst R_3 was large compared with R_1. The equivalent resistance was thus

$$R_1 R_2/(R_1 + R_2 + R_3).$$

This method was much improved by F.E. Smith. After a series of experiments the result was stated in the form that the ohm was represented by a column of mercury at 0°C of 14.4446 ± 0.0006 gm in mass of 1 sq. mm. cross section area and of length 106.245 ± 0.004 cm.

2. Determination of current in Absolute Units by Rayleigh's Currents balance method. If two current carrying coils are placed in such a way that they are parallel and their axes are coincident, a force will exist between the coils. This force is proportional to the product of the two currents in the coils and can be measured with a balance.

Lord Rayleigh was the first who could measure the current with a highest accuracy. The Rayleigh's balance consists of four stationary coils and two smaller coils which form part of the beam of a balance. The currents through the larger coils are arranged such that the magnetic lines of induction enter the smaller coils at the top and leave at the bottom. Thus the lines of induction are everywhere at right angles to the current and the force will act on the smaller coils (E and F) in vertical direction.

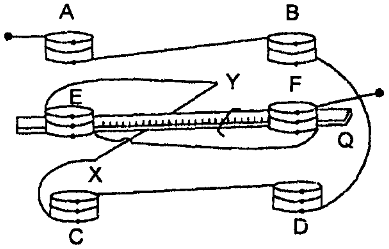


Fig. 19.2. Rayleigh's current balance.

The balance beam is suspended at its mid point by two ribbons of very fine flexible copper wires, one on each side of the beam.

The mechanical force on the small coils is measured by sliding a rider of known weight on the beam. The weight can be calibrated as the square of the current.

The theory of the force exerted is best presented after that of Ayrton Mather and Smith. If a current i is passing through the coils, then

The force in dynes on the small coil = i × number of turns per cm of small coil n × number of lines of induction entering the coil,

$$\text{i.e.,} \quad F = ni\,[\pi r_2^2\,(dB/dx) \times \text{length of the small coil}]$$
$$= i\pi r_2^2 \times (dB/dx) \times \text{total number of the turns on small coil } n_2.$$

where r_2 is the radius of the smaller coil and dB/dx is the average rate of change of the field due to one of the big stationary coils along the line joining the centres.

We know that the magnetic induction B on the axis of a coil of n_1 turns, radius r_1 at a distance x from its centre is

$$B = 2\pi n_1\, i r_1^2/(r_1^2 + x^2)^{3/2}.$$

$$\therefore \quad dB/dx = 2\pi n_1 i r_1^2\,(-3x)/(r_1^2 + x^2)^{5/2}.$$

We know that dB/dx is maximum when $x = \frac{1}{2} r_1$. Hence at this position

$$dB/dx = 96\pi n_1 i/25\sqrt{5}r_1^2.$$ Hence the total force on the small coil is

$$F = \pi r_2^2 n_2 i \times 96\pi n_1 i/25\,\sqrt{5}r_1^2 = 96 \times 5^{-5/2}\,\pi^2 n_1 n_2 i^2\,(r_2^2/r_1^2). \quad \text{...(12)}$$

The value of this force will remain almost independent of the small variations in the vertical position of the small coil. This force also depends on direct counts of the number of turns and on the ratio of the radii of the two coils. In this way current can be measured in terms of force, *i.e.*, the weight placed on the beam and the displacement from the point of suspension. With the improved Rayleigh balance it is possible to compare the currents to an accuracy of two in a million.

19.7 RATIONALISED MKS SYSTEM OF UNITS

In 1901 Giorgi, pursuing an idea originally due to Maxwell, considered that a great simplification of the units in electricity could be brought about by adopting meter and kilogram as the units of length and mass. He pointed out that if a unit of electrical measurement was added as a fourth basic unit, a system could be constructed which would be satisfactory for uniform application in most sciences and technologies. The International Electrotechnical Commission in 1950 recommended ampere as the fourth unit. The important features of this system are as follows :

1. The dyne is so small that 981 dynes act on our hand when we hold one gramme. A unit of force which makes one kilogram to accelerate by one meter/sec^2 is therefore much more reasonable in size. This new unit is called '*newton*'. Similarly unit of work done joule, is somewhat practicable.
2. The outstanding advantage of this system lies in the sphere of electrical units, as it ends the use of *emu* and *esu*. All experiments, whether purely electric, purely magnetic, or a combination use the same units.
3. In most cases the quantities used in the *mks* system are identical with the practical quantities of electricity and magnetism, such as amperes, volts, ohms and farads which are generally used as practical units since a long time.
4. The introduction of a factor 4π into the fundamental equations converts the system into rationlized *mks* system. It makes simpler form to the frequency used derived equations.

The equations expressing Coulomb's law in electricity and magnetism are

$$F = q_1q_2/4\pi\varepsilon_0 r^2 \text{ and } F = \mu_0\, m_1m_2/4\pi r^2.$$

The ampere's law of force between two current elements is

$$F = \mu_0 i_1 i_2\, dl_1\, dl_2/4\pi r^2. \qquad ...(13)$$

These equations show that the introduction of a factor 4π constitutes the process of '*rationalization*'. It is obvious that we can not just put the factor 4π into the constants ε_0, μ_0, as we also have relations without any factor 4π, such as

$$\mathbf{D} = \varepsilon_0\mathbf{E} \text{ and } \mathbf{B} = \mu_0\mathbf{H}. \qquad ...(14)$$

International System of Units (SI). In 1960, International general conference of weight and measures recommended that if the MKSA system were enlarged by the addition of basic units *candela* for light measurement and *kelvin* for temperature, then this would be acceptable for all branches of science. This new system was named as **International system of units,** abreviated as *SI*. Instead of six basic units (meter, kilogramme, second, ampere, candela, kelvin), two supplementary units are also used.

radian (rad) for plane angle, steradian(sr) for solid angle,

SI is rational and coherent. It is rational in that it admits only one measurement unit for any one physical quantity. It is coherent in that every unit used in science can be derived from six basic units and two supplementary units.

Dimensions of Electrical Units in the MKSA system

Charge : q = current × time = ampere sec = coulomb

$$[q] = [i] \times [T] = [IT].$$

Potential Difference : V = work done per unit charge = joule/coul = volt.

$$[V] = \frac{[W]}{[q]} = \frac{[ML^2T^{-2}]}{[IT]} = [I^{-1}ML^2T^{-3}]$$

Electric field strength : $E = -dV/dx$ = volt/meter

$$[E] = [V]/[L] = [I^{-1}MLT^{-3}]$$

Electric displacement : $\int \mathbf{D} \cdot d\mathbf{S} = q$ or D = coulomb/meter^2.

$$[D] = [q]/[L^2] = [IL^{-2}T]$$

$$\textit{Permittivity} : \varepsilon_0 = \frac{D}{E} = \frac{\text{coulomb/meter}^2}{\text{volt/meter}} = \frac{\text{coulomb}}{\text{volt.meter}}$$

$$[\varepsilon_0] = \frac{[D]}{[E]} = \frac{[IL^{-2}T]}{[I^{-1}MLT^{-3}]} = [I^2M^{-1}L^{-3}T^4].$$

Resistance : R = Potential/current = volt/ampere = ohm.

$$[R] = [V]/[I] = [I^{-2}ML^2T^{-3}].$$

Table 19.1. Electromagnetic Quantities and their units

Physical concept	*S.I. units*	*Gaussian units*	*S.I. / Gaussian*
Current i	Ampere (A)	Statampere	3×10^9
		Abampere	10^{-1}
Charge q	Coulomb (C) = As	Statcoulomb	3×10^9
Electric field intensity E	Newton/coulomb = volt/meter	Dyne/statcoulomb = Stat volt/cm.	$1/3 \times 10^4$
Potential V, emf $\mathcal{E}$	Volt (V) = Joule/coul	Statvolt = erg/stat coul.	1/300
Capacitance C	Farad (F) = coul/volt	Statfarad, cm	9×10^{11}
Electric dipole moment p	Meter-coulomb	Cm-stat coul	3×10^{11}
Electric polarization P	Coul./m^2	Stat coul/cm^2 = Stat volt/cm.	3×10^5
Permittivity ε, ε_0	Coul^2/newton m^2 = farad/meter	Not used	.—
Displacement D	Coul./m^2	Dyne/stat coul.	$12\pi \times 10^5$
Resistance R	Ohm Ω = volt/amp	Stat ohm = stat volt/stat amp.	$1/9 \times 10^{11}$
Resistivity ρ	Ohm-meter	Stat ohm-cm	$1/9 \times 10^9$
Magnetic flux Φ	Weber (Wb)	Maxwell, gauss-cm^2	10^8
Magnetic induction B	Weber/m^2 = Tesla (T) = newton/amp. m	Gauss = μ_0 (oersted)	10^4
Permeability μ, μ_0	Weber/m.amp = henry/m. = newton/amp^2.	Gauss/oersted	$10^7/4\pi$
Inductance L	Henry (H)	Erg/stat amp^2	9×10^{11}
	= web./amp.	Stat henry	$1/9 \times 10^{11}$
Magnetic dipole moment p_m	Amp – meter^2 = Weber-meter	Gauss cm^2 = erg/Oersted = erg/gauss	10^3
Magnetization M	Amp/meter	Oersted	10^{-3}
Magnetic intensity H	Amp/meter	Oersted	$4\pi \times 10^{-3}$
Pole strength g_m	Amp. meter	Gauss cm^2	$(10^8/4\pi)$
Conductance Y	Mho = amp/volt.	Stat mho	9×10^{11}
Magnetic vector Potential A	Weber/meter	Gauss cm	10^6
Conductivity	Mho/meter	Stat mho/cm	9×10^{-9}

Capacitance : C = Charge/Potential = coulomb/volt = farad.

$$[C] = \frac{[q]}{[V]} = \frac{[IT]}{[I^{-1}ML^2T^{-3}]} = [I^2M^{-1}L^{-2}T^4]$$

Inductance : L = Induced emf/(di/dt) = volt sec/amp = henry.

$$[L] = [V]/[IT^{-1}] = [I^{-2}ML^2T^{-2}].$$

Magnetic field strength : $\int \mathbf{H} \cdot d\mathbf{l} = i$ or H = ampere turn/meter.

$$[H] = [I]/[L] = [IL^{-1}]$$

Magnetic flux : Induced emf $\mathcal{E}$ = –Rate of change of flux ($d\Phi/dt$).

$$\therefore \quad \Phi = \text{volt. sec per turn} = \text{weber}.$$

$$[\Phi] = [V]\,[T] = [I^{-1}ML^2T^{-2}].$$

Magnetic Induction B = magnetic flux/area = weber/meter^2.

$$[B] = [\Phi]/[L]^2 = [I^{-1}MT^{-2}].$$

Table 19.2. Equations of electromagnetism for SI and Gaussian units

Equations content	*Equation for SI units*	*Equation for Gaussian units*
Coulomb's law	$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1q_2\hat{r}}{r^2}$	$F = \frac{q_1q_2}{r^2}\hat{r}$
Gauss's law	$\oint \mathbf{E} \cdot d\mathbf{S} = q/\varepsilon_0$ or $\nabla \times \mathbf{E} = \rho/\varepsilon_0$	$\oint \mathbf{E} \cdot d\mathbf{S} = 4\pi q$ $\nabla \times \mathbf{E} = 4\pi\rho$
Electric field near a conductor	$E = \sigma/\varepsilon_0$	$E = 4\pi\sigma$
Potential of point charge	$V = q/4\pi\varepsilon_0 r$	$V = q/r$
Magnetic force on a moving charge	$\mathbf{F} = q(\mathbf{v} \times \mathbf{B})$	$\mathbf{F} = \frac{q}{c}(\mathbf{v} \times \mathbf{B})$
Magnetic force on a current element	$d\mathbf{F} = i(d\mathbf{l} \times \mathbf{B})$	$d\mathbf{F} = \frac{i}{c}(d\mathbf{l} \times \mathbf{B})$
Force per unit length on parallel current	$\frac{dF}{dl} = \frac{\mu_0}{2\pi}\frac{i_1i_2}{d}$	$\frac{dF}{dl} = \frac{2i_1i_2}{c^2d}$
Magnetic field created by a moving charge	$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{q}{r^3}(\mathbf{v} \times \mathbf{r})$	$\mathbf{B} = \frac{q}{cr^3}(\mathbf{v} \times \mathbf{r})$
Magnetic field created by a current element	$d\mathbf{B} = \frac{\mu_0 i}{4\pi r^3}(d\mathbf{l} \times \mathbf{r})$	$d\mathbf{B} = \frac{i}{cr^3}(d\mathbf{l} \times \mathbf{r})$
Magnetic field of long wire	$B = \mu_0 i/2\pi r$	$B = 2i/cr$
Magnetic field within a long solenoid	$B = \mu_0 ni$	$B = 4\pi ni/c.$
Faraday's law of electromagnetic induction	$\int \mathbf{E} \cdot d\mathbf{l} = \frac{d\Phi}{dt}$ or $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\int \mathbf{E} \cdot d\mathbf{l} = \frac{1}{c}\frac{d\Phi}{dt}$ $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$
Energy density of electromagnetic force	$u = \frac{1}{2}\varepsilon_0 E^2 + \frac{B^2}{2\mu_0}$	$u = \frac{1}{8\pi}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$
Magnetic moment of circling particle	$\mathbf{m} = \frac{q}{2m}\mathbf{L}$	$\mathbf{m} = \frac{q}{2mc}\mathbf{L}.$
Magnetic moment of current loop	$m = iA$	$m = iA/c$
Electric displacement	$\mathbf{D} = \varepsilon_0\mathbf{E} + \mathbf{P}$	$\mathbf{D} = \mathbf{E} + 4\pi\mathbf{P}$
Magnetic intensity	$\mathbf{H} = \mathbf{B}/\mu_0 - \mathbf{M}$	$\mathbf{H} = \mathbf{B} - 4\pi\mathbf{M}$
Poynting vector	$\mathbf{S} = (\mathbf{E} \times \mathbf{B})/\mu_0$	$\mathbf{S} = \frac{c}{4\pi}(\mathbf{E} \times \mathbf{H})$

$$\textit{Permeability}\ \mu_0 = \frac{B}{H} = \frac{\text{volt sec/meter}^2}{\text{ampere/meter}} = \frac{\text{volt.sec}}{\text{amp.meter}} = \frac{\text{henry}}{\text{meter}}.$$

$$[\mu_0] = \frac{[B]}{[H]} = \frac{[I^{-1}MT^{-2}]}{[IL^{-1}]} = [I^{-2}MLT^{-2}].$$

$$[\varepsilon_0\mu_0]^{1/2} = [I^2M^{-1}L^{-3}T^4I^{-2}MLT^{-2}]^{1/2} = [L^{-2}T^2]^{1/2} = [LT^{-1}]^{-1}.$$

19.8 FUNDAMENTAL PHYSICAL CONSTANTS

In the following table the unified atomic mass scale ($^{12}C \cong 12$) has been used and the values are given for *SI* units with the following abbreviations :

C = coulomb, *m* = meter *T* = tesla
J = joule mol = molecules *u* = atomic mass unit
K = degree kelvin *N* = newton
kg = kilogram *s* = second

Constant	*Symbol*	*Value in SI units*
Avogadro's number	N_A	6.022045×10^{23} mol^{-1}
Bohr magneton	$\mu_B = e\hbar/2m_e$.	$9.274078 \times 10^{-24}\ JT^{-1}$
Boltzmann constant	k	$1.380662 \times 10^{-23}\ JK^{-1}$
Electron rest mass	m_e	9.109534×10^{-31} kg, $5.48586 \times 10^{-4}\ u$
Elementary charge	e	$1.602192 \times 10^{-19}\ C$
Fine structure constant	$1/\alpha = 4\pi\varepsilon_0\ \hbar c/e^2$	137.03604
First Bohr Orbit	$a_0 = 4\pi\varepsilon_0\hbar^2/m_e e^2$	$5.291771 \times 10^{-11}\ m$
Mass unit (unified mass scale)	u	$1.66057 \times 10^{-27}\ kg$
Neutron rest mass	m_n	$1.00867\ u$
Nuclear magneton	μ_N	$5.05082 \times 10^{-27}\ JT^{-1}$
Permeability of free space	μ_0	$4\pi \times 10^{-7}\ NA^{-2}$
Permittivity of free space	ε_0	$8.854188 \times 10^{-12}\ C^2N^{-1}\ m^{-2}$
Planck constant	h	$6.62618 \times 10^{-34}\ Js$
Proton rest mass	m_p	1.672649×10^{-27} kg, $1.00728\ u$
Rydberg constant	R_∞	$1.09737 \times 10^{7}\ m^{-1}$
Speed of light in vacuum	c	$2.997925 \times 10^{8}\ ms^{-1}$
Faraday constant of electrolysis		$9.64867 \times 10^{7}\ Ck\ mol^{-1}$
Bohr magneton	μ_B	$9.27410 \times 10^{-24}\ JT^{-1}$
Gas constant	R_0	$8.3143 \times 10^{3}\ JK^{-1}\ k$mol^{-1}.
Compton wavelength of electron	λ_c	2.426310×10^{-12} m

Energy Conversion Factors

1 Joule = 10^7 ergs = $6.70043 \times 10^9\ u$ = $6.24146 \times 10^{18}\ eV$
1 *kg* = $6.02205 \times 10^{26}\ u$ = $5.60955 \times 10^{35}\ eV$
1°*K* = $9.25103 \times 10^{-14}\ u$ = $8.61735 \times 10^{-5}\ eV$
1 cal = $2.803460 \times 10^{10}\ u$ = $4.186\ J$ = $2.61143 \times 10^{19}\ eV$
1 *eV* = $1.602192 \times 10^{-19}\ J$,
1 British Thermal Unit (BTU) = 1055 *J*
1 Kilowatt hour (kWh) = $3.6 \times 10^6\ J$.
1 Atomic mass unit *u* = 931.481 MeV

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